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SOME ESTIMATES FOR THE ANISOTROPIC NAVIER-STOKES EQUATIONS AND FOR THE HYDROSTATIC APPROXIMATION (*)

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Abstract — *This paper is devoted to the study of the Navier-Stokes equations describing the flow of an incompressible fluid in a shallow domain and to the hydrostatic approximation of these equations We first study the behaviour of solutions of the Navier-Stokes equations when the depth of the domain tends to zero We then derive the existence of solutions for the hydrostatic approximation*

Résumé — *Ce papier est consacré à l'étude des équations de Navier-Stokes décrivant l'écoulement d'un liquide incompressible dans un bassin peu profond et à l'approximation hydrostatique de ces équations Nous étudions tout d'abord le comportement des solutions des équations de Navier-Stokes lorsque la profondeur du bassin tend vers zéro, puis nous en déduisons l'existence de solutions pour l'approximation hydrostatique*

1. INTRODUCTION

In geophysical fluid dynamics the anisotropic Navier-Stokes equations are widely used (see e.g. [Pe]). They describe the movement of a fluid by mean of a turbulent viscosity model. When the horizontal and vertical dimensions of the domain in mind are very different, the turbulent viscosity coefficient has no reason to be isotropic. We then get the anisotropic Navier-Stokes equations. Moreover when our domain is shallow, the hydrostatic approximation of the Navier-Stokes equations is a basic model, in current use for the description of the fluid motion (see e.g. [Pe], [Pi]), and many programmes have been developed to solve these equations (see for example [ZD], and the references in [Pi], [Sc], [FEF], ...).

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In this paper we study both of these models. In [BLT] the Stokes case was investigated and in [La] some numerical experiments have been performed. This section is devoted to some notations, in paragraph 2 we study the anisotropic Navier-Stokes equations in a shallow domain and finally in paragraph 3 we give the existence of a solution for the hydrostatic approximation.

1.1. Basic equations

Let us first introduce some notations. Let Ω_0 be the domain in $\mathbf{R}^3$ with Lipschitz boundary (see [Ne]) defined by

$$\Omega_0 = \left\{\xi = (\xi_i) \in \mathbf{R}^3,\ (\xi_1, \xi_2) \in G_0,\ -h(\xi_1, \xi_2) < \xi_3 < 0\right\}$$

where G_0 is a bounded domain in $\mathbf{R}^2$ and h is a positive Lipschitz map from G_0 into $\mathbf{R}$. We denote by $G_b = \partial\Omega_0 - G_0$ where $\partial\Omega_0$ is the boundary of Ω_0, and we assume that $h_0 = \sup\left\{h(\xi_1, \xi_2),\ (\xi_1, \xi_2) \in G_0\right\}$ is small compared to the diameter d of G_0 and we put

$$\varepsilon = h_0/d\,,$$

with $0 < \varepsilon < 1$.

In this paper we study the motion of some incompressible fluid in Ω_0, governed by the anisotropic stationary Navier-Stokes equations and subject to Coriolis forces :

Find $v = (v_i)$, $v_i : \Omega_0 \to \mathbf{R}$, $i = 1, 2, 3$ and $p : \Omega_0 \to \mathbf{R}$ such that

$$\rho\,(v\,|\,\nabla)\,v - \Delta_\eta\,v + 2\,\rho\,\omega \times v + \nabla p = \rho g \tag{1a}$$

$$\operatorname{div} v = 0 \tag{1b}$$

where the operators $(v\,|\,\nabla)$ and Δ_η are defined by

$$(v\,|\,\nabla) = \sum_{i=1}^{3} v_i\,\partial_i \quad \text{and} \quad \Delta_\eta = \sum_{i=1}^{3} \eta_i\,\partial_{ii}^2$$

and where ρ is the density, $\eta = (\eta_1, \eta_2, \eta_3)$ is the dynamic turbulent viscosity vector, $-2\,\rho\,\omega \times v$ is the Coriolis force where $\omega = (0, 0, \omega_3)$ denotes is the angular velocity induced by the rotation of the Earth and $g = (0, 0, g_3)$ is the gravity vector.

Moreover the velocity v is subject to the boundary conditions

$$v = 0 \quad \text{on} \quad G_b \tag{1c}$$

$$\eta_3\,\partial_3 v_1 = \rho\,\tau_1\,, \quad \eta_3\,\partial_3 v_2 = \rho\,\tau_2\,, \quad v_3 = 0 \quad \text{on}\ G_0 \tag{1d}$$

where τ_1 and τ_2 are given functions on G_0.

1.2. Transformation of the problem

Let us do the following change of variables and functions

$$x_1 = \xi_1/d, \quad x_2 = \xi_2/d, \quad x_3 = \xi_3/h_0 \tag{2a}$$

$$u_1(x) = v_1(\xi), \quad u_2(x) = v_2(\xi), \quad u_3(x) = v_3(\xi)/\varepsilon,$$
$$P(x) = p(\xi)/\rho - g_3\,\xi_3\,. \tag{2b}$$

Then if

$$\Gamma_0 = \left\{x = (x_1, x_2) \in \mathbf{R}^2, (\xi_1, \xi_2) \in G_0\right\}$$

we get the problem :

Let Ω be the domain in $\mathbf{R}^3$, independent of ε, with Lipschitz boundary $\partial\Omega$ defined by

$$\Omega = \left\{x = (x_i) \in \mathbf{R}^3, (x_1, x_2) \in \Gamma_0, -y(x_1, x_2) < x_3 < 0\right\}$$

with $\|x\| \leq 1$ and $y = h/h_0$, so $0 \leq y(x_1, x_2) \leq 1$ for all $(x_1, x_2) \in \Gamma_0$. The bottom Γ_b is defined by $\Gamma_b = \partial\Omega - \Gamma_0$ (see *fig.* 1).

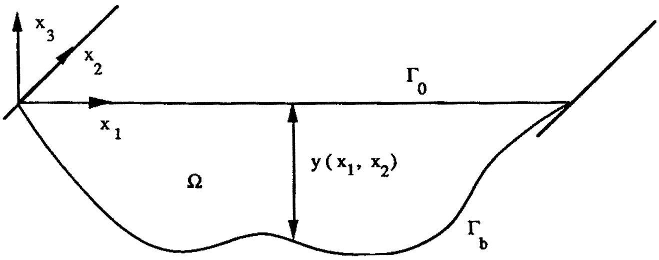

Figure 1. — Schematic representation of the domain Ω.

If $\nu = (\nu_1, \nu_2, \nu_3)$ is the cinematic turbulent viscosity vector with $\nu_i = \eta_i/(\rho d)$, we let $\nu^\varepsilon = (\nu_1, \nu_2, \nu_3/\varepsilon^2)$. With the notations (2*a*) and (2*b*), the problem (1*a*)-(1*f*) becomes : Given τ_1, τ_2, we seek $u = (u_i)$, $u_i : \Omega \to \mathbf{R}$ and $P : \Omega \to \mathbf{R}$ such that

$$(u\,|\,\nabla u_1) - \Delta_{\nu^\varepsilon} u_1 - f u_2 + \partial_1 P = 0 \tag{3a}$$

$$(u\,|\,\nabla u_2) - \Delta_{\nu^\varepsilon} u_2 + f u_1 + \partial_2 P = 0 \tag{3b}$$

$$\varepsilon^2[(u\,|\,\nabla u_3) - \Delta_{\nu^\varepsilon} u_3] + \partial_3 P = 0 \tag{3c}$$

$$\operatorname{div} u = 0 \tag{3d}$$

with $f = 2\,d\omega_3$. The boundary conditions are

$$u = 0 \quad \text{on} \quad \Gamma_b \tag{3e}$$

$$\nu_3\, \partial_3 u_1 = \varepsilon \tau_1, \quad \nu_3\, \partial_3 u_2 = \varepsilon \tau_2, \quad u_3 = 0 \quad \text{on}\ \Gamma_0 . \tag{3f}$$

1.3. The hydrostatic approximation

The hydrostatic approximation of the Navier-Stokes equations in the domain Ω has the following form. Let $\mu = (\mu_1, \mu_2, \mu_3)$ be a turbulent cinematic viscosity vector, given τ_1, τ_2, we seek $u = (u_i)$, $u_i : \Omega \to \mathbf{R}$ and $P : \Omega \to \mathbf{R}$ such that

$$(u \,|\, \nabla u_1) - \Delta_\mu u_1 - f u_2 + \partial_1 P = 0 \tag{4a}$$

$$(u \,|\, \nabla u_2) - \Delta_\mu u_2 + f u_1 + \partial_2 P = 0 \tag{4b}$$

$$\partial_3 P = 0 \tag{4c}$$

$$\operatorname{div} u = 0 \tag{4d}$$

with the boundary conditions

$$u_1 = 0, \quad u_2 = 0, \quad u_3 n_3 = 0 \quad \text{on}\ \Gamma_b \tag{4e}$$

$$\mu_3\, \partial_3 u_1 = \tau_1, \quad \mu_3\, \partial_3 u_2 = \tau_2, \quad u_3 = 0 \quad \text{on}\ \Gamma_0 . \tag{4f}$$

Our aim in this paper is to explain how we can get problem (4) from problem (3) when $\varepsilon \to 0$.

1.4. Basic functional spaces

As usual $W^{s,p}(\Omega)$ denotes the Sobolev space of order s in $L^p(\Omega)$ with norm $\| \cdot \|_{W^{s,p}(\Omega)}$. In particular $H^s(\Omega) = W^{s,2}(\Omega)$, $H^{1/2}(\partial\Omega)$ is the space of traces of functions in $H^1(\Omega)$ and $H_0^1(\Omega)$ is the closure of $D(\Omega)$ in $H^1(\Omega)$ etc... (*cf.* [Ad]).

Let $V_1 = \{\varphi \in H^1(\Omega),\ \varphi = 0 \text{ on } \Gamma_b\}$, $V_2 = V_1$ and $V_3 = H_0^1(\Omega)$. These spaces are equipped with the norm

$$|\varphi|_{H^1(\Omega)} = \left(\sum_{i=1}^{3} \|\partial_i \varphi\|_{L^2(\Omega)}^2 \right)^{1/2}$$

which is equivalent to the usual norm. Then we put $V = V_1 \oplus V_2 \oplus V_3$ and

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega),\ \int_\Omega q\, dx = 0 \right\} .$$

Finally for $i = 1, 2, 3$, let $H(\partial_i, \Omega)$ be the Hilbert space

$$H(\partial_i, \Omega) = \{\varphi \in L^2(\Omega),\ \partial_i \varphi \in L^2(\Omega)\}$$

equipped with the graph norm and let

$$H_0(\partial_i, \Omega) = \left\{\varphi \in H(\partial_i, \Omega),\ \int_\Omega (\varphi\, \partial_i \psi + \psi\, \partial_i \varphi)\, dx = 0\ \forall \psi \in H(\partial_i, \Omega)\right\}$$

which can be interpreted as

$$H_0(\partial_i, \Omega) = \{\varphi \in H(\partial_i, \Omega),\ \varphi n_i = 0 \quad \text{on } \partial\Omega\}$$

where $n = (n_i)$ is the outward unit vector on $\partial\Omega$.

1.5. Weak formulation of the anisotropic Navier-Stokes equations

Consider now the following functionals

$$A_\varepsilon(\varphi, \psi) = \sum_{i=1}^{3} \nu_i^\varepsilon \int_\Omega \partial_i \varphi\, \partial_i \psi\, dx \quad (\varphi, \psi \in H^1(\Omega))$$

where ν^ε is as above.

$$B(u, \varphi, \psi) = \int_\Omega \varphi (u | \nabla \psi)\, dx \quad (u \in V,\ \varphi \in L^2(\Omega),\ \psi \in H^1(\Omega))$$

$$C(\varphi, \psi) = f \int_\Omega \varphi \psi\, dx \quad (\varphi, \psi \in L^2(\Omega))$$

$$D_i(\varphi, \psi) = \int_\Omega \varphi\, \partial_i \psi\, dx \quad (\varphi \in L^2(\Omega),\ \psi \in H(\partial_i, \Omega))$$

$$E(\varphi, \psi) = \int_{\partial\Omega} \varphi \psi\, d\gamma \quad (\varphi, \psi \in L^2(\partial\Omega)).$$

Then a weak formulation of problem (3) is :

Given $\tau_1, \tau_2 \in H^{1/2}(\partial\Omega)$, $\varepsilon > 0$, find $u = (u_i) \in V$ and $P \in L_0^2(\Omega)$ such that

$$A_\varepsilon(u_1, v_1) - B(u, u_1, v_1) - C(u_2, v_1) - D_1(P, v_1) = E\left(\tau_1, \frac{v_1}{\varepsilon}\right) \quad (5a)$$

$$A_\varepsilon(u_2, v_2) - B(u, u_2, v_2) + C(u_1, v_2) - D_2(P, v_2) = E\left(\tau_2, \frac{v_2}{\varepsilon}\right) \quad (5b)$$

$$\varepsilon^2(A_\varepsilon(u_3, v_3) - B(u, u_3, v_3)) - D_3(P, v_3) = 0 \quad (5c)$$

$$\int_\Omega q \operatorname{div} u\, dx = 0 \quad (5d)$$

for all $v = (v_i) \in V$ and for all $q \in L_0^2(\Omega)$.

Using similar arguments as in [Li, GR], one can prove that for any τ_1, $\tau_2 \in H^{1/2}(\partial\Omega)$ and any $\varepsilon > 0$, there exists at least one pair $(u, P) \in V \times L_0^2(\Omega)$ which satisfy equations (5a) to (5d)

2. SOME ESTIMATES OF THE SOLUTION OF THE ANISOTROPIC NAVIER-STOKES EQUATIONS

We now give some sufficient conditions to « neglect » the velocity terms in equation (3c) We assume that ν_1, $\nu_2 > 0$ are constant and $\nu_3 = \nu_0\,\varepsilon^{2s} > 0$ Moreover we assume that there exist τ_1^0, $\tau_2^0 \in H^{1/2}(\partial\Omega)$ such that $\tau_i = \varepsilon^t \tau_i^0$, $i = 1, 2$

In the sequel, we denote by C some constants which do not depend on ε

2.1. Estimate of the viscosity terms

Let $v = u$ and $q = P$ in equations (5), then, using equation (5d), after the summation of equations (5a) to (5c) we get

$$\begin{aligned}
&\nu_1\|\partial_1 u_1\|^2_{L^2(\Omega)} + \nu_2\|\partial_2 u_1\|^2_{L^2(\Omega)} + \nu_0\,\varepsilon^{2s-2}\|\partial_3 u_1\|^2_{L^2(\Omega)} + \\
&\quad + \nu_1\|\partial_1 u_2\|^2_{L^2(\Omega)} + \nu_2\|\partial_2 u_2\|^2_{L^2(\Omega)} + \nu_0\,\varepsilon^{2s-2}\|\partial_3 u_2\|^2_{L^2(\Omega)} \\
&\quad + \nu_1\,\varepsilon^2\|\partial_1 u_3\|^2_{L^2(\Omega)} + \nu_2\,\varepsilon^2\|\partial_2 u_3\|^2_{L^2(\Omega)} + \nu_0\,\varepsilon^{2s}\|\partial_3 u_3\|^2_{L^2(\Omega)} \\
&\quad \leq \varepsilon^{t-1}(E(\tau_1^0, u_1) + E(\tau_2^0, u_2))
\end{aligned} \tag{6}$$

But from the inequality

$$\left|\int_{\partial\Omega} \varphi\,\tau\,d\gamma\right| \leq C\,\|\partial_3\varphi\|_{L^2(\Omega)}\|\tau\|_{L^2(\partial\Omega)}$$

for any $\varphi \in V_1$ and $\tau \in H^{1/2}(\partial\Omega)$, we deduce

$$E(\tau_1^0, u_1) + E(\tau_2^0, u_2) \leq C\,(\|\partial_3 u_1\|^2_{L^2(\Omega)} + \|\partial_3 u_2\|^2_{L^2(\Omega)})^{1/2}$$

Therefore using equation (6) we get

$$\begin{aligned}
&\nu_1\|\partial_1 u_1\|^2_{L^2(\Omega)} + \nu_2\|\partial_2 u_1\|^2_{L^2(\Omega)} + \nu_0\,\varepsilon^{2s-2}\|\partial_3 u_1\|^2_{L^2(\Omega)} + \\
&\quad + \nu_1\|\partial_1 u_2\|^2_{L^2(\Omega)} + \nu_2\|\partial_2 u_2\|^2_{L^2(\Omega)} + \nu_0\,\varepsilon^{2s-2}\|\partial_3 u_2\|^2_{L^2(\Omega)} \\
&\quad + \nu_1\,\varepsilon^2\|\partial_1 u_3\|^2_{L^2(\Omega)} + \nu_2\,\varepsilon^2\|\partial_2 u_3\|^2_{L^2(\Omega)} + \nu_0\,\varepsilon^{2s}\|\partial_3 u_3\|^2_{L^2(\Omega)} \\
&\quad \leq C\,\varepsilon^{2t-2s}
\end{aligned} \tag{7}$$

From this inequality we obtain

LEMMA 1 : *We have the following estimates*

(a) $\|\partial_1 u_i\|_{L^2(\Omega)} \leq C\,\varepsilon^{t-s}$, $i = 1, 2$ *and* $\|\partial_1 u_3\|_{L^2(\Omega)} \leq C\,\varepsilon^{t-1-s}$,
(b) $\|\partial_2 u_i\|_{L^2(\Omega)} \leq C\,\varepsilon^{t-s}$, $i = 1, 2$ *and* $\|\partial_2 u_3\|_{L^2(\Omega)} \leq C\,\varepsilon^{t-1-s}$,
(c) $\|\partial_3 u_i\|_{L^2(\Omega)} \leq C\,\varepsilon^{t+1-2s}$, $i = 1, 2$ *and* $\|\partial_3 u_3\|_{L^2(\Omega)} \leq C\,\varepsilon^{t-s}$.

Proof : Inequalities (a) and (b) as well as the first of (c) are a straightforward consequence of (7). We deduce the last inequality of (c) from (a), (b) and from equation $\operatorname{div} u = 0$. ■

2.2. Estimate of the convective terms

Let us first estimate the velocity in $L^p(\Omega)$ with $p = 2$, 3, and 6. With the above notations we have

LEMMA 2 :

(a) $\|u_i\|_{L^2(\Omega)} \leq C\,\varepsilon^{t+1-2s}$, $i = 1, 2$ *and* $\|u_3\|_{L^2(\Omega)} \leq C\,\varepsilon^{t-s}$,
(b) $\|u_i\|_{L^3(\Omega)} \leq C\,\varepsilon^{(3t+2-5s)/3}$, $i = 1, 2$ *and* $\|u_3\|_{L^3(\Omega)} \leq C\,\varepsilon^{(3t-1-3s)/3}$,
(c) $\|u_i\|_{L^6(\Omega)} \leq C\,\varepsilon^{(3t+1-4s)/3}$, $i = 1, 2$ *and* $\|u_3\|_{L^6(\Omega)} \leq C\,\varepsilon^{(3t-2-3s)/3}$.

Proof : The assertion (a) is a straightforward consequence of lemma 1 (c) and of the inequality

$$\|\varphi\|_{L^2(\Omega)} \leq C\ \|\partial_3 \varphi\|_{L^2(\Omega)}$$

for all $\varphi \in V_1$.

For the assertion (c) we use the Gagliardo inequality (see e.g [Ne])

$$\|v\|^3_{L^{3/2}(\Omega)} \leq C \prod_{i=1}^{3} \|\partial_i v\|_{L^1(\Omega)},$$

which gives, with $v = \varphi^4$ and owing to the Cauchy-Schwarz inequality

$$\|\varphi\|_{L^6(\Omega)} \leq C \prod_{i=1}^{3} \|\partial_i \varphi\|^{1/3}_{L^2(\Omega)}$$

and the result follows from lemma 1.

We deduce the part (b) from (a) and (c) and from

$$\|v\|_{L^3(\Omega)} \leq \|v\|^{1/2}_{L^2(\Omega)}\ \|v\|^{1/2}_{L^6(\Omega)}\,.$$ ■

We are now ready to study the non linear-terms.

LEMMA 3 (a) *For $i = 1, 2$ we have*

$$\| (u \mid \nabla u_i) \|_{H^{-1}(\Omega)} \leq C\,(\varepsilon^{2t+1-3s} + \varepsilon^{(6t-7s)/3})$$

and

$$\| (u \mid \nabla u_i) \|_{W^{-1,3/2}(\Omega)} \leq C\,(\varepsilon^{(6t+4-10s)/3} + \varepsilon^{(6t+1-7s)/3}) .$$

(b) *We have the estimates for u_3*

$$\| (u \mid \nabla u_3) \|_{H^{-1}(\Omega)} \leq C\,(\varepsilon^{(6t-7s)/3} + \varepsilon^{(6t-1-6s)/3}) .$$

Proof : Let us remark first that

$$(u \mid \nabla u_i) = \sum_{j=1}^{3} \partial_j (u_j\, u_i) ,$$

thus

$$\| (u \mid \nabla u_i) \|_{H^{-1}(\Omega)} \leq \sum_{j=1}^{3} \| u_i\, u_j \|_{L^2(\Omega)} \leq \sum_{j=1}^{3} \| u_j \|_{L^6(\Omega)} \| u_i \|_{L^3(\Omega)} .$$

Since $H_0^1(\Omega) \subset L^6(\Omega)$ we have also

$$\| \partial_3 (u_3)^2 \|_{H^{-1}(\Omega)} \leq C\, \| \partial_3 (u_3)^2 \|_{L^{6/5}(\Omega)} \leq 2\,C\, \| u_3 \|_{L^3(\Omega)} \| \partial_3 u_3 \|_{L^2(\Omega)} .$$

In the same way we get

$$\| (u \mid \nabla u_i) \|_{W^{-1,3/2}(\Omega)} \leq \sum_{j=1}^{3} \| u_i\, u_j \|_{L^{3/2}(\Omega)} \leq \sum_{j=1}^{3} \| u_j \|_{L^2(\Omega)} \| u_i \|_{L^6(\Omega)} .$$

thus lemma 3 follows from lemma 2. ∎

2.3. Behavior of the equations when $\varepsilon \to 0$

We are now able to study the behavior of equation (3) when $\varepsilon \to 0$. Indeed since

$$\| \partial_{33}^2 u_3 \|_{H^{-1}(\Omega)} = \| \partial_1 (\partial_3 u_1) + \partial_2 (\partial_3 u_2) \|_{H^{-1}(\Omega)} \leq \| \partial_3 u_1 \|_{L^2(\Omega)} + \| \partial_3 u_2 \|_{L^2(\Omega)} ,$$

we deduce from equation (3)

$$\| \partial_3 P \|_{H^{-1}(\Omega)} \leq \varepsilon^2 \nu_1 \| \partial_1 u_3 \|_{H^{-1}(\Omega)} + \varepsilon^2 \nu_2 \| \partial_2 u_3 \|_{H^{-1}(\Omega)} + \\ + \nu_3 (\| \partial_3 u_1 \|_{L^2(\Omega)} + \| \partial_3 u_2 \|_{L^2(\Omega)}) + \varepsilon^2 \| (u \mid \nabla u_3) \|_{H^{-1}(\Omega)} ,$$

so by lemma 1 and 3 there exists a constant C such that

$$\|\partial_3 P\|_{H^{-1}(\Omega)} \leq C\left(\varepsilon^{t+1-s} + \varepsilon^{t+1} + \varepsilon^{(6t+6-7s)/3} + \varepsilon^{(6t+5-6s)/3}\right).$$

In the same way, using the estimate

$$\|f u_i\|_{H^{-1}(\Omega)} \leq C \varepsilon^{t+1-2s}$$

for the Coriolis term which follows from lemma 2, there exists a constant C such that

$$\|\partial_i P\|_{H^{-1}(\Omega)} \leq C\left(\varepsilon^{t-s} + \varepsilon^{t-1} + \varepsilon^{t+1-2s} + \varepsilon^{2t+1-3s} + \varepsilon^{(6t-7s)/3}\right), \quad i = 1, 2$$

$$\|\partial_i P\|_{W^{-1,3/2}(\Omega)} \leq C\left(\varepsilon^{t-s} + \varepsilon^{t-1} + \varepsilon^{t+1-2s} + \varepsilon^{(6t+4-10s)/3} + \varepsilon^{(6t+1-7s)/3}\right), \quad i = 1, 2 .$$

In particular if $s = t = 1$ we have

$$\|\nabla P\|_{W^{-1,3/2}(\Omega)} \leq C$$

and therefore by [AG], P is uniformly bounded in $L^{3/2}(\Omega)$. Thus we have proved the

THEOREM 1 : (a) *Assume that in equation* (3), $\nu_3 = \nu_0\, \varepsilon^{2s}$, $s > 0$, *and* $\tau_i = \varepsilon^t \tau_i^0$, $i = 1, 2$ *with* $t > -1$ *and* $s < 5/6(t+1)$, *then*

$$\|\partial_3 P\|_{H^{-1}(\Omega)} \to 0 \quad as \quad \varepsilon \to 0 .$$

(b) *If moreover* $s = t = 1$ *then* P *is bounded in* $L^{3/2}(\Omega)$ *uniformly in* ε. ■

3. EXISTENCE OF SOLUTIONS FOR THE HYDROSTATIC APPROXIMATION

Let us now derive the existence of a solution for the hydrostatic approximation (4). Let

$$W = V_1 \oplus V_2 \oplus H_0(\partial_3, \Omega)$$

and

$$W_0 = \{v \in W ; \operatorname{div} v = 0\} .$$

Finally we define

$$Z = \left\{v = (v_i) \in W,\ \partial_3 v_i \in L^3(\Omega),\ i = 1, 2, \text{ and } \operatorname{div} v \in L^3(\Omega)\right\} .$$

From now we assume that $s = t = 1$ in the previous section and we define $\mu_1 = \nu_1$, $\mu_2 = \nu_2$, $\mu_3 = \nu_0$ and

$$A(\varphi, \psi) = \sum_{i=1}^{3} \mu_i \int_\Omega \partial_i \varphi \, \partial_i \psi \, dx \quad (\varphi, \psi \in H^1(\Omega)) .$$

For any $\varepsilon > 0$ let $u^\varepsilon = (u_i^\varepsilon)$ and P^ε be a solution of problem (5). By lemma 1 and 2, u^ε is bounded in W and P^ε in $L^{3/2}(\Omega)$, so we can assume that u^ε converges weakly in W to some $u \in W_0$, that u_i^ε converges strongly to u_i in $L^4(\Omega)$ $(i = 1, 2)$ and P^ε converges weakly to P in $L^{3/2}(\Omega)$ when $\varepsilon \to 0$. Hence by lemma 1, 2 and 3 for any $v \in Z$ we get

$$A(u_1, v_1) - B(u, u_1, v_1) - C(u_2, v_1) + A(u_2, v_2) - B(u, u_2, v_2) + \\ + C(u_1, v_2) - \int_\Omega P \operatorname{div} v \, dx = E(\tau_1^0, v_1) + E(\tau_2^0, v_2) \quad (8)$$

which is the weak formulation of the hydrostatic approximation.

We have thus proved the theorem.

THEOREM 2 : *For any* τ_1^0, $\tau_2^0 \in H^{1/2}(\partial\Omega)$ *there exists* $u \in W_0$ *and* $P \in L^{3/2}(\Omega)$ *which satisfies equation* (8). *Moreover this solution is the limit when* $\varepsilon \to 0$ *of a solution of problem* (5). ∎

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REFERENCES

[Ad] R. A. ADAMS, Sobolev Spaces ; Academic Press, New York (1975).

[AG] C. AMROUCHE, V. GIRAULT, Propriétés fonctionnelles d'opérateurs, Application au problème de Stokes en dimension quelconque ; *Preprint* Université P. & M. Curie, Laboratoire d'Analyse Numérique (1990).

[BLT] O. BESSON, M. R. LAYDI, R. TOUZANI, Un modèle asymptotique en océanographie ; *C. R. Acad. Sci. Paris,* t. 310, Série I, p. 661-665 (1990).

[FEF] Finite elements in fluids ; Gallagher & al. (Ed.) J. Wiley, Chichester.

[GR] V. GIRAULT, P. A. RAVIART, Finite element methods for Navier-Stokes equations, theory and algorithms ; *Springer-Verlag,* Berlin (1986).

[La] M. R. LAYDI, Sur l'influence de l'hypothèse hydrostatique dans la modélisation du lac de Neuchâtel ; *Rapport interne* Université de Neuchâtel (1989).

[Li] J. L. LIONS, Quelques méthodes de résolution des problèmes aux limites non linéaires ; *Dunod Gauthier-Villars,* Paris (1969).

[Ne] J. NEČAS, Les méthodes directes en théorie des équations elliptiques ; *Masson,* Paris (1967).

[Pe] J. PEDLOSKY, Geophysical fluid dynamics ; *Springer-Verlag,* New York (1987).

[Pi] O. PIRONNEAU, Méthodes des éléments finis pour les fluides ; *Masson,* Paris (1988).

[Sc] Schweizerische Zeitschrift für Hydrologie, Vol. 45, no. 1 (1983).

[Te] R. TEMAM, Navier-Stokes equations ; *North-Holland,* Amsterdam (1985).

[ZD] E. ZUUR, D. E. DIETRICH, The SOMS model and its application to lake Neuchâtel ; *Aquatic Sci.* **52** (1990) 115-129.