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A CONSERVATIVE PARTICLE APPROXIMATION FOR A BOUNDARY ADVECTION-DIFFUSION PROBLEM (*)

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Abstract — *We present and analyse a purely deterministic particle method for a model advection-diffusion problem with Dirichlet boundary conditions In this method, particles are convected by the vector field and the boundary condition effects, as well as the diffusion effects, are taken into account by a modification of the weights of the particles The order of convergence of the method is of the same kind as in the case of the whole space*

Résumé — *Nous présentons et analysons une méthode particulaire déterministe pour un problème modèle d'advection-diffusion avec conditions aux limites de Dirichlet Dans cette méthode, les particules sont convectées par le champ de vitesse, les conditions aux limites et les effets de diffusion sont pris en compte par une modification des poids des particules L'ordre de convergence de la méthode est du même type que dans le cas de l'espace tout entier*

INTRODUCTION

The vortex method is now commonly used to solve problems at very high Reynolds number for incompressible fluid flows. The method consists in the application of the particle method to the vorticity equation of either the Euler or the Navier-Stokes system. In the inviscid case in two dimensions, the vorticity is convected by the flow at the fluid velocity with no other effect ; the method reduces then to the discretisation of the vorticity into vorticity elements and convection of these elements by the flow.

One of the main interests of the vortex method lies in the fact that very little numerical diffusion is added when compared to other methods such as finite difference or finite element. The method is very well adapted to inviscid flows and the first problem arises with the treatment of viscous

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terms A first answer was given by A Chorin [4] in terms of a random walk method which is based on the addition of a brownian part to the movement of the particles This method is very easy to implement but is noisy and not very accurate, on the other hand it has been possible to extend it to the case of boundary conditions (see C Anderson [1], J Goodman [9]) Particle in cell type methods, based on a coupling of the vortex method with a finite difference method, have also been derived (see S Huberson-A Jollès [11] and G H Cottet [5] for example)

A purely deterministic approach was introduced and studied (see S Huberson [10], G H Cottet-S Mas-Gallic [6] and [7], P Degond-S Mas-Gallic [8]) In contrast to the random walk approach, the basic idea of this method is that the vorticity carried by each particle evolves in time in order to take into account the viscous effects For two-dimensional computations of the Navier-Stokes equation we refer to J P Choquin-S Huberson [2] and B Lucquin-Desreux [13] and to J P Choquin-B Lucquin-Desreux [3] for a comparison between the deterministic and the random methods (see also A Leonard-G Winckelmans [12] for computations in three dimensions)

The aim of this paper is to present and analyse a conservative two-dimensional extension of the deterministic method to the case of Dirichlet boundary conditions (for one dimensional case see also S Mas-Gallic [16] and B Lucquin-Desreux [14] respectively for non conservative and conservative methods) The basic idea of the method is to add to the usual vorticity an extra term with support in a neighbourhood of the boundary The vorticity creation due to the boundary is modelled by an increase of the weights of the existing particles rather than by a creation of new particles A boundary integral equation formulation is used to construct the method, and an auxiliary unknown which represents the normal derivative of the vorticity on the boundary is introduced Although the analysis is presented here in the two dimensional case, it would be analogous in any dimension larger than one Let us notice that it is possible to treat the case of Neumann boundary conditions by the same kind of method and this will be the purpose of a forthcoming work

An outline of the paper is as follows We consider a convection-diffusion equation with boundary conditions and we want to solve it by a particle method The first Section is devoted to the study of the continuous problem , we introduce an integral approximation of Laplace's operator with Dirichlet boundary conditions in which the kernel depends on a regularisation parameter ε and on cut-off functions We prove its conservation property, then its consistency with the diffusion model, and its stability under hypotheses on the cut-off functions, examples of which can be found at the end of Section I 2 Finally the convergence of the integro-differential equation solution towards the convection-diffusion equation solution is proved in Theorem I.1 under a stability condition relating the parameter ε to the square root of the viscosity ν. The same condition was already appearing in the whole space case except for non negative kernels. In the present case, it seems difficult to avoid this condition on account of the correcting term due to the boundary. However, partial results can be obtained in a non negative case with no stability condition. In Section 2, we introduce the particle method and follow the same outline ; the conservation property of the scheme is verified, then the consistency with the integral model, the stability and the convergence of the scheme are proved. The error estimate of Theorem II.1 is similar to the one obtained in the whole space [8]. Let us finally mention that numerical tests have been recently obtained by F. Pépin [18], with an approach which is very close to the one presented here.

Let us now introduce some notations. For a given domain Ω of R^n, we shall later on work in the standard Sobolev spaces

$$W^{m,p}(\Omega) = \{\phi \in L^p(\Omega) ; \partial^\alpha \phi \in L^p(\Omega), |\alpha| \leq m\} ,$$

where m is a non negative integer and $p > 1$ is real. The space $W^{m,p}(\Omega)$ is provided with the norm

$$\|\phi\|_{m,p,\Omega} = \left(\sum_{|\alpha| \leq m} \int_\Omega |\partial^\alpha \phi(x)|^p \, dx\right)^{\frac{1}{p}}$$

and semi-norm

$$|\phi|_{m,p,\Omega} = \left(\sum_{|\alpha| = m} \int_\Omega |\partial^\alpha \phi(x)|^p \, dx\right)^{\frac{1}{p}} .$$

I. THE CONTINUOUS PROBLEM

In the case of the whole space, the approximation of the Laplace operator defined in [8] can be interpreted as an approximated integral representation of the solution u of :

$$(I - \varepsilon^2 \Delta) u = f \quad \text{in} \quad \mathbb{R}^2 .$$

Following the same idea, we derive (see [16]) from the problem

$$(I - \varepsilon^2 \Delta) u = f \quad \text{in} \quad \mathbb{R}^2 , \quad u(x_1, 0) = g(x_1) , \quad x_1 \in \mathbb{R}$$

an approximation of the Laplace operator with Dirichlet boundary conditions. We now describe the method in the particular case $g = 0$, although the analysis could be achieved in the non homogeneous case.

Let u be the solution of the following advection-diffusion problem in the two dimensional half-space, with initial data u_0 and homogeneous Dirichlet boundary condition

$$\frac{\partial u}{\partial t} + \operatorname{div}(au) - \nu\,\Delta u = 0 \quad \text{in} \quad \mathbb{R}_+^2 \times (0, T)\,, \tag{1.1}$$

$$u(\,.,0) = u_0\,, \tag{1.2}$$

$$u(x_1, 0, t) = 0\,, \quad x_1 \in \mathbb{R}\,, \quad t \in (0, T)\,, \tag{1.3}$$

where T is a non negative number and $a = (a_1, a_2)$ a vector field such that

$$a_2(\,.,0,\,.\,) = 0\,. \tag{1.4}$$

The function u is approximated by the solution u_ε of the integro-differential problem

$$\frac{\partial u_\varepsilon}{\partial t} + \operatorname{div}(au_\varepsilon) - \nu \Delta_\varepsilon u_\varepsilon = 0 \quad \text{in} \quad \mathbb{R}_+^2 \times (0, T)\,, \tag{1.5}$$

$$u_\varepsilon(\,.,0) = u_0\,, \tag{1.6}$$

where Δ_ε is an integral approximation of the Laplace operator with Dirichlet boundary condition. In order to define this operator, we first introduce some notations. Let η be a function, we denote by η_ε the function defined by the usual scaling in two dimensions, that means :

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^2}\,\eta\left(\frac{x}{\varepsilon}\right). \tag{1.7}$$

We then introduce a function $P\eta$ on $\mathbb{R}_+$, as well as its associated function $P\eta_\varepsilon$, by setting :

$$P\eta\,(x_2) = \int_{\mathbb{R}_+^2} \eta\,(y_1, x_2 + y_2)\,dy\,, \quad P\eta_\varepsilon(x_2) = \frac{1}{\varepsilon} P\eta\left(\frac{x_2}{\varepsilon}\right). \tag{1.8}$$

We need two other functions ζ and θ, and the functions ζ_ε, θ_ε obtained by the previous scaling (1.7). The operator Δ_ε is thus defined by

$$\Delta_\varepsilon v = \Lambda_\varepsilon(v, p(v))\,, \tag{1.9}$$

where

$$\Lambda_\varepsilon(v, p)(x) = \frac{1}{\varepsilon^2}\left[\int_{\mathbb{R}_+^2} (v(y) - v(x))\,\eta_\varepsilon(x-y)\,dy - \varepsilon v(x)\,P\eta_\varepsilon(x_2) - \right.$$
$$\left. - \varepsilon^2 \int_{\mathbb{R}} p(y_1)\,\zeta_\varepsilon(x_1 - y_1, x_2)\,dy_1\right], \tag{1.10}$$

and $p(v)$ is a function defined on $\mathbb{R}$ by the integral equation :

$$\int_0^{+\infty} \int_{\mathbb{R}} v(y_1, y_2)\, \theta_\varepsilon(x_1 - y_1, -y_2)\, dy_1\, dy_2 = \\ = \varepsilon^2 \int_{\mathbb{R}} p(v)(y_1)\, \zeta_\varepsilon(x_1 - y_1, 0)\, dy_1 . \quad (1.11)$$

The problem is now to choose the functions η, ζ and θ in such a way that the integral operator Δ_ε is a « good » approximation of Δ. Let us remark that, in the non conservative approach, it is possible to reduce the number of cut-off functions by choosing $\theta = \eta$; in the present case, one more degree of freedom is necessary, for the function θ will be determined, at least « in normal variable », by the conservation relation.

We shall now first examine the conservativity of this approximation, then its consistency and stability, and finally its convergence.

I.1. The conservativity of the integral approximation

The exact solution u of (1.1)-(1-3) satisfies the following relation of conservation

$$\frac{d}{dt} \int_{\mathbb{R}^2_+} u(x, t)\, dx + \nu \int_{\mathbb{R}} q(x_1, t)\, dx_1 = 0 ,$$

where :

$$q(x_1, t) = \frac{\partial u}{\partial x_2}(x_1, 0, t) .$$

We establish now a similar relation of conservation for the solution u_ε of the approximated problem (1 5)-(1 6), (1.9)-(1.11) We set

$$C_\zeta = \left(\int_{\mathbb{R}^2_+} \zeta(x)\, dx \right) / \left(\int_{\mathbb{R}} \zeta(x_1, 0)\, dx_1 \right) .$$

LEMMA I.1 : *We suppose that η is an even function and :*

$$\int_{\mathbb{R}^2_+} \zeta(x)\, dx = 1/2 . \quad (1.12)$$

Then, for any function θ such that

$$x_2 > 0 , \quad \int_{\mathbb{R}} \theta(y_1, -x_2)\, dy_1 = \frac{1}{C_\zeta} P\eta(x_2) , \quad (1.13)$$

we have

$$\frac{d}{dt}\int_{\mathbb{R}^2_+} u_\varepsilon(x,t)\,dx + \nu\int_{\mathbb{R}} q_\varepsilon(x_1,t)\,dx_1 = 0 \tag{1.14}$$

where $q_\varepsilon = p(u_\varepsilon)$, *so that the approximated problem* (1.5)-(1.6), (1.9)-(1.11) *satisfies the same relation of conservation as the continuous one.*

Proof : We first integrate the equation (1.5) over $\mathbb{R}^2_+$:

$$0 = \frac{d}{dt}\int_{\mathbb{R}^2_+} u_\varepsilon(x,t)\,dx - \int_{\mathbb{R}} (a_2 u_\varepsilon)(x_1,0,t)\,dx_1 -$$

$$-\frac{\nu}{\varepsilon^2}\int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+} (u_\varepsilon(y,t) - u_\varepsilon(x,t))\,\eta_\varepsilon(x-y)\,dy\,dx$$

$$+\frac{\nu}{\varepsilon}\int_{\mathbb{R}^2_+} u_\varepsilon(x,t)\,P\eta_\varepsilon(x_2)\,dx + \nu\left(\int_{\mathbb{R}^2_+}\zeta_\varepsilon(x)\,dx\right)\left(\int_{\mathbb{R}} q_\varepsilon(y_1,t)\,dy_1\right).$$

Since η is an even function, the third integral is zero. Let us estimate the fourth one. We integrate the boundary integral equation (1.11) over $\mathbb{R}$ and get :

$$\int_{\mathbb{R}^2_+} u_\varepsilon(y,t)\left(\int_{\mathbb{R}}\theta_\varepsilon(x_1-y_1,-y_2)\,dx_1\right)dy =$$

$$= \varepsilon^2\int_{\mathbb{R}}\zeta_\varepsilon(x_1,0)\,dx_1\int_{\mathbb{R}} q_\varepsilon(y_1,t)\,dy_1\,.$$

On the other hand, for any θ satisfying (1.13), we have :

$$\int_{\mathbb{R}}\theta_\varepsilon(x_1-y_1,-y_2)\,dx_1 = \frac{1}{C_\zeta}P\eta_\varepsilon(y_2)\,.$$

It follows thus that :

$$\frac{1}{\varepsilon}\int_{\mathbb{R}^2_+} u_\varepsilon(x,t)\,P\eta_\varepsilon(x_2)\,dx = \int_{\mathbb{R}^2_+}\zeta_\varepsilon(z)\,dz\int_{\mathbb{R}} q_\varepsilon(y_1,t)\,dy_1\,,$$

and we get :

$$\int_{\mathbb{R}}(a_2 u_\varepsilon)(x_1,0,t)\,dx_1 =$$

$$= \frac{d}{dt}\int_{\mathbb{R}^2_+} u_\varepsilon(x,t)\,dx + 2\nu\int_{\mathbb{R}^2_+}\zeta_\varepsilon(z)\,dz\int_{\mathbb{R}} q_\varepsilon(y_1,t)\,dy_1\,.$$

Since $a_2(\,.,\,0,\,.\,) = 0$, the hypotheses made on ζ finally allow to conclude. $\square$

Remarks :

— Notice that we have assumed that the velocity field is tangential to the boundary since this is the case when considering the Navier-Stokes system ; in addition, this hypothesis allows to obtain the stability of the solution of the starting problem (1.1)-(1.3) in terms of initial data, as will be shown in the remark following the theorem I.1.

In case this condition is not satisfied, the following equation has to be added to (1.5)-(1.6)

$$u_\varepsilon(x_1, 0, t) = 0\,, \quad x_1 \in \mathbb{R}\,, \quad \text{if} \quad a_2(x_1, 0, t) > 0\,, \quad t \in (0, T)\,.$$

— The function θ is not entirely defined by the relation (1.13), but only determined in « normal variable », contrarily to the one dimensional case [16]. This function needs only be defined for $x_2 < 0$, and can be extended by parity to $\mathbb{R}^2$.

— The conservativity does not need any modification in any of the two integral equations (1.10) or (1.11), contrarily to the one dimensional case, where the boundary integral equation had to be modified [14].

— Let G be the one dimensional kernel of $I - \Delta$, that is : $G(x) = \exp(-|x|)/2$. Then the functions $\eta = \zeta = \theta = G_2$, with $G_2(x) = G(x_1)\,G(x_2)$, for $x = (x_1, x_2)$ satisfy the hypotheses of lemma I.1 (with $C_\zeta = 1$ and $P\eta = G$).

The convergence of the particle approximation of the continuous problem (1.5)-(1.6), (1.9)-(1.11) is obtained under the following assumption

$$\zeta(x_1, x_2) = \zeta^{(1)}(x_1)\,\zeta^{(2)}(x_2)\,. \tag{1.15}$$

In that case equations (1.10), (1.11) can be simplified, in so far as the auxiliary unknown p is eliminated between these two equations, and the operator Δ_ε is equivalently defined by :

$$\Delta_\varepsilon v(x) = \frac{1}{\varepsilon^2}\left[\int_{\mathbb{R}^2_+} (v(y) - v(x))\,\eta_\varepsilon(x-y)\,dy - \varepsilon v(x)\,P\eta_\varepsilon(x_2) - \right.$$
$$\left. - \varepsilon\,\frac{\zeta^{(2)}_\varepsilon(x_2)}{\zeta^{(2)}(0)}\int_{\mathbb{R}^2_+} v(y)\,\theta_\varepsilon(x_1 - y_1, -y_2)\,dy\right], \tag{1.16}$$

where :

$$\zeta^{(2)}_\varepsilon(x_\varepsilon) = \frac{1}{\varepsilon}\,\zeta^{(2)}\left(\frac{x_2}{\varepsilon}\right)\,. \tag{1.17}$$

From now on, we therefore suppose condition (1.15) satisfied, although the analysis in the continuous case could be made in the general case.

I.2. The consistency of the integral approximation

We give a first consistency result concerning particular integral kernels We set

$$G_2(x_1, x_2) = G(x_1)\, G(x_2)\,, \quad \text{with} \quad G(x) = \exp(-\ |x|\ )/2$$

PROPOSITION I 1 *Let $p > 1$ We suppose that $\eta = G_2$, $\zeta^{(2)} = G$ and*

$$\theta(x_1, x_2) = 6\, G_2(x_1, x_2) - 2\, G_2\left(x_1, \frac{x_2}{2}\right) + \frac{1}{3} G_2\left(x_1, \frac{x_2}{3}\right)$$

Then, there exists a constant $C = C(p, \eta, \theta, \zeta) > 0$, such that, for any $v \in W^{4\ p}(\mathbb{R}^2_+)$, verifying $v(\,.,\,0) = 0$, we have

$$\|\Delta_\varepsilon v - \Delta v\|_{0\ p\ \mathbb{R}^2_+} \leqslant C\, \varepsilon^2 \|v\|_{4\ p\ \mathbb{R}^2_+}$$

Proof Let us denote by Pv the extension of v in the negative half space defined by

$$Pv(x_1, x_2) = -\,6\, v(x_1, -\,x_2) + 4\, v(x_1, -\,2\,x_2) - v(x_1, -\,3\,x_2)\,, \quad \text{for } x_2 < 0$$

It follows that, setting $I(\eta) = \int_{\mathbb{R}^2} \eta(x)\, dx$, we have

$$\Delta_\varepsilon v(x) = \frac{1}{\varepsilon^2}\left[(Pv * \eta_\varepsilon - I(\eta)\, v)(x)\right]$$

Moreover we have $Pv \in W^{4\ p}(\mathbb{R}^2)$, and the estimate follows from the results of [8] □

Let us now give, in the more general case, an estimate of the consistency error that will be less accurate but sufficient for the convergence of the method

PROPOSITION I 2 *Let $p > 1$ and p^* defined by $1/p + 1/p^* = 1$ We suppose the conditions* (1 12) *and* (1 13) *satisfied, that η is an even function of each variable, and that, for $i = 1, 2$, we have*

$$\int_{\mathbb{R}^2} x_i^2\, \eta(x)\, dx = 2 \tag{1 18}$$

Moreover, we assume that

$$\left|\int_{\mathbb{R}^2_+} z_2\, \eta(z_1, z_2)\, dz\right| < +\infty\,,$$

$$0 < \left|\int_{\mathbb{R}^2_+} \int_0^{+\infty} z_2\, \eta(y_1, y_2 + z_2)\, dz_2\, dy\right| < +\infty\,,$$

Finally, let us suppose that the following conditions are satisfied

— *if* $p = +\infty$, *then for* $ij = 1, 2$:

$$\int_{\mathbb{R}_+^2} |z_i z_j| |\eta(z)| \, dz < +\infty , \quad \int_{\mathbb{R}_+^2} |z_i z_j| |\theta(z_1, -z_2)| \, dz < +\infty \quad (1.19)$$

$$\zeta^{(2)} \in L^{\infty}(\mathbb{R}_+)$$

— *if* $p < +\infty$, *then* $\theta(x_1, x_2) = \theta^{(1)}(x_1) \, \theta^{(2)}(x_2)$ *and for* $ij = 1, 2$:

$$\int_{\mathbb{R}_+^2} |z_i z_j|^{p^*} |\eta(z)| \, dz < +\infty ,$$

$$\int_{\mathbb{R}_+^2} |z_i z_j|^{p^*} |\theta^{(1)}(z_1)| \, |\theta^{(2)}(-z_2)|^{p^*} dz < +\infty \quad (1.20)$$

$$\eta \in L^1(\mathbb{R}_+^2) , \quad \zeta^{(2)} \in L^p(\mathbb{R}_+) , \quad \theta^{(1)} \in L^1(\mathbb{R}) .$$

Then, if the function ζ_2 *is defined, up to a multiplicative constant* λ, *by*

$$\zeta^{(2)}(x_2) = \lambda \frac{\displaystyle\int_{\mathbb{R}_+^2} z_2 \, \eta(z_1, x_2 + z_2) \, dz}{\displaystyle\int_{\mathbb{R}_+^2} \int_0^{+\infty} z_2 \, \eta(y_1, y_2 + z_2) \, dz_2 \, dy} . \quad (1.21)$$

there exists a constant $C = C(p, \eta, \theta, \zeta) > 0$, *such that, for any* $v \in W^{2,p}(\mathbb{R}_+^2)$, *verifying :* $v(\,.,0) = 0$, *we have :*

$$\|\Delta_\varepsilon v - \Delta v\|_{0, p, \mathbb{R}_+^2} \leq C \, \|v\|_{2, p, \mathbb{R}_+^2} .$$

Proof : Let us denote by Pv the odd extension of v in the negative half space, that means :

$$Pv(x_1, x_2) = -v(x_1, -x_2) , \quad \text{for} \quad x_2 < 0 .$$

It follows that, setting again $I(\eta) = \int_{\mathbb{R}^2} \eta(x) \, dx$, we have

$$\Delta_\varepsilon v(x) = \frac{1}{\varepsilon^2} [(Pv * \eta_\varepsilon - I(\eta) \, v)(x) + Q_\varepsilon v(x)] ,$$

where

$$Q_\varepsilon v(x) =$$

$$= \int_{\mathbb{R}_+^2} v(y) \left[\eta_\varepsilon(x_1 - y_1, x_2 + y_2) - \varepsilon \frac{\zeta_\varepsilon^{(2)}(x_2)}{\zeta^{(2)}(0)} \, \theta_\varepsilon(x_1 - y_1, -y_2) \right] dy .$$

We set $x_1 - y_1 = \varepsilon z_1$, $y_2 = \varepsilon z_2$, so that :

$$Q_\varepsilon v(x) = \\ = \int_{\mathbb{R}^2_+} v(x_1 - \varepsilon z_1, \varepsilon z_2)\left[\eta\left(z_1, \frac{x_2}{\varepsilon} + z_2\right) - \varepsilon \frac{\zeta_\varepsilon^{(2)}(x_2)}{\zeta^{(2)}(0)} \theta(z_1, -z_2)\right] dz .$$

As $v(\,.,0) = 0$, the Taylor expansion with integral remainder of v around the point $(x_1, 0)$ gives

$$Q_\varepsilon v(x) = \varepsilon \frac{\partial v}{\partial x_2}(x_1, 0) \times \\ \times \int_{\mathbb{R}^2_+} z_2\left[\eta\left(\frac{x_1}{\varepsilon} - z_1, \frac{x_2}{\varepsilon} + z_2\right) - \varepsilon \frac{\zeta_\varepsilon^{(2)}(x_2)}{\zeta^{(2)}(0)} \theta\left(\frac{x_1}{\varepsilon} - z_1, -z_2\right)\right] dz + \\ + \varepsilon^2 \int_0^1 (1-\lambda) \sum_{i=1}^{i=2}\sum_{j=1}^{j=2}\left[\psi^\lambda_{\varepsilon,i,j} v(x) - \varepsilon \frac{\zeta_\varepsilon^{(2)}(x_2)}{\zeta^{(2)}(0)} \phi^\lambda_{\varepsilon,i,j} v(x_1)\right] d\lambda ,$$

where :

$$\psi^\lambda_{\varepsilon,i,j} v(x) = (-1)^{j-i} \int_{\mathbb{R}^2_+} \frac{\partial^2 v}{\partial x_i \partial x_j}(x_1 - \varepsilon\lambda z_1, \varepsilon\lambda z_2) \times \\ \times z_i z_j \eta\left(z_1, \frac{x_2}{\varepsilon} + z_2\right) dz ,$$

$$\phi^\lambda_{\varepsilon,i,j} v(x_1) = (-1)^{j-i} \int_{\mathbb{R}^2_+} \frac{\partial^2 v}{\partial x_i \partial x_j}(x_1 - \varepsilon\lambda z_1, \varepsilon\lambda z_2) z_i z_j \theta(z_1, -z_2) dz .$$

According to (1.15) we have $C_\zeta = \left(\int_0^{+\infty} \zeta^{(2)}(x_2)\, dx_2\right) / \zeta^{(2)}(0)$. According to (1.21) and (1.13), the ε term in the previous expansion of $Q_\varepsilon v$ is zero. It follows from the hypothesis (1.19) that :

$$\|Q_\varepsilon v\|_{0,\infty,\mathbb{R}^2_+} \leq C\varepsilon^2 \|v\|_{2,\infty,\mathbb{R}^2_+} .$$

This estimate and the results of [8], for which the hypotheses (1.18) on η are needed, allow to conclude in the case $p = +\infty$. For $p < +\infty$, we estimate separately the two terms $\psi^\lambda_{\varepsilon,i,j} v$ and $\phi^\lambda_{\varepsilon,i,j} v$. By Hölder's inequality, we have

$$|\psi^\lambda_{\varepsilon,i,j} v(x)|^p \leq \\ \leq C_{p,i,j}(\eta) \int_{\mathbb{R}^2_+} \left|\frac{\partial^2 v}{\partial x_i \partial x_j}(x_1 - \varepsilon\lambda z_1, \varepsilon\lambda z_2)\right|^p \left|\eta\left(z_1, \frac{x_2}{\varepsilon} + z_2\right)\right| dz ,$$

where, according to (1.20) :

$$C_{p,i,j}(\eta) = \left[\int_{\mathbb{R}_+^2} |y_i\, y_j|^{p^*} \, |\eta(y_1, y_2)| \, dy\right]^{p/p^*} < +\infty ,$$

and integrating over $\mathbb{R}_+^2$ yields :

$$\int_{\mathbb{R}_+^2} |\psi_{\varepsilon,i,j}^{\lambda} v(x)|^p \, dx \leqslant C_{p,i,j}(\eta) \times$$

$$\times \int_{\mathbb{R}_+^2} \left(\int_{\mathbb{R}} \left|\frac{\partial^2 v}{\partial x_i\, \partial x_j}(x_1 - \varepsilon\lambda z_1, \varepsilon\lambda z_2)\right|^p dx_1\right)$$

$$\times \left(\int_0^{+\infty} \left|\eta\left(z_1, \frac{x_2}{\varepsilon} + z_2\right)\right| dx_2\right) dz .$$

We use the change of variables $x_1 - \varepsilon\lambda z_1 = y_1$ in the x_1 integral and $x_2/\varepsilon + z_2 = y_2$ in the x_2 one, to get :

$$\int_{\mathbb{R}_+^2} |\psi_{\varepsilon,i,j}^{\lambda} v(x)|^p \, dx \leqslant$$

$$\leqslant \varepsilon C_{p,i,j}(\eta) \int_{\mathbb{R}_+^2} \left|\frac{\partial^2 v}{\partial x_i\, \partial x_j}(y_1, \varepsilon\lambda z_2)\right|^p dy_1\, dz_2 \int_{\mathbb{R}_+^2} |\eta(z_1, y_2)| \, dz_1\, dy_2 .$$

Finally, by setting $\varepsilon\lambda y_2 = z_2$ in the first integral, we obtain :

$$\|\psi_{\varepsilon,i,j}^{\lambda} v\|_{0,p,\mathbb{R}_+^2} \leqslant [C_{p,i,j}(\eta) \|\eta\|_{0,1,\mathbb{R}_+^2}]^{1/p} \frac{1}{\lambda^{1/p}} \|v\|_{2,p,\mathbb{R}_+^2}$$

As $p > 1$, the function $\lambda^{-1/p}$ is integrable over $(0,1)$, and the estimate of the first term is thus finished by use of (1.20). Using Hölder inequality for the second term, and the definition (1.20) of θ, we have

$$|\phi_{\varepsilon,i,j}^{\lambda} v(x_1)|^p \leqslant C'_{p,i,j}(\theta) \int_{\mathbb{R}_+^2} \left|\frac{\partial^2 v}{\partial x_i\, \partial x_j}(x_1 - \varepsilon\lambda z_1, \varepsilon\lambda z_2)\right|^p |\theta^{(1)}(z_1)| \, dz,$$

with, according to (1.20) :

$$C'_{p,i,j}(\theta) = \left[\int_{\mathbb{R}_+^2} |z_i\, z_j|^{p^*} \, |\theta^{(1)}(z_1)| \, |\theta^{(2)}(-z_2)|^{p^*} dz\right]^{p/p^*} < +\infty .$$

We integrate over $\mathbb{R}$ and set $y_1 = x_1 - \varepsilon\lambda z_1$ in the x_1 integral, which becomes thus independent of z_1, and obtain :

$$\int_{\mathbb{R}} |\phi^{\lambda}_{\varepsilon, i, j} v(x_1)|^p \, dx_1 \leq C'_{p, i, j}(\theta) \int_0^{+\infty} \left(\int_{\mathbb{R}} \left| \frac{\partial^2 v}{\partial x_1 \, \partial x_2} (y_1, \varepsilon\lambda z_2) \right|^p dy_1 \right) \times \times \left(\int_{\mathbb{R}} |\theta^{(1)}(z_1)| \, dz_1 \right) dz_2 .$$

We do the change of variable $y_2 = \varepsilon\lambda z_2$, so that :

$$\int_{\mathbb{R}} |\phi^{\lambda}_{\varepsilon, i, j} v(x_1)|^p \, dx_1 \leq$$

$$\leq \frac{1}{\varepsilon\lambda} C'_{p, i, j}(\theta) \| \theta^{(1)} \|_{0, 1, \mathbb{R}} \left(\int_{\mathbb{R}^2_+} \left| \frac{\partial^2 v}{\partial x_1 \, \partial x_2} (y_1, y_2) \right|^p dy \right) .$$

This, with the estimate

$$\| \zeta^{(2)}(\cdot / \varepsilon) \|_{0, p, \mathbb{R}_+} \leq C \varepsilon^{1/p} ,$$

and the hypotheses (1.20) finally gives :

$$\left\| \varepsilon \frac{\zeta^{(2)}_{\varepsilon}(\cdot)}{\zeta^{(2)}(0)} \phi^{\lambda}_{\varepsilon, i, j} v \right\|_{0, p, \mathbb{R}^2_+} \leq C \frac{1}{\lambda^{1/p}} \| v \|_{2, p, \mathbb{R}^2_+} ,$$

and this concludes the proof. □

Remark 1 : The estimate obtained in the previous proposition does not tend to zero with ε ; anyway, we shall need later on a stability condition of the type

$$\nu \leq C \varepsilon^2 ,$$

which ensures that the source term $\sigma = \nu (\Delta u - \Delta_{\varepsilon} u)$ in the equation satisfied by the error $u - u_{\varepsilon}$ (see Theorem I.1) tends to zero with ε.

Remark 2 : It is possible to obtain similar estimates as those obtained in proposition I.1, under the following assumptions :

$$\theta (x_1, x_2) = 6 \, \tilde{\theta} (x_1, - x_2) - 2 \, \tilde{\theta} \left(x_1, \frac{- x_2}{2} \right) + \frac{1}{3} \tilde{\theta} \left(x_1, \frac{- x_2}{3} \right) ,$$

$$\zeta^{(2)}(x_2) = \lambda \frac{\displaystyle\int_{\mathbb{R}^2_+} z_2^k \, \eta (z_1, x_2 + z_2) \, dz}{\displaystyle\int_{\mathbb{R}^2_+} \int_0^{+\infty} z_2^k \, \eta (y_1, y_2 + z_2) \, dz_2 \, dy} , \quad \text{for} \quad k = 1, 2, 3 .$$

One can check that it implies : $\int_{\mathbb{R}} \eta(x_1, .)\, dx_1 = G$.

Remark 3 : Let us now give some examples of cut-off functions which fulfill the previous assumptions. Rewritting the consistency condition (1.21), and assuming that

$$\int_{\mathbb{R}} \zeta^{(1)}(t)\, dt = 1\,, \quad \eta(x_1, x_2) = \eta^{(1)}(x_1)\, \eta^{(2)}(x_2)\,,$$

we get, for $x_2 > 0$,

$$\zeta^{(2)}(x_2) = \frac{1}{2} \frac{\displaystyle\int_0^{+\infty} z_2\, \eta^{(2)}(x_2 + z_2)\, dz_2}{\displaystyle\int_0^{+\infty}\int_0^{+\infty} z_2\, \eta^{(2)}(y_2 + z_2)\, dy_2\, dz_2}\,.$$

We consider first the so-called step and hat functions, respectively given by

$$\eta^{(2)}(t) = 3 \quad \text{if} \quad |t| \leqslant 1\,;\, 0 \quad \text{otherwise}$$

and

$$\eta^{(2)}(t) = 12(1 - |t|)^+\,;$$

we get respectively

$$\zeta^{(2)}(t) = \frac{3}{2}(1 - |t|)(1 - |t|)^+$$

and

$$\zeta^{(2)}(t) = 2(1 - |t|)^2\,(1 - |t|)^+\,.$$

Notice that the functions $\eta^{(2)}$ and $\zeta^{(2)}$ are both piecewise polynomial but that the degree of $\zeta^{(2)}$ is increased by 2 with respect to that of $\eta^{(2)}$. Notice also that the step function does not fulfill the required smoothness assumption. In fact, convergence can be proved under weaker conditions than those assumed here with a lot more technics (see [19] for example). In the case of the gaussian function

$$\eta^{(2)}(t) = \frac{1}{\sqrt{4\,\pi}} \exp\left(-\frac{t^2}{4}\right),$$

we get

$$\zeta^{(2)}(t) = \frac{1}{3\sqrt{4\,\pi}}\left(2\exp\left(-\frac{t^2}{4}\right) - |t| \int_{|t|}^{+\infty} \exp\left(-\frac{s^2}{4}\right) ds\right).$$

Let us remark that the functions $\eta^{(2)}$ and $\zeta^{(2)}$ have similar expressions and that if $\eta^{(2)}$ is compactly supported, so is $\zeta^{(2)}$.

I.3. The stability of the integral approximation

We recall the approximation (1.16) of Δ

$$\Delta_\varepsilon v(x) = \frac{1}{\varepsilon^2}\left[\int_{\mathbb{R}^2_+} v(y)\,\eta_\varepsilon(x-y)\,dy - v(x)\,I(\eta) - \right.$$
$$\left. - \varepsilon\,\frac{\zeta_\varepsilon^{(2)}(x_2)}{\zeta^{(2)}(0)}\int_{\mathbb{R}^2_+} v(y)\,\theta_\varepsilon(x_1-y_1,-y_2)dy\right],$$

with $I(\eta) = \int_{\mathbb{R}^2} \eta(x)\,dx$, and we easily deduce the following stability result :

PROPOSITION I.3 : *Let $p \geqslant 1$ and p^* defined by : $1/p + 1/p^* = 1$. We suppose that : $\eta \in L^1(\mathbb{R}^2)$, $\zeta^{(2)} \in L^p(\mathbb{R}_+)$, $\theta \in L^1(\mathbb{R}, L^{p^*}(\mathbb{R}_-))$; then the operator Δ_ε is continuous from $L^p(\mathbb{R}^2_+)$ into $L^p(\mathbb{R}^2_+)$, and we have*

$$\|\Delta_\varepsilon v\|_{0,p,\mathbb{R}^2_+} \leqslant \frac{1}{\varepsilon^2} C(\eta, \zeta, \theta)\,\|v\|_{0,p,\mathbb{R}^2_+},$$

where :

$$C(\eta,\zeta,\theta) = \|\eta\|_{0,1,\mathbb{R}^2_+} + |I(\eta)| + \frac{\|\zeta^{(2)}\|_{0,p,\mathbb{R}_+}}{|\zeta^{(2)}(0)|}\,\|\theta\|_{L^1(\mathbb{R},L^{p^*}(\mathbb{R}_-))}.$$

Later on, for the particle method, we shall need $W^{m,p}$ stability results, so that we state now, by simple derivation of (1.16), the :

PROPOSITION I.4 : *Let $p \geqslant 1$, $m \geqslant 0$ and p^* defined by : $1/p + 1/p^* = 1$. We suppose that : $\eta \in W^{m,1}(\mathbb{R}^2_+)$, $\zeta^{(2)} \in W^{m,p}(\mathbb{R}_+)$, $\theta \in W^{m,1}(\mathbb{R}, L^{p^*}(\mathbb{R}_-))$; then there exists a constant $C = C(\eta, \zeta, \theta, m)$ such that :*

$$\|\Delta_\varepsilon v\|_{m,p,\mathbb{R}^2_+} \leqslant \frac{1}{\varepsilon^2} C\left(\|v\|_{m,p,\mathbb{R}^2_+} + \varepsilon^{-m}\right)\|v\|_{0,p,\mathbb{R}^2_+}).$$

The following corollary can thus be immediately deduced.

COROLLARY I.1 : *Let $1 < p \leqslant +\infty$, $m \geqslant 0$ and p^* defined by : $1/p + 1/p^* = 1$. Let us suppose that $a \in L^\infty(0, T\,;\, W^{m+1,\infty}(\mathbb{R}^2_+))$ and*

$$\nu \leqslant C_{st,m}\,\varepsilon^{2+m}. \tag{1.22}$$

If the hypotheses of the previous proposition are satisfied, the solution u_ε *of* (1 5)-(1 6), (1 16) *is in* $L^\infty(0, T, W^{m\,p}(\mathbb{R}^2_+))$, *and there is a constant* $C = C(\eta, \zeta, \theta, a, C_{st\,m}, T)$, *such that, for* $t \leqslant T$

$$\|u_\varepsilon(\cdot, t)\|_{m\,p\,\mathbb{R}^2_+} \leqslant C\,\|u_0\|_{m\,p\,\mathbb{R}^2_+}$$

This quite classical result can be proved, using hypothesis $a_2(\cdot, 0, \cdot) = 0$, by energy inequalities in the L^p case for $p < +\infty$ (see Theorem I 1), and by similar arguments to those developed in [8] in the L^∞ case

Remark Concerning the non negative case We set

$$\Delta_\varepsilon v(x) = \frac{1}{\varepsilon^2}\left[K^\varepsilon v(x) - I(\eta)\,v(x)\right],$$

where the integral operator K^ε is defined by

$$K^\varepsilon v(x) = \int_{\mathbb{R}^2_+} \sigma^\varepsilon(x, y)\,v(y)\,dy,$$

$$\sigma^\varepsilon(x, y) = \eta_\varepsilon(x - y) - \varepsilon\,\frac{\zeta_\varepsilon^{(2)}(x_2)}{\zeta^{(2)}(0)}\,\theta_\varepsilon(x_1 - y_1, -y_2)$$

Then, if for any $(x, y) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+$ we have

$$\sigma^1(x, y) \geqslant 0, \quad \eta(x) \geqslant 0, \quad \eta \neq 0,$$

we obtain, under the condition

$$\|K^1\| = \sup_{x \in \mathbb{R}^2_+} \int_{\mathbb{R}^2_+} \sigma^1(x, y)\,dy \leqslant 1,$$

the following estimate

$$\|u_\varepsilon\|_{L^\infty((0\,T) \times \mathbb{R}^2_+)} \leqslant C\,\|u_0\|_{L^\infty((0\,T) \times \mathbb{R}^2_+)}$$

where the constant C only depends on η, ζ, θ, a, T The stability is thus obtained with no stability condition of the type (1 22) Let us finally remark that the previous conditions are satisfied in the particular case $\eta = \theta = G_2$ and $\zeta^{(2)} = G$

I.4. The convergence of the integral approximation

Let $e_\varepsilon = u - u_\varepsilon$ be the error made in the approximation defined above This error is solution of the integro-differential system

$$\frac{\partial e_\varepsilon}{\partial t} + \operatorname{div}(a e_\varepsilon) - \nu\,\Delta_\varepsilon e_\varepsilon = \nu\,(\Delta u - \Delta_\varepsilon u) \quad \text{in} \quad \mathbb{R}^2_+ \times (0, T), \quad (1\ 23)$$

$$e_\varepsilon(\cdot, 0) = 0 \quad (1\ 24)$$

THEOREM I.1 : *Let* $1 < p \leqslant +\infty$ *and* p^* *defined by* : $1/p + 1/p^* = 1$. *We suppose the hypotheses of propositions* I.2 *and* I.3 *satisfied and that* a *in* $L^\infty(0, T ; W^{1,\infty}(\mathbb{R}^2_+))$. *We suppose the stability condition*

$$\nu \leqslant C_{st,0}\, \varepsilon^2 \tag{1.25}$$

satisfied, then there exists a constant $C = C(\eta, \theta, \zeta, a, p, T, C_{st,0})$, *such that, if* u *solution of* (1.1)-(1.3) *belongs to* $L^\infty(0, T ; W^{2,p}(\mathbb{R}^2_+))$ *and if* u_ε *solution of* (1.5)-(1.6) *and* (1.16) *is in* $L^\infty(0, T ; L^p(\mathbb{R}^2_+))$, *we have* (*for* $t \leqslant T$) :

$$\|(u - u_\varepsilon)(\,.,\, t)\|_{0,p,\mathbb{R}^2_+} \leqslant C\varepsilon^2 \|u_0\|_{2,p,\mathbb{R}^2_+}\,.$$

Proof : Let us consider the case $p < +\infty$. We multiply the equation (1.23) by $|e_\varepsilon|^{p-2}\, e_\varepsilon$ and integrate over $\mathbb{R}^2_+$

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2_+} |e_\varepsilon(x, t)|^p\, dx + \int_{\mathbb{R}^2_+} (\operatorname{div}(a e_\varepsilon)|e_\varepsilon|^{p-2}\, e_\varepsilon)(x, t)\, dx -$$
$$- \nu \int_{\mathbb{R}^2_+} (\Delta_\varepsilon e_\varepsilon |e_\varepsilon|^{p-2}\, e_\varepsilon)(x, t)\, dx$$
$$= \int_{\mathbb{R}^2_+} \sigma(x, t)(|e_\varepsilon|^{p-2}\, e_\varepsilon)(x, t)\, dx\,,$$

where $\sigma = \nu(\Delta u - \Delta_\varepsilon u)$ is proportional to the consistency error of the integral approximation Δ_ε of Δ. We integrate two times by parts in the second integral and, since $a_2(\,.,\, 0,\, .\,) = 0$, we obtain :

$$\int_{\mathbb{R}^2_+} (\operatorname{div}(a\, e_\varepsilon)|e_\varepsilon|^{p-2}\, e_\varepsilon)(x, t)\, dx = -\int_{\mathbb{R}} (a_2 |e_\varepsilon|^p)(x_1, 0, t)\, dx_1 -$$
$$- \frac{p-1}{p}\int_{\mathbb{R}^2_+} \left(\sum_{i=1}^{i=2} a_i \frac{\partial |e_\varepsilon|^p}{\partial x_i}\right)(x, t)\, dx$$
$$= \frac{p-1}{p}\int_{\mathbb{R}^2_+} (\operatorname{div} a |e_\varepsilon|^p)(x, t)\, dx\,.$$

So that we have, by Hölder's inequality :

$$\frac{d}{dt}\|e_\varepsilon(\,.,\, t)\|^p_{0,p,\mathbb{R}^2_+} \leqslant (p-1)\|\operatorname{div} a(\,.,\, t)\|_{0,\infty,\mathbb{R}^2_+}\, \|e_\varepsilon(\,.,\, t)\|^p_{0,p,\mathbb{R}^2_+} +$$
$$+ \nu p\, \|\Delta_\varepsilon e_\varepsilon(\,.,\, t)\|_{0,p,\mathbb{R}^2_+}\, \|e_\varepsilon(\,.,\, t)\|^{p-1}_{0,p,\mathbb{R}^2_+}$$
$$+ p\, \|\sigma(\,.,\, t)\|_{0,p,\mathbb{R}^2_+}\, \|e_\varepsilon(\,.,\, t)\|^{p-1}_{0,p,\mathbb{R}^2_+}\,.$$

The stability result of proposition I 3 gives, setting $C_1 = (p-1)/p \| \operatorname{div} a \|_{L^\infty((0\ T \times \mathbb{R}_+^2))}$

$$\frac{d}{dt} \| e_\varepsilon(\,.,\, t) \|_{0\ p\ \mathbb{R}_+^2} \leqslant \left[C_1 + \frac{\nu}{\varepsilon^2} C(\eta,\, \zeta,\, \theta) \right] \| e_\varepsilon(\,.,\, t) \|_{0\ p\ \mathbb{R}_+^2} + \\ + \| \sigma(\,.,\, t) \|_{0\ p\ \mathbb{R}_+^2}$$

By Gronwall's inequality, and the fact that $e_\varepsilon(\,.,\, 0) = 0$, we obtain, under condition (1 25)

$$\| e_\varepsilon(\,.,\, t) \|_{0\ p\ \mathbb{R}_+^2} \leqslant C\,(\eta,\, \theta,\, \zeta,\, a,\, p,\, T,\, C_{st\ 0}) \| \sigma \|_{L^\infty(0\ T\ L^p(\mathbb{R}_+^2))}$$

Finally, proposition I 2 completes the proof in the L^p case, $p < +\infty$ The proof in the L^∞ case is omitted here, see for example [8] □

Remark A classical result of regularity proves that if the velocity field $a \in L^\infty(0, T, W^{1\ \infty}(\mathbb{R}_+^2))$ and if $u_0 \in W^{2\ p}(\mathbb{R}_+^2)$, the solution u of (1 1)-(1 3) is in $L^\infty(0, T, W^{2\ p}(\mathbb{R}_+^2))$ The conclusion of the previous theorem follows then from the estimate

$$\| u \|_{L^\infty(0\ T\ W^{2\ p}(\mathbb{R}_+^2))} \leqslant C\, \| u_0 \|_{2\ p\ \mathbb{R}_+^2}, \tag{1 26}$$

where the constant C does not depend on ν In order to obtain estimates which do not depend on ν, we make use of the positivity of the operator $-\Delta$ rather than of its smoothing effect

Let us prove this result in the L^2 case By multiplication of equation (1 1) by u and integration with respect to x we get

$$\frac{1}{2} \frac{d}{dt} \| u(\,.,\, t) \|_{L^2(\mathbb{R}_+^2)}^2 + \int_{\mathbb{R}_+^2} \operatorname{div}\,(au)(x,\, t)\, u(x,\, t)\, dx - \\ - \nu \int_{\mathbb{R}_+^2} u(x,\, t)\, \Delta u(x,\, t)\, dx = 0$$

Now, since $a_2(\,.,\, 0,\, .\,) = 0$, we have

$$\int_{\mathbb{R}_+^2} \operatorname{div}\,(au)(x,\, t)\, u(x,\, t)\, dx = -\frac{1}{2} \int_{\mathbb{R}_+^2} a(x,\, t)\,.\,\operatorname{grad}\,(u^2)(x,\, t)\, dx \\ = \frac{1}{2} \int_{\mathbb{R}_+^2} \operatorname{div} a(x,\, t)\, u^2(x,\, t)\, dx$$

and we verify that since the function u satisfies a homogeneous Dirichlet boundary condition, we also have

$$-\int_{\mathbb{R}_+^2} u(x,\, t)\, \Delta u(x,\, t)\, dx = \int_{\mathbb{R}_+^2} |\operatorname{grad} u(x,\, t)|^2\, dx \geqslant 0$$

We combine these results and find,

$$\frac{1}{2}\frac{d}{dt}\|u(\,.,\,t)\|_{0,2,\mathbb{R}_+^2}^2 \leqslant \|\operatorname{div} a\|_{0,\infty}\,\|u(\,.,\,t)\|_{0,2,\mathbb{R}_+^2}^2 .$$

Application of Gronwall's lemma yields then

$$\|u\|_{L^\infty(0,T,L^2(\mathbb{R}_+^2))} \leqslant C\,\|u_0\|_{0,2,\mathbb{R}_+^2},$$

where the constant C does not depend on ν. The previous estimate is based on two essential properties : first the fact that $a_2(\,.,\,0,\,.\,)=0$ and secondly the positivity of the operator $-\Delta$ with homogeneous Dirichlet or homogeneous Neumann boundary conditions. Now the first and second derivatives of u satisfy the same type of equation with precisely one of these boundary conditions. The estimate (1.26) can thus be obtained by similar arguments in the case $p=2$ and generalized without difficulty to the case $p<+\infty$. The L^∞ case can be derived from application of the maximum principle.

Otherwise, if the condition $a_2(\,.,\,0,\,.\,)=0$ is not fulfilled, we only have :

$$\|(u-u_\varepsilon)(\,.,\,t)\|_{0,p,\mathbb{R}_+^2} \leqslant C\,\varepsilon^2\|u\|_{L^\infty(0,T,W^{2,p}(\mathbb{R}_+^2))} .$$

II. THE DISCRETIZED PROBLEM

II.1. A conservative particle approximation

The solution u_ε of the continuous problem (1.5)-(1.6), (1.16) is approximated by a particle method. At initial time, let a uniform grid of size h be given on $\mathbb{R}_+^2$. We choose a quadrature rule on $(0,1)^2$ from which we derive a quadrature rule on each cell of the grid. The whole set of points thus defined constitutes the set of particles centered at x_p with weights ω_p, where $p=(p_1,p_2)$, p_1 is an integer and p_2 a non negative integer :

$$\int_{\mathbb{R}_+^2} f(x)\,dx \approx \sum_p \omega_p f(x_p) . \tag{2.1}$$

The particles evolve in time under the vector field a action. Their positions $X_p(t)=X(t,x_p,0)$ and weights $\omega_p(t)$ at time t are given by

$$\frac{\partial X}{\partial t}(t,x,s)=a(X(t,x,s),t)\,,\quad X(s,x,s)=x\,; \tag{2.2}$$

$$\frac{d\omega_p}{dt}(t)=\operatorname{div} a(X_p(t),t)\,\omega_p(t)\,,\quad \omega_p(0)=\omega_p . \tag{2.3}$$

We denote by $J(t, x, s)$ the Jacobian of the transformation between the Eulerian coordinates x and the Lagrangian one $X(t, x, s)$. We shall see later that, under some hypotheses on a, the time dependent particles still define a quadrature rule, that means ·

$$\int_{\mathbb{R}^2_+} f(x)\, dx \approx \sum_p \omega_p(t)\, f(X_p(t))\,.$$

The solution u_ε of (1.5)-(1.6), (1.16) is approximated by :

$$u^h_\varepsilon(x, t) = \sum_p \omega_p(t)\, u_p(t)\, \chi_\varepsilon(x - X_p(t))\,. \tag{2.4}$$

where χ is a function of integral 1 over $\mathbb{R}^2$, and χ_ε is defined by the usual scaling (1.7) in two dimensions. In order to define the coefficients $u_p(t)$, we first give a particle approximation of Δ_ε. According to (1.16), we can write

$$\Delta_\varepsilon v(x) = \frac{1}{\varepsilon^2}\left[\int_{\mathbb{R}^2_+} [v(y) - v(x)]\, \eta_\varepsilon(x - y)\, dy - \right.$$

$$\left. - \varepsilon \frac{\zeta^{(2)}_\varepsilon(x_2)}{\zeta^{(2)}(0)} \int_{\mathbb{R}^2_+} [v(y) - v(x)]\, \theta_\varepsilon(x_1 - y_1, - y_2)\, dy - v(x)\, \phi\left(\frac{x_2}{\varepsilon}\right)\right], \tag{2.5}$$

where :

$$\phi(x_2) = P\,\eta(x_2) + \frac{\zeta^{(2)}(x_2)}{\zeta^{(2)}(0)} \int_{\mathbb{R}^2_+} \theta(y_1, - y_2)\, dy. \tag{2.6}$$

The integral operator Δ_ε is approximated by

$$\Delta^h_\varepsilon(t)\, v(x) = \frac{1}{\varepsilon^2}\left\{\sum_q \omega_q(t) [v(X_q(t)) - v(x)]\, \eta_\varepsilon(x - X_q(t)) - \right.$$

$$- \varepsilon \frac{\zeta^{(2)}_\varepsilon(x_2)}{\zeta^{(2)}(0)} \sum_q \omega_q(t) [v(X_q(t)) - v(x)]$$

$$\left. \times\, \theta_\varepsilon(x_1 - X^1_q(t), - X^2_q(t)) - v(x)\, \phi\left(\frac{x_2}{\varepsilon}\right)\right\}, \tag{2.7}$$

with the notations : $X_q(t) = (X_q^1(t), X_q^2(t))$. The coefficients $u_p(t)$ are thus solutions of the following differential system

$$0 = \frac{du_p}{dt}(t) + \operatorname{div} a(X_p(t), t)\, u_p(t)$$
$$- \frac{\nu}{\varepsilon^2} \Big\{ \sum_q \omega_q(t) [u_q(t) - u_p(t)]\, \eta_\varepsilon(X_p(t) - X_q(t))$$
$$- \varepsilon \frac{\zeta_\varepsilon^{(2)}(X_p^2(t))}{\zeta^{(2)}(0)} \sum_q \omega_q(t) [u_q(t) - u_p(t)]$$
$$\times\, \theta_\varepsilon(X_p^1(t) - X_q^1(t), -X_q^2(t)) - u_p(t) \phi\left(\frac{X_p^2(t)}{\varepsilon}\right)\Big\},$$
$$u_p(0) = u_0(x_p) \qquad (2.8)$$

In order to study the conservativity, we first need a particle approximation of the integral term $\int_{\mathbb{R}} q_\varepsilon(x_1, t)\, dx_1$. On account of condition (1.12) made on ζ, we have seen in Lemma I.1 that :

$$\int_{\mathbb{R}} q_\varepsilon(x_1, t)\, dx_1 = \frac{2}{\varepsilon} \int_{\mathbb{R}_+^2} u_\varepsilon(x, t)\, P\eta_\varepsilon(x_2)\, dx\,. \qquad (2.9)$$

Thus we define the desired approximation by :

$$I^h(t)(q_\varepsilon)(t) = \frac{2}{\varepsilon} \sum_q \omega_q(t)\, u_q(t)\, P\eta_\varepsilon(X_q^2(t))\,. \qquad (2.10)$$

LEMMA II.1 *We suppose the conditions* (1.12) *and* (1.15) *satisfied We assume moreover that $\zeta^{(2)}$ is an even function, and that*

$$\zeta^{(2)}(0) = \int_{\mathbb{R}^2} \theta(x)\, dx\,, \quad \zeta^{(2)} = P\eta\,, \quad \theta(x) = \theta^{(1)}(x_1)\, \zeta^{(2)}(x_2)\,, \qquad (2.11)$$

where the function $\theta^{(1)}$ is even Then the particle approximation u_ε^h satisfies the following relation of conservation

$$\frac{d}{dt}\left[\sum_p \omega_p(t)\, u_p(t)\right] + \nu I^h(t)(q_\varepsilon)(t) = 0\,. \qquad (2.12)$$

Proof : By (2.3), we have (we suppress the variable t for simplification) :

$$\frac{d}{dt}\left(\sum_p \omega_p u_p\right) = \sum_p \left(\frac{d\omega_p}{dt} u_p + \omega_p \frac{du_p}{dt} \right) =$$

$$= \sum_p \left(\operatorname{div} a(X_p)\, \omega_p u_p + \omega_p \frac{du_p}{dt} \right).$$

Using (2.8), it follows that :

$$\frac{d}{dt}\left(\sum_p \omega_p u_p\right) = \frac{\nu}{\varepsilon^2}\Bigg[\sum_p \sum_q \omega_p\, \omega_q (u_q - u_p)\, \eta_\varepsilon(X_p - X_q) -$$

$$- \frac{\varepsilon}{\zeta^{(2)}(0)} \sum_p \sum_q \omega_p\, \omega_q (u_q - u_p)\, \theta_\varepsilon(X_p^1 - X_q^1, -X_q^2)\, \zeta_\varepsilon^{(2)}(X_p^2)$$

$$- \sum_p \omega_p u_p\, \phi\left(\frac{X_p^2(t)}{\varepsilon}\right)\Bigg].$$

Since η is an even function, the first sum is zero, while the second one disappears on account of (2.11). Finally by (2.11), we have :

$$\phi(x_2) = P\eta(x_2) + \zeta^{(2)}(x_2) = 2\, P\eta(x_2)\,,$$

and relation (2.12) follows then the definition of $I^h(t)(q_\varepsilon)$. □

Remark : From hypotheses (1.12), (1.13) and (2.11) we derive :

$$\zeta^{(2)}(0) = 1/2\,, \quad \int_{\mathbb{R}_+^2} \zeta(x)\, dx = \int_{\mathbb{R}_+^2} \theta(x)\, dx = 1/2\,,$$

$$\int_{\mathbb{R}} \theta^{(1)}\, dx_1 = \int_{\mathbb{R}} \zeta^{(1)}(x_1)\, dx_1\,. \tag{2.13}$$

Moreover, we have : $C_\zeta = \left(\int_{\mathbb{R}} \zeta^{(1)}(x_1)\, dx_1\right)^{-1}$. Notice that the function $G_2(x) = G(x_1)\, G(x_2)$, with $G(x) = \exp(-|x|)/2$, still satisfies the whole set of hypotheses.

II.2. The consistency of the particle approximation

We define the quadrature error at time t by setting :

$$E_t(f) = \int_{\mathbb{R}_+^2} f(x)\, dx - \sum_p \omega_p(t)\, f(X_p(t))\,, \tag{2.14}$$

where :

$$\omega_p(t) = \omega_p J(t, x_p, 0), \quad X_p(t) = X(t, x_p, 0). \tag{2.15}$$

We suppose that this quadrature rule (2.14) is of order $m \geq 1$ at initial time, that means that there exists a constant $C > 0$ such that :

$$|E_0(f)| \leq C h^m \|f\|_{m\,1, \mathbb{R}_+^2}, \quad \text{for any} \quad f \in W^{m\,1}(\mathbb{R}_+^2)$$

It is then of easy check to see that, under hypotheses (1.4) and

$$a \in L^\infty(0, T; W^{m\,\infty}(\mathbb{R}_+^2)), \tag{2 16}$$

$$\operatorname{div} a \in L^\infty(0, T; W^{m,\infty}(\mathbb{R}_+^2)), \tag{2.17}$$

the domain of computation $\mathbb{R}_+^2$ is globally invariant with time, and the particles still define at any time $t \in [0, T]$ a quadrature rule of same accuracy ; more precisely, there is a constant $C = C(T)$ such that :

$$|E_t(f)| \leq C h^m \|f\|_{m\,1\, \mathbb{R}_+^2}, \quad \text{for any} \quad f \in W^{m\,1}(\mathbb{R}_+^2). \tag{2.18}$$

Let us now introduce some notations · we denote by l_r, for $1 < r < +\infty$, the space of sequences $\bar{v} = (v_p)_p$, such that $\sum_p \omega_p(t)|v_p|^r < +\infty$, and we denote by $\|\cdot\|_r$ its norm. (We remark that this norm depends on time.) Similarly, we denote by l_∞, the space of sequences $\bar{v} = (v_n)_n$, such that $\sup\{|v_p|, p\} < +\infty$, and we denote by $\|\cdot\|_\infty$ its norm. Finally, we call $\pi(t)$ the operator that associates to any function v the sequence $(v(X_p(t)))_p$. We then have the following consistency result :

PROPOSITION II.1 : *We assume that the quadrature rule* (2.14) *is of order* $m \geq 1$ *at initial time, and that the conditions* (2.16) *and* (2.17) *on* a *are satisfied Let* $r > 1$ *and* r^* *defined by* $1/r + 1/r^* = 1$. *We suppose* (1.15) *and* (2.11) *satisfied, and the functions* $\eta \in W^{m\,1}(\mathbb{R}^2)$, $\zeta^{(2)} \in L^\infty(\mathbb{R}_+)$, $\theta \in W^{m\,1}(\mathbb{R}^2)$ *compactly supported We suppose moreover that, if* $r < +\infty$,

$$\eta \in W^{m\,\infty}(\mathbb{R}^2), \quad \theta \in W^{m\,\infty}(\mathbb{R}^2). \tag{2.19}$$

Then there exists a constant $C = C(\eta, \zeta, \theta) > 0$, *such that, for any* $v \in W^{m,r}(\mathbb{R}_+^2)$, *we have*

$$\left\| \pi(t)(\Delta_\varepsilon - \Delta_\varepsilon^h(t))\, v \right\|_r \leq C \frac{h^m}{\varepsilon^{m+1}} \|v\|_{m\,r\,\mathbb{R}_+^2}.$$

Proof We have, by (2 5) and (2 7)

$$\varepsilon^2(\Delta_\varepsilon - \Delta_\varepsilon^h(t))\, v(x) = (\Delta_1 + \Delta_2)(x, t),$$

where :

$$\Delta_1(x, t) = \int_{\mathbb{R}^2_+} [v(y) - v(x)]\, \eta_\varepsilon(x - y)\, dy - \\ - \sum_q \omega_q(t) [v(X_q(t)) - v(x)]\, \eta_\varepsilon(x - X_q(t)),$$

$$\Delta_2(x, t) = -\varepsilon \frac{\zeta_\varepsilon^{(2)}(x_2)}{\zeta^{(2)}(0)} \left[\int_{\mathbb{R}^2_+} [v(y) - v(x)]\, \theta_\varepsilon(x_1 - y_1, -y_2)\, dy - \right. \\ \left. - \sum_q \omega_q(t) [v(X_q(t)) - v(x)]\, \theta_\varepsilon(x_1 - X_q^1(t), -X_q^2(t)) \right].$$

We first remark that, since $\zeta^{(2)}$ is compactly supported, included in a sphere of radius R for example, then for x_2 greater than $R\varepsilon$, $\Delta_2(\,.\,, x_2, t) = 0$.

Using the previous notations, we have

$$\Delta_1(x, t) = E_t(y \to \Phi_1(x, y)), \quad \Delta_2(x, t) = -\varepsilon \frac{\zeta_\varepsilon^{(2)}(x_2)}{\zeta^{(2)}(0)} E_t(y \to \Phi_2(x, y)),$$

with :

$$\Phi_1(x, y) = [v(y) - v(x)]\, \eta_\varepsilon(x - y),$$

$$\Phi_2(x, y) = [v(y) - v(x)]\, \theta_\varepsilon(x_1 - y_1, -y_2).$$

By (2.18), it follows that :

$$|E_t(\Phi_i(x, .))| \leqslant C h^m \| \Phi_i(x, .) \|_{m, 1, \mathbb{R}^2_+}, \quad \text{for} \quad i = 1, 2.$$

We have now to estimate separately these two quantities ; the proof of the first one (corresponding to $i = 1$) is omitted here, since it mimics that of the second one $\| \Phi_2(x, .) \|_{m, 1, \mathbb{R}^2_+}$ and is even simpler. Let $n = (n_1, n_2)$ be an index of derivation, with $|n| = n_1 + n_2 \leqslant m$. In order to simplify the notations, we shall denote, for an index k of derivation with $k = (k_1, k_2)$, by « $k < n$ » the fact that :

$$k_1 \leqslant n_1,\ k_2 \leqslant n_2,\ k_1 + k_2 < |n|.$$

We first compute the derivates of Φ_2.

$$\partial_y^n \Phi_2(x, y) = \sum_{k = (k_1, k_2),\, 0 \leqslant k < n} c_k\, \phi_k(x, y) + \phi_n(x, y),$$

with

$$\phi_k(x, y) = \frac{\partial^k \theta_\varepsilon}{\partial y_1^{k_1} \partial y_2^{k_2}} (x_1 - y_1, - y_2) \frac{\partial^{n-k} v}{\partial y_1^{n_1 - k_1} \partial y_2^{n_2 - k_2}} (y) ,$$

$$\phi_n(x, y) = \frac{\partial^n \theta_\varepsilon}{\partial y_1^{k_1} \partial y_2^{k_2}} (x_1 - y_1, - y_2) [v(y) - v(x)]$$

We estimate each term ϕ_k of the sum like a convolution product in tangential variable, and like a product in normal variable, and obtain

$$\| \phi_k(x, \cdot) \|_{0,1,\mathbb{R}^2_+} \leq \frac{1}{\varepsilon^{m-1}} \| \theta^{(1)} \|_{m-1,1,\mathbb{R}} \| \zeta^{(2)} \|_{m,1,1,\mathbb{R}} \| v \|_{m,\infty,\mathbb{R}^2_+}$$

For the last term ϕ_n, we use a Taylor expansion with integral remainder, and conclude in the same way

$$\| \phi_n(x, \cdot) \|_{0,1,\mathbb{R}^2_+} \leq (\| y_1 \to y_1 \, \partial^n \theta_\varepsilon(y) \|_{0,1,\mathbb{R}^2} +$$

$$+ \| y_2 \to (y_2 - x_2) \, \partial^n \theta_\varepsilon(y) \|_{0,1,\mathbb{R}^2}) \| v \|_{1,\infty,\mathbb{R}^2_+}$$

In fact, recalling that $0 \leq x_2 \leq R\varepsilon$, and that the functions are compactly supported, we have

$$\| y_2 \to (y_2 - x_2) \, \partial^{n_2} \zeta_\varepsilon^{(2)}(y_2) \|_{0,1,\mathbb{R}} \leq C \frac{1}{\varepsilon^{n_2}} \varepsilon ,$$

so that

$$\| \phi_n(x, \cdot) \|_{0,1,\mathbb{R}^2_+} \leq C \frac{1}{\varepsilon^{m-1}} \| v \|_{1,\infty,\mathbb{R}^2_+}$$

This, with the estimate

$$\left| \varepsilon \frac{\zeta_\varepsilon^{(2)}(x_2)}{\zeta^{(2)}(0)} \right| = \left| 2 \, \zeta^{(2)}\left(\frac{x_2}{\varepsilon} \right) \right| \leq C ,$$

due to (2 13) and hypotheses on $\zeta^{(2)}$, completes the proof in the case $r = +\infty$ For $r < +\infty$, we have, by (2 18)

$$\left(\sum_p \omega_p(t) | \Delta_2(X_p(t), t) |^r \right)^{1/r} \leq$$

$$\leq 2 C h^m \left(\sum_p \omega_p(t) \left| \zeta^{(2)}\left(\frac{X_p^2(t)}{\varepsilon} \right) \right|^r \| \Phi_2(X_p(t), \cdot) \|_{m,1}^r \right)^{1/r} ,$$

so that we have to estimate

$$\int_{\mathbb{R}_+^2} |\partial_y^n \Phi_2(X_p(t), y)| \, dy \leq \sum_{k = (k_1, k_2),\, 0 \leq k < n} c_k \, \psi_k(X_p(t)) + \psi_n(X_p(t)) ,$$

with, for all k :

$$\psi_k(X_p(t)) = \int_{\mathbb{R}_+^2} |\phi_k(X_p(t), y)| \, dy .$$

We estimate the first terms of the sum ($k < n$). By Hölder's inequality, we have :

$$|\psi_k(X_p(t))|^r \leq \left(\int_{\mathbb{R}_+^2} \left| \frac{\partial^k \theta_\varepsilon}{\partial y_1^{k_1} \partial y_2^{k_2}} (X_p^1(t) - y_1, - y_2) \right| dy \right)^{r/r^*} \times$$

$$\times \left(\int_{\mathbb{R}_+^2} \left| \frac{\partial^k \theta_\varepsilon}{\partial y_1^{k_1} \partial y_2^{k_2}} (X_p^1(t) - y_1, - y_2) \right| \left| \frac{\partial^{n-k} v}{\partial y_1^{n_1 - k_1} \partial y_2^{n_2 - k_2}} (y) \right|^r dy \right) .$$

We use the change of variable $z_1 = X_p^1(t) - y_1$ in the first integral, which becomes thus independent of $X_p(t)$, and get

$$\sum_p \omega_p(t) \left| \zeta^{(2)} \left(\frac{X_p^2(t)}{\varepsilon} \right) \psi_k(X_p(t)) \right|^r \leq$$

$$\leq \left(\frac{1}{\varepsilon^{|k|}} \|\theta\|_{|k|, 1, \mathbb{R}^2} \right)^{r/r^*} \int_{\mathbb{R}_+^2} \sigma_k(t, y) \left| \frac{\partial^{n-k} v}{\partial y_1^{n_1 - k_1} \partial y_2^{n_2 - k_2}} (y) \right|^r dy ,$$

with :

$$\sigma_k(t, y) = \sum_p \omega_p(t) \left| \zeta^{(2)} \left(\frac{X_p^2(t)}{\varepsilon} \right) \right|^r \left| \frac{\partial^k \theta_\varepsilon}{\partial y_1^{k_1} \partial y_2^{k_2}} (X_p^1(t) - y_1, - y_2) \right| .$$

The functions θ and $\zeta^{(2)}$ being compactly supported, the number of terms in the preceding sum is, for any $y \in \mathbb{R}_+^2$, bounded by $C \varepsilon^2/h^2$; moreover, we have $\omega_p(t) \leq C h^2$, so that :

$$\sigma_k(t, y) \leq C \frac{1}{\varepsilon^{|k|}} \|\theta\|_{|k|, \infty, \mathbb{R}^2} \|\zeta^{(2)}\|_{0, \infty, \mathbb{R}}^r .$$

Using the hypotheses of regularity made on these functions, it follows that :

$$\sum_p \omega_p(t) \left| \zeta^{(2)} \left(\frac{X_p^2(t)}{\varepsilon} \right) \psi_k(X_p(t)) \right|^r \leq C \left(\frac{1}{\varepsilon^{m-1}} \right)^{r/r^* + 1} \|v\|_{m, r, \mathbb{R}_+^2}^r ,$$

and this gives the first part of the estimate of $\| \pi(t)(\Delta_2)(\cdot, t) \|_r$. Let us estimate the last term ψ_n.

$$\psi_n(X_p(t)) = \int_{\mathbb{R}^2_+} \left| \frac{\partial^n \theta_\varepsilon}{\partial y_1^{n_1} \partial y_2^{n_2}} (X_p^1(t) - y_1, -y_2) \right| |v(y) - v(X_p(t))| \, dy .$$

We use a Taylor expansion with integral remainder and get

$$\psi_n(X_p(t)) \leq \int_0^1 \sum_{\iota = 1}^{2} \psi^\iota_{n, \lambda}(X_p(t)) \, d\lambda ,$$

where :

$$\psi^\iota_{n, \lambda}(X_p(t)) = \int_{\mathbb{R}^2_+} \left| \frac{\partial^n \theta_\varepsilon}{\partial y_1^{n_1} \partial y_2^{n_2}} (X_p^1(t) - y_1, -y_2) \right| \times$$
$$\times \left| \frac{\partial v}{\partial z_\iota} (X_p(t) + \lambda (y - X_p(t)))(y_\iota - X_p^\iota(t)) \right| dy .$$

Using Hölder's inequality, we obtain :

$$|\psi^\iota_{n, \lambda}(X_p(t))|^r \leq I_\iota \int_{\mathbb{R}^2_+} \left| \frac{\partial^n \theta_\varepsilon}{\partial y_1^{n_1} \partial y_2^{n_2}} (X_p^1(t) - y_1, -y_2) \right| \times$$
$$\times \left| \frac{\partial v}{\partial z_\iota} (X_p(t) + \lambda (y - X_p(t))) \right|^r dy ,$$

with :

$$I_\iota = \left(\int_{\mathbb{R}^2_+} |y_\iota - X_p^\iota(t)|^{r^*} \left| \frac{\partial^n \theta_\varepsilon}{\partial y_1^{n_1} \partial y_2^{n_2}} (X_p^1(t) - y_1, -y_2) \right| dy \right)^{\frac{r}{r^*}} .$$

We do the change of variables in the two integrals I_ι

$$z_1 = \frac{X_p^1(t) - y_1}{\varepsilon} , \quad z_2 = \frac{-y_2}{\varepsilon} ,$$

so that

$$I_1 \leq \varepsilon^{r\left(1 - \frac{|n|}{r^*}\right)} \left(\int_{\mathbb{R}^2_-} |z_1|^{r^*} \left| \frac{\partial^n \theta}{\partial y_1^{n_1} \partial y_2^{n_2}} (z) \right| dz \right)^{r/r^*} ,$$

$$I_2 \leq \varepsilon^{r\left(1 - \frac{|n|}{r^*}\right)} \left(\int_{\mathbb{R}^2_-} \left| z_2 + \frac{X_p^2(t)}{\varepsilon} \right|^{r^*} \left| \frac{\partial^n \theta}{\partial y_1^{n_1} \partial y_2^{n_2}} (z) \right| dz \right)^{r/r^*} .$$

As remarked at the beginning of this proof, the function $\zeta^{(2)}$ being compactly supported, we have : $|X_p^2(t)| \leqslant R\varepsilon$, so that :

$$I_2 \leqslant \varepsilon^{r\left(1-\frac{|n|}{r^*}\right)}\left(\int_{\mathbb{R}_-^2}(|z_2|+R)^{r^*}\left|\frac{\partial^n\theta}{\partial y_1^{n_1}\partial y_2^{n_2}}(z)\right|dz\right)^{r/r^*}.$$

We use the change of variables $z = X_p(t) + \lambda(y - X_p(t))$ in the remaining integral and get, for $i = 1, 2$:

$$|\psi_{n,\lambda}^i(X_p(t))|^r \leqslant C\,\frac{\varepsilon^{r\left(1-\frac{|n|}{r^*}\right)}}{\lambda^2\,\varepsilon^{|n|+2}}\times$$

$$\times\int_{\mathbb{R}_+^2}\left|\frac{\partial^n\theta}{\partial y_1^{n_1}\partial y_2^{n_2}}\left(\frac{X_p^1(t)-z_1}{\varepsilon\lambda},\frac{(1-\lambda)X_p^2(t)-z_2}{\varepsilon\lambda}\right)\right|\left|\frac{\partial v}{\partial z_i}(z)\right|dz\,.$$

It follows that

$$\sum_p \omega_p(t)|\psi_{n,\lambda}^i(X_p(t))|^r\left|\zeta^{(2)}\left(\frac{X_p^2(t)}{\varepsilon}\right)\right|^r \leqslant$$

$$\leqslant C\,\frac{\varepsilon^{r-|n|\,r}}{(\varepsilon\lambda)^2}\int_{\mathbb{R}_+^2}\sigma(t,z)\left|\frac{\partial v}{\partial z_i}(z)\right|^r dz\,,$$

where :

$$\sigma(t,z) =$$

$$\sum_p \omega_p(t)\left|\frac{\partial^n\theta}{\partial y_1^{n_1}\partial y_2^{n_2}}\left(\frac{X_p^1(t)-z_1}{\varepsilon\lambda},\frac{(1-\lambda)X_p^2(t)-z_2}{\varepsilon\lambda}\right)\right|\left|\zeta^{(2)}\left(\frac{X_p^2(t)}{\varepsilon}\right)\right|^r.$$

For any $z \in \mathbb{R}_+^2$, the functions $\theta^{(1)}$ and $\zeta^{(2)}$ being compactly supported, the number of particles contained in the previous sum is bounded by $\left(\frac{\varepsilon\lambda}{h}\right)\left(\frac{\varepsilon}{h}\right)$, so that :

$$\sigma(t,z) \leqslant Ch^2\left(\frac{\varepsilon\lambda}{h}\right)\left(\frac{\varepsilon}{h}\right)\|\theta\|_{|n|,\infty,\mathbb{R}^2}\|\zeta^{(2)}\|_{0,\infty,\mathbb{R}}^r\,.$$

We then deduce :

$$\left(\sum_p \omega_p(t)|\psi_n(X_p(t))|^r\left|\zeta^{(2)}\left(\frac{X_p^2(t)}{\varepsilon}\right)\right|^r\right)^{1/r} \leqslant$$

$$\leqslant C\,\varepsilon^{1-|n|}\left(\int_0^1\frac{1}{\lambda^{1/r}}d\lambda\right)\|v\|_{1,r,\mathbb{R}_+^2}\,,$$

which leads to the estimate of $\|\pi(t)(\Delta_2)(\,.\,,t)\|_r$, in the case $r < +\infty$.

□

II.3. Stability of the particle approximation

We define a discrete operator on the sequences $\bar{v} = (v_p)_p$, in the following way

$$\bar{\Delta}_\varepsilon^h(t)\,\bar{v} = ((\bar{\Delta}_\varepsilon^h(t)\,\bar{v})_p)_p\,,$$

where, using (2.11) :

$$\begin{aligned}(\bar{\Delta}_\varepsilon^h(t)\,\bar{v})_p = \frac{1}{\varepsilon^2}\Big\{\sum_q \omega_q(t)\,[\eta_\varepsilon(X_p(t) - X_q(t)) - \\ - 2\,\varepsilon P\,\eta_\varepsilon(X_p^2(t))\,\theta_\varepsilon(X_p^1(t) - X_q^1(t), - X_q^2(t))](v_q - v_p) - \\ - 2\,v_p\,\zeta^{(2)}\left(\frac{X_p^2(t)}{\varepsilon}\right)\Big\}\end{aligned} \tag{2.20}$$

We give now a stability result :

PROPOSITION II.2 : *Let $r > 1$ be given. We assume that the functions η, θ, $\zeta^{(2)}$ are compactly supported, and that $a \in L^\infty(0, T\,;\, W^{1,\infty}(\mathbb{R}_+^2))$. Moreover, we suppose that $\eta \in L^\infty(\mathbb{R}^2)$, $\theta \in L^\infty(\mathbb{R}^2)$, $\zeta^{(2)} \in L^\infty(\mathbb{R}_+)$. Then there exists a constant $C = C(\eta,\ \theta,\ \zeta^{(2)},\ a)$, $C > 0$ such that, for any sequence $\bar{v} \in l_r$, we have :*

$$\left\|\bar{\Delta}_\varepsilon^h(t)\,\bar{v}\right\|_r \leqslant \frac{C}{\varepsilon^2}\,\|\bar{v}\|_r\,.$$

Proof : We write :

$$\varepsilon^2\left\|\bar{\Delta}_\varepsilon^h(t)\,\bar{v}\right\|_r^r \leqslant C\sum_{\iota=1}^{\iota=5}\sigma_\iota\,,$$

where :

$$\sigma_1 = \sum_p \omega_p(t)\left(\sum_q \omega_q(t)\,|\,\eta_\varepsilon(X_p(t) - X_q(t))\,v_q\,|\right)^r,$$

$$\sigma_2 = \sum_p \omega_p(t)\,|2\,\varepsilon\zeta_\varepsilon^{(2)}(X_p^2(t))|^r \times$$

$$\times\left(\sum_q \omega_q(t)\,|\,\theta_\varepsilon(X_p^1(t) - X_q^1(t), - X_q^2(t))\,v_q\,|\right)^r,$$

$$\sigma_3 = \sum_p \omega_p(t)\,|v_p|^r\left(\sum_q \omega_q(t)\,|\,\eta_\varepsilon(X_p(t) - X_q(t))\,|\right)^r,$$

$$\sigma_4 = \sum_p \omega_p(t)|v_p|^r \; |2\, \varepsilon \zeta_\varepsilon^{(2)}(X_p^2(t))|^r \times$$

$$\times \left(\sum_q \omega_q(t) | \theta_\varepsilon(X_p^1(t) - X_q^1(t), - X_q^2(t)) | \right)^r ,$$

$$\sigma_5 = \sum_p \omega_p(t)|v_p|^r \left| \zeta^{(2)}\left(\frac{X_p^2(t)}{\varepsilon} \right) \right|^r .$$

Since the functions η, θ are compactly supported, and $a \in L^\infty(0, T; W^{1,\infty}(\mathbb{R}_+^2))$, for $\psi(X_q(t)) = \eta_\varepsilon(X_p(t) - X_q(t))$ or $\theta_\varepsilon(X_p^1(t) - X_q^1(t), - X_q^2(t))$, the number of terms in each sum

$$\sum_q \omega_q(t)\; \psi(X_q(t)) ,$$

is bounded (for any fixed p) by : C_ε^2/h^2 ; since $\omega_p(t) \leqslant Ch^2$ we deduce that, for any $t \in (0, T)$

$$\sum_q \omega_q(t)\; \psi(X_q(t)) \leqslant C \;\|\psi\|_\infty ,$$

and :

$$|\sigma_i| \leqslant C \;\|\bar{v}\|_r^r , \quad \text{for} \quad i = 3, 4, 5 .$$

For the first term σ_1, we use Hölder's inequality and get :

$$\left(\sum_q \omega_q(t) | \eta_\varepsilon(X_p(t) - X_q(t))\, v_q | \right)^r \leqslant$$

$$\leqslant \left(\sum_q \omega_q(t) | \eta_\varepsilon(X_p(t) - X_q(t)) | \right)^{r/r^*} \times$$

$$\times \left(\sum_q \omega_q(t) | \eta_\varepsilon(X_p(t) - X_q(t)) |\, |v_q|^r \right)$$

$$\leqslant C \left(\sum_q \omega_q(t) | \eta_\varepsilon(X_p(t) - X_q(t)) |\, |v_q|^r \right) ,$$

which leads to :

$$\sigma_1 \leqslant \sum_q \omega_q(t) |v_q|^r \left(\sum_p \omega_p(t) | \eta_\varepsilon(X_p(t) - X_q(t)) | \right) \leqslant C' \sum_q \omega_q(t) |v_q|^r .$$

We obtain as well for the second term σ_2

$$\left(\sum_q \omega_q(t)\left|\theta_\varepsilon(X_p^1(t)-X_q^1(t), -X_q^2(t))\,v_q\right|\right)^r \leqslant$$

$$\leqslant \left(\sum_q \omega_q(t)\left|\theta_\varepsilon(X_p^1(t)-X_q^1(t), -X_q^2(t))\right|\right)^{r/r^*} \times$$

$$\times \left(\sum_q \omega_q(t)\left|\theta_\varepsilon(X_p^1(t)-X_q^1(t), -X_q^2(t))\right| |v_q|^r\right)$$

$$\leqslant C\left(\sum_q \omega_q(t)\left|\theta_\varepsilon(X_p^1(t)-X_q^1(t), -X_q^2(t))\right| |v_q|^r\right),$$

so that

$$|\sigma_2| \leqslant 2\,C \sum_q \omega_q(t)|v_q|^r \times$$

$$\times \left(\sum_p \omega_p(t)\left|\theta_\varepsilon(X_p^1(t)-X_q^1(t), -X_q^2(t))\right| \left|\zeta^{(2)}\left(\frac{X_p^2(t)}{\varepsilon}\right)\right|^r\right)$$

$$\leqslant C' \sum_q \omega_q(t)|v_q|^r,$$

and this ends the proof, in the case $r < +\infty$. The case $r = +\infty$ can be easily checked as well. □

II.4. The convergence of the particle approximation

We recall the notation introduced in (2.4)

$$u_\varepsilon^h(x, t) = \sum_p \omega_p(t)\,u_p(t)\,\chi_\varepsilon(x - X_p(t)),$$

the regularized particle approximation of u_ε. More generally, for any function v, let us denote by $\pi_\varepsilon^h(t)\,v$ the regularized projection of v on the particles, that means :

$$\pi_\varepsilon^h(t)\,v(x) = \sum_p \omega_p(t)\,v(X_p(t))\,\chi_\varepsilon(x - X_p(t)). \tag{2.21}$$

Finally, we define the sequences :

$$\bar{u}^h(t) = (u_p(t))_p,\ \bar{u}_\varepsilon(t) = \pi(t)\,u_\varepsilon(\,.,\,t) = (u_\varepsilon(X_p(t), t))_p\,.$$

THEOREM II.1 : *Let* $1 < r \leqslant +\infty$ *be given and* r^* *defined by :* $1/r + 1/r^* = 1$. *We suppose that the quadrature error* (2.14) *is of order* $m \geqslant 1$ *at initial time. We assume that* $\chi \in W^{m,1}(\mathbb{R}^2) \cap C^1(\mathbb{R}^2)$ *is compactly supported, and that :*

$$\int_{\mathbb{R}^2} \chi(x)\, dx = 1\,,$$
$$\int_{\mathbb{R}^2} x^\alpha \chi(x)\, dx = 0\,, \quad \alpha \in \mathbb{N}^2\,, \quad 1 \leqslant |\alpha| \leqslant s-1\,, \quad (s \in \mathbb{N} - (0, 1))\,. \tag{2.22}$$

We suppose moreover that $a \in L^\infty(0, T; W^{m+1,\infty}(\mathbb{R}^2_+))$, *that the hypotheses of the propositions* II.1 *and* II.2 *on the functions* η, $\zeta^{(2)}$ *and* θ *are verified, and that :*

$$\eta \in W^{m,1}(\mathbb{R}^2_+)\,, \quad \zeta^{(2)} \in W^{m,r}(\mathbb{R}_+)\,, \quad \theta \in W^{m,1}(\mathbb{R}, L^{r^*}(\mathbb{R}_-))\,.$$

Let us denote by $s^* = \max(m, s)$. *Then, if the following stability condition is satisfied*

$$\nu \leqslant C_{st, s^*}\, \varepsilon^{2+s^*}\,, \tag{2.23}$$

there exists a constant $C = C(a, T, C_{st,0}, \eta, \theta, \zeta^{(2)}, \chi) > 0$, *such that for* $t \in (0, T)$, *we have :*

$$\|(u_\varepsilon - u_\varepsilon^h)(\cdot, t)\|_{0, r, \mathbb{R}^2_+} \leqslant C \left(\frac{h^m}{\varepsilon^m} + \varepsilon^s + \frac{\nu h^m}{\varepsilon^{m+1}} \right) \|u_0\|_{s^*, r, \mathbb{R}^2_+}\,.$$

Proof : We write

$$(u_\varepsilon - u_\varepsilon^h)(x, t) = (u_\varepsilon - \pi_\varepsilon^h(t)\, u_\varepsilon) \times (x, t) + \\ + \sum_p \omega_p(t) [u_\varepsilon(X_p(t), t) - u_p(t)]\, \chi_\varepsilon(x - X_p(t))\,,$$

and

$$\|(u_\varepsilon - \pi_\varepsilon^h(t)\, u_\varepsilon)(\cdot, t)\|_{0, r, \mathbb{R}^2_+} \leqslant \\ \leqslant C \left[\frac{h^m}{\varepsilon^m} \|u_\varepsilon(\cdot, t)\|_{m, r, \mathbb{R}^2_+} + \varepsilon^s \|u_\varepsilon(\cdot, t)\|_{s, r, \mathbb{R}^2_+} \right],$$

by a result of approximation due to [19]. We now estimate the second term. Let us set : $\bar{e}^h(t) = \bar{u}^h(t) - \bar{u}_\varepsilon(t)$ the error made on the particles :

$$\bar{e}^h(t) = (e_p(t))_p\,, \quad e_p(t) = u_p(t) - u_\varepsilon(X_p(t), t)\,.$$

The sequence $\bar{e}^h(t)$ satisfies the differential system

$$\frac{de_p(t)}{dt} + \operatorname{div} a(X_p(t), t)\, e_p(t) -$$

$$- \frac{\nu}{\varepsilon^2} \left\{ \sum_q \omega_q(t) [e_q(t) - e_p(t)] [\eta_\varepsilon(X_p(t) - X_q(t))] \right.$$

$$- 2\, \zeta^{(2)}\left(\frac{X_p^2(t)}{\varepsilon} \right) \theta_\varepsilon(X_p^1(t) - X_q^1(t), - X_q^2(t))]$$

$$\left. - 2\, e_p(t)\, \zeta^{(2)}\left(\frac{X_p^2(t)}{\varepsilon} \right) \right\} = \sigma_p(t)\,,\ e_p(0) = u_p(0) - u_\varepsilon(x_p, 0) = 0\,,$$

where $\sigma_p(t) = \nu(\Delta_\varepsilon^h(t) - \Delta_\varepsilon)(u_\varepsilon)(X_p(t), t)$ is proportional to the consistency error. Let us now consider the case $r < +\infty$. We multiply the former equation by $\omega_p(t) |e_p(t)|^{r-2} e_p(t)$ and sum over the indices p :

$$\frac{1}{r} \left(\frac{d}{dt} \sum_p \omega_p(t) |e_p(t)|^r \right) + \left(1 - \frac{1}{r} \right) \sum_p (\operatorname{div} (a(X_p(t), t))\, \omega_p(t)\, |e_p(t)|)^r$$

$$- \nu \sum_p (\bar{\Delta}_\varepsilon^h(t)\, \bar{e}^h(t))_p\, |e_p(t)|^{r-2} e_p(t)$$

$$= \sum_p \omega_p(t)\, \sigma_p(t) |e_p(t)|^{r-2} e_p(t)\,.$$

Let us denote by : $\bar{\sigma}^h(t) = (\sigma_p(t))_p = \nu \pi(t)(\Delta_\varepsilon^h(t) - \Delta_\varepsilon)(u_\varepsilon(\,.,\, t))$. By Hölder's inequality, we get :

$$\frac{d}{dt} \|\bar{e}^h(t)\|_r^r \leqslant (r-1) \|\operatorname{div} a(\,.,\, t)\|_{0,\infty,\mathbb{R}_+^2} \|\bar{e}^h(t)\|_r^r +$$

$$+ \nu r \left(\sum_p \omega_p(t) \left| (\bar{\Delta}_\varepsilon^h(t)\, \bar{e}^h(t))_p \right|^r \right)^{\frac{1}{r}} \|\bar{e}^h(t)\|_r^{r-1} + r \|\bar{\sigma}^h(t)\|_r \|\bar{e}^h(t)\|_r^{r-1}\,.$$

We apply the stability result proved in proposition II.2 in order to evaluate the second term, and this leads to :

$$\frac{d}{dt} \|\bar{e}^h(t)\|_r \leqslant \left(\frac{r-1}{r} \|\operatorname{div} a(\,.,\, t)\|_{0,\infty,\mathbb{R}_+^2} + C \frac{\nu}{\varepsilon^2} \right) \|\bar{e}^h(t)\|_r + \|\bar{\sigma}^h(t)\|_r\,.$$

Using the stability condition (1.25) included in (2.23), we deduce, by Gronwall's inequality, and the fact that $\bar{e}^h(0) = 0$, that :

$$\|\bar{e}^h(t)\|_r \leqslant C\,(T, C_{st,0}) \|\bar{\sigma}^h\|_{L^\infty(0,T;\, l_r)}\,.$$

On account of the proposition II.1, this finally gives :

$$\left\|\bar{e}^h(t)\right\|_r \leqslant C \, \frac{\nu h^m}{\varepsilon^{m+1}} \left\|u_\varepsilon\right\|_{L^\infty(0, T, W^{m, r}(\mathbb{R}_+^2))} .$$

Moreover, we have

$$(u_\varepsilon^h - \pi_\varepsilon^h(t)\, u_\varepsilon)(x, t) = \sum_p \omega_p(t)\, e_p(t)\, \chi_\varepsilon(x - X_p(t)) ,$$

and

$$\left|\sum_p \omega_p(t)\, e_p(t)\, \chi_\varepsilon(x - X_p(t))\right|^r \leqslant$$

$$\leqslant \sigma(x, t) \left(\sum_p \omega_p(t) \left|e_p(t)\right|^r \left|\chi_\varepsilon(x - X_p(t))\right|\right) ,$$

with :

$$\sigma(x, t) = \left(\sum_p \omega_p(t) \left|\chi_\varepsilon(x - X_p(t))\right|\right)^{\frac{r}{r^*}} .$$

Since χ is compactly supported, $\sigma(\,.,\, t)$ is bounded, and we get

$$\int_{\mathbb{R}_+^2} \left|\sum_p \omega_p(t)\, e_p(t)\, \chi_\varepsilon(x - X_p(t))\right|^r dx \leqslant$$

$$\leqslant C \sum_p \omega_p(t) \left|e_p(t)\right|^r \int_{\mathbb{R}^2} \chi_\varepsilon(x - X_p(t))\, dx ,$$

so that

$$\left\|(u_\varepsilon^h \quad \pi_\varepsilon^h(t)\, u_\varepsilon)(\,.,\, t)\right\|_{0, r, \mathbb{R}_+^2} \leqslant C \left\|\bar{e}_h(t)\right\|_r .$$

Using the previous estimate on $\left\|\bar{e}^h(t)\right\|_r$, and the $W^{m,p}$ stability result contained in corollary I.1, which gives a bound to $\left\|u_\varepsilon\right\|_{L^\infty(0, T, W^{s^*, r}(\mathbb{R}_+^2))}$ in terms of initial data u_0 ends the proof for $r < +\infty$. We refer again to [8] for the L^∞ case. □

Remark : Only the weaker stability condition (1.25) is needed, instead of (2.23), in order to obtain an error estimate in terms of u_ε :

$$\left\|(u_\varepsilon - u_\varepsilon^h)(\,.,\, t)\right\|_{0, r, \mathbb{R}_+^2} \leqslant C \left(\frac{h^m}{\varepsilon^m} + \varepsilon^s + \frac{\nu h^m}{\varepsilon^{m+1}}\right) \left\|u_\varepsilon\right\|_{L^\infty(0, T, W^{s^*, r}(\mathbb{R}_+^2))} .$$

The final convergence result follows then immediately.

COROLLARY II.2 : *Let* $1 < r \leqslant +\infty$, $m \geqslant 1$ *and* r^* *defined by :* $1/r + 1/r^* = 1$. *We suppose the hypotheses of theorem* I.1 *and of theorem* II.1 *(for* $s = 2$*) satisfied. We set :* $s^* = \max(m, 2)$. *Then there is a constant* C *depending on* a, T, C_{st, s^*}, η, θ, $\zeta^{(2)}$, χ, *such that : for* $t \leqslant T$, *and for any function* $u_0 \in W^{s^*, r}(\mathbb{R}^2_+)$, *we have :*

$$\|(u - u_\varepsilon^h)(\cdot, t)\|_{0, r, \mathbb{R}^2_+} \leqslant C\left(\frac{h^m}{\varepsilon^m} + \varepsilon^2 + \frac{\nu h^m}{\varepsilon^{m+1}}\right)\|u_0\|_{s^*, r, \mathbb{R}^2_+}.$$

Considering the integral operator as a bounded operator, we have been able to obtain the whole set of previous results. Unfortunately, they are all based on the assumption that the numerical viscosity is related to the physical viscosity by condition (1.25). Although this constraint is not too drastic, other results, which are based on the positivity of the integral operator, and for which no condition on the viscosity is needed, can be obtained as well.

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