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MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
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ON A TWO-DIMENSIONAL MAGNETOHYDRODYNAMIC PROBLEM I. MODELLING AND ANALYSIS (*)

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Abstract. — *We consider a two-dimensional Magnetohydrodynamic system of equations describing the motion of a conducting fluid in which eddy currents flow. The mathematical model is derived and existence of solutions is proved by using fixed point techniques. Uniqueness is obtained under restrictive conditions on the involved physical parameters.*

Résumé. — *Nous considérons un problème de magnétohydrodynamique bidimensionnelle décrivant le mouvement d'un fluide conducteur soumis à des courants de Foucault. Le modèle mathématique est obtenu et l'existence de solutions est démontrée en utilisant des techniques de point fixe. L'unicité est obtenue dans un certain nombre de situations physiques.*

1. INTRODUCTION

This paper is devoted to the study of a system of partial differential equations modelling the two-dimensional motion of an electrically conducting incompressible fluid in presence of an electromagnetic field produced by eddy currents. Such a system involves a coupling between Maxwell's equations in the whole space and Navier-Stokes equations in the region occupied by the fluid. This problem is encountered in a large variety of industrial applications such as electromagnetic casting (See [1, 2] for instance) and electromagnetic stirring installations.

In this first part we are concerned with the existence and uniqueness of solutions of such a problem. In a second part, we shall investigate its numerical analysis.

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An outline of the paper is as follows : in Section 2, we describe the setting of the physical problem and the derivation of the equations. In Section 3, we introduce some notations and briefly recall some basic tools. Section 4 is devoted to the proof of the existence of solutions using a fixed point theorem. Uniqueness of such solutions is proved in Section 5 under restrictions on the physical parameters.

2. THE MATHEMATICAL MODEL

Consider a set of three infinite cylindrical conductors Λ_0, Λ_1, Λ_2 the intersections of which with the plane Ox_1x_2 are denoted by Ω_0, Ω_1, Ω_2 respectively and such that their generating lines are parallel to the x_3-axis. The conductor Λ_0 is assumed to be made of an electrically conducting fluid while the conductors Λ_1, Λ_2 stand for one inductor surrounding the fluid, the loop being « closed at infinity ». The main goal of such a set-up is to generate Lorentz forces that either maintain the fluid in levitation and/or create an electromagnetic stirring motion. The domains Ω_0, Ω_1, Ω_2 are assumed to be bounded, connected and of disjoined closures, i.e., $\bar{\Omega}_i \cap \bar{\Omega}_j = \emptyset$ for $i \neq j$, and their respective boundaries Γ_0, Γ_1, Γ_2 are assumed to be smooth (C^1 say, *cf.* [3] for further precision). We shall denote in the sequel

$$\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2\,, \quad \Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$$

and we assume that the domain Ω_0 is given once for all, i.e., we do not treat a free-boundary problem. Let us mention that we have restricted ourselves to three conductors for the sake of simplicity ; the model and the results presented in this paper can be generalized however to several situations where we have a large number of conductors.

Let us assume now that an alternating current parallel to the x_3-axis flows in the inductor. This current is of a given frequency $\omega/2\pi$ and total intensity $J \geqslant 0$. The frequency is assumed sufficiently small with respect to the diameter of the conductors making it possible to neglect the current displacements in the conductors.

The commonly used physical fields for such problems are the magnetic induction $\mathbf{b}$, the current density $\mathbf{j}$, the electric field $\mathbf{e}$, the velocity of the fluid $\mathbf{u}$ and its pressure p. Because of the geometry, we consider x_3-invariant solutions, i.e., all the fields $\mathbf{b}$, $\mathbf{j}$, $\mathbf{e}$, $\mathbf{u}$, ... are spatially depending only on $x = (x_1, x_2) \in \mathbb{R}^2$. We seek all electromagnetic fields of the form :

$$\begin{aligned}
\mathbf{b}(x, t) &= \operatorname{Re}\ (e^{i\omega t}\mathbf{b}(x))\,, \\
\mathbf{j}(x, t) &= \operatorname{Re}\ (e^{i\omega t}\mathbf{j}(x))\,, \\
\mathbf{e}(x, t) &= \operatorname{Re}\ (e^{i\omega t}\mathbf{e}(x))\,,
\end{aligned}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and t denotes the time variable and $\mathbf{b}(x)$, $\mathbf{j}(x)$, $\mathbf{e}(x)$ are complex-valued functions. By contrast, we assume that the frequency is large enough so that we can admit that only a time-averaged Lorentz force is responsible for the fluid motion. In this way, $\mathbf{u}$, p can be assumed time-independent and real-valued.

In order to obtain the electromagnetic model we use Maxwell's equations and Ohm's law in each connected component of Ω. If we assume that the current density $\mathbf{j}$, which is vanishing outside Ω, is parallel to the x_3-axis in all the domains Ω_k, $k = 0, 1, 2$, we deduce from the Biot-Savart's law that the magnetic induction $\mathbf{b}$ has no x_3-component and its behaviour at infinity is an $O(|x|^{-1})$ with $|x| = (x_1^2 + x_2^2)^{1/2}$. Let ε_3 denote the unit vector in the x_3-direction, we can write :

$$\mathbf{j}(x) = j(x)\, \varepsilon_3, \quad \mathbf{b}(x) = (b_1(x), b_2(x)), \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

where b_1 and b_2 are the two components of $\mathbf{b}$ in the x_1 and x_2-direction respectively.

From the Maxwell's equation :

$$\operatorname{div} \mathbf{b} = 0 \quad \text{in} \quad \mathbb{R}^3,$$

we deduce the existence of a scalar field $\phi : \mathbb{R}^2 \to \mathbb{C}$ such that

$$\mathbf{b}(x) = \mathbf{curl}\, \phi(x), \quad x \in \mathbb{R}^2, \tag{2.1}$$

where the two-dimensional vector curling operator is defined by :

$$\mathbf{curl}\, \phi \overset{\text{def}}{=} \left(\frac{\partial \phi}{\partial x_2}, -\frac{\partial \phi}{\partial x_1} \right).$$

From the Maxwell's equation :

$$\mathbf{curl}\, \mathbf{b} = \mu_0 \mathbf{j} \quad \text{in} \quad \mathbb{R}^3,$$

in which we have neglected current displacements and assumed that the magnetic permeability μ is constant and equal to the one of the vacuum μ_0, we deduce that

$$-\Delta \phi(x) = \mu_0 j(x), \quad x \in \mathbb{R}^2. \tag{2.2}$$

If $x \notin \bar{\Omega}$, then $j(x) = 0$ and the function ϕ is harmonic outside the domain $\bar{\Omega}$. It remains to establish a relationship between j and ϕ inside the domain Ω.

The velocity $\mathbf{u}$ is assumed to have no x_3-component and we set $\mathbf{u}(x) = (u_1(x), u_2(x))$ the two components of $\mathbf{u}$. By using the Ohm's law

$$\mathbf{j} = \sigma(\mathbf{e} + \mathbf{u} \times \mathbf{b}), \quad \text{in} \quad \Lambda_k, \quad k = 0, 1, 2,$$

where $\sigma = \sigma_k$ is the electric conductivity of the domain Λ_k, $k = 0, 1, 2$ and $\mathbf{u} = 0$ in Λ_1 and Λ_2, we conclude that $\mathbf{e}$ is parallel to the x_3-axis, i.e., $\mathbf{e} = e(x)\,\boldsymbol{\varepsilon}_3$ and with (2.1) :

$$j = \sigma(e - \mathbf{u}\,.\,\nabla\phi) \quad \text{in } \Omega\,. \tag{2.3}$$

Clearly, to link j to ϕ it remains to derive a relationship between e and ϕ. To do this, we use the Maxwell's equation :

$$\mathbf{curl}\,\mathbf{e} + i\,\omega\,\mathbf{b} = 0 \quad \text{in} \quad \Lambda_k\,, \quad k = 0, 1, 2\,,$$

in order to obtain

$$\mathbf{curl}\,(e + i\,\omega\,\phi) = 0 \quad \text{in } \Omega\,.$$

This implies there are complex constants C_k such that

$$e + i\,\omega\,\phi = C_k \quad \text{in } \Omega_k\,, \quad k = 0, 1, 2\,. \tag{2.4}$$

Combining relationships (2.2), (2.3) and (2.4) yields :

$$\Delta\phi = 0 \quad \text{in} \quad \mathbb{R}^2\backslash\bar{\Omega}\,, \tag{2.5}$$

$$-\Delta\phi + \mu_0\,\sigma_k(\mathbf{u}\,.\,\nabla\phi + i\,\omega\,\phi - C_k) = 0 \quad \text{in} \quad \Omega_k\,, \quad k = 0, 1, 2\,. \tag{2.6}$$

The next step of the construction of the mathematical model consists in relating the unknown constants C_k, $k = 0, 1, 2$ in (2.6) to the given total current intensity J. Let us impose that if the total current in Ω_1 is J then its value is $-J$ in Ω_2 (which corresponds to a difference of π in phase). Moreover, we impose that the total current in Ω_0 is zero. More precisely, we set

$$J = \int_{\Omega_1} j(x)\,dx = -\int_{\Omega_2} j(x)\,dx \tag{2.7}$$

and

$$\int_{\Omega_0} j(x)\,dx = 0\,. \tag{2.8}$$

From (2.7), (2.3) with $\mathbf{u} = 0$ in $\Omega_1 \cup \Omega_2$, and (2.4) we obtain

$$\sigma_k\,C_k = \frac{1}{|\Omega_k|}\left((-1)^{k+1}J + i\,\omega\int_{\Omega_k}\phi(x)\,dx\right)\,, \quad k = 1, 2\,, \tag{2.9}$$

where $|\Omega_k|$ is the measure of Ω_k.

Now, assuming that div $\mathbf{u} = 0$ in Ω_0 and $\mathbf{u} = 0$ on Γ_0 we have

$$\int_{\Omega_0}\mathbf{u}\,.\,\nabla\phi\,dx = 0$$

and, from (2.8), (2.3), (2.4) we obtain

$$\sigma_0 C_0 = \frac{i\omega}{|\Omega_0|} \int_{\Omega_0} \phi(x)\, dx . \tag{2.10}$$

Let us define

$$I_k(\phi) = \frac{1}{|\Omega_k|} \int_{\Omega_k} \phi(x)\, dx \tag{2.11}$$

and

$$J_k \overset{\text{def}}{=} \begin{cases} \dfrac{(-1)^{k+1} J}{|\Omega_k|} & \text{if } k = 1, 2, \\ 0 & \text{if } k = 0 . \end{cases} \tag{2.12}$$

If we eliminate C_k in (2.6) by using (2.9) and (2.10), we obtain

$$-\Delta\phi + \mu_0 \sigma_k \mathbf{u} \cdot \nabla\phi + i\mu_0 \sigma_k \omega(\phi - I_k(\phi)) = \\ = \mu_0 J_k \quad \text{in} \quad \Omega_k, \quad k = 0, 1, 2 . \tag{2.13}$$

It remains, to achieve the electromagnetic model, to give boundary conditions.

The interface conditions are deduced by interpreting the two first Maxwell's equations considered here in the sense of distributions and by assuming the non-existence of surface currents. Namely :

$$[\phi] = \left[\frac{\partial \phi}{\partial n} \right] = 0 \quad \text{on} \quad \Gamma_k, \quad k = 0, 1, 2 , \tag{2.14}$$

where $[\psi] \overset{\text{def}}{=} \psi_{\text{ext}} - \psi_{\text{int}}$ denotes the jump of the function ψ on Γ and $\dfrac{\partial}{\partial n}$ stands for the outward normal derivative, the vector $\mathbf{n}$ being the outward unit normal to Γ. The behaviour at infinity is deduced from $\mathbf{b}(x) = O(|x|^{-1})$. Hence $\phi(x) = O(\log |x|)$ when $|x| \to +\infty$. From potential theory we get :

$$\phi(x) = \alpha \log |x| + \beta + O(|x|^{-1}), \quad |x| \to +\infty , \tag{2.15}$$

$$\nabla\phi(x) = \alpha \frac{x}{|x|^2} + O(|x|^{-2}), \quad |x| \to +\infty , \tag{2.16}$$

where α, β are complex numbers.

The electromagnetic model is given by equations (2.5), (2.13), (2.14) and (2.15). If $\mathbf{u}$ is given, the unknowns are ϕ, α and β. Remark that if

(ϕ, α, β) is a solution of (2.5), (2.13), (2.14) and (2.15) then $(\phi + \gamma, \alpha, \beta + \gamma)$ is another one for all $\gamma \in \mathbb{C}$. This is natural since the function ϕ has to be known only up to an additive constant. In order to fix this constant, we set

$$\int_{\Omega_0} \phi(x)\, dx = 0\,, \tag{2.17}$$

and we have, by using (2.10) and (2.4) :

$$e = -i\omega\phi \quad \text{in} \quad \Omega_0\,. \tag{2.18}$$

Let us consider now the fluid flow problem in Ω_0. The fluid motion is governed by stationary incompressible Navier-Stokes equations where the body force term $\mathbf{f}$ is given by the Lorentz forces (*cf.* [4]). After time-averaging on a period $T = 2\pi/\omega$, and using (2.1), (2.3) and (2.18), we easily verify that :

$$\begin{aligned}
\mathbf{f}(x) \;&\overset{\text{def}}{=}\; \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \operatorname{Re}\,(e^{i\omega t} j(x)\,\varepsilon_3) \times \operatorname{Re}\,(e^{i\omega t}\mathbf{b}(x))\,dt \\
&= \frac{\sigma_0\,\omega}{2}\,(\phi_I\,\nabla\phi_R - \phi_R\,\nabla\phi_I) \\
&\quad - \frac{\sigma_0}{2}\,((\mathbf{u}\,.\,\nabla\phi_R)\,\nabla\phi_R + (\mathbf{u}\,.\,\nabla\phi_I)\,\nabla\phi_I)
\end{aligned} \tag{2.19}$$

where ϕ_R, ϕ_I denote respectively the real and the imaginary part of ϕ. We then obtain for the fluid motion equations :

$$\begin{aligned}
-\nu\,\Delta\mathbf{u} + \mathbf{u}\,.\,\nabla\mathbf{u} + \nabla p &= \\
&= \frac{\sigma_0\,\omega}{2\rho}\,(\phi_I\,\nabla\phi_R - \phi_R\,\nabla\Phi_I) \\
&\quad - \frac{\sigma_0}{2\rho}\,((\mathbf{u}\,.\,\nabla\phi_R)\,\nabla\phi_R + (\mathbf{u}\,.\,\nabla\phi_I)\,\nabla\phi_I) \quad \text{in} \quad \Omega_0\,, && (2.20)\\
\operatorname{div}\mathbf{u} &= 0 \quad \text{in} \quad \Omega_0\,, && (2.21)\\
\mathbf{u} &= 0 \quad \text{on} \quad \Gamma_0\,, && (2.22)
\end{aligned}$$

where $\nu > 0$ is the kinematic viscosity of the fluid and ρ its density. Note that depending on physical requirements, condition (2.22) may be replaced by a slip boundary condition (vanishing normal velocity and tangential traction). Such a condition would be more adapted to a free boundary problem. The results stated in this paper can be straightforwardly extended to such conditions.

Reporting equations (2.5), (2.13)-(2.15) and (2.20)-(2.22), the mathematical model finally consists in seeking $\phi : \mathbb{R}^2 \to \mathbb{C}$, $\mathbf{u} : \Omega_0 \to \mathbb{R}^2$, $p : \Omega_0 \to \mathbb{R}$ and $(\alpha, \beta) \in \mathbb{C}^2$ such that :

$$- \Delta\phi + \mu_0 \sigma_k \mathbf{u} \,.\, \nabla\phi + i \mu_0 \omega \sigma_k (\phi - I_k(\phi)) = \mu_0 J_k \quad \text{in} \quad \Omega_k , \quad k = 0, 1, 2 , \tag{2.23}$$

$$\Delta\phi = 0 \quad \text{in} \quad \Omega' = \mathbb{R}^2 \backslash \bar{\Omega} , \tag{2.24}$$

$$\int_{\Omega_0} \phi \, dx = 0 , \tag{2.25}$$

$$\phi(x) = \alpha \log |x| + \beta + O(|x|^{-1}) \quad |x| \to + \infty , \tag{2.26}$$

$$[\phi] = \left[\frac{\partial\phi}{\partial n} \right] = 0 \qquad \text{on} \quad \Gamma_k , \quad k = 0, 1, 2 , \tag{2.27}$$

$$- \nu \, \Delta\mathbf{u} + \mathbf{u} \,.\, \nabla\mathbf{u} + \nabla p - \frac{\sigma_0 \, \omega}{2 \rho} (\phi_I \nabla\phi_R - \phi_R \nabla\Phi_I) + \frac{\sigma_0}{2 \rho} ((\mathbf{u} \,.\, \nabla\phi_R) \nabla\phi_R + (\mathbf{u} \,.\, \nabla\phi_I) \nabla\phi_I) = 0 \quad \text{in} \quad \Omega_0 , \tag{2.28}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad \Omega_0 , \tag{2.29}$$

$$\mathbf{u} = 0 \quad \text{on} \quad \Gamma_0 ; \tag{2.30}$$

we recall that in (2.23) we set $\mathbf{u} = 0$ in $\Omega_1 \cup \Omega_2$, I_k and J_k are given by (2.11), (2.12) in which J is a data of the problem.

Remark 2.1 : We have restricted ourselves to the case of three conductors in order to simplify the presentation. the presented model can be straightforwardly generalized to the case of a set of N conductors where some of which are made of liquid metal, the central hypothesis being that the integral of the current density j over the whole plane $\mathbb{R}^2$ is zero.

3. NOTATIONS AND SOME BASIC TOOLS

If D is an open subset of $\mathbb{R}^2$ and if $v : D \to \mathbb{C}$ is a complex-valued function, we denote by v^* its complex conjugate and by $|v|$ its modulus. The Sobolev spaces $W^{m,p}(D)$, $H^m(D)$, $L^p(D)$ and the corresponding norms $\| \cdot \|_{m,p,D}$, $\| \cdot \|_{m,D}$, $\| \cdot \|_{0,p,D}$ and semi-norms $| \cdot |_{m,p,D}$, $| \cdot |_{m,D}$ have their usual meanings for real or complex-valued functions.

Since we shall deal with partial differential equations that are formulated in $\mathbb{R}^2$, we shall make use of the following weighted Sobolev space :

$$W_0^1(\mathbb{R}^2) = \{ \psi : \mathbb{R}^2 \to \mathbb{C} \, ; \, \xi\psi \in L^2(\mathbb{R}^2), \, \nabla\psi \in L^2(\mathbb{R}^2)^2 \} \tag{3.1}$$

equipped with the norm

$$\|\psi\|_{W_0^1(\mathbb{R}^2)} \stackrel{\text{def}}{=} (\|\xi\psi\|_{0,\mathbb{R}^2}^2 + \|\nabla\psi\|_{0,\mathbb{R}^2}^2)^{1/2} \tag{3.2}$$

where $\xi(x) = (1 + |x|)^{-1}(1 + \log(2 + |x|))^{-1}$.

For a study of the space $W_0^1(\mathbb{R}^2)$ one can consult [5] for instance. In particular, it is known that the constant functions belong to $W_0^1(\mathbb{R}^2)$ which is a Banach space when it is equipped with the norm (3.2). Moreover, the semi-norm $\|\nabla\psi\|_{0,\mathbb{R}^2}$ is a norm on the quotient space $W_0^1(\mathbb{R}^2)/\mathbb{C}$, equivalent to the quotient norm. It follows that if Ω_0 is an open bounded connected domain of $\mathbb{R}^2$ and if

$$\|\psi\| \stackrel{\text{def}}{=} (\|\psi\|_{0,\Omega_0}^2 + \|\nabla\psi\|_{0,\mathbb{R}^2}^2)^{1/2} \quad \text{for} \quad \psi \in W_0^1(\mathbb{R}^2),$$

then $\|\cdot\|$ and $\|\cdot\|_{W_0^1(\mathbb{R}^2)}$ are equivalent norms on $W_0^1(\mathbb{R}^2)$ (the proof is easily obtained *ab absurdo*). Since the norm and semi-norm $\|\cdot\|_{1,\Omega_0}$ and $|\cdot|_{1,\Omega_0}$ are equivalent on

$$\tilde{H}^1(\Omega_0) \stackrel{\text{def}}{=} \left\{g \in H^1(\Omega_0)\,;\ \int_{\Omega_0} g\,dx = 0\right\},$$

it is easy to prove the following result.

THEOREM 3.1 : *Let Ω_0 be an open bounded connected domain of $\mathbb{R}^2$ and let*

$$\tilde{W}_0^1(\mathbb{R}^2) \stackrel{\text{def}}{=} \left\{\psi \in W_0^1(\mathbb{R}^2)\,;\ \int_{\Omega_0} \psi\,dx = 0\right\}.$$

Then, the semi-norm $\|\nabla\psi\|_{0,\mathbb{R}^2}$ is a norm on $\tilde{W}_0^1(\mathbb{R}^2)$ which is equivalent to the norm $\|\psi\|_{W_0^1(\mathbb{R}^2)}$. □

Now, since we shall make use of functions satisfying relationship (2.25), we shall introduce the following notation. If X is a space of functions defined on a domain of $\mathbb{R}^2$ containing Ω_0, $\tilde{X}$ will denote the subspace of functions of X satisfying (2.25), i.e.

$$\tilde{X} = \left\{u \in X\,;\ \int_{\Omega_0} u\,dx = 0\right\}.$$

By tradition, we still denote by $L_0^2(\Omega_0) = \tilde{L}^2(\Omega_0)$.

Finally, we recall a Leray-Schauders's homotopy lemma which constitutes the main tool for proving the existence of a solution for problem (2.23)-(2.30).

THEOREM 3.2 [6] : *Let X be a Banach space of norm* $\| \cdot \|$ *and let T stand for a continuous and compact operator from X into X. Assume there exists a constant C such that each solution of the equation*

$$u = \lambda T(u), \quad 0 \leqslant \lambda \leqslant 1, \tag{3.3}$$

satisfies the bound

$$\|u\| \leqslant C. \tag{3.4}$$

Then, T has at least one fixed point in X. □

In all the following C, C_1, C_2, ... will denote generic positive constants and will have no connection with the constants C_0, C_1, C_2 introduced in Section 2.

4. EXISTENCE OF SOLUTIONS

In this section, existence of a solution of problem (2.23)-(2.30) is proved by using Theorem 3.2. Naturally, the main task we are assigned is to choose a Banach space X and a compact operator T corresponding to our problem such that estimate (3.4) is satisfied.

We have chosen to seek solutions $(\phi, \mathbf{u}, p)$ of Problem (2.23)-(2.30) such that $(\phi, \mathbf{u}, p) \in W^{1,4}(\Omega_0) \times H_0^1(\Omega_0)^2 \times L_0^2(\Omega_0)$. For this choice we shall show hereafter that Problem (2.23)-(2.30) is well-posed. The proof will contain the following steps : we first state some properties of the electromagnetic problem (2.23)-(2.27) for a given velocity field. Required *a priori* estimates are next given for Problem (2.23)-(2.30). Finally, compactness arguments allow us to apply Theorem 3.2.

Let us define the Hilbert space

$$V = \{\mathbf{v} \in H_0^1(\Omega_0)^2 ; \operatorname{div} \mathbf{v} = 0\}, \tag{4.1}$$

and consider the following auxiliary problem :

Given $\mathbf{u} \in V$, $f \in L_0^2(\Omega)$;

find $\phi \in H_{\mathrm{loc}}^2(\mathbb{R}^2)$, $(\alpha, \beta) \in \mathbb{C}^2$ such that

$$-\Delta\phi + i\omega\mu_0\sigma_0\phi + \mu_0\sigma_0\mathbf{u}\cdot\nabla\phi = f, \quad \text{in } \Omega_0, \tag{4.2}$$

$$-\Delta\phi + i\omega\mu_0\sigma_k(\phi - I_k(\phi)) = f, \quad \text{in } \Omega_k,\ k = 1, 2, \tag{4.3}$$

$$\Delta\phi = 0, \quad \text{in } \Omega' = \mathbb{R}^2\backslash\bar{\Omega}, \tag{4.4}$$

$$\phi(x) = \alpha \log|x| + \beta + O(|x|^{-1}), \quad |x| \to +\infty, \tag{4.5}$$

$$\int_{\Omega_0} \phi(x)\, dx = 0. \tag{4.6}$$

THEOREM 4.1 : *Let* $\mathbf{u} \in V$, $f \in L_0^2(\Omega)$ *be given. Then problem* (4.2)-(4.6) *has a unique solution* (ϕ, α, β). *Moreover,* $\phi \in W_0^1(\mathbb{R}^2)$ *which implies* $\alpha = 0$. *In addition, there exists a constant* $C > 0$, *independent of* $\mathbf{u}$ *and* f, *such that*

$$\| \phi \|_{1, \Omega_0} \leq C \| f \|_{0, \Omega} . \tag{4.7}$$

Proof : Let (ϕ, α, β) be a solution of Problem (4.2)-(4.6). Clearly, from the potential theory, there exist $R > 0$, $C > 0$ and $\psi \in H^2_{\mathrm{loc}}(\mathbb{R}^2)$ such that for $|x| > R$:

$$\phi(x) = \alpha \log |x| + \beta + \psi(x) ,$$

with
$$|\psi(x)| \leq \frac{C}{|x|}, \quad |\nabla \psi(x)| \leq \frac{C}{|x|^2} .$$

Let B_r denote the ball centered in 0 and of radius $r \geq R$ (we assume that R is large enough in order to have $\bar{\Omega} \subset B_R$). Integrating equations (4.2), (4.3), (4.4) on B_r yields :

$$- \int_{B_r} \Delta \phi \, dx + \mu_0 \sigma_0 \int_{\Omega_0} \mathbf{u} \cdot \nabla \phi \, dx = \int_{\Omega} f \, dx .$$

By using the Green's formula together with the fact that $f \in L_0^2(\Omega)$, $\operatorname{div} \mathbf{u} = 0$ in Ω_0 and $\mathbf{u} = 0$ on Γ_0, we obtain

$$\int_{\partial B_r} \frac{\partial \phi}{\partial n} \, ds = 0 .$$

It follows that $2 \pi \alpha = O\left(\frac{1}{r}\right)$ and letting $r \to +\infty$, we obtain $\alpha = 0$. In other words, we have proved that all the solutions (ϕ, α, β) of (4.2)-(4.6) are such that $\alpha = 0$. We are then allowed to seek ϕ in the space $W_0^1(\mathbb{R}^2)$.

In Theorem 3.1, we have seen that the space $\tilde{W}_0^1(\mathbb{R}^2)$, equipped with the norm $|\cdot|_{1, \mathbb{R}^2}$ is a Hilbert space. Let us now define the following sesquilinear form on $W_0^1(\mathbb{R}^2)$:

$$a(\phi, \psi) = \int_{\mathbb{R}^2} \nabla \phi \cdot \nabla \psi^* \, dx + i \omega \mu_0 \sum_{k=0}^{2} \sigma_k \times$$
$$\times \left(\int_{\Omega_k} \phi \psi^* \, dx - I_k(\phi) \int_{\Omega_k} \psi^* \, dx \right) + \mu_0 \sigma_0 \int_{\Omega_0} (\mathbf{u} \cdot \nabla \phi) \psi^* \, dx .$$

We set the following problem :

Find $\phi \in \tilde{W}_0^1(\mathbb{R}^2)$ such that

$$a(\phi, \psi) = (f, \psi), \quad \forall \psi \in \tilde{W}_0^1(\mathbb{R}^2), \tag{4.8}$$

where $(., .)$ denotes the $L^2(\Omega)$-inner product.

If (ϕ, α, β) is a solution of Problem (4.2)-(4.6) then $\alpha = 0$ and clearly ϕ is a solution of (4.8). Conversely, if ϕ is a solution of Problem (4.8) then (4.2)-(4.4) and (4.6) are satisfied and ϕ belongs to $H^2_{\mathrm{loc}}(\mathbb{R}^2)$. Since ϕ is harmonic outside the compact domain $\bar{\Omega}$ and because $\phi \in W_0^1(\mathbb{R}^2)$, there exists $\beta \in \mathbb{C}$ such that $\phi(x) = \beta + O(|x|^{-1})$ when $|x| \to \infty$ and $(\phi, 0, \beta)$ is a solution of Problem (4.2)-(4.6). In this sense, Problem (4.8) is equivalent to Problem (4.2)-(4.6).

We have from the definition of a and since $\mathbf{u} \in V$:

$$\begin{aligned}\operatorname{Re}\,(a(\phi, \phi)) &= |\phi|^2_{1, \mathbb{R}^2} + \mu_0\,\sigma_0 \operatorname{Re}\left(\int_{\Omega_0} (\mathbf{u} \,.\, \nabla\phi)\,\phi^*\,dx\right) \\ &= |\phi|^2_{1, \mathbb{R}^2}, \quad \text{for all} \quad \phi \in \tilde{W}_0^1(\mathbb{R}^2)\,.\end{aligned}$$

Moreover, the continuity of the form a is obvious. By Lax-Milgram's theorem we then conclude to the existence and uniqueness of a solution of (4.8). The estimate (4.7) follows by putting $\psi = \phi$ in (4.8) and taking the real part of the resulting equality together with Cauchy-Schwarz inequality and the equivalence of norms $|\,.\,|_{1, \Omega_0}$ and $\|\,.\,\|_{1, \Omega_0}$ in $\tilde{H}^1(\Omega_0) = \left\{g \in H_0^1(\Omega_0)\,;\, \int_{\Omega_0} g\,dx = 0\right\}$. □

Remark 4.1 . The above existence result is not necessary for the proof of the existence of solutions of (2.23)-(2.30). We have mentioned it here for the sake of completeness. The estimate (4.7) is however basic for the sequel. □

Let us now define the operator

$$T_1 : g \in L_0^2(\Omega) \mapsto T_1\,g = \phi\,|_{\Omega_0} \in \tilde{H}^2(\Omega_0)$$

where ϕ is the solution of Problem (4.2)-(4.6) with $\mathbf{u} = 0$ and $f = g$. By classical interior regularity results for elliptic problems (*cf.* [7]) we deduce from (4.7) that $T_1 :\, L_0^2(\Omega) \to \tilde{H}^2(\Omega_0)$ is continuous. Furthermore, since the imbedding $\tilde{H}^2(\Omega_0) \to \tilde{W}^{1,4}(\Omega_0)$ is compact, the operator $T_1 : L_0^2(\Omega) \to \tilde{W}^{1,4}(\Omega_0)$ is compact.

Notice that if $\mathbf{u} \in V$, we can extend it by zero to $\Omega \backslash \Omega_0$, and if $\phi \in W^{1,4}(\Omega_0)$ we have $\mathbf{u} \cdot \nabla\phi \in L_0^2(\Omega)$ since $\operatorname{div} \mathbf{u} = 0$ in Ω_0 and $\mathbf{u} = 0$ on Γ_0. Therefore, if $f \in L_0^2(\Omega)$ and $\mathbf{u} \in V$ are given, the problem :

$$\text{Find } \phi \in \tilde{W}^{1,4}(\Omega_0) \text{ such that } \phi = T_1(f - \sigma \mathbf{u} \cdot \nabla\phi) \tag{4.9}$$

is meaningful. Moreover, if ϕ is solution of Problem (4.2)-(4.6) then $\phi|_{\Omega_0}$ is solution of Problem (4.9). Conversely, if ϕ is solution of (4.9) then $\phi \in \tilde{H}^2(\Omega_0)$ and there exists ψ solution of (4.2)-(4.6) such that $\phi = \psi|_{\Omega_0}$. In this sense, Problems (4.2)-(4.6) and (4.9) are equivalent.

Let us consider now the fluid flow problem. Given a function $\mathbf{g} \in L^{4/3}(\Omega_0)^2$, we consider the Stokes problem :

Find $(\mathbf{w}, p) \in H_0^1(\Omega_0)^2 \times L_0^2(\Omega_0)$ such that

$$-\nu \Delta\mathbf{w} + \nabla p = \mathbf{g}, \quad \text{in } \Omega_0, \tag{4.10}$$

$$\operatorname{div} \mathbf{w} = 0 \quad \text{in } \Omega_0. \tag{4.11}$$

It is clear that Problem (4.10)-(4.11) has a unique solution (*cf.* [8]). Moreover, using regularity results for the Stokes problem (*cf.* [8], the boundary Γ_0 being of class C^1), we deduce that the operator

$$T_2 : \mathbf{g} \in L^{4/3}(\Omega_0)^2 \mapsto T_2\,\mathbf{g} = \mathbf{w} \in V$$

is linear and compact. Now, since $\mathbf{u} \in V$ and $\phi \in W^{1,4}(\Omega_0)$ we have from Hölder's inequalities that the functions $(\mathbf{u} \cdot \nabla)\mathbf{u}$, $(\mathbf{u} \cdot \nabla\phi_R)\nabla\phi_R$, $(\mathbf{u} \cdot \nabla\phi_I)\nabla\phi_I$, $\phi_I \nabla\phi_R$ and $\phi_R \nabla\phi_I$ belong to $L^{4/3}(\Omega_0)^2$. Problem (2.23)-(2.30) is therefore equivalent to the problem :

Find $(\phi, \mathbf{u}) \in \tilde{W}^{1,4}(\Omega_0) \times V$ such that

$$\phi = \lambda T_1 f_1(\phi, \mathbf{u}), \tag{4.12}$$

$$\mathbf{u} = \lambda T_2 \mathbf{f}_2(\phi, \mathbf{u}), \tag{4.13}$$

where

$$\lambda = 1,$$

$$f_1(\phi, \mathbf{u}) = \mu_0(J_k - \sigma_k \mathbf{u} \cdot \nabla\phi) \text{ in } \Omega_k,\ k = 0, 1, 2, \ (\mathbf{u} = 0 \text{ in } \Omega_1 \cup \Omega_2),$$

$$\mathbf{f}_2(\phi, \mathbf{u}) = -\mathbf{u} \cdot \nabla\mathbf{u} + \frac{\sigma_0 \omega}{2\rho}(\phi_I \nabla\phi_R - \phi_R \nabla\Phi_I) - \frac{\sigma_0}{2\rho}((\mathbf{u} \cdot \nabla\phi_R)\nabla\phi_R + (\mathbf{u} \cdot \nabla\phi_I)\nabla\phi_I).$$

The main estimate is given in the following result.

THEOREM 4.2 : *There exists a positive constant C, independent of J and ν, such that* $\forall \lambda \in [0, 1]$, *if* $(\phi, \mathbf{u})$ *is solution of* (4.12)-(4.13), *then*

$$\sqrt{\nu}\,\|\mathbf{u}\|_{1, \Omega_0} + \|\mathbf{u} \cdot \nabla\phi_R\|_{0, \Omega_0} + \|\mathbf{u} \cdot \nabla\phi_I\|_{0, \Omega_0} + \|\phi\|_{1, 4, \Omega_0} \leq CJ\,. \quad (4.14)$$

Proof : Let $(\phi, \mathbf{u})$ be a solution of Problem (4.12)-(4.13). By Theorem 4.1, where $\mathbf{u}$ and f are respectively replaced by $\lambda\mathbf{u}$ and λg, with $g = \mu_0 J_k$ in Ω_k, $k = 0, 1, 2$, we have

$$\|\phi\|_{1, \Omega} \leq CJ\,, \quad (4.15)$$

where C is independent of λ. Now, equation (4.13) implies there exists a unique $p \in L_0^2(\Omega_0)$ such that $(\mathbf{u}, p)$ satisfies :

$$\begin{aligned} -\nu\,\Delta\mathbf{u} + \nabla p &= \lambda\,\mathbf{f}_2(\phi, \mathbf{u}) && \text{in } \Omega_0\,, \\ \operatorname{div}\mathbf{u} &= 0 && \text{in } \Omega_0\,, \\ \mathbf{u} &= 0 && \text{on } \Gamma_0\,. \end{aligned}$$

Hence, by multiplying by $\mathbf{u}$ and using Green's formula, we obtain

$$\nu\,\|\mathbf{u}\|_{1, \Omega_0}^2 + \frac{\sigma_0\,\lambda}{2\,\rho}\,\|\mathbf{u} \cdot \nabla\phi_R\|_{0, \Omega_0}^2 + \frac{\sigma_0\,\lambda}{2\,\rho}\,\|\mathbf{u} \cdot \nabla\phi_I\|_{0, \Omega_0}^2 =$$
$$= -\frac{\sigma_0\,\omega\,\lambda}{2\,\rho}\,((\mathbf{u} \cdot \nabla\phi_I, \phi_R) - (\mathbf{u} \cdot \nabla\phi_R, \phi_I))\,,$$

where $(\,.,\,.\,)$ denotes the inner product in $L^2(\Omega_0)$. The Cauchy-Schwarz inequality yields :

$$\begin{aligned} \omega\,|(\mathbf{u} \cdot \nabla\phi_I, \phi_R)| &\leq \omega\,\|\mathbf{u} \cdot \nabla\phi_I\|_{0, \Omega_0}\,\|\phi_R\|_{0, \Omega_0} \\ &\leq \frac{1}{2}\,\|\mathbf{u} \cdot \nabla\phi_I\|_{0, \Omega_0}^2 + \frac{\omega^2}{2}\,\|\phi_R\|_{0, \Omega_0}^2 \\ &\leq \frac{1}{2}\,\|\mathbf{u} \cdot \nabla\phi_I\|_{0, \Omega_0}^2 + \frac{\omega^2}{2}\,\|\phi_R\|_{1, \Omega_0}^2\,. \end{aligned}$$

Similarly, we have

$$\omega\,|(\mathbf{u} \cdot \nabla\phi_R, \phi_I)| \leq \frac{1}{2}\,\|\mathbf{u} \cdot \nabla\phi_R\|_{0, \Omega_0}^2 + \frac{\omega^2}{2}\,\|\phi_I\|_{1, \Omega_0}^2\,.$$

Therefore, by (4.15) :

$$\nu\,\|\mathbf{u}\|_{1, \Omega_0}^2 + \frac{\sigma_0\,\lambda}{4\,\rho}\,(\|\mathbf{u} \cdot \nabla\phi_R\|_{0, \Omega_0}^2 + \|\mathbf{u} \cdot \nabla\phi_I\|_{0, \Omega_0}^2) \leq$$
$$\leq \frac{\sigma_0\,\lambda\,\omega^2}{4\,\rho}\,(\|\phi_R\|_{1, \Omega_0}^2 + \|\phi_I\|_{1, \Omega_0}^2) \leq C\,\lambda\,J^2\,. \quad (4.16)$$

Inequality (4.16) shows that the three first terms of (4.14) are bounded by CJ. To obtain the bound for the last term, we first remark that the estimate (4.16) implies

$$\|f_1(\phi, \mathbf{u})\|_{0, \Omega_0} \leq CJ .$$

The continuity of the operator $T_1 : L_0^2(\Omega) \rightarrow \tilde{W}^{1,4}(\Omega_0)$ implies then :

$$\| \phi \|_{1, 4, \Omega_0} \leq CJ , \tag{4.17}$$

which finally gives the desired result. □

We are now ready to prove the existence theorem.

THEOREM 4.3 : *Problem* (2.23)-(2.30) *has at least one solution.*

Proof : Clearly, it is sufficient to prove that Problem (4.12)-(4.13) has at least one solution for $\lambda = 1$. This is done by applying Theorem 3.2 with

$$X = \tilde{W}^{1,4}(\Omega_0) \times V , \quad T = \begin{pmatrix} T_1 f_1 \\ T_2 \mathbf{f}_2 \end{pmatrix} .$$

Here X is considered as a real Banach space. The continuity of T is obvious. Furthermore, it is clear that the image of each bounded subset of $\tilde{W}^{1,4}(\Omega_0) \times V$ by the mapping

$$(\phi, \mathbf{u}) \mapsto (f_1(\phi, \mathbf{u}), \mathbf{f}_2(\phi, \mathbf{u}))$$

is bounded in $L_0^2(\Omega) \times L^{4/3}(\Omega_0)^2$. The compactness of the operators T_1 and T_2 implies then the compactness of $T : X \rightarrow X$ which, by Theorems 3.2 and 4.2, yields the existence of a solution of Problem (4.12)-(4.13) for any $\lambda \in [0, 1]$. □

5. A UNIQUENESS RESULT

In this section we seek conditions under which uniqueness of the solutions of Problem (2.23)-(2.30) can be guaranteed. To do this, a scaling is performed by setting $\hat{\phi} = \nu^{-1/2} \phi$. Problem (4.12)-(4.13) with $\lambda = 1$ can then be transformed into the problem to find $(\hat{\phi}, \mathbf{u}) \in X \overset{\text{def}}{=} \tilde{W}^{1,4}(\Omega_0) \times V$ such that :

$$\hat{\phi} = T_1 \hat{f}_1(\hat{\phi}, \mathbf{u}), \mathbf{u} = T_2 \hat{\mathbf{f}}_2(\hat{\phi}, \mathbf{u}) \tag{5.1}$$

where

$$\hat{f}_1(\hat{\phi}, \mathbf{u}) = \mu_0(J_k\, \nu^{-1/2} - \sigma_k\, \mathbf{u} \,.\, \nabla\phi) \quad \text{in } \Omega_k, \quad k = 0, 1, 2\,, \tag{5.2}$$

$$\hat{\mathbf{f}}_2(\hat{\phi}, \mathbf{u}) = -\mathbf{u} \,.\, \nabla\mathbf{u} - \frac{\sigma_0\, \nu}{2\,\rho}\,((\mathbf{u} \,.\, \nabla\hat{\phi}_R)\,\nabla\hat{\phi}_R + (\mathbf{u} \,.\, \nabla\hat{\phi}_I)\,\nabla\hat{\phi}_I) - \frac{\sigma_0\, \omega\, \nu}{2\,\rho}\,(\hat{\phi}_R\,\nabla\hat{\phi}_I - \hat{\phi}_I\,\nabla\hat{\phi}_R)\,. \tag{5.3}$$

If $\hat{T} : (\hat{\phi}, \mathbf{u}) \in X \mapsto \hat{T}(\hat{\phi}, \mathbf{u}) \in X$ is the mapping defined by

$$\hat{T}(\hat{\phi}, \mathbf{u}) = (T_1\, \hat{f}_1(\hat{\phi}, \mathbf{u}), T_2\, \hat{\mathbf{f}}_2(\hat{\phi}, \mathbf{u}))\,,$$

then Problem (5.1) is equivalent to find the fixed points of $\hat{T}$. Moreover, it is easy to see that Problem (2.23)-(2.30) has a unique solution if and only if $\hat{T}$ has a unique fixed point. In order to find sufficient conditions to obtain the uniqueness of a fixed point of $\hat{T}$, we calculate $\|D\hat{T}(\hat{\phi}, \mathbf{u})\|_{\mathscr{L}(X)}$ where $D\hat{T}(\hat{\phi}, \mathbf{u})$ is the Fréchet derivative of $\hat{T}$ at a point $(\hat{\phi}, \mathbf{u}) \in X$ and $\|\,.\,\|_{\mathscr{L}(X)}$ is the norm of continuous linear operators from X into X.

THEOREM 5.1 : *There is a constant C, independent of ν and J, such that for all $(\hat{\phi}, \mathbf{u}) \in X$ we have :*

$$\|D\hat{T}(\hat{\phi}, \mathbf{u})\|_{\mathscr{L}(X)} \leq C\left((1 + \nu^{-1})(\|\mathbf{u}\|_{1,\Omega_0} + \|\mathbf{u}\|^2_{1,\Omega_0}) + \|\hat{\phi}\|_{1,4,\Omega_0} + \|\hat{\phi}\|^2_{1,4,\Omega_0}\right)\,. \tag{5.4}$$

Proof : We denote by $D_{\hat{\phi}}$, D_u the partial derivatives with respect to $\hat{\phi}$ and $\mathbf{u}$ and we obtain for all pairs $(\hat{\phi}, \mathbf{u})$, $(\hat{\psi}, \mathbf{v}) \in X$:

$$D_{\hat{\phi}}\, T_1\, \hat{f}_1(\hat{\phi}, \mathbf{u})\,\hat{\psi} = -\mu_0\, T_1(-\sigma_0\, \mathbf{u} \,.\, \nabla\hat{\psi})\,;$$

$$D_u\, T_1\, \hat{f}_1(\hat{\phi}, \mathbf{u})\,\mathbf{v} = -\mu_0\, T_1(-\sigma_0\, \mathbf{v} \,.\, \nabla\hat{\phi})\,;$$

$$D_{\hat{\phi}}\, T_2\, \hat{f}_2(\hat{\phi}, \mathbf{u})\,\hat{\psi} = T_2\Big[-\frac{\sigma_0\, \nu}{2\,\rho}\,((\mathbf{u} \,.\, \nabla\hat{\phi}_R)\,\nabla\hat{\psi}_R + (\mathbf{u} \,.\, \nabla\hat{\phi}_I)\,\nabla\hat{\psi}_I + (\mathbf{u} \,.\, \nabla\hat{\psi}_R)\,\nabla\hat{\phi}_R + (\mathbf{u} \,.\, \nabla\hat{\psi}_I)\,\nabla\hat{\phi}_I) - \frac{\sigma_0\, \omega\, \nu}{2\,\rho}\,(\hat{\phi}_R\,\nabla\hat{\psi}_I - \hat{\phi}_I\,\nabla\hat{\psi}_R + \hat{\psi}_R\,\nabla\hat{\phi}_I - \hat{\psi}_I\,\nabla\hat{\phi}_R)\Big]\,;$$

$$D_u\, T_2\, \hat{f}_2(\hat{\phi}, \mathbf{u})\,\mathbf{v} = T_2\Big[-\mathbf{u} \,.\, \nabla\mathbf{v} - \mathbf{v} \,.\, \nabla\mathbf{u} - \frac{\sigma_0\, \nu}{2\,\rho}\,((\mathbf{v} \,.\, \nabla\hat{\phi}_R)\,\nabla\hat{\phi}_R + (\mathbf{v} \,.\, \nabla\hat{\phi}_I)\,\nabla\hat{\phi}_I)\Big]\,.$$

If, for a continuous linear operator $A : Y \to Z$ where Y and Z are two Banach spaces, we denote by $\|A\|_{\mathscr{L}(Y, Z)}$ its norm, we easily prove by using Hölder's inequalities that :

$$\left\|D\hat{T}(\hat{\phi}, \mathbf{u})(\hat{\psi}, \mathbf{v})\right\|_X \leqslant C\Big[\|T_1\|_{\mathscr{L}(L^2(\Omega_0), W^{1,4}(\Omega_0))} \left(\|\mathbf{u}\|_{1,\Omega_0} \|\hat{\psi}\|_{1,4,\Omega_0} + \|\mathbf{v}\|_{1,\Omega_0} \|\hat{\phi}\|_{1,4,\Omega_0} \right)$$
$$+ \|T_2\|_{\mathscr{L}(L^{4/3}(\Omega_0)^2, V)} \Big\{ \nu \Big(\|\mathbf{u}\|_{1,\Omega_0} \|\hat{\phi}\|_{1,4,\Omega_0} \|\hat{\psi}\|_{1,4,\Omega_0}$$
$$+ \|\hat{\phi}\|_{1,4,\Omega_0} \|\hat{\psi}\|_{1,4,\Omega_0} + \|\mathbf{v}\|_{1,\Omega_0} \|\hat{\phi}\|^2_{1,4,\Omega_0} \Big) + \|\mathbf{u}\|_{1,\Omega_0} \|\mathbf{v}\|_{1,\Omega_0} \Big\} \Big]$$

where C is independent of J and ν. By noticing that

$$\|T_1\|_{\mathscr{L}(L^2(\Omega_0), W^{1,4}(\Omega_0))} \leqslant C \quad \text{and} \quad \|T_2\|_{\mathscr{L}(L^{4/3}(\Omega_0)^2, V)} \leqslant C\nu^{-1}$$

where C is independent of J and ν, we finally obtain for all $(\hat{\phi}, \mathbf{u}) \in X$:

$$\left\|D\hat{T}(\hat{\phi}, \mathbf{u})\right\|_{\mathscr{L}(X)} \leqslant C\Big[(1 + \nu^{-1}) \|\mathbf{u}\|_{1,\Omega_0} + \|\hat{\phi}\|_{1,4,\Omega_0}$$
$$+ \|\mathbf{u}\|_{1,\Omega_0} \|\hat{\phi}\|_{1,4,\Omega_0} + \|\hat{\phi}\|^2_{1,4,\Omega_0} \Big] .$$

By using the inequality $ab \leqslant \frac{1}{2}(a^2 + b^2)$ we thus have proven (5.4). □

THEOREM 5.2 : *Let* σ_k, $k = 0, 1, 2$, ρ *and* ω *be given positive real numbers. Then, for all* $\nu^* > 0$ *there exists a positive real number* γ *such that for all* $\nu \geqslant \nu^*$ *and all* $J \geqslant 0$ *satisfying* $J \leqslant \gamma \nu^{1/2}$ *Problem* (2.23)-(2.30) *has a unique solution.*

Proof : Clearly, Problem (2.23)-(2.30) has a unique solution if and only if Problem (4.12)-(4.13) has a unique solution $(\phi, \mathbf{u})$ for $\lambda = 1$ or, equivalently, if and only if $\hat{T}$ has a unique fixed point $(\hat{\phi}, \mathbf{u})$ with $\hat{\phi} = \nu^{-1/2}\phi$.

From Theorem 4.2 we deduce that if $(\hat{\phi}, \mathbf{u})$ is a fixed point of $\hat{T}$, then

$$\|\mathbf{u}\|_{1,\Omega_0} + \|\hat{\phi}\|_{1,4,\Omega_0} \leqslant \hat{C} J \nu^{-1/2}, \tag{5.5}$$

where $\hat{C}$ is independent of J, ν and $(\hat{\phi}, \mathbf{u})$. We prove, from Theorem 5.1, that for all $\nu^* > 0$ there exists $\varepsilon = \varepsilon(\nu^*) > 0$ such that for all $(\hat{\psi}, \mathbf{v}) \in X$ satisfying $\|\mathbf{v}\|_{1, \Omega_0} + \|\hat{\psi}\|_{1, 4, \Omega_0} \leq \varepsilon$ and for all $\nu > \nu^*$:

$$\left\|D\hat{T}(\hat{\psi}, \mathbf{v})\right\|_{\mathscr{L}(X)} < 1 . \tag{5.6}$$

By choosing $\gamma = \varepsilon/\hat{C}$ we deduce from (5.5) and (5.6) that if $\nu > \nu^*$ and $J \leq \gamma \nu^{1/2}$, then $\hat{T}$ has *at most* one fixed point. The existence is proven in Theorem 4.3. □

We conclude this paper by some remarks.

Remark 5.1 : Let us fix all the physical parameters in Problem (2.23)-(2.30) except J and look for the solutions $(\phi, \mathbf{u}, p, \alpha, \beta)$ of this problem as functions of J. Trivially, if $J = 0$ we have the unique solution (0, 0, 0, 0, 0). The proof of Theorem 5.2 shows that for small J we can apply the implicit function theorem in order to obtain the existence of a solution branch $(\phi, \mathbf{u}, p, \alpha, \beta)$ parametrized by J, starting from the trivial solution.

Remark 5.2 : The arbitrary choice of J and ν as parameters in the statement of the uniqueness result is made in Theorem 5.2 to simplify the presentation. We can prove, for example, that if ν and J are given, uniqueness holds if σ_0 is small enough.

Remark 5.3 : As we have mentioned it in Section 2, prescribing a slip boundary condition for $\mathbf{u}$ instead of Dirichlet boundary conditions can also be considered. In this case, the Stokes problem can be written in terms of a stream-function/vorticity formulation whose standard regularity results (*cf.* [7]) yield the same properties for the operator T_2 as those used for the proof of existence and uniqueness of solutions.

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