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ANALYSIS OF THE SCHWARZ ALGORITHM FOR MIXED FINITE ELEMENTS METHODS (*)

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Abstract — *The Schwarz alternating algorithm [25, 21, 4, 27] is considered in this paper for mixed finite element methods for second-order elliptic equations General convergence estimates similar to [4] are established In particular, we determine a uniform convergence rate for the algorithm associated with a fine-coarse domain decomposition [17]*

Résumé — *L'algorithme alterné de Schwarz [25, 21, 4, 27] est considéré dans cet article pour des méthodes d'éléments finis mixtes pour des équations elliptiques du second ordre Des estimations de convergence générale semblables à [4] sont établies En particulier, nous déterminons un taux de convergence uniforme pour l'algorithme associé à un domaine de décomposition fin-grossier [17]*

1. INTRODUCTION

Our object in this paper is to study the convergence of the Schwarz alternating algorithm applied to mixed finite element methods for second-order elliptic equations. To illustrate the method, we take as our model the homogeneous Neumann boundary value problem

$$\begin{aligned} -\nabla \,.\, (a(x)\,\nabla p) &= f\,, \quad \text{in} \quad \Omega\,, \\ a(x)\,\nabla p \,.\, \nu &= 0\,, \quad \text{on} \quad \partial\Omega\,, \end{aligned} \tag{1.1}$$

where Ω is a polygonal domain in $\mathbf{R}^2$ and ν denotes the unit outward normal vector to the boundary $\partial\Omega$. ∇ and $\nabla\,.$ indicate the gradient and the divergence operators, respectively. The flux variable,

$$\mathbf{u} = -\,a\,\nabla p\,, \tag{1.2}$$

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is of interest in many physical problems and will be introduced in order to be approximated directly. The function p will be called the « pressure » throughout this paper.

Let $c(x) = a(x)^{-1}$. Let $(.,.)$ denote the inner product in $L^2(\Omega)$ or $L^2(\Omega)^2$. Set

$$\mathbf{V} = H_0(\operatorname{div}; \Omega) = \{\mathbf{v} \in L^2(\Omega)^2 ; \nabla . \mathbf{v} \in L^2(\Omega) \text{ and } \mathbf{v} . \nu = 0 \text{ on } \partial\Omega\} ;$$

this is clearly a Hilbert space for the norm :

$$\|\mathbf{v}\|_H = (\|\mathbf{v}\|_0^2 + \|\nabla . \mathbf{v}\|_0^2)^{1/2} .$$

Also, let $W = L_0^2(\Omega)$ be the closed subspace of $L^2(\Omega)$ having functions with vanishing mean value. The weak form of (1.1) that leads to the mixed finite element method is given by seeking $(\mathbf{u} ; p) \in \mathbf{V} \times W$ satisfying the equations

$$\begin{aligned} (c\mathbf{u}, \mathbf{v}) - (\nabla . \mathbf{v}, p) &= 0 , & \mathbf{v} &\in \mathbf{V} , \\ (\nabla . \mathbf{u}, w) &= (f, w) , & w &\in W . \end{aligned} \tag{1.3}$$

The first equation in (1.3) stems from testing (1.2), divided by $a(x)$, against $\mathbf{V}$, and the second from testing (1.1), after substitution using (1.2), against W.

The mixed finite element discretization of (1.3) seeks $(\mathbf{u}_h ; p_h)$ from $\mathbf{V}^h \times W^h$, a mixed finite element space associated with a prescribed triangulation $\mathcal{T}_h$, satisfying

$$\begin{aligned} (c\mathbf{u}_h, \mathbf{v}) - (\nabla . \mathbf{v}, p_h) &= 0 , & \mathbf{v} &\in \mathbf{V}^h , \\ (\nabla . \mathbf{u}_h, w) &= (f, w) , & w &\in W^h . \end{aligned} \tag{1.4}$$

A number of known families of mixed finite element spaces, which yield adequate approximate solutions when used in (1.4), have been constructed in several papers. For completeness, we shall outline some of the families in § 2.

Many physical problems, e.g., petroleum reservoir simulation, modeling of ground-water contamination, elasticity problems, and seismic exploration, involve the need for very accurate determination of the flux function. More accurate approximations of the flux can be achieved through the use of mixed finite element methods particularly in the context of discontinuous coefficients $a(x)$, since the flux is introduced as an independent variable in the method. However, the technique of the mixed method leads to saddle point problems whose numerical solutions have been quite difficult. Thus, fast and efficient algorithms for solving the discretization problem are very important for the application of the mixed method.

The iterative methods to be considered in this paper bear this consideration in mind and provide efficient algorithms for obtaining the numerical solution. The method fits the general strategy of the product iterative method proposed in [4, 27, 28], which is designed for elliptic problems. However, due to the saddle point property of the mixed finite element method, the product algorithm can not be applied directly here. In light of [20, 22], we reduce the saddle point problem to an elliptic problem by eliminating the pressure through the use of substructures of the domain. Thus, the standard iterative methods for elliptic problems can be applied to the reduced problem for the flux and yield a fast solution algorithm.

A general theory analogous to those presented in [4, 27] is established for the convergence of the Schwarz alternating algorithm. In particular, we determine a uniform bound for the fine-coarse domain decomposition structure.

The paper is organized as follows. In § 2, we review some of the known families of mixed finite element spaces. The extension of the Schwarz alternating method is discussed in § 3. In § 4, we establish the theory for the convergence of the method.

2. MIXED FINITE ELEMENT METHODS

It is known that an adequate approximation can be provided by (1.4) if the finite element space $\mathbf{V}^h \times W^h$ satisfies the Babuška-Brezzi stability conditions (*cf.* [1, 6]). More precisely, assume that the bilinear form $(c\,.,\,.\,)$ is coercive in the discrete divergence-free subspace and there exists a positive constant β independent of the mesh size h of $\mathcal{T}_h$ such that

$$\sup_{\mathbf{v} \in \mathbf{V}^h} \frac{(\nabla \cdot \mathbf{v}, w)}{\|\mathbf{v}\|} \geqslant \beta \|w\|^2, \quad w \in W^h. \tag{2.1}$$

A theory of Fortin [19] shows that the stability condition is equivalent to the existence of a locally-defined projection operator Π_h

$$\Pi_h : \quad \mathbf{V} \cap H^1(\Omega)^2 \to \mathbf{V}^h$$

satisfying the commutation property

$$Q_h \nabla \cdot = \nabla \cdot \Pi_h, \quad \text{on} \quad \mathbf{V} \cap H^1(\Omega)^2, \tag{2.2}$$

where Q_h is the local L^2 projection operator from W onto W^h. The general theory of the mixed finite element method also relies on the relation (2.2) and the local nature of Π_h and Q_h (*cf.* [10, 19, 24, 18, 29]).

Our object in this section is to introduce some of the known families of finite element spaces that satisfy (2.2), and hence (2.1). We concentrate on regularly-defined triangulations $\mathcal{T}_h$; the stability analysis for each family to be introduced has been studied in [13] on locally-refined grids. First of all, we review the construction of two families defined on rectangles.

RT Rectangular Elements : The Raviart-Thomas (RT) space [24] of index r on rectangle K for the flux is defined by

$$\mathbf{V}^k(K) = Q_{r+1,r}(K) \times Q_{r,r+1}(K);$$

the corresponding space for the pressure is

$$W^h(K) = Q_{r,r}(K),$$

where $Q_{i,j}(K)$ denotes the polynomials of degree i in the x variable and degree j in the y variable. The projection operator Π_h satisfying (2.2) is defined element-wise by the following degrees of freedom :

$$\begin{aligned} &\langle \mathbf{v} \cdot \nu, p \rangle_e, \quad p \in P_r(e), \text{ all four edges}, \\ &(\mathbf{v}, \boldsymbol{\phi})_K, \quad \boldsymbol{\phi} \in Q_{r-1,r}(K) \times Q_{r,r-1}(K). \end{aligned} \tag{2.3}$$

BDFM Elements : The Brezzi-Douglas-Fortin-Marini (BDFM) spaces (*cf.* [8]) are modifications of the rectangular RT spaces. The space of index j for the flux variable is defined by

$$\mathbf{V}^h(K) = P_j(K) \backslash \{y^j\} \times P_j(K) \backslash \{x^j\};$$

and the corresponding space for the pressure is defined by

$$W^h(K) = P_{j-1}(K),$$

where $P_i(K)$ denotes the polynomials of total degree no larger than i. The projection operator Π_h can be defined similarly using the following degrees of freedom :

$$\begin{aligned} &\langle \mathbf{v} \cdot \nu, p \rangle_e, \quad p \in P_{j-1}(e), \text{ all four edges}, \\ &(\mathbf{v}, \boldsymbol{\phi})_K, \quad \boldsymbol{\phi} \in P_{j-2}(K)^2. \end{aligned}$$

We now turn to some families of the finite element space defined on triangles.

RT Triangular Elements : Let $\mathbf{x} = (x, y)$ be the space variable. The RT space [24] of index j on the triangle K for the flux is defined by

$$\mathbf{V}^h(K) = P_j(K) \oplus \mathbf{x}\hat{P}_j(K),$$

where $\hat{P}_j(K)$ is the space of homogeneous polynomials of degree j on K. The corresponding space for the pressure is given by

$$W^h(K) = P_j(K).$$

The projection operator Π_h can be defined by the following degrees of freedom :

$$\langle \mathbf{v} \cdot \nu, p \rangle_e, \quad p \in P_j(e), \text{ all three edges },$$
$$(\mathbf{v}, \boldsymbol{\phi})_K, \quad \boldsymbol{\phi} \in P_{j-1}(K)^2 .$$

BDM Elements : The Brezzi-Douglas-Marini (BDM) space [9] of index j for the flux variable is defined by

$$\mathbf{V}^h(K) = P_j(K) \times P_j(K) ;$$

and the corresponding pressure space is defined by

$$W^h(K) = P_{j-1}(K) .$$

The projection operator Π_h can be defined element-wise by the following degrees of freedom (*cf.* [9]) :

$$\begin{aligned} \langle \Pi_h \mathbf{v}|_K \cdot \nu, p \rangle_e &= \langle \mathbf{v} \cdot \nu, p \rangle_e, & p &\in P_j(e), \text{ all three edges },\\ \langle \Pi_h \mathbf{v}|_K, \nabla w \rangle_K &= (\mathbf{v}, \nabla w)_K, & w &\in P_{j-1}(K) ,\\ (\Pi_h \mathbf{v}|_K, \mathbf{curl}\, \phi)_K &= (\mathbf{v}, \mathbf{curl}\, \phi)_K, & \phi &\in B_{j+1}(K) , \end{aligned}$$

where $B_{j+1} = \lambda_1 \lambda_2 \lambda_3 P_{j-2}(K)$, and λ_i are the barycentric coordinates of K.

The iterative methods proposed in the next section are applied to a positive definite problem defined on a subspace $\mathscr{H}^h$ of $\mathbf{V}^h$. The subspace $\mathscr{H}^h$ consists of those discrete fluxes that are divergence free ; i.e.,

$$\mathscr{H}^h = \{\mathbf{v} \in \mathbf{V}^h ; \quad \nabla \cdot \mathbf{v} = 0\} .$$

Thus, any flux $\mathbf{v} \in \mathscr{H}^h$ can be expressed as the **curl** of a stream function $\phi \in H^1(\Omega)$. Furthermore, the stream function ϕ is uniquely determined in $H_0^1(\Omega)$, since the flux has zero boundary values in the normal direction to $\partial\Omega$. Denote by $\mathscr{S}^h$ the set of stream functions with vanishing boundary value. The space $\mathscr{S}^h$ shall be termed the stream-function space of the mixed finite element method. Note that any stream function ψ is a continuous piecewise polynomial. Thus, $\mathscr{S}^h$ is a finite element space of C^0-piecewise polynomials associated with the triangulation $\mathcal{T}_h$.

Remark 2.1 : In general, let $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Assume that the discrete flux $\mathbf{v}$ is divergence free and such that

$$\mathbf{v} \cdot \nu = g \quad \text{on} \quad \Gamma_1 .$$

Let $\psi \in H^1(\Omega)$ be the stream function of v. Note that ψ is a piecewise polynomial. Then, as ψ is continuous on $\bar{\Omega}$, ψ can be uniquely determined by fixing its value at one point of $\bar{\Omega}$. At first, we set $\psi(x_0) = 0$, where x_0 is an arbitrary point of Γ_1 (for instance). Next, we choose a continuous piecewise polynomial χ on Γ_1 which satisfies

$$\frac{\partial \chi}{\partial \tau} = g \quad \text{on} \quad \Gamma_1, \quad \text{and} \quad \chi(x_0) = 0,$$

where $\dfrac{\partial}{\partial \tau}$ denotes the tangential partial derivative on Γ_1. Now the stream function ψ can be defined uniquely by assigning χ to ψ on Γ_1.

The stream-function space for the families mentioned above can be characterized as follows.

THEOREM 2.1 : *Let $\mathcal{S}^h$ denote the stream-function space. Then,*

1. *For the rectangular RT element of index r, we have*

$$\mathcal{S}^h = \left\{\phi \in C^0(\Omega)\,;\, \phi|_K \in Q_{r+1,r+1}(K),\, K \in \mathcal{T}_h\right\}.$$

2. *For the BDFM element of index j, we have*

$$\mathcal{S}^h = \left\{\phi \in C^0(\Omega)\,;\, \phi|_K \in P_{j+1}(K)\backslash\{x^{j+1}, y^{j+1}\},\, K \in \mathcal{T}_h\right\}.$$

3. *For the triangular RT element of index $j \geq 0$, we have*

$$\mathcal{S}^h = \left\{\phi \in C^0(\Omega)\,;\, \phi|_K \in P_{j+1}(K),\, K \in \mathcal{T}_h\right\}.$$

4. *For the BDM element of index $j \geq 1$, we have*

$$\mathcal{S}^h = \left\{\phi \in C^0(\Omega)\,;\, \phi|_K \in P_{j+1}(K),\, K \in \mathcal{T}_h\right\}.$$

Proof : We illustrate the proof for the BDFM and the triangular RT elements only ; the analysis for other families is similar.

First, we consider the triangular RT element of index j. Let $\mathcal{S}^h$ be defined as in the theorem. It is obvious that **curl** ϕ is a discrete flux in the RT space of index j. Further, it is divergence free. Thus, $\mathcal{S}^h$ is a subspace of the stream-function space for the RT element of index j. Conversely, for any $\mathbf{v} \in \mathcal{H}^h$, let $\phi \in H_0^1(\Omega)$ be the stream function of v. Since v is divergence free, we know that $\mathbf{v}|_K \in P_j(K)^2$ on any $K \in \mathcal{T}_h$. Thus, ϕ is a continuous piecewise polynomial of order $j+1$, which implies $\phi \in \mathcal{S}^h$.

Secondly, let

$$\mathcal{S}^h = \left\{\phi \in C^0(\Omega)\,;\, \phi|_K \in P_{j+1}(K)\backslash\{x^{j+1}, y^{j+1}\},\, K \in \mathcal{T}_h\right\}. \quad (2.4)$$

The space $\mathscr{S}^h$ is well defined by (2.4). Actually, a polynomial $\phi \in P_{j+1}(\hat{K}) \setminus \{x^{j+1}, y^{j+1}\}$ is uniquely determined by the following degrees of freedom on the reference element $\hat{K}$:

$$\phi(-1, m_i),\ \phi(1, m_i),\ \phi(m_i, -1),\ \phi(m_i, 1)\,,$$

$$m_i = -1 + 2\,i/j\,, \quad i = 0, 1, \ldots, j$$

$$(\phi, \theta)_{\hat{K}}\,, \quad \theta \in P_{j-3}(\hat{K})\,.$$

Thus, reasoning the same as above shows that $\mathscr{S}^h$ defined by (2.4) is the stream-function space for the BDFM elements. □

Remark 2.2 : The stream-function space and its applications in domain decomposition and multilevel decomposition iterative methods are discussed in [12] for the Douglas-Wang (DW) elements.

3. SCHWARZ ALTERNATING ALGORITHM

Assume that we have an overlapping domain decomposition for Ω which aligns with $\mathcal{T}_h$ on the boundary ; i.e., there exist subdomains $\Omega_i \subset \Omega$, for $i = 1, \ldots, J$, such that

$$\Omega = \bigcup_{i=1}^{J} \Omega_i\,.$$

Further, for any element $K \in \mathcal{T}_h$ and index i, K either is entirely in Ω_i or has an empty intersection with Ω_i. Thus, the restriction of $\mathcal{T}_h$ on Ω_i provides a regularly-defined triangulation $\mathcal{T}_i$ for Ω_i. Let $\mathbf{V}_i^h \times W_i^h$ be the corresponding finite element space associated with $\mathcal{T}_i$. Analogously, set

$$\mathscr{H}_i^h = \left\{ \mathbf{v} \in \mathbf{V}_i^h\,;\, \nabla \cdot \mathbf{v} = 0 \right\}.$$

The first step in the Schwarz alternating method involves seeking a discrete flux $\mathbf{u}^* \in \mathbf{V}^h$ such that

$$\nabla \cdot \mathbf{u}^* = f^h\,, \tag{3.1}$$

where $f^h \in W^h$ is the discretization of f defined by

$$(f^h, w) = (f, w)\,, \quad w \in W^h;$$

i.e., f^h is the local L^2-projection of f in W^h. To obtain such a flux $\mathbf{u}^*$, let $\mathcal{T}_0 = \{K_i\}_{i=1}^{L}$ be a « coarse » triangulation of Ω whose elements align with those of $\mathcal{T}_h$ on the boundary. Hence, $\mathcal{T}_h$ can be regarded as a refinement of $\mathcal{T}_0$. As before, let $\tilde{\mathbf{V}}_i^h \times \tilde{W}_i^h$ be the finite element space associated with the triangulation $K_{i,h}$, which is the restriction of $\mathcal{T}_h$ on K_i. Let f_0^h be the L^2 projection of f^h in the space $\tilde{W}_0^h$ and $f_i^h \in \tilde{W}_i^h$ be the restriction of $f^h - f_0^h$ on K_i. It follows that

$$f^h = f_0^h + \sum_{i=1}^{L} f_i^h . \tag{3.2}$$

Schwarz Algorithm (Part 1) :

(1) For each i, where $0 \leq i \leq L$, find $(\mathbf{u}_i^* ; p_i^*) \in \tilde{\mathbf{V}}_i^h \times \tilde{W}_i^h$ such that

$$\begin{aligned} (\tilde{c}\mathbf{u}_i^*, \mathbf{v}) - (\nabla . \mathbf{v}, p_i^*) &= 0 , & \mathbf{v} \in \tilde{\mathbf{V}}_i^h , \\ (\nabla . \mathbf{u}_i^*, w) &= (f_i^h, w) , & w \in \tilde{W}_i^h , \end{aligned} \tag{3.3}$$

where $\tilde{c}$ is an arbitrary positive function on Ω.

(2) Set $\mathbf{u}^* = \sum_{i=0}^{L} \mathbf{u}_i^*$.

Remark 3.1 : As mentioned in (3.3), the coefficient $\tilde{c}$ is quite general in the computation. This is because we only care to have some discrete flux satisfying the second equation of (3.3). Therefore, one may, for instance, take $\tilde{c} = 1$ or $\tilde{c} = c$ for the sake of convenience in the real computation.

THEOREM 3.1 : *Let the discrete flux* $\mathbf{u}^*$ *be obtained as above. Then,*

$$\nabla . \mathbf{u}^* = f^h .$$

Proof : It follows from (3.3) that

$$\nabla . \mathbf{u}_i^* = f_i^h ,$$

for $i = 0, 1, \dots, L$. This, together with (3.2), completes the proof of the theorem. □

Now the saddle point problem (1.4) can be reduced to a positive definite problem as follows : By setting

$$\hat{\mathbf{u}}^h = \mathbf{u}_h - \mathbf{u}^* ,$$

we see that (1.4) is equivalent to seeking $(\hat{\mathbf{u}}^h ; p_h)$ satisfying

$$\begin{aligned} (c\hat{\mathbf{u}}^h, \mathbf{v}) - (\nabla . \mathbf{v}, p_h) &= -(c\mathbf{u}^*, \mathbf{v}) , & \mathbf{v} \in \mathbf{V}^h , \\ (\nabla . \hat{\mathbf{u}}^h, w) &= 0 , & w \in W^h . \end{aligned} \tag{3.4}$$

Thus, the new flux $\hat{\mathbf{u}}^h$ is divergence free and can be identified by the following problem :

$$(c\hat{\mathbf{u}}^h, \mathbf{v}) = -(c\mathbf{u}^*, \mathbf{v}), \quad \mathbf{v} \in \mathcal{H}^h. \tag{3.5}$$

It is clear that (3.5) is selfadjoint and positive definite, and hence the standard Schwarz alternating method can be applied. Let P_i be the projection operator from $\mathcal{H}^h$ to $\mathcal{H}_i^h$ defined by

$$(cP_i \xi, \mathbf{v}) = (c\xi, \mathbf{v}), \quad \xi \in \mathcal{H}^h, \quad \mathbf{v} \in \mathcal{H}_i^h. \tag{3.6}$$

Assume, throughout this paper, that ω is a real number in (0, 2).

Schwarz Algorithm-1 (Part 2) : Given $\hat{\mathbf{u}}_n^h \in \mathcal{H}^h$, an approximation to (3.5), we seek the next approximate solution $\hat{\mathbf{u}}_{n+1}^h \in \mathcal{H}^h$ as follows :

(1) Let $Z_0 = \hat{\mathbf{u}}_n^h$ and define $Z_i \in \mathcal{H}^h$, for $i = 1, \dots, J$, by

$$Z_i = Z_{i-1} + \omega P_i(\hat{\mathbf{u}}^h - Z_{i-1}).$$

(2) Set $\hat{\mathbf{u}}_{n+1}^h = Z_J$.

The substructure $\{\Omega_i\}_{i=1}^J$ used in the Schwarz algorithm-1 (Part 2) is quite general in the construction. A convergence estimate for the general substructure will be established in § 4 under some assumptions. However, there are some practically important substructures for which a better rate of convergence is possible. Let us consider the « two level » domain decomposition which was described in [16, 17]. Starting from a « coarse » triangulation $\mathcal{T}_0$ of mesh size h_0, which could be the one that was used to construct $\mathbf{u}^*$ in Part 1 (for instance), we construct subdomains Ω_i by expanding the element $K_i \in \mathcal{T}_0$ by a prescribed distance $d = O(h_0)$; the part outside Ω will be omitted. It follows that $\{\Omega_i\}_{i=1}^J$ forms an overlapping domain decomposition of Ω. The Schwarz algorithm-1 (Part 2) can be applied to this substructure and, as in the case for second-order elliptic equation, yields a convergence rate bounded by $1 - O(h_0^2)$ (see Theorem 4.2). In light of the multigrid method, we make use of the « coarse » triangulation $\mathcal{T}_0$. Let $\mathbf{V}_0^h \times W_0^h$ be the finite element space associated with $\mathcal{T}_0$ and $\mathcal{H}_0^h$ be a subspace of $\mathbf{V}_0^h$ consisting of flux elements with divergence free. Then, a slight modification of the Schwarz algorithm-1 (Part 2) can be stated as follows :

Schwarz Algorithm-2 (Part 2) : Given $\hat{\mathbf{u}}_n^h \in \mathcal{H}^h$, an approximate solution from (3.4), we seek the next approximate solution $\hat{\mathbf{u}}_{n+1}^h \in \mathcal{H}^h$ as follows :

(1) Let $Z_{-1} = \hat{\mathbf{u}}_n^h$ and define $Z_i \in \mathcal{H}^h$, for $i = 0, 1, \dots, J$, by

$$Z_i = Z_{i-1} + \omega P_i(\hat{\mathbf{u}}^h - Z_{i-1}).$$

(2) Set $\hat{\mathbf{u}}_{n+1}^h = Z_J$.

The Schwarz Algorithm (Part 2) involves some projection operator P_i onto the subspace $\mathscr{H}_i^h$. Since it is generally very hard to find a nodal basis for $\mathscr{H}_i^h$, a direct computation of P_i v is almost impossible in practice. We propose two approaches which lead to an easy determination of P_i v. The first is based on the stream-function space and can be illustrated as follows. Let $\mathscr{S}_i^h$ be the corresponding stream-function space over Ω_i. Denote by $\tilde{a}(\cdot, \cdot)$ the bilinear form defined by

$$\tilde{a}(\phi, \psi) = (c\,\mathbf{curl}\,\phi, \mathbf{curl}\,\psi), \quad \phi, \psi \in \mathscr{S}_i^h .$$

THEOREM 3.2 : *For any $\boldsymbol{\xi} \in \mathscr{H}^h$, let $\eta_i \in \mathscr{S}_i^h$ be defined by*

$$\tilde{a}(\eta_i, \psi) = (c\boldsymbol{\xi}, \mathbf{curl}\,\psi), \quad \psi \in \mathscr{S}_i^h . \tag{3.7}$$

Then,

$$P_i\,\boldsymbol{\xi} = \mathbf{curl}\,\eta_i . \tag{3.8}$$

Proof : Note that the operator **curl** is bijective from $\mathscr{S}_i^h$ to $\mathscr{H}_i^h$. Then, as $\tilde{a}(\cdot, \cdot) = (c\,\cdot, \cdot)$, (3.7) is equivalent to

$$(c\,\mathbf{curl}\,\eta_i, \mathbf{v}) = (c\boldsymbol{\xi}, \mathbf{v}), \quad \mathbf{v} \in \mathscr{H}_i^h ,$$

which, together with (3.6), implies (3.8). □

Remark 3.1 : Theorem 3.2 shows that the action $P_i\,\boldsymbol{\xi}$ can be calculated through the computation of a second-order elliptic problem in the standard Galerkin finite element space. This idea can obviously be applied to the computation of $\hat{\mathbf{u}}^h$. Actually, the same reasoning shows that

$$\hat{\mathbf{u}}^h = \mathbf{curl}\,\eta , \tag{3.9}$$

for some $\eta \in \mathscr{S}^h$ defined by

$$\tilde{a}(\eta, \psi) = -(c\mathbf{u}^*, \mathbf{curl}\,\psi), \quad \psi \in \mathscr{S}^h , \tag{3.10}$$

where, as before, $\tilde{a}(\eta, \psi) = (c\,\mathbf{curl}\,\eta, \mathbf{curl}\,\psi)$. It is clear that (3.10) is the standard Galerkin method for a second-order elliptic equation. Thus, the reduced mixed finite element method is equivalent to a standard Galerkin method and all the existing results in domain decomposition and preconditioning techniques are applicable. However, the problem (3.7) is different from the standard Galerkin method applied directly to (1.1) for the pressure only. The elliptic problem (3.7) is equivalent to the mixed finite element method for (1.1), and hence provides a more accurate approximate flux, especially for problems with discontinuous coefficient $a(x)$.

Remark 3.2 : The technique developed in this section can be extended to problems with mixed Dirichlet-Neumann boundary values for the second-order elliptic equation. To see this, one can use a similar Schwarz algorithm (Part 1) to achieve a reduced problem (3.5) in which the new flux $\hat{\mathbf{u}}^h$ takes a given boundary value in the outward normal direction to the Neumann boundary. Then, by applying Remark 2.1, the reduced problem is equivalent to (3.10) in which η is known on the Neumann boundary and arbitrary on the Dirichlet boundary. We emphasize that the test function ψ should be zero on the Neumann boundary as well.

The second approach to the computation of $P_i\,\boldsymbol{\xi}$, as suggested in [22], can be obtained by solving a saddle point problem on Ω_i as follows. Let $(\boldsymbol{\xi}_i^h\,;\,\theta_i^h) \in \mathbf{V}_i^h \times W_i^h$ be defined such that

$$\begin{aligned} (c\boldsymbol{\xi}_i^h, \mathbf{v}) - (\nabla\,.\,\mathbf{v}, \theta_i^h) &= (c\boldsymbol{\xi}, \mathbf{v})\,, \quad && \mathbf{v} \in \mathbf{V}_i^h\,, \\ (\nabla\,.\,\boldsymbol{\xi}_i^h, w) &= 0\,, && w \in W_i^h\,. \end{aligned} \tag{3.11}$$

Then, it is obvious that

$$\boldsymbol{\xi}_i^h = P_i\,\boldsymbol{\xi}\,.$$

4. CONVERGENCE ANALYSIS

In this section, we establish the convergence of the Schwarz alternating algorithm proposed in Section 3. Note that the methods are essentially applied to a selfadjoint and positive definite problem. The general result developed in [4, 27] (see also [26, 28]) can be employed to yield some estimates.

For completeness, we cite the result of [27] (see also [4]) as follows : Let $a(\,.,\,.\,)$ be a symmetric and coercive bilinear form defined on a Hilbert space V. Assume that V_i, $i = 1, \ldots, J$, are closed subspaces of V satisfying

$$V = \sum_{i=1}^{J} V_i\,.$$

Let P_i be the projection operator onto V_i with respect to the form $a(\,.,\,.\,)$. The main result in [27] is concerned with the norm estimate of the product operator E :

$$E = (I - \omega P_J)(I - \omega P_{J-1}) \ldots (I - \omega P_1)\,,$$

where ω is any real number in (0, 2).

Assume that for any $v \in V$ there exist $v_i \in V_i$, for $i = 1, \ldots, J$, such that $v = \sum_{i=1}^{J} v_i$ satisfying

$$\sum_{i=1}^{J} \|v_i\|^2 \leqslant C_1 \|v\|^2\,, \tag{4.1}$$

$$\sum_{i=2}^{J} \|P_i\, w_i\|^2 \leqslant C_0 \|v\|^2\,, \tag{4.2}$$

and

$$\sum_{j=1}^{J-1} \|P_j w_{j+1}\|^2 \leq C_2 \|v\|^2 \tag{4.3}$$

for some constants C_1, C_0, and C_2, where $w_j = \sum_{k=j}^{J} v_k$ and $\| \cdot \| = a(\cdot, \cdot)^{1/2}$. The estimate of the norm of E is stated as follows.

THEOREM 4.1 : *Assume that* (4.1), (4.2), *and* (4.3) *hold. Then*

$$\|Eu\|^2 \leq \gamma \|u\|^2 , \quad u \in V, \tag{4.4}$$

where

$$\gamma = 1 - \frac{\omega (2 - \omega)}{2(\omega^2 C_2 + C_1)} \tag{4.5}$$

or

$$\gamma = 1 - \frac{1}{C_0} \quad if \quad \omega = 1 . \tag{4.6}$$

We now apply Theorem 4.1 to mixed finite elements. Since no real iteration is performed in the Schwarz Algorithm (Part 1), we shall consider the convergence of the method of Part 2 only. Note that the algorithm is a particular case of the general product algorithm (*cf.* [4, 5, 26, 27, 28]), then we have :

LEMMA 4.1 : *Let* $(\hat{\mathbf{u}}^h, p_h)$ *be the solution of* (3.4) *and* $\hat{\mathbf{u}}_n^h$ *be an approximate flux given by the Schwarz Algorithm*-1 *(Part* 2*). Let* $e_n = \hat{\mathbf{u}}^h - \hat{\mathbf{u}}_n^h$ *be the error at step n. Then,*

$$e_{n+1} = Ee_n , \tag{4.7}$$

where

$$E = (I - \omega P_J) \dots (I - \omega P_1) .$$

Now we apply Theorem 4.1 to the product operator E of (4.7). It is clear that it suffices to check the assumptions of Theorem 4.1. Note that, in this application, the Hilbert space $V = \mathscr{H}^h$ and the bilinear form is given by $a(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{v})_c$. Thus, we naturally have $\|\mathbf{v}\| = (\mathbf{v}, \mathbf{v})_c^{1/2}$ in this section. Assume that there exist functions $\phi_i(x) \in W^{1,\infty}(\Omega)$ such that $\phi_i \geq 0$ and $\phi_i = 0$ on $\Omega \backslash \Omega_i$ for $i = 1, \dots, J$. Further, assume that $\sum_{i=1}^{J} \phi_i \equiv 1$ holds on Ω and there exists a constant C satisfying

$$|\nabla \phi_i| \leq \frac{C}{d} \tag{4.8}$$

and

$$|\nabla \psi_i| \leq \frac{C}{d}, \tag{4.9}$$

where $\psi_i = \sum_{k=i}^{J} \phi_k$ and d is a parameter which, in general, characterizes the size of the overlapped subdomain.

LEMMA 4.2 : *Assume the existence of a partition of unity* $\{\phi_i\}_{i=1}^{J}$ *satisfying* (4.8) *and* (4.9). *Then for any* $\mathbf{v} \in \mathscr{H}^h$, *there exists a partition* $\mathbf{v} = \sum_{i=1}^{J} \mathbf{v}_i$, $\mathbf{v}_i \in \mathscr{H}_i^h$, *such that* (4.1), (4.2), *and* (4.3) *are valid for some constants* C_1, C_0, *and* C_2. *Furthermore,*

$$C_i = O(J/d^2), \quad i = 0, 1, 2. \tag{4.10}$$

Proof: For any $\mathbf{v} \in \mathscr{H}^h$, let σ be the stream function of $\mathbf{v}$; i.e., $\sigma \in \mathscr{S}^h$ and satisfies

$$\mathbf{curl}\, \sigma = \mathbf{v}. \tag{4.11}$$

Set

$$\mathbf{v}_i = \mathbf{curl}\, I_h(\phi_i \sigma), \tag{4.12}$$

where I_h is the nodal interpolation operator onto the stream-function space $\mathscr{S}^h$. It follows that $\mathbf{v}_i \in \mathscr{H}_i^h$ and

$$\mathbf{v} = \sum_{i=1}^{J} \mathbf{v}_i. \tag{4.13}$$

We now estimate the constants C_0, C_1, and C_2 for the decomposition (4.13). We take as our model the estimate of C_0 from (4.2). Set

$$\mathbf{w}_i = \sum_{k=i}^{J} \mathbf{v}_k = \mathbf{curl}\, I_h(\psi_i \sigma).$$

It follows that (*cf.* [17])

$$\begin{aligned} \|P_i \mathbf{w}_i\|^2 &\leq \int_{\Omega_i} c(x)|\mathbf{w}_i|^2\, dx \\ &= \int_{\Omega_i} c(x)\left|\nabla(I_h(\psi_i \sigma))\right|^2 dx \\ &\leq C\left(\int_{\Omega_i} c(x)|\nabla\sigma|^2\, dx + d^{-2}\int_{\Omega_i} c(x)|\sigma|^2\, dx\right) \end{aligned} \tag{4.14}$$

for some constant C. Summing (4.14) over the index i yields

$$\sum_{i=1}^{J} \|P_i \mathbf{w}_i\| \leq C \left(\sum_{i=1}^{J} \int_{\Omega_i} c(x) |\nabla \sigma|^2 \, dx + d^{-2} \sum_{i=1}^{J} \int_{\Omega_i} c(x) |\sigma|^2 \, dx \right)$$
$$\leq CJ/d^2 \int_{\Omega} c(x) |\nabla \sigma|^2 \, dx \leq CJ/d^2 \|\mathbf{v}\|^2 ,$$

which completes the proof of the Lemma. □

Thus, combining Theorem 4.1 and Lemma 4.2 yields the following result.

THEOREM 4.2 : *Under the assumptions of Lemma* 4.2, *there is a constant* $\tilde{C}$ *such that the convergence of the Schwarz algorithm-1 (Part* 2) *is bounded by*

$$\gamma_0 = 1 - \frac{\omega (2 - \omega) d^2}{\tilde{C} J} . \tag{4.15}$$

We see from (4.15) that the convergence rate for the Schwarz algorithm-1 (Part 2) has an upper bound dependent upon two parameters d and J, which characterize the size of the overlapped subdomain and the number of subdomains, respectively. We emphasize that, in the estimate (4.15), J could be replaced by N_0 defined by

$$N_0 = \max_{x \in \Omega} N_x , \tag{4.16}$$

where N_x denotes the number of subdomains containing $x \in \Omega$. The number N_0 is apparently bounded from above by J. However, in some important applications, the number N_0 could be independent of the number of subdomains J. We consider, for example, the substructure $\{\Omega_i\}_{i=1}^{J}$ obtained by expanding each element of the coarse level $\mathcal{T}_0$ by the prescribed distance $d = O(h_0)$, which was used to define the Schwarz algorithm-2 (Part 2). It is clear that the number N_0 is a constant independent of J. However, the parameter d, which is proportional to h_0, contributes a negative effect to the convergence as a small number. As in the case for second-order elliptic problem, the use of the coarse level can balance this negative effect and yield uniform convergence for the method. The result is presented as follows.

LEMMA 4.3 : *Let* $\{\Omega_i\}_{i=1}^{J}$ *be the substructure obtained from the coarse triangulation* $\mathcal{T}_0$ *by expanding each element with a prescribed distance* $d = O(h_0)$. *Then, for any* $\mathbf{v} \in \mathcal{H}^h$, *there exists a partition* $\mathbf{v} = \sum_{i=0}^{J} \mathbf{v}_i$ *with*

$\mathbf{v}_i \in \mathscr{H}_i^h$ *such that* (4.1), (4.2), *and* (4.3) *hold for some constants. Further, we have*

$$C_i = O(1), \quad i = 0, 1, 2. \tag{4.17}$$

Proof : The proof is similar to that of Lemma 4.2. For completeness, we outline the idea as follows : Let $\mathbf{v} \in \mathscr{H}^h$ and σ be its stream function. Let σ_0 be the L^2-projection of σ in S_0^h and set

$$\hat{\sigma} = \sigma - \sigma_0 .$$

Analogously to (4.12), set $\mathbf{v}_i = \mathbf{curl}\, I_h(\phi_i\, \hat{\sigma})$. It follows that

$$\mathbf{v} = \sum_{i=0}^{J} \mathbf{v}_i , \tag{4.18}$$

where $\mathbf{v}_0 = \mathbf{curl}\, \sigma_0$. Note that, for this substructure, the existence of a partition of unity $\{\phi_i\}_{i=1}^{J}$ satisfying (4.8) and (4.9) is straightforward. To estimate the constants C_i, $i = 0, 1, 2$, for the partition (4.18), we use (4.14), with σ substituted by $\hat{\sigma}$, to obtain

$$\|P_i \mathbf{w}_i\|^2 \leqslant C \left(\int_{\Omega_i} c(x) |\nabla \hat{\sigma}|^2 \, dx + h_0^{-2} \int_{\Omega_i} c(x) |\hat{\sigma}|^2 \, dx \right), \tag{4.19}$$

for $i \geqslant 1$. As for $P_0 \mathbf{w}_0$, observe that $\mathbf{w}_0 = \mathbf{v}_0 + \mathbf{w}_1$. Thus,

$$\begin{aligned} \|P_0 \mathbf{w}_0\|^2 &\leqslant \\ &\leqslant 2(\|\mathbf{v}_0\|^2 + \|\mathbf{w}_1\|^2) \\ &\leqslant C \left(\int_{\Omega} c(x) |\nabla \sigma_0|^2 \, dx + \int_{\Omega} c(x) |\nabla \hat{\sigma}|^2 \, dx + h_0^{-2} \int_{\Omega} c(x) |\hat{\sigma}|^2 \, dx \right). \end{aligned} \tag{4.20}$$

Combining (4.19) and (4.20), together with the fact that Ω_i overlapes with only a fixed number of Ω_k's (i.e., N_0 is an integer independent of J), yields

$$\begin{aligned} \sum_{i=0}^{J} \|P_i \mathbf{w}_i\|^2 &\leqslant \\ &\leqslant C \left(\int_{\Omega} c(x) |\nabla \sigma_0|^2 \, dx + \int_{\Omega} c(x) |\nabla \hat{\sigma}|^2 \, dx + h_0^{-1} \int_{\Omega} c(x) |\hat{\sigma}|^2 \, dx \right) \\ &\leqslant C \int_{\Omega} c(x) |\nabla \sigma|^2 \, dx \leqslant C \|\mathbf{v}\|^2 . \end{aligned} \tag{4.21}$$

Here we have used the estimates

$$\int_{\Omega} c(x)\,|\nabla\sigma_0|^2\,dx \leq C \int_{\Omega} c(x)\,|\nabla\sigma|^2\,dx$$

and

$$\int_{\Omega} c(x)\,|\nabla\hat{\sigma}|^2\,dx \leq C h_0^2 \int_{\Omega} c(x)\,|\nabla\sigma|^2\,dx$$

in the derivation of (4 21) This completes the proof of the lemma □

THEOREM 4 3 *Let $\{\Omega_i\}_{i=1}^{J}$ be the substructure of Ω described in Lemma* 4 3 *Then, there exists a constant C such that the convergence of the Schwarz algorithm-2 (Part* 2) *is bounded by*

$$\gamma_1 = 1 - \frac{\omega(2-\omega)}{C}$$

Remark 4 1 The Lemmas 4 2 and 4 3 are essential to the establishment of Theorems 4 2 and 4 3 We point out that a different decomposition of $\mathbf{v} \in \mathscr{H}^h$ may lead to the same result as well To illustrate this, let

$$\mathbf{v}_i = \Pi_h\,\mathbf{curl}\,(\phi_i\,\hat{\sigma}),$$

where Π_h is the locally-defined projection operator onto $\mathbf{V}^h$ described in § 2 It is obvious that (4 18) is still valid and such that

$$C_i = O(1), \quad i = 0, 1, 2$$

Remark 4 2 Some numerical experiments have been conducted to illustrate the efficiency of the algorithms We refer to [14] for details

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