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L. SONKE

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**ANALYSIS OF DOMAIN DECOMPOSITION
FOR NON SYMMETRIC PROBLEMS : APPLICATION
TO THE NAVIER-STOKES EQUATIONS (*)**

L. SONKE ⁽¹⁾

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Résumé — Cet article est consacré aux fondements mathématiques d'un algorithme de gradient conjugué pour la solution de problèmes non symétriques par décomposition de domaine, proposé par Sonké et al [5] et appliqué à la simulation d'écoulements laminares dans une cavité en L.

L'analyse est effectuée pour un problème modèle de Dirichlet associé à un opérateur linéaire général du second ordre, non symétrique.

En utilisant une décomposition de domaine sans recouvrement, nous prouvons l'existence et l'unicité de la solution du problème de décomposition-coordination équivalent, au moyen de l'opérateur de Steklov-Poincaré. Une technique de symétrisation est ensuite appliquée au problème de décomposition coordination et permet de construire un algorithme de gradient-conjugué pour le calcul de la solution.

L'application de la méthode à une forme linéarisée des équations de Navier-Stokes est expliquée à la fin de l'article.

D'un point de vue mathématique, les conditions de cette application sont assez restrictives. Cependant, les résultats numériques présentés dans [5] suggèrent que ces conditions ne sont pas optimales. Elles sont donc susceptibles d'amélioration.

Abstract — A Dirichlet problem with a general second order and non symmetric linear operator is solved via a domain decomposition method without overlapping existence and uniqueness of solution for the equivalent decomposition-coordination problem is proved, using Steklov-Poincaré operator. A symmetrization technique is applied to obtain a conjugate gradient algorithm for computation of solution. Application to a linearized form of the Navier Stokes equations is explained.

Key Words — Domain decomposition, Steklov-Poincaré operator, symmetrization technique, conjugate gradient, Navier-Stokes equations.

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(¹) LIMSI-CNRS, BP 103, 91403 Orsay Cedex, and Laboratoire d'Analyse Numérique Université Paris-Sud, 91405 Orsay Cedex, France

1. INTRODUCTION

During the last decade, scientific calculation has witnessed an increasing interest in domain decomposition methods. Development of parallel calculation has favoured numerical experiments in this scope of science. Besides these experiments, numerous theoretical investigations have been devoted to domain decomposition methods without overlapping. It can be referred to Glowinski [1] as one of pioneer works accomplished in domain decomposition without overlapping. Among recent studies, reader can consult Glowinski and Wheeler [2], Chan [3], Chan and Resasco [4], Sonké *et al.* [5]. Except [5], these investigations have been restricted to symmetric problems. Iterative methods based on the classical conjugate gradient method are often used. Most of non symmetric problems are solved by numerical analogs to the Schwarz alternating method [6]. In [1], a method based on a combination of optimal control and domain decomposition for symmetric problems is proposed for the study of nonlinear problems. The same idea can be applied to non symmetric problems and utilization of preconditionners [3], [4] can increase performance of such procedure. Nevertheless, this method is expensive, for each iteration of the least squares algorithm [1] needs several domain decomposition procedures. The first conjugate gradient algorithm for domain decomposition methods for non symmetric problems was proposed in [5] and applied to the study of fluid motion in an L-shaped cavity.

This paper is devoted to the mathematical background of the non symmetric method mentioned above. Using domain decomposition without overlapping, we prove the existence and uniqueness of solution for a Dirichlet problem with a linear and non symmetric general second order operator. For computation of this solution, we propose a conjugate gradient algorithm based on a « symmetrization » technique introduced by Sonké [7].

2. MATHEMATICAL ANALYSIS

2.1. Model problem

Let Ω be an open bounded set of R^N , with regular boundary Γ . Denote A , B and D operators defined by

$$\begin{aligned} Au &= \alpha_0 u - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(\alpha_{ij} \frac{\partial u}{\partial x_j} \right) \\ Bu &= \sum_{i=1}^N \beta_i \frac{\partial u}{\partial x_i} \quad Du = Au + Bu \end{aligned}$$

where α_0 belongs to $L^\infty(\Omega)$, the α'_{ij} s $1 \leq i, j \leq N$ belong to $L^\infty(\Omega)$ and the β_i 's belong to $L^\infty(\Omega)$. Suppose f belongs to $L^2(\Omega)$, take g in $H^{1/2}(\Gamma)$ and consider the following global problem

$$\begin{cases} Du = f & \text{in } \Omega \\ u = g & \text{on } \Gamma. \end{cases} \quad (1)$$

Problem (1) is non symmetric, because of operator B , and also because we do not suppose the usual symmetry conditions $\alpha_{ij} = \alpha_{ji}$.

Denote $\beta = (\beta_1, \beta_2, \dots, \beta_N)^T$. Under the following conditions

$$\operatorname{div} \beta = 0 \quad (2)$$

$$\exists \tau > 0 \text{ such that } \forall \xi \in R^N, \sum_{i,j=1}^N \alpha_{ij}(x) \xi_i \xi_j \geq \tau |\xi|^2 \text{ a.e. in } \Omega, \quad (3)$$

$$\alpha_0(x) \geq \tau_0 \text{ a.e. in } \Omega, \quad (4)$$

with $\tau_0 > -\tau/C_\Omega^2$, where C_Ω is the Poincaré constant of Ω .

Sonké [7] proved the existence and uniqueness of solution for problem (1) using the Lax-Milgram theorem [8] and the following lemma of Temam [9].

LEMMA 1: Suppose u , v and w are vector functions with $\nabla \cdot u = 0$. Operator $b(\cdot, \cdot, \cdot)$ defined by

$$b(u, v, w) = \sum_{i,j=1}^N \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx$$

is a continuous trilinear form on $H_0^1(\Omega) \times H_0^1(\Omega) \times (H_0^1(\Omega) \cap L^N(\Omega))$ and one has the following estimate

$$b(u, v, w) \leq c(N) \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} \|w\|_{H_0^1(\Omega) \cap L^N(\Omega)}.$$

Now consider a family of non-intersecting open sets Ω_k , $n = 1, 2, \dots, n$ ($n \geq 2$), which is a partition of Ω . For simplicity, we limit our study to the case $n = 2$. Ω is the union of Ω_1 , Ω_2 and γ , Ω_1 and Ω_2 are non-intersecting domains and γ is the intersection of their closures.

$$\begin{aligned} \Omega &= \Omega_1 \cup \Omega_2 \\ \Omega_1 \cap \Omega_2 &= \emptyset. \end{aligned}$$

Denote

$$\begin{aligned} \Gamma_k &= \Gamma \cap \partial\Omega_k \quad k = 1, 2 \\ \gamma &= \partial\Omega_1 \cap \partial\Omega_2. \end{aligned}$$

According to distribution theory [10], the global problem (1) is equivalent to the following set of problems

$$\begin{cases} Du_k = f_k & \text{in } \Omega_k \\ u_k = g_k & \text{on } \Gamma_k \end{cases} \quad (5)$$

$$u_1 = u_2 \quad \text{on } \gamma \quad (6)$$

$$\frac{\partial u_1}{\partial \nu_A^{(1)}} + \frac{\partial u_2}{\partial \nu_A^{(2)}} - (\beta^{(1)} \cdot \nu^{(1)}) u_1 - (\beta^{(2)} \cdot \nu^{(2)}) u_2 = 0 \quad \text{on } \gamma$$

$\beta^{(k)}$, u_k , f_k (resp. g_k) denote the restriction of β , u , f (resp. g) on Ω_k (resp. Γ_k). $\nu^{(k)}$ is the unit normal vector directed out of Ω_k .

$$\frac{\partial u}{\partial \nu_A^{(k)}} = \sum_{i,j=1}^N \alpha_{ij} \frac{\partial u}{\partial x_j} \nu_i^{(k)}$$

is the conormal derivative associated to operator A , directed out of Ω_k . $(\beta^{(k)} \cdot \nu^{(k)})$ denotes the inner product of $\beta^{(k)}$ and $\nu^{(k)}$ in R^N .

From the coordination problem (6), we can rewrite problem (5)-(6) in the following primal approach

$$\text{Find } \lambda \text{ defined on } \gamma \text{ such that} \quad (7)$$

$$\frac{\partial u_1(\lambda)}{\partial \nu_A^{(1)}} + \frac{\partial u_2(\lambda)}{\partial \nu_A^{(2)}} - (\beta^{(1)} \cdot \nu^{(1)}) u_1(\lambda) - (\beta^{(2)} \cdot \nu^{(2)}) u_2(\lambda) = 0 \quad \text{on } \gamma$$

where $u_k(\lambda)$ is the solution of (5) with

$$u_k(\lambda)|_{\gamma} = \lambda. \quad (8)$$

We are now going to prove the existence and uniqueness of solution for problem (7)-(8).

2.2. Solution of problem (7)-(8)

Define the following spaces

$$\Lambda_0 = \{ \mu \in L^2(\gamma) \text{ such that } \mu = \tilde{\mu}|_{\gamma} \text{ where } \tilde{\mu} \in H_0^1(\Omega) \}$$

$$\Lambda_g =$$

$$= \{ \mu \in L^2(\gamma) \text{ such that } \mu = \tilde{\mu}|_{\gamma} \text{ where } \tilde{\mu} \in H^1(\Omega) \text{ and } \tilde{\mu} = g \text{ on } \Gamma \}.$$

For $\mu \in \Lambda_0$, we need $\tilde{\mu}_k$, $k = 1, 2$, which is an extension of μ in

$$H_k = \{ u_k \in H^1(\Omega_k) \text{ such that } u_k = 0 \text{ on } \Gamma_k \}.$$

This extension can be defined as the unique element $\tilde{\mu}_k \in K_k$ satisfying

$$\tilde{\mu}_k|_{\gamma} = \mu$$

with

$$H_k = H_0^1(\Omega) \oplus K_k, \quad k = 1, 2$$

A_0 is a Hilbert space with norm $\|\mu\|_{A_0} = \|\tilde{\mu}\|_{H_0^1}$ and the induced inner product. We now give a weak formulation of problem (7)-(8).

Define functions $u_k(\lambda)$, u_k^λ and u_k^0 , $k = 1, 2$ by

$$u_k(\lambda) = u_k^\lambda + u_k^0, \quad k = 1, 2$$

where $u_k(\lambda)$ is the solution of problem (5) with boundary conditions (8) u_k^λ is the solution of the following problem.

Find $u_k^\lambda \in H_k$, $u_k^\lambda = \lambda$ on γ

$$\begin{aligned} \int_{\Omega_k} \left(\sum_{i,j=1}^N \alpha_{ij} \frac{\partial u_k^\lambda}{\partial x_j} \frac{\partial v_k}{\partial x_i} \right) dx + \int_{\Omega_k} \alpha_0 u_k^\lambda v_k dx + \\ + \sum_{i=1}^N \int_{\Omega_k} \beta_i \frac{\partial u_k^\lambda}{\partial x_i} v_k dx = 0 \quad \forall v_k \in H_0^1(\Omega_k) \end{aligned} \quad (9)$$

u_k^0 is the solution of the following problem.

Find $u_k^0 \in H^1(\Omega_k)$, $u_k^0 = g_k$ on Γ_k , $u_k^0 = 0$ on γ

$$\begin{aligned} \int_{\Omega_k} \left(\sum_{i,j=1}^N \alpha_{ij} \frac{\partial u_k^0}{\partial x_j} \frac{\partial v_k}{\partial x_i} \right) dx + \int_{\Omega_k} \alpha_0 u_k^0 v_k dx + \\ + \sum_{i=1}^N \int_{\Omega_k} \beta_i \frac{\partial u_k^0}{\partial x_i} v_k dx = \int_{\Omega_k} f_k v_k dx \quad \forall v_k \in H_0^1(\Omega_k). \end{aligned} \quad (10)$$

Define the non symmetric Steklov-Poincaré operator T [11] on A_0 by

$$T\lambda = \frac{\partial u_1^\lambda}{\partial \nu_A^{(1)}} + \frac{\partial u_2^\lambda}{\partial \nu_A^{(2)}} - (\beta^{(1)} \cdot \nu^{(1)}) u_1^\lambda - (\beta^{(2)} \cdot \nu^{(2)}) u_2^\lambda. \quad (11)$$

For $\mu \in A_0$, we have

$$\begin{aligned} \int_{\gamma} T\lambda \cdot \mu d\gamma + \int_{\gamma} \left(\frac{\partial u_1^0}{\partial \nu_A^{(1)}} + \frac{\partial u_2^0}{\partial \nu_A^{(2)}} \right) - (\beta^{(1)} \cdot \nu^{(1)}) u_1^0 - \\ - ((\beta^{(2)} \cdot \nu^{(2)}) u_2^0) \mu d\gamma = 0. \end{aligned}$$

We have then the following variational formulation for problem (7)-(8)

$$\begin{cases} \text{Find } \lambda \in A_0 \text{ such that} \\ t(\lambda, \mu) + l(\mu) = 0 \quad \forall \mu \in A_0 \end{cases} \quad (12)$$

where $t(., .)$ is the bilinear form defined on A_0 by

$$\begin{aligned} t(., .) : A_0 \times A_0 &\rightarrow R \\ (\lambda, \mu) &\rightarrow t(\lambda, \mu) = \int_{\gamma} T\lambda \cdot \mu \, d\gamma \end{aligned}$$

and $l(.)$ is the linear form defined on A_0 by

$$\begin{aligned} l(.) : A_0 &\rightarrow R \\ \mu &\rightarrow l(\mu) = \\ &= \int_{\gamma} \left(\frac{\partial u_1^0}{\partial \nu_A^{(1)}} + \frac{\partial u_2^0}{\partial \nu_A^{(2)}} - (\beta^{(1)} \cdot \nu^{(1)}) u_1^0 - (\beta^{(2)} \cdot \nu^{(2)}) u_2^0 \right) \mu \, d\gamma. \end{aligned}$$

We have the following result.

THEOREM 1 : *Suppose conditions (2), (3) and (4) are verified. Therefore, problem (12) has a unique solution.*

Demonstration of theorem 1 : Denote

$$||\alpha|| = \max \{ ||\alpha_{ij}||_{\infty}, 1 \leq i, j \leq N \}$$

where

$$||\alpha_{ij}||_{\infty} = \sup_{x \in \Omega} \{ \alpha_{ij}(x) \}$$

i) $t(., .)$ is continuous on $A_0 \times A_0$.

Let $(\lambda, \mu) \in A_0 \times A_0$. Using Green formulas, we obtain

$$\begin{aligned} t(\lambda, \mu) &= \sum_{k=1}^2 \left(\int_{\Omega_k} \left(\sum_{i,j=1}^N \alpha_{ij} \frac{\partial u_k^{\lambda}}{\partial x_j} \frac{\partial \tilde{\mu}_k}{\partial x_i} \right) dx \right. \\ &\quad \left. + \int_{\Omega_k} \alpha_0 u_k^{\lambda} \tilde{\mu}_k \, dx - \int_{\Omega} \sum_{i=1}^N \beta_i \frac{\partial \tilde{\mu}_k}{\partial x_i} u_k^{\lambda} \, dx \right) \\ &= \sum_{k=1}^2 \left(\int_{\Omega_k} \left(\sum_{i,j=1}^N \alpha_{ij} \frac{\partial u_k^{\lambda}}{\partial x_j} \frac{\partial u_k^{\mu}}{\partial x_i} \right) dx \right. \\ &\quad \left. + \int_{\Omega_k} \alpha_0 u_k^{\lambda} u_k^{\mu} \, dx - \int_{\Omega} \sum_{i=1}^N \beta_i \frac{\partial u_k^{\mu}}{\partial x_i} u_k^{\lambda} \, dx \right). \end{aligned}$$

Therefore

$$|t(\lambda, \mu)| \leq \left| \sum_{k=1}^2 \left(\int_{\Omega_k} \left(\sum_{i,j=1}^N \alpha_{ij} \frac{\partial u_k^\lambda}{\partial x_j} \frac{\partial u_k^\mu}{\partial x_i} \right) dx + \int_{\Omega_k} \alpha_0 u_k^\lambda u_k^\mu dx \right) \right| \\ + \left| \sum_{k=1}^2 \int_{\Omega} \sum_{i=1}^N \beta_i \frac{\partial u_k^\mu}{\partial x_i} u_k^\lambda dx \right|.$$

From Lemma 1, we obtain

$$|t(\lambda, \mu)| \leq \left(N \cdot \text{Cte} \|\alpha\| \int_{\Omega_k} \sum_{i=1}^N \left| \frac{\partial u_k^\lambda}{\partial x_i} \right| \left| \frac{\partial u_k^\mu}{\partial x_i} \right| dx \right. \\ \left. + \|\alpha_0\|_\infty \int_{\Omega_k} |u_k^\lambda| |u_k^\mu| dx \right) \\ + \sum_{k=1}^2 c(N) \cdot \|\beta\|_\infty \|u_k^\lambda\|_{H_0^1} \|u_k^\mu\|_{H_0^1} \\ \leq \max(N \cdot \text{Cte} \|\alpha\|, \|\alpha_0\|_\infty, \|\beta\|_\infty) \\ \times \left(\sum_{k=1}^2 \|u_k^\lambda\|_{H^1}^2 \right)^{1/2} \left(\sum_{k=1}^2 \|u_k^\mu\|_{H^1}^2 \right)^{1/2}$$

thus $t(., .)$ is continuous.

ii) $t(., .)$ is A_0 -elliptic.

Let $\lambda \in A_0$. We have

$$\int_{\Omega} \sum_{i=1}^N \beta_i \frac{\partial u_k^\lambda}{\partial x_i} u_k^\lambda dx = 0.$$

We then obtain from Green formulas

$$t(\lambda, \lambda) = \sum_{k=2}^N \int_{\Omega_k} \left(\sum_{i,j=1}^N \alpha_{ij} \frac{\partial u_k^\lambda}{\partial x_j} \frac{\partial u_k^\lambda}{\partial x_i} + \alpha_0 (u_k^\lambda)^2 \right) dx.$$

Using conditions (3)-(4), we obtain

$$t(\lambda, \lambda) \geq \sum_{k=2}^N (\tau \|\nabla u_k^\lambda\|_{L^2}^2 + \tau_0 \|u_k^\lambda\|_{L^2}^2) \\ \geq \min(\tau, \tau_0) \sum_{k=1}^N \|u_k^\lambda\|_{H^1}^2$$

$t(., .)$ is therefore A_0 -elliptic.

The linear form $\ell(\cdot)$ being continuous, we conclude that problem (12) has a unique solution by the Lax-Milgram theorem.

3. ALGORITHMIC APPROACH

Our objective in this section is the construction of an algorithm for the solution of problem (12). Recall the following result.

Consider the following problem

$$\begin{cases} \text{Find } u \in V & \text{such that} \\ a(u, v) = L(v), & \forall v \in V \end{cases} \quad (13)$$

where

- i) V is a Hilbert space for $(\cdot, \cdot)$ and $\|\cdot\|$,
- ii) $a : V \times V \rightarrow R$ is bilinear, continuous, V -elliptic,
- iii) $L : V \rightarrow R$ is continuous.

Problem (13) has a unique solution. More, if $a(\cdot, \cdot)$ is symmetric, therefore (13) is equivalent to the following minimization problem

$$\begin{cases} \text{Find } u \in V & \text{such that} \\ J(u) \leq J(v), & \forall v \in V \end{cases} \quad (14)$$

where

$$J(v) = \frac{1}{2} a(v, v) - L(v).$$

A conjugate gradient algorithm for solution of (14) is given in [2], but we cannot use this algorithm to compute the solution of (12), since this problem is not symmetric. In order to find an algorithm for solution of (12), we are going to apply the symmetrization technique described in [7].

3.1. « Symmetrization »

The principle of the symmetrization technique is described below.

Given a non symmetric linear problem (P) , which solution belongs to a Hilbert space V , we construct in $V = V \times V$, a symmetric positive definite problem (P') which contains problem (P) . Iterative methods of the conjugate gradient type can then be used to solve (P') . Solution of (P) is deduced from the solution of (P') .

Now define operators $\bar{A}$, $\bar{B}$ and $\bar{D}$ by

$$\begin{aligned} \bar{A}\psi &= \alpha_0 \psi - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(\bar{\alpha}_{ij} \frac{\partial \psi}{\partial x_j} \right) \quad \text{where} \quad \bar{\alpha}_{ij} = \alpha_{ji}, \\ \bar{B}\psi &= \sum_{i=1}^N \bar{\beta}_i \frac{\partial \psi}{\partial x_i} \quad \text{where} \quad \bar{\beta}_i = -\beta_i \quad \text{and} \quad \bar{D}\psi = \bar{A}\psi + \bar{B}\psi. \end{aligned}$$

We use large characters to denote product spaces, for example $\mathbf{H}^1(\Omega) = (H^1(\Omega))^2$.

Define $F = (f, f)^T$, $G = (g, g)^T$ and operator $P : \mathbf{H}^1(\Omega) \rightarrow \mathbf{H}^{-1}(\Omega)$ by

$$\Phi = (\phi, \psi)^T \rightarrow P \Phi = (\bar{D}\psi, D\phi)^T.$$

Consider the following problem, which is a juxtaposition of two independent problems similar to (1).

$$\begin{cases} P \Phi = F & \text{in } \Omega \\ \Phi = G & \text{on } \Gamma. \end{cases} \quad (15)$$

The following result is proved in [7].

PROPOSITION 1: Denote $\beta = (\beta_1, \beta_2, \dots, \beta_N)^T$, $U = (u_1, u_2)^T$ and $V = (v_1, v_2)^T$. Define operator $\sigma(., .)$ on $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$ by

$$\sigma(U, V) = a(U, V) + b(U, V)$$

where

$$\begin{aligned} a(U, V) = \int_{\Omega} \left(\sum_{i,j=1}^N \left(\alpha_{ij} \frac{\partial u_1}{\partial x_j} \frac{\partial v_2}{\partial x_i} + \bar{\alpha}_{ij} \frac{\partial u_2}{\partial x_j} \frac{\partial v_1}{\partial x_i} \right) \right) dx + \\ + \int_{\Omega} \alpha_0 (u_1 v_2 + u_2 v_1) dx \end{aligned}$$

$$b(U, V) = b_{\beta}(u_1, v_2) + b_{\bar{\beta}}(u_2, v_1) = \sum_{i=1}^N \int_{\Omega} \beta_i \frac{\partial u}{\partial x_i} v dx.$$

We suppose hypothesis (2).

Therefore $\sigma(., .)$ is a bilinear symmetric form on $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$.

One can also prove (see [7]) that $\sigma(., .)$ is the bilinear form of the variational formulation of (15), $\sigma(., .)$ is continuous on $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$ and one has the following estimate

$$|\sigma(U, V)| \leq 4 \cdot \max(N \cdot \text{Cte } \|\alpha\|, \|\alpha_0\|_{\infty}, \|\beta\|_{\infty}) \|U\|_{\mathbf{H}^1} \|V\|_{\mathbf{H}^1}. \quad (16)$$

In the following, we suppose that $\sigma(., .)$ is coercive.

Global problem (15) is equivalent to the family of problems

$$\begin{cases} P \Phi_k = F_k & \text{in } \Omega \\ \Phi_k = G_k & \text{on } \Gamma \end{cases} \quad (17)$$

with the coordination problem

$$\begin{cases} \Phi_1 = \Phi_2 & \text{on } \gamma \\ \frac{D\Phi_1}{D\nu^{(1)}} + \frac{D\Phi_2}{D\nu^{(2)}} - \nu_\beta^{(1)} \cdot \Phi_1 - \nu_\beta^{(2)} \cdot \Phi_2 = 0 & \text{on } \gamma \end{cases} \quad (18)$$

where for $k = 1, 2$.

Φ_k and F_k (resp. G_k) are restrictions of Φ and F (resp. G) to Ω_k (resp. Γ_k), $\frac{D}{D\nu^{(k)}}$ is defined by

$$\Phi = (\phi, \psi)^T \rightarrow \frac{D\Phi}{D\nu^{(k)}} = \left(\frac{\partial \psi}{\partial \nu_A^{(k)}}, \frac{\partial \phi}{\partial \nu_A^{(k)}} \right)^T$$

and $\nu_\beta^{(k)} \cdot \Phi = ((\bar{\beta}^{(k)}, \nu^{(k)}) \psi, (\beta^{(k)}, \nu^{(k)}) \phi)^T$.

We have the following formulation of (17)-(18) in the primal approach.

$$\begin{cases} \text{Find } M \text{ such that} \\ \frac{D\Phi_1(M)}{D\nu^{(1)}} + \frac{D\Phi_2(M)}{D\nu^{(2)}} - \nu_\beta^{(1)} \cdot \Phi_1(M) - \nu_\beta^{(2)} \cdot \Phi_2(M) = 0 & \text{on } \gamma \end{cases} \quad (19)$$

where

$$\Phi_k(M)|_\gamma = M, \quad k = 1, 2. \quad (20)$$

3.2. Solution of problem (19)-(20)

Define the following spaces

$$A_0 = \left\{ M = (\mu, \mu)^T \in L^2(\gamma) \times L^2(\gamma) \text{ such that } M = \tilde{M}|_\gamma, \right. \\ \left. \tilde{M} \in \mathbf{H}_0^1(\Omega) \right\}$$

$$A_g = \left\{ M \in L^2(\gamma) \times L^2(\gamma) \text{ such that } M = \tilde{M}|_\gamma, \right. \\ \left. \tilde{M} \in \mathbf{H}^1(\Omega) \text{ and } \tilde{M} = G \text{ on } \Gamma \right\}$$

A_0 is a Hilbert space with norm $\|M\|_{A_0} = \|\tilde{M}\|_{\mathbf{H}_0^1}$, where $\tilde{M}$ is the unique element of

$$\mathbf{H}_k = \{U_k \in \mathbf{H}^1(\Omega_k) \text{ such that } U_k = 0 \text{ on } \Gamma_k\}$$

defined by its restrictions $\tilde{M}_k = \tilde{M}|_{\Omega_k} \in \mathbf{K}_k$ such that

$$\tilde{M}_k|_\gamma = M \text{ with } \mathbf{H}_k = \mathbf{H}_0^1(\Omega) \oplus \mathbf{K}_k, \quad k = 1, 2.$$

Consider the following decomposition

$$\Phi_k(M) = \Phi_k^M + \Phi_k^0 \quad (21)$$

where $\Phi_k(M)$ is the solution of problem (17) with boundary conditions (20), Φ_k^M is the solution of the following variational problem.

Find $\Phi_k^M = (\phi_k^\mu, \psi_k^\mu)^T \in \mathbf{H}_k$, $\Phi_k^M = M = (\mu, \mu)^T$ on γ

$$\begin{aligned} & \int_{\Omega_k} \left(\sum_{i,j=1}^N \left(\alpha_{ij} \frac{\partial \phi_k^\mu}{\partial x_j} \frac{\partial v_k}{\partial x_i} + \bar{\alpha}_{ij} \frac{\partial \psi_k^\mu}{\partial x_j} \frac{\partial u_k}{\partial x_i} \right) \right) dx + \int_{\Omega_k} \alpha_0 (\phi_k^\mu v_k + \psi_k^\mu u_k) dx \\ & \sum_{i=1}^N \int_{\Omega_k} \beta_i \frac{\partial \phi_k^\mu}{\partial x_i} v_k dx + \sum_{i=1}^N \int_{\Omega_k} \bar{\beta}_i \frac{\partial \psi_k^\mu}{\partial x_i} u_k dx \\ & = 0 \quad \forall U_k = (u_k, v_k)^T \in \mathbf{H}_0^1(\Omega_k) \quad (22) \end{aligned}$$

Φ_k^0 is the solution of the following variational problem.

Find $\Phi_k^0 = (\phi_k^0, \psi_k^0) \in \mathbf{H}^1(\Omega)$, $\Phi_k^0 = G_k$ on Γ_k , $\Phi_k^0 = 0$ on γ

$$\begin{aligned} & \int_{\Omega_k} \left(\sum_{i,j=1}^N \left(\alpha_{ij} \frac{\partial \phi_k^0}{\partial x_j} \frac{\partial v_k}{\partial x_i} + \bar{\alpha}_{ij} \frac{\partial \psi_k^0}{\partial x_j} \frac{\partial u_k}{\partial x_i} \right) \right) dx + \int_{\Omega_k} \alpha_0 (\phi_k^0 v_k + \psi_k^0 u_k) dx \\ & \sum_{i=1}^N \int_{\Omega_k} \beta_i \frac{\partial \phi_k^0}{\partial x_i} v_k dx + \sum_{i=1}^N \int_{\Omega_k} \bar{\beta}_i \frac{\partial \psi_k^0}{\partial x_i} u_k dx \\ & = \int_{\Omega_k} (f u_k + f v_k) dx \quad \forall U_k = (u_k, v_k)^T \in \mathbf{H}_0^1(\Omega_k). \quad (23) \end{aligned}$$

Define the following Steklov-Poincaré operator Ξ

$$\Xi M = \frac{D \Phi_1^M}{D \nu^{(1)}} + \frac{D \Phi_2^M}{D \nu^{(2)}} - \nu_{\beta}^{(1)} \cdot \Phi_1^M - \nu_{\beta}^{(2)} \cdot \Phi_2^M. \quad (24)$$

Define the linear form L by

$$\begin{aligned} & L: A_0 \rightarrow R \\ & M \rightarrow L(M) = \int_{\gamma} \left(\frac{D \Phi_1^0}{D \nu^{(1)}} + \frac{D \Phi_2^0}{D \nu^{(2)}} - \nu_{\beta}^{(1)} \cdot \Phi_1^0 - \nu_{\beta}^{(2)} \cdot \Phi_2^0 \right) \cdot M d\gamma. \end{aligned}$$

Define the bilinear form $\zeta(.,.)$ by

$$\begin{aligned} & \zeta: A_0 \times A_0 \rightarrow R \\ & (M, M') \rightarrow \zeta(M, M') = \int_{\gamma} (\Xi M) \cdot M' d\gamma. \end{aligned}$$

We obtain the following variational formulation for problem (19)-(20)

$$\begin{cases} \text{Find } M = (\lambda, \lambda)^T \in \Lambda_0 \text{ such that} \\ \zeta(M, M') + L(M') = 0 \quad \forall M' = (\mu, \mu)^T \in \Lambda_0 \end{cases} \quad (25)$$

PROPOSITION 2 : The bilinear form $\zeta(., .)$ is symmetric

Demonstration of proposition 2 : Take a couple $(M, M')^T = ((\lambda, \lambda)^T, (\mu, \mu)^T)$ in Λ_0 . We have

$$\begin{aligned} \zeta(M, M') &= \int_{\gamma} (\Xi M) \cdot M' d\gamma \\ &= \sum_{k=1}^2 \int_{\gamma} \left(\frac{D\Phi_k^M}{D\nu^{(k)}} - \nu_{\beta}^{(k)} \cdot \Phi_k^M \right) \cdot M' d\gamma \\ &= \sum_{k=1}^2 \int_{\gamma} \left(\frac{\partial \psi_k^\lambda}{\partial \nu_{\bar{A}}} \mu + \frac{\partial \phi_k^\lambda}{\partial \nu_A} \mu \right) d\gamma \\ &\quad - \sum_{k=1}^2 ((\bar{\beta}_k \cdot \nu^{(k)}) \psi_k^\lambda \mu + \beta_k \cdot \nu^{(k)} \phi_k^\lambda \mu) d\gamma. \end{aligned}$$

Using Green formulas, we obtain

$$\begin{aligned} \zeta(M, M') &= \sum_{k=1}^2 \left(\int_{\Omega_k} \left(\sum_{i,j=1}^N \bar{\alpha}_{ij} \frac{\partial \psi_k^\lambda}{\partial x_j} \frac{\partial \hat{\mu}}{\partial x_i} \right) dx \right. \\ &\quad \left. + \int_{\Omega_k} \alpha_0 \psi_k^\lambda \hat{\mu} dx - \sum_{i=1}^N \int_{\Omega_k} \bar{\beta}_i \frac{\partial \hat{\mu}}{\partial x_i} \psi_k^\lambda dx \right) \\ &\quad + \sum_{k=1}^2 \left(\int_{\Omega_k} \left(\sum_{i,j=1}^N \alpha_{ij} \frac{\partial \phi_k^\lambda}{\partial x_j} \frac{\partial \hat{\mu}}{\partial x_i} \right) dx \right. \\ &\quad \left. + \int_{\Omega_k} \alpha_0 \phi_k^\lambda \hat{\mu} dx - \sum_{i=1}^N \int_{\Omega_k} \beta_i \frac{\partial \hat{\mu}}{\partial x_i} \phi_k^\lambda dx \right) \\ &= \sum_{k=1}^2 \left(\int_{\Omega_k} \left(\sum_{i,j=1}^N \alpha_{ij} \frac{\partial \phi_k^\lambda}{\partial x_j} \frac{\partial \psi_k^\mu}{\partial x_i} + \bar{\alpha}_{ij} \frac{\partial \psi_k^\lambda}{\partial x_j} \frac{\partial \phi_k^\mu}{\partial x_i} \right) dx \right. \\ &\quad \left. + \int_{\Omega_k} \alpha_0 (\phi_k^\lambda \psi_k^\mu + \psi_k^\lambda \phi_k^\mu) dx \right) \\ &\quad - \sum_{k=1}^2 \left(\sum_{i=1}^N \int_{\Omega_k} \beta_i \frac{\partial \psi_k^\mu}{\partial x_i} \phi_k^\lambda dx \sum_{i=1}^N \int_{\Omega_k} \bar{\beta}_i \frac{\partial \phi_k^\mu}{\partial x_i} \psi_k^\lambda dx \right) \end{aligned}$$

prolonging functions ϕ_k^λ , ψ_k^λ , Φ_k^M , and $\Phi_k^{M'}$ on $\Omega - \Omega_k$ by 0, we can extend integrals on Ω . We obtain

$$\begin{aligned} \zeta(M, M') = & \sum_{k=1}^2 \left(\sum_{i=1}^N \int_{\Omega} \beta_i \frac{\partial \phi_k^\lambda}{\partial x_i} \psi_k^\mu dx \sum_{i=1}^N \int_{\Omega} \bar{\beta}_i \frac{\partial \psi_k^\lambda}{\partial x_i} \phi_k^\mu dx \right) \\ & + \sum_{k=1}^2 \left(\int_{\Omega} \left(\sum_{i,j=1}^N \alpha_{ij} \frac{\partial \phi_k^\lambda}{\partial x_j} \frac{\partial \psi_k^\mu}{\partial x_i} + \bar{\alpha}_{ij} \frac{\partial \psi_k^\lambda}{\partial x_j} \frac{\partial \phi_k^\mu}{\partial x_i} \right) dx \right. \\ & \left. + \int_{\Omega} \alpha_0 (\phi_k^\lambda \psi_k^\mu + \psi_k^\lambda \phi_k^\mu) dx \right) = \sum_{k=1}^2 \sigma(\Phi_k^M, \Phi_k^{M'}) \end{aligned}$$

where $\sigma(., .)$ is the symmetric bilinear form defined in proposition 1. We conclude that $\zeta(., .)$ is symmetric.

THEOREM 2 : *Under hypothesis (2), (3), (4), with $\tau_0 > 0$, and the following additional condition*

$$\exists \tau_1 > 0, \quad |\alpha_0| \leq \tau_1 \quad \text{and} \quad \tau + (\tau_0 - 2\tau_1) C_\Omega^2 \geq \alpha > 0, \quad (26)$$

solution of problem (12) can be computed by means of a conjugate gradient algorithm.

Démonstration of Theorem 2 : Since problem (25) contains problem (12), we just have to prove that one can compute solution of (25) using a conjugate gradient algorithm.

i) $\zeta(., .)$ is continuous.

From the proof of proposition 2, we have

$$\zeta(M, M') = \sigma(\Phi_1^M, \Phi_1^{M'}) + \sigma(\Phi_2^M, \Phi_2^{M'}).$$

We have then

$$|\zeta(M, M')| \leq |\sigma(\Phi_1^M, \Phi_1^{M'})| + |\sigma(\Phi_2^M, \Phi_2^{M'})|.$$

From estimate (16) and Cauchy-Schwarz inequalities, we obtain

$$\begin{aligned} |\zeta(M, M')| & \leq 4 \cdot \max(N \cdot \text{Cte } \|\alpha\|, \|\alpha_0\|_\infty, \|\beta\|_\infty) \times \\ & \quad \times \sum_{k=1}^2 \|\Phi_k^M\|_{H^1} \|\Phi_k^{M'}\|_{H^1} \\ & \leq 4 \cdot \max(N \cdot \text{Cte } \|\alpha\|, \|\alpha_0\|_\infty, \|\beta\|_\infty) \times \\ & \quad \times \left(\sum_{k=1}^2 \|\Phi_k^M\|_{H^1}^2 \right)^{1/2} \left(\sum_{k=1}^2 \|\Phi_k^{M'}\|_{H^1}^2 \right)^{1/2} \end{aligned}$$

$\zeta(., .)$ is thus continuous.

ii) Coercivity.

We suppose that $\sigma(., .)$ is coercive. Actually, this can be proved under condition (26).

Then, relation

$$\zeta(M, M) = \sigma(\Phi_1^M, \Phi_1^M) + \sigma(\Phi_2^M, \Phi_2^M)$$

and the fact that the A_0 -norm is induced by the H^1 -norm prove that $\zeta(., .)$ is coercive. Proposition 2 completes this demonstration.

Remark : One can prove that $\zeta(., .)$ is coercive, under the more precise following condition

$$\min(\tau + (\tau_0 - 2\tau_1)C_{\Omega_1}^2, \tau + (\tau_0 - 2\tau_1)C_{\Omega_2}^2) \geq \alpha > 0, \quad (27)$$

where $\Omega_{i'} = \Omega_i \cup \gamma \cup W_{ij}$, $W_{ij} = \Omega_{i'} \cap \Omega_j$ is a very thin open set.

3.3. Algorithm

We apply an algorithm of minimization for the solution of problem (25), see Glowinski [12]. From this algorithm, we obtain the following conjugate gradient algorithm for the solution of (12).

Initialization.

Choose λ^0 in A_g compute ρ^0 .

For $k = 1, 2$.

Find $u_k^0 \in H^1(\Omega_k)$, $u_k^0 = g_k$ on Γ_k , $u_k^0 = \lambda^0$ on γ such that

$$\begin{aligned} \int_{\Omega_k} \left(\sum_{i,j=1}^N \alpha_{ij} \frac{\partial u_k^0}{\partial x_j} \frac{\partial v_k}{\partial x_i} \right) dx + \int_{\Omega_k} \alpha_0 u_k^0 v_k dx + \\ + \sum_{k=1}^N \int_{\Omega_k} \beta_i \frac{\partial u_k^0}{\partial x_i} v_k dx = \int_{\Omega_k} f_k v_k dx \quad \forall v_k \in H_0^1(\Omega_k). \end{aligned} \quad (28)$$

Find $\rho^0 \in A_0$ such that

$$\begin{aligned} \sum_{k=1}^2 \left(\int_{\Omega_k} \left(\nabla \tilde{\rho}_k^0 \nabla \tilde{\mu}_k dx + \int_{\Omega_k} \tilde{\rho}_k^0 \tilde{\mu}_k dx \right) \right) = \\ = \sum_{k=1}^2 \left(\int_{\Omega_k} \left(\sum_{i,j=1}^N \alpha_{ij} \frac{\partial u_k^0}{\partial x_j} \frac{\partial \tilde{\mu}_k}{\partial x_i} \right) dx \right. \\ + \int_{\Omega_k} \alpha_0 u_k^0 \tilde{\mu}_k dx + \sum_{i=1}^N \int_{\Omega_k} \beta_i \frac{\partial u_k^0}{\partial x_i} \tilde{\mu}_k dx \\ \left. - \int_{\Omega_k} f_k \tilde{\mu}_k dx \right) \quad \forall \mu \in A_0. \end{aligned} \quad (29)$$

Take

$$\bar{\delta}^0 = \delta^0 = \bar{\rho}^0 = \rho^0. \quad (30)$$

Iterations.

$n \geq 0$, λ^n , ρ^n , $\bar{\rho}^n$, δ^n et $\bar{\delta}^n$ known, compute λ^{n+1} , ρ^{n+1} , $\bar{\rho}^{n+1}$, δ^{n+1} et $\bar{\delta}^{n+1}$.

For $k = 1, 2$.

Find $v_k^n \in H_k$, $v_k^n = \delta^n$ on γ such that

$$\begin{aligned} \int_{\Omega_k} \left(\sum_{i,j=1}^N \alpha_{ij} \frac{\partial v_k^n}{\partial x_j} \frac{\partial z_k}{\partial x_i} \right) dx + \int_{\Omega_k} \alpha_0 v_k^n z_k dx + \\ + \sum_{i=1}^N \int_{\Omega_k} \beta_i \frac{\partial v_k^n}{\partial x_i} z_k dx = 0 \quad \forall z_k \in H_0^1(\Omega_k). \end{aligned} \quad (31)$$

Find $w_k^n \in H_k$, $w_k^n = \bar{\delta}^n$ on γ such that

$$\begin{aligned} \int_{\Omega_k} \left(\sum_{i,j=1}^N \bar{\alpha}_{ij} \frac{\partial w_k^n}{\partial x_j} \frac{\partial y_k}{\partial x_i} \right) dx + \int_{\Omega_k} \alpha_0 w_k^n y_k dx + \\ + \sum_{i=1}^N \int_{\Omega_k} \beta_i \frac{\partial w_k^n}{\partial x_i} y_k dx = 0 \quad \forall y_k \in H_0^1(\Omega_k). \end{aligned} \quad (32)$$

Compute η^n and $\bar{\eta}^n$ as solutions of the following variational problems.

Find $\eta^n \in \Lambda_0$ such that

$$\begin{aligned} \sum_{k=1}^2 \left(\int_{\Omega_k} \nabla \tilde{\eta}_k^n \cdot \nabla \tilde{\zeta}_k dx + \int_{\Omega_k} \tilde{\eta}_k^n \tilde{\zeta}_k dx \right) = \\ = \sum_{k=1}^2 \left(\int_{\Omega_k} \left(\sum_{i,j=1}^N \tilde{\alpha}_{ij} \frac{\partial w_k^n}{\partial x_j} \frac{\partial \tilde{\zeta}_k}{\partial x_i} \right) dx + \int_{\Omega_k} \alpha_0 w_k^n \tilde{\zeta}_k dx \right. \\ \left. + \sum_{i=1}^N \int_{\Omega_k} \beta_i \frac{\partial w_k^n}{\partial x_i} \tilde{\zeta}_k dx \right) \quad \forall \zeta \in \Lambda_0. \end{aligned} \quad (33)$$

Find $\bar{\eta}^n \in \Lambda_0$ such that

$$\begin{aligned} \sum_{k=1}^2 \left(\int_{\Omega_k} \nabla \tilde{\bar{\eta}}_k^n \cdot \nabla \tilde{\zeta}_k dx + \int_{\Omega_k} \tilde{\bar{\eta}}_k^n \tilde{\zeta}_k dx \right) = \\ = \sum_{k=1}^2 \left(\int_{\Omega_k} \left(\sum_{i,j=1}^N \alpha_{ij} \frac{\partial v_k^n}{\partial x_j} \frac{\partial \tilde{\zeta}_k}{\partial x_i} \right) dx + \int_{\Omega_k} \alpha_0 v_k^n \tilde{\zeta}_k dx \right. \\ \left. + \sum_{i=1}^N \int_{\Omega_k} \beta_i \frac{\partial v_k^n}{\partial x_i} \tilde{\zeta}_k dx \right) \quad \forall \zeta \in \Lambda_0 \end{aligned} \quad (34)$$

$\varphi_n =$

$$\frac{\sum_{k=1}^2 \int_{\Omega_k} (|\nabla \tilde{\rho}_k^n|^2 + |\tilde{\rho}_k^n|^2) dx + \sum_{k=1}^2 \int_{\Omega_k} (|\nabla \tilde{\tilde{\rho}}_k^n|^2 + |\tilde{\tilde{\rho}}_k^n|^2) dx}{\sum_{k=1}^2 \int_{\Omega_k} (\nabla \tilde{\eta}_k^n \cdot \nabla v_k^n + \tilde{\eta}_k^n v_k^n) dx + \sum_{k=1}^2 \int_{\Omega_k} (\nabla \tilde{\tilde{\eta}}_k^n \cdot \nabla v_k^n + \tilde{\tilde{\eta}}_k^n v_k^n) dx} \quad (35)$$

$$\lambda^{n+1} = \lambda^n + \varphi_n \delta^n \quad (36)$$

$$\rho^{n+1} = \rho^n - \varphi_n \eta^n \quad (37)$$

$$\bar{\rho}^{n+1} = \bar{\rho}^n - \varphi_n \bar{\eta}^n \quad (38)$$

$\rho_n =$

$$\frac{\sum_{k=1}^2 \int_{\Omega_k} (|\nabla \tilde{\rho}_k^{n+1}|^2 + |\tilde{\rho}_k^{n+1}|^2) dx + \sum_{k=1}^2 \int_{\Omega_k} (|\nabla \tilde{\tilde{\rho}}_k^{n+1}|^2 + |\tilde{\tilde{\rho}}_k^{n+1}|^2) dx}{\sum_{k=1}^2 \int_{\Omega_k} (|\nabla \tilde{\rho}_k^n|^2 + |\tilde{\rho}_k^n|^2) dx + \sum_{k=1}^2 \int_{\Omega_k} (|\nabla \tilde{\tilde{\rho}}_k^n|^2 + |\tilde{\tilde{\rho}}_k^n|^2) dx} \quad (39)$$

$$\delta^{n+1} = \rho^{n+1} + \rho_n \delta^n \quad (40)$$

$$\bar{\delta}^{n+1} = \bar{\rho}^{n+1} + \rho_n \bar{\delta}^n. \quad (41)$$

Do $n = n + 1$, go to (31).

4. APPLICATION TO THE 2D NAVIER-STOKES EQUATIONS

The 2D Navier-Stokes equations for laminar unsteady flow of an incompressible fluid are considered in the velocity-vorticity formulation [13]. We have the following system

$$\frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} - \frac{1}{\text{Re}} \nabla^2 \omega = 0 \quad (42)$$

$$\nabla^2 u = \frac{\partial \omega}{\partial y} \quad (43)$$

$$\nabla^2 v = -\frac{\partial \omega}{\partial x} \quad (44)$$

plus boundary and initial conditions.

Where Re is the Reynolds number, u and v are velocity components and vorticity is defined by

$$\omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}.$$

Continuity equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ is implicitly taken into account, see Fasel [14].

Denote $U = (u, v)^T$ the velocity vector. At each time step, velocity components can be computed by a conjugate gradient algorithm for symmetric problems with domain decomposition, proposed by Sonké *et al.* [15]. The same algorithm can be applied for vorticity computation, *via* the following time discretization

$$\frac{3\omega^{n+1} - 4\omega^n + \omega^{n-1}}{2\Delta t} + 2(U \cdot \nabla \omega)^n - (U \cdot \nabla \omega)^{n-1} = \frac{1}{\text{Re}} \nabla^2 \omega^{n+1} \quad (45)$$

which is obtained using an explicit scheme of Adams-Bashforth type for non linear term [16]. We have then to solve an Helmholtz problem at each time step.

For reasons mentioned in [5], we avoid scheme (45). Using a semi-implicit two-steps second order accurate θ -scheme [17] for vorticity, we obtain at each time step the following problem

$$\left\{ \begin{array}{l} \left(\frac{I}{\Delta t} + \theta \left[(\tilde{U}^{n+1} \cdot \nabla) - \frac{1}{\text{Re}} \nabla^2 \right] \right) \omega^{n+1} = \\ = \left(\frac{I}{\Delta t} - (1 - \theta) \left[(U^n \cdot \nabla) - \frac{1}{\text{Re}} \nabla^2 \right] \right) \omega^n \quad \text{in } \Omega \\ \omega^{n+1} = \frac{\partial \tilde{u}^{n+1}}{\partial y} - \frac{\partial \tilde{v}^{n+1}}{\partial x} \quad \text{on } \Gamma, \quad 0 \leq \theta \leq 1. \end{array} \right. \quad (46)$$

This problem is similar to the model problem (1), operator A is replaced by

$$\frac{I}{\Delta t} - \frac{\theta}{\text{Re}} \nabla^2$$

where I is the identity operator, that is

$$\alpha_0 = \frac{1}{\Delta t} \quad \text{and} \quad \alpha_{ij} = \frac{\theta}{\text{Re}} \delta_{ij}$$

δ_{ij} is the Kronecker symbol, operator B is replaced by $\theta(\tilde{U}^{n+1} \cdot \nabla)$, where $\tilde{U}^{n+1}$ is a second order prediction of U^{n+1} .

Evolution of fluid motion can then be studied by solving system (43), (44), (46).

Condition (2) is satisfied, since continuity equation is implicitly verified [14].

Conditions (3) and (4) are satisfied with $\tau = \frac{\theta}{\text{Re}}$ and $\tau_0 = \frac{1}{\Delta t}$.

Now, taking $\tau_1 = \frac{1}{\Delta t}$, condition (27) can be expressed in the following form

$$\min \left(\frac{\theta}{\text{Re}} - \frac{C_{\Omega_1}^2}{\Delta t}, \frac{\theta}{\text{Re}} - \frac{C_{\Omega_2}^2}{\Delta t} \right) \geq \alpha > 0$$

that is

$$\text{Re} < \frac{\theta \cdot \Delta t}{\max(C_{\Omega_1}^2, C_{\Omega_2}^2)} \quad (47)$$

Recall that if l is the thickness of a bounded set O , then one has $C_O^2 \approx \frac{1}{2} l^2$, see Raviart and Thomas [18]

Scheme (46) is unconditionally stable for $1/2 \leq \theta \leq 1$ and one can choose quite large time steps, e.g. $\Delta t = 1/10$

Suppose Ω is decomposed into subdomains of thickness $\leq 1/10$. Then for a fully implicit scheme, that is $\theta = 1$, condition (47) is satisfied for $\text{Re} \leq 20$

The discrete version of algorithm presented in this paper and numerical results of its application to problem (46) can be seen in [5]. This application shows that condition (27) is restrictive, since successful results have been obtained in simulation of fluid motion for $\text{Re} = 100$, computational domain being decomposed into subdomains of thickness $l = 1/2$

REFERENCES

- [1] R. GLOWINSKI, J. PERIAUX and Q. V. DIHN, Domain Decomposition Methods for Nonlinear Problems in Fluid Dynamics, *INRIA report* 147, July 1982
- [2] R. GLOWINSKI and M. F. WHEELER, Domain Decomposition and Mixed Finite Element Methods for Elliptic Problems, *Proc. of the First International Symposium on Domain Decomposition Methods for Partial Differential Equations Paris*, January 1987, SIAM 1988
- [3] T. F. CHAN, Analysis of preconditioners for domain decomposition, *SIAM J Numer Anal.*, 24/2, 1987
- [4] T. F. CHAN and D. C. RESASCO, A Framework for Analysis and Construction of Domain Decomposition Preconditioners, *Proc. of the First International Symposium on Domain Decomposition Methods for Partial Differential Equations Paris*, January 1987, SIAM 1988
- [5] L. SONKE, P. LE QUÈRE and T. A. PHUOC LOC, Domain decomposition and multigradient methods for the Navier-Stokes equations, *Proc. 6th International Conference on Numerical Methods in Laminar and Turbulent Flow, Swansea, U.K.* July 1989

- [6] K. MILLER, Numerical analogs to the Schwarz alternating procedure, *Numer. Math.* 1965.
- [7] L. SONKÉ, Solution algorithmique de certaines équations aux dérivées partielles non symétriques : application aux équations de Navier-Stokes, *in preparation*.
- [8] R. DAUTRAY and J. L. LIONS, Analyse mathématique et calcul numérique pour les sciences et les techniques, *Masson*, Paris 1985.
- [9] R. TEMAM, Navier-Stokes equations, *North Holland*, 1977.
- [10] L. SCHWARTZ, Théorie des distributions, *Hermann*, Paris, 1957.
- [11] P. S. VASSILEVSKI, Poincaré-Steklov operators for elliptic difference problems, *C. R. Acad. Bulgare Sci.*, 38, no. 5, 543-546, 1985.
- [12] R. GLOWINSKI, Numerical Methods for nonlinear variational problems, *Springer*, 1984.
- [13] C. G. SPEZIALE, On the Advantages of the Vorticity-Velocity Formulation of the Equations of Fluid Dynamics, *J. Comp. Phys.* 73, 476-480, 1987.
- [14] H. F. FASEL, Numerical solution of the complete Navier-Stokes equations for the simulation of unsteady flows. Approximation Methods For Navier-Stokes Problem, *Proc. Paderborn, Germany*, 177-191, *Springer-Verlag*, 1979.
- [15] L. SONKÉ, P. LE QUÉRÉ and TA PHUOC LOC, Domain decomposition and Velocity-vorticity formulation for fluids flows in multiply-connected regions, *in preparation*.
- [16] J. M. VANEL, R. PEYRET and P. BONToux, A pseudo-spectral solution of vorticity-stream function equation using the influence matrix technique, *Numer. Meth. Fluid Dynamics II*, 463-472, *Clarendon Press, Oxford*, 1986.
- [17] R. PEYRET and T. D. TAYLOR, Computational methods for fluid flow, *Springer-Verlag*, 1985.
- [18] P. A. RAVIART and J. M. THOMAS, Introduction à l'analyse numérique des équations aux dérivées partielles. *Masson*, Paris, 1984.