RAIRO
MODÉLISATION MATHÉMATIQUE
ET ANALYSE NUMÉRIQUE

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RAIRO – Modélisation mathématique et analyse numérique, tome 27, nº 1 (1993), p. 65-76.

<http://www.numdam.org/item?id=M2AN_1993__27_1_65_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE
(Vol. 27, n° 1, 1993, p. 65 à 76)

HOMOGENIZATION OF FREE BOUNDARY OSCILLATIONS OF AN INVISCID FLUID IN A POROUS RESERVOIR (*)

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Communicated by E. S. PALENCIA

Abstract. — *We consider the linearised free boundary oscillations of a heavy inviscid incompressible fluid in an open bounded cylindrical reservoir, containing many vertical solid tubes, distributed in a periodic manner. The aim of the paper is to investigate limits of eigenfrequencies and eigenmodes, when the periodicity ε of the structure tends to zero.*

Résumé. — *Nous considérons les oscillations linéarisées de la frontière libre d'un fluide pesant non visqueux incompressible dans un réservoir qui contient de nombreux tubes verticaux distribués d'une manière périodique. Le but de l'article est d'examiner les limites des fréquences propres et des modes propres lorsque la périodicité ε de la structure tend vers zéro.*

1. INTRODUCTION

In this paper we consider the linearised free boundary oscillations of a heavy inviscid incompressible fluid in a cylindrical domain, containing many cylindrical solid parts of a small diameter, distributed in a periodic manner with a small period $\varepsilon > 0$.

Let $G \subset Y =]0, 1[^2$ be an open regular set, strictly included in Y, and $Y^* = Y \backslash \bar{G}$. For $k \in Z^2$ we define $Y_k = Y + k$, $G_k = G + k$. Let $\Gamma \subset \mathbb{R}^2$ be a bounded regular domain and

$$\Omega = \Gamma \times]0, 1[, \tag{1.1}$$

$$\Sigma = \partial\Gamma \times]0, 1[, \tag{1.2}$$

$$\Gamma_0 = \Gamma \times \{0\} , \quad \Gamma_1 = \Gamma \times \{1\} . \tag{1.3}$$

(*) Manuscript received December 1991.

The work was supported by a Grant of Ministry of Science of Republic Croatia.

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For sufficiently small $\varepsilon > 0$ we consider the sets

$$I_\varepsilon = \{k \in Z^2 : \varepsilon Y_k \subset \Gamma\} , \tag{1.4}$$

$$G^\varepsilon = \bigcup_{k \in I_\varepsilon} (\varepsilon G_k) , \quad \Gamma^\varepsilon = \Gamma \backslash \bar{G}^\varepsilon \tag{1.5}$$

and define

$$\Omega^\varepsilon = \Gamma^\varepsilon \times]0, 1[, \quad \Sigma^\varepsilon = \partial G^\varepsilon \times]0, 1[, \tag{1.6}$$

$$\Gamma_0^\varepsilon = \Gamma^\varepsilon \times \{0\} , \quad \Gamma_1^\varepsilon = \Gamma^\varepsilon \times \{1\} . \tag{1.7}$$

Obviously,

$$\partial \Omega^\varepsilon = \Sigma \cup \Sigma^\varepsilon \cup \Gamma_0^\varepsilon \cup \Gamma_1^\varepsilon . \tag{1.8}$$

We consider Γ_1^ε to be the free boundary of the fluid, being in the equilibrium state in the reservoir Ω^ε with the solid walls Σ, Σ^ε and Γ_0^ε. Assuming that diam $\Gamma \ll 1$, one can consider small potential oscillations of the fluid around the equilibrium state. Let $\Phi^\varepsilon(x_1, x_2, x_3, t)$ and $U^\varepsilon(x_1, x_2, t)$ $((x_1, x_2) \in \Gamma^\varepsilon$, $x_3 \in]0, 1[$, $t \in \mathbb{R})$ denote, respectively, the velocity potential and the free surface perturbation (in the direction opposite to the gravity). Looking for the oscillations of the form

$$\Phi^\varepsilon(x_1, x_2, x_3, t) = \varphi^\varepsilon(x_1, x_2, x_3) \sin \omega_\varepsilon t , \tag{1.9}$$

$$U^\varepsilon(x_1, x_2, t) = u^\varepsilon(x_1, x_2) \cos \omega_\varepsilon t , \tag{1.10}$$

we have the following spectral problem [*cf.* 3] :

$$\Delta \varphi^\varepsilon = 0 \quad \text{in} \quad \Omega^\varepsilon , \tag{1.11}$$

$$\frac{\partial \varphi^\varepsilon}{\partial \nu} = 0 \quad \text{on} \quad \Sigma \cup \Sigma^\varepsilon \cup \Gamma_0^\varepsilon , \tag{1.12}$$

$$\frac{\partial \varphi^\varepsilon}{\partial x_3} + \omega_\varepsilon u^\varepsilon = 0 \quad \text{on} \quad \Gamma_1^\varepsilon , \tag{1.13}$$

$$\frac{\omega_\varepsilon}{g} \varphi^\varepsilon + u^\varepsilon = 0 \quad \text{on} \quad \Gamma_1^\varepsilon . \tag{1.14}$$

Here ν denotes the outward normal on $\partial \Omega^\varepsilon$; g is the gravity acceleration.

With the help of homogenization method [1], [5] and an abstract theorem of G. A. Yosifian, O. A. Oleinik and A. S. Shamaev [4], we shall find out the limits of the eigenvalues ω_ε and eigenmodes u^ε, when the periodicity ε tends to zero.

2. THE CONVERGENCE THEOREM

Let q^i $(i = 1, 2)$ be a solution to the problem

$$\Delta q^i = 0 \quad \text{in} \quad \mathbb{R}^2 , \tag{2.1}$$

$$\frac{\partial q^i}{\partial \nu} = - \nu_i \quad \text{on} \quad \partial G , \tag{2.2}$$

$$q^i \text{ is } Y\text{-periodic} \tag{2.3}$$

(where ν denotes the outward normal on ∂G), and

$$\Theta = |Y^*| , \quad a_{ij} = \Theta \delta_{ij} + \int_{Y^*} \frac{\partial q^i}{\partial y_j} dy , \quad i, j = 1, 2 . \tag{2.4}$$

The matrix $(a_{ij})_{i,j=1,2}$ is symmetric and positive definite [2]. With the help of two-scale asymptotic expansion [1], [5], one can get easily the homogenized problem, corresponding to the problem (1.11)-(1.14) :

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial \varphi}{\partial x_j} \right) + \Theta \frac{\partial^2 \varphi}{\partial x_3^2} = 0 \quad \text{in} \quad \Omega , \tag{2.5}$$

$$\sum_{i,j=1}^{2} a_{ij} \frac{\partial \varphi}{\partial x_j} \nu_i = 0 \quad \text{on} \quad \Sigma , \tag{2.6}$$

$$\frac{\partial \varphi}{\partial x_3} = 0 \quad \text{on} \quad \Gamma_0 , \tag{2.7}$$

$$\frac{\partial \varphi}{\partial x_3} + \omega u = 0 \quad \text{on} \quad \Gamma_1 , \tag{2.8}$$

$$\frac{\omega}{g} \varphi + u = 0 \quad \text{on} \quad \Gamma_1 . \tag{2.9}$$

Let us introduce the Hilbert spaces

$$H^\varepsilon = \left\{ v \in L^2(\Gamma_1^\varepsilon) : \int_{\Gamma_1^\varepsilon} v \, dx_1 \, dx_2 = 0 \right\} , \tag{2.10}$$

$$V^\varepsilon = \{ v \in H^1(\Omega^\varepsilon) : \gamma^\varepsilon \in H^\varepsilon \} \tag{2.11}$$

$$H = \left\{ v \in L^2(\Gamma_1) : \int_{\Gamma_1} v \, dx_1 \, dx_2 = 0 \right\} , \tag{2.12}$$

$$V = \{ v \in H^1(\Omega) ; \gamma v \in H \} \tag{2.13}$$

equipped, respectively, with the norms

$$\|v\|_{H^\varepsilon} = \left(\int_{\Gamma_1^\varepsilon} v^2\, dx_1\, dx_2 \right)^{1/2}, \tag{2.14}$$

$$\|v\|_{V^\varepsilon} = \left(\int_{\Omega^\varepsilon} |\nabla v|^2\, dx \right)^{1/2}, \tag{2.15}$$

$$\|v\|_{H} = \left(\Theta \int_{\Gamma_1} v^2\, dx_1\, dx_2 \right)^{1/2}, \tag{2.16}$$

$$\|v\|_{V} = \left(\int_{\Omega} \left(\left(\sum_{i,j=1}^{2} a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} + \Theta \left(\frac{\partial v}{\partial x_3} \right)^2 \right) dx \right)^{1/2} \right). \tag{2.17}$$

Let $K^\varepsilon : H^\varepsilon \to V^\varepsilon$ and $K : H \to V$ be the linear operators, defined, respectively, by the conditions

$$(K^\varepsilon f, v)_{V^\varepsilon} = (f, \gamma^\varepsilon v)_{H^\varepsilon}, \quad f \in H^\varepsilon, \quad v \in V^\varepsilon, \tag{2.18}$$

$$(Kf, v)_V = (f, \gamma v)_H, \quad f \in H, \quad v \in V, \tag{2.19}$$

where $\gamma^\varepsilon : H^1(\Omega^\varepsilon) \to L^2(\Gamma_1^\varepsilon)$ and $\gamma : H^1(\Omega) \to L^2(\Gamma_1)$ denote the trace operators, and let

$$A^\varepsilon = \gamma^\varepsilon K^\varepsilon, \quad A = \gamma K. \tag{2.20}$$

Then the problems (1.11)-(1.14) and (2.5)-(2.9) are equivalent, respectively, to the equations

$$A^\varepsilon u^\varepsilon = \mu^\varepsilon u, \quad \varphi = -\omega_\varepsilon K^\varepsilon u^\varepsilon, \tag{2.21}$$

and

$$Au = \mu u, \quad \varphi = -\omega K u, \tag{2.22}$$

where $\mu^\varepsilon = g/\omega_\varepsilon^2$, $\mu = g/\omega^2$. It is easy to see that the operators $A^\varepsilon : H^\varepsilon \to H^\varepsilon$ and $A : H \to H$ are symmetric, compact and strictly positive. Consequently, the problems (2.21) and (2.22) have numerable sets of eigenvalues

$$\mu_1^\varepsilon \geqslant \mu_2^\varepsilon \geqslant \cdots, \tag{2.23}$$

$$\mu_1 \geqslant \mu_2 \geqslant \cdots, \tag{2.24}$$

and correspondent eigenfunctions

$$u_k^\varepsilon,\ k = 1, 2, \ldots,\ (u_k^\varepsilon, u_{k'}^\varepsilon)_{H^\varepsilon} = \delta_{kk'}, \tag{2.25}$$

$$u_k,\ k = 1, 2, \ldots,\ (u_k, u_{k'})_H = \delta_{kk'}, \tag{2.26}$$

respectively. Each eigenvalue in (2.23) and (2.24) is counted as many times as its multiplicity.

For $f \in H$, let

$$R^\varepsilon f = f - \frac{1}{|\Gamma^\varepsilon|} \int_{\Gamma_1^\varepsilon} f \, dx_1 \, dx_2 \,. \tag{2.27}$$

Obviously, $R^\varepsilon : H \to H^\varepsilon$.

THEOREM 2.1 : *The following conclusions hold true :*

$$\lim_{\varepsilon \to 0} \mu_k^\varepsilon = \mu_k \,, \quad k = 1, 2, \dots ; \tag{2.28}$$

if the multiplicity of μ_{k+1} is equal m and if w is an element of the corresponding eigenspace, then there exists a linear combination $\bar{u}$ of the functions $u_{k+1}^\varepsilon, u_{k+2}^\varepsilon, \dots, u_{k+m}^\varepsilon$, such that

$$\lim_{\varepsilon \to 0} \|\bar{u}^\varepsilon - R^\varepsilon w\|_{H^\varepsilon} = 0 \,. \tag{2.29}$$

3. PROPERTIES OF THE OPERATORS A^ε

LEMMA 3.1 : *For $f \in H$, it holds*

$$\lim_{\varepsilon \to 0} \|R^\varepsilon f\|_{H^\varepsilon} = \|f\|_H \,. \tag{3.1}$$

Proof : We have

$$\|R^\varepsilon f\|_{H^\varepsilon}^2 = \int_{\Gamma_1^\varepsilon} f^2 \, dx_1 \, dx_2 - \frac{1}{|\Gamma^\varepsilon|} \left(\int_{\Gamma_1^\varepsilon} f \, dx_1 \, dx_2 \right)^2 . \tag{3.2}$$

Let χ^ε be the characteristic function of the set Γ^ε; it is known that

$$\chi^\varepsilon \to \Theta \quad \text{weakly in} \quad L^2(\Gamma) \,, \tag{3.3}$$

when ε tends to zero. With the help of it, we obtain

$$\lim_{\varepsilon \to 0} \int_{\Gamma_1^\varepsilon} f^2 \, dx_1 \, dx_2 = \lim_{\varepsilon \to 0} \int_{\Gamma_1} \chi^\varepsilon f^2 \, dx_1 \, dx_2 = \|f\|_H^2 \,, \tag{3.4}$$

$$\lim_{\varepsilon \to 0} |\Gamma^\varepsilon| = \int_\Gamma \chi^\varepsilon \, dx_1 \, dx_2 = \Theta |\Gamma| \,, \tag{3.5}$$

$$\lim_{\varepsilon \to 0} \int_{\Gamma_1^\varepsilon} f \, dx_1 \, dx_2 = \lim_{\varepsilon \to 0} \int_{\Gamma_1} \chi^\varepsilon f \, dx_1 \, dx_2 = 0 \tag{3.6}$$

and hence (3.1).

In what follows, $C > 0$ denotes a generic constant, not depending on ε and having possibly different values at different places.

It is known [2] that there exists a linear extension operator $P^{\varepsilon}: H^1(\Omega^{\varepsilon}) \to H^1(\Omega)$, with the property

$$\|\nabla(P^{\varepsilon} v)\|_{L^2(\Omega)} \leq C \|\nabla v\|_{L^2(\Omega^{\varepsilon})}, \quad v \in H^1(\Omega^{\varepsilon}). \tag{3.7}$$

LEMMA 3.2 : *It holds the inequality*

$$\|P^{\varepsilon} v\|_{L^2(\Gamma_1)} \leq C \|v\|_{V^{\varepsilon}}, \quad v \in V^{\varepsilon}. \tag{3.8}$$

Proof : Obviously,

$$P^{\varepsilon} v - \frac{1}{|\Gamma|} \int_{\Gamma_1} P^{\varepsilon} v \, dx_1 \, dx_2 \in V. \tag{3.9}$$

Therefore it holds

$$\left\| P^{\varepsilon} v - \frac{1}{|\Gamma|} \int_{\Gamma_1} P^{\varepsilon} v \, dx_1 \, dx_2 \right\|_{L^2(\Gamma_1)} \leq C \|\nabla(P^{\varepsilon} v)\|_{L^2(\Omega)}. \tag{3.10}$$

Using (3.7), we obtain

$$\|P^{\varepsilon} v\|^2_{L^2(\Gamma_1)} \leq C \|v\|^2_{V^{\varepsilon}} + (1 - \Theta)\|P^{\varepsilon} v\|^2_{L^2(\Gamma_1)}, \tag{3.11}$$

and hence (3.8).

LEMMA 3.3 : *It holds the inequality*

$$\|A^{\varepsilon} f\|_{H^{\varepsilon}} \leq C \|f\|_{H^{\varepsilon}}, \quad f \in H^{\varepsilon}. \tag{3.12}$$

Proof : We have

$$\|A^{\varepsilon} f\|_{H^{\varepsilon}} \leq \|P^{\varepsilon} K^{\varepsilon} f\|_{L^2(\Gamma_1)} \tag{3.13}$$

and, using Lemma 3.2,

$$\|A^{\varepsilon} f\|_{H^{\varepsilon}} \leq C \|K^{\varepsilon} f\|_{V^{\varepsilon}}; \tag{3.14}$$

because of (2.18), it holds

$$\|K^{\varepsilon} f\|^2_{V^{\varepsilon}} = (f, A^{\varepsilon} f)_{H^{\varepsilon}} \leq \|f\|_{H^{\varepsilon}} \cdot \|A^{\varepsilon} f\|_{H^{\varepsilon}}, \tag{3.15}$$

and hence (3.12).

LEMMA 3.4 : *Let* $f \in H$ *and* $\psi^{\varepsilon} = K^{\varepsilon} R^{\varepsilon} f$. *Then there exists a subsequence of* $\{\psi^{\varepsilon}\}$ *(denoted again by* $\{\psi^{\varepsilon}\}$*) and a function* $\psi \in V$, *such that*

$$P^{\varepsilon} \psi^{\varepsilon} \rightarrow \psi \quad \text{weakly in} \quad H^1(\Omega), \tag{3.16}$$

$$\text{strongly in} \quad L^2(\Omega) \text{ and} \tag{3.17}$$

$$\text{strongly in} \quad L^2(\Gamma_1). \tag{3.18}$$

Proof: Because of (2.18), it holds

$$\|\psi^{\varepsilon}\|_{V^{\varepsilon}}^2 = (R^{\varepsilon} f, \psi^{\varepsilon})_{H^{\varepsilon}} \leqslant \|R^{\varepsilon} f\|_{H^{\varepsilon}} \cdot \|P^{\varepsilon} \psi^{\varepsilon}\|_{L^2(\Gamma_1)}; \tag{3.19}$$

taking into account Lemmas 3.1 and 3.2, we obtain

$$\|\psi^{\varepsilon}\|_{V^{\varepsilon}}^2 \leqslant C \|\psi^{\varepsilon}\|_{V^{\varepsilon}}, \tag{3.20}$$

and hence

$$\|\psi^{\varepsilon}\|_{V^{\varepsilon}} \leqslant C. \tag{3.21}$$

Using (3.7), we have

$$\|\nabla(P^{\varepsilon} \psi^{\varepsilon})\|_{L^2(\Omega)} \leqslant C \|\psi^{\varepsilon}\|_{V^{\varepsilon}} \leqslant C. \tag{3.22}$$

Using Lemma 3.2 again, (3.21) and (3.22), one can easy conclude that

$$\|P^{\varepsilon} \psi^{\varepsilon}\|_{H^1(\Omega)} \leqslant C, \tag{3.23}$$

and hence (3.16)-(3.18). Finally, from (3.3), (3.18) and the equality

$$\int_{\Gamma_1} \chi^{\varepsilon} \cdot (P^{\varepsilon} \psi^{\varepsilon})\, dx_1\, dx_2 = 0, \tag{3.24}$$

we obtain $\psi \in V$.

LEMMA 3.5 : *Under the notation of Lemma* 3.4, *let*

$$\tau^{\varepsilon} = \begin{cases} \nabla \psi^{\varepsilon} & \text{in} \quad \Omega^{\varepsilon} \\ 0 & \text{otherwise}. \end{cases} \tag{3.25}$$

Then there exists a subsequence of $\{\tau^{\varepsilon}\}$ (denoted again by $\{\tau^{\varepsilon}\}$) and a function $\tau \in (L^2(\Omega))^3$, such that

$$\tau^{\varepsilon} \rightarrow \tau \quad \text{weakly in} \quad (L^2(\Omega))^3. \tag{3.26}$$

For each $v \in V$, it holds

$$\int_{\Omega} \tau \cdot \nabla v\, dx = \Theta \int_{\Gamma_1} f \cdot v\, dx_1\, dx_2. \tag{3.27}$$

Proof : The conclusion (3.26) follows immediately from (3.21). Because of (2.18), for $v \in V$ it holds

$$\int_{\Omega} \tau^{\varepsilon} \cdot \nabla v \, dx = \int_{\Gamma_1^{\varepsilon}} f \cdot v \, dx_1 \, dx_2 - \frac{1}{|\Gamma^{\varepsilon}|} \int_{\Gamma_1^{\varepsilon}} f \, dx_1 \, dx_2 \cdot \int_{\Gamma_1^{\varepsilon}} v \, dx_1 \, dx_2 \, . \tag{3.28}$$

Taking into account (3.26) and (3.4)-(3.6), we obtain (3.27).

LEMMA 3.6 : *The functions* ψ *and* τ, *defined, respectively, by Lemmas* 3.4 *and* 3.5, *satisfy the equations*

$$\tau_i = \sum_{j=1}^{2} a_{ij} \frac{\partial \psi}{\partial x_j}, \quad i = 1, 2 , \tag{3.29}$$

$$\tau_3 = \Theta \frac{\partial \psi}{\partial x_3} . \tag{3.30}$$

Proof : Let

$$q^{i,\varepsilon}(x_1, x_2) = (P q^i)\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon} \right), \quad (x_1, x_2) \in \Gamma , \quad i = 1, 2 , \tag{3.31}$$

where $P : H^1(Y^*) \to H^1(Y)$ is a linear extension operator with the property [2]

$$\|\nabla (P v)\|_{L^2(Y)} \leqslant C \, \|\nabla v\|_{L^2(Y^*)}, \quad v \in H^1(Y^*) . \tag{3.32}$$

Let

$$\eta^i(y) = \begin{cases} \nabla (q^i(y) + y_i), & y \in Y^* \\ 0, & y \in G \end{cases} , \; i = 1, 2 , \tag{3.33}$$

and

$$\eta^{i,\varepsilon}(x_1, x_2) = \eta^i \left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon} \right) (x_1, x_2) \in \Gamma , \quad i = 1, 2 . \tag{3.34}$$

From (2.1)-(2.3) it follows the inequality

$$\|q^{i,\varepsilon}\|_{L^2(\Gamma)} \leqslant C . \tag{3.35}$$

The functions (3.34) satisfy the equations

$$\operatorname{div} \eta^{i,\varepsilon} = 0 \quad \text{in} \quad \Omega , \quad i = 1, 2 . \tag{3.36}$$

Let $v \in \mathscr{D}(\Omega)$; multiplying (3.36) by $(P^{\varepsilon} \psi^{\varepsilon}) v$, after integration by parts we obtain

$$\int_{\Omega} \eta^{i,\varepsilon} . \nabla (P^{\varepsilon} \psi^{\varepsilon}) . v \, dx + \int_{\Omega} \eta^{i,\varepsilon} . (P^{\varepsilon} \psi^{\varepsilon}) . \nabla v \, dx = 0 , \quad (3.37)$$

or

$$\int_{\Omega} \eta^{i,\varepsilon} \cdot \tau^{\varepsilon} \cdot v \, dx = - \sum_{j=1}^{2} \int_{\Omega} \eta_j^{i,\varepsilon} \cdot (P^{\varepsilon} \psi^{\varepsilon}) \cdot \frac{\partial v}{\partial x_j} dx . \quad (3.38)$$

Let $\{\psi^{\varepsilon}, \tau^{\varepsilon}\}$ be a subsequence satisfying (3.16)-(3.18) and (3.26). Taking into account the fact that

$$\eta_j^{i,\varepsilon} \to a_{ij} \quad \text{weakly in} \quad L^2(\Gamma) , \quad i, j = 1, 2 , \quad (3.39)$$

we obtain

$$\lim_{\varepsilon \to 0} \int_{\Omega} \eta^{i,\varepsilon} . \tau^{\varepsilon} . v \, dx = \sum_{j=1}^{2} \int_{\Omega} a_{ij} \frac{\partial \psi}{\partial x_j} v \, dx . \quad (3.40)$$

Setting in (3.28) the function $(\varepsilon q^{i,\varepsilon} + x_i)$ $(i = 1, 2)$ instead of v, we have

$$\int_{\Omega} \tau^{\varepsilon} . \eta^{i,\varepsilon} . v \, dx + \int_{\Omega} \tau^{\varepsilon} . (\varepsilon q^{i,\varepsilon} + x_i) . \nabla v \, dx = 0 . \quad (3.41)$$

Taking into account (3.26), (3.35) and (3.40), we find that

$$\sum_{j=1}^{2} \int_{\Omega} \left(a_{ij} \frac{\partial \psi}{\partial x_j} v + \tau_j x_i \frac{\partial v}{\partial x_j} \right) dx + \int_{\Omega} \tau_3 x_i \frac{\partial v}{\partial x_3} dx = 0 , \quad i = 1, 2 . \quad (3.42)$$

Setting in (3.27) the function $x_i v$ $(i = 1, 2)$ instead of v, we have

$$\sum_{j=1}^{2} \int_{\Omega} \tau_j x_i \frac{\partial v}{\partial x_j} dx + \int_{\Omega} \tau_3 x_i \frac{\partial v}{\partial x_3} dx + \int_{\Omega} \tau_i v \, dx = 0 , \quad i = 1, 2 . \quad (3.43)$$

From (3.42) and (3.43) it follows

$$\sum_{j=1}^{2} \int_{\Omega} a_{ij} \frac{\partial \psi}{\partial x_j} v \, dx - \int_{\Omega} \tau_i v \, dx = 0 , \quad i = 1, 2 , \quad (3.44)$$

and hence (3.29). Finally, we have

$$\int_{\Omega} \tau_3^{\varepsilon} v \, dx = \int_{\Omega^{\varepsilon}} \frac{\partial \psi^{\varepsilon}}{\partial x_3} v \, dx = - \int_{\Omega^{\varepsilon}} \psi^{\varepsilon} \frac{\partial v}{\partial x_3} dx = - \int_{\Omega} \chi^{\varepsilon} \cdot (P^{\varepsilon} \psi^{\varepsilon}) \cdot \frac{\partial v}{\partial x_3} dx \quad (3.45)$$

Taking into account (3.3), (3.17) and (3.26), we obtain

$$\int_{\Omega} \tau_3 \, v \, dx = - \Theta \int_{\Omega} \psi \frac{\partial v}{\partial x_3} dx \,, \tag{3.46}$$

and hence (3.30).

LEMMA 3.7 : *The function ψ, defined by Lemma 3.4, doesn't depend on the subsequence $\{\psi^{\varepsilon}\}$ and satisfies the equality*

$$\psi = Kf \,. \tag{3.47}$$

Proof : The conclusion follows immediately from (2.19) and Lemmas 3.5 and 3.6.

LEMMA 3.8 : *For $f \in H$, it holds*

$$\lim_{\varepsilon \to 0} \|A^{\varepsilon} R^{\varepsilon} f - R^{\varepsilon} A f\|_{H^{\varepsilon}} = 0 \,. \tag{3.48}$$

Proof : We have

$$\|A^{\varepsilon} R^{\varepsilon} f - R^{\varepsilon} A f\|_{H^{\varepsilon}} \leqslant$$

$$\leqslant \|\psi^{\varepsilon} - Kf\|_{L^2(\Gamma_1^{\varepsilon})} + \frac{1}{|\Gamma^{\varepsilon}|^{1/2}} \left| \int_{\Gamma_1^{\varepsilon}} Kf \, dx_1 \, dx_2 \right|$$

$$\leqslant \|P^{\varepsilon} \psi^{\varepsilon} - Kf\|_{L^2(\Gamma_1)} + \frac{1}{|\Gamma^{\varepsilon}|^{1/2}} \left| \int_{\Gamma_1^{\varepsilon}} Kf \, dx_1 \, dx_2 \right| \,. \tag{3.49}$$

Taking into account (3.5), (3.6), (3.18) and (3.47), we obtain (3.48).

LEMMA 3.9 : *If $\{f^{\varepsilon}\}$ is a sequence of functions $f^{\varepsilon} \in H^{\varepsilon}$ with the property*

$$\|f^{\varepsilon}\|_{H^{\varepsilon}} \leqslant C \,, \tag{3.50}$$

then there exists its subsequence (denoted again by $\{f^{\varepsilon}\}$) and a function $w \in H$, such that

$$\lim_{\varepsilon \to 0} \|A^{\varepsilon} f^{\varepsilon} - R^{\varepsilon} w\|_{H^{\varepsilon}} = 0 \,. \tag{3.51}$$

Proof : Because of (2.18) and Lemma 3.2, we have

$$\|K^{\varepsilon} f^{\varepsilon}\|_{V^{\varepsilon}}^2 = (f^{\varepsilon}, \gamma^{\varepsilon} K^{\varepsilon} f^{\varepsilon})_{H^{\varepsilon}} \leqslant C \|P^{\varepsilon} K^{\varepsilon} f^{\varepsilon}\|_{L^2(\Gamma_1)} \leqslant C \|K^{\varepsilon} f^{\varepsilon}\|_{V^{\varepsilon}} \tag{3.52}$$

and hence

$$\|K^{\varepsilon} f^{\varepsilon}\|_{V^{\varepsilon}} \leq C . \tag{3.53}$$

Using (3.7) and Lemma 3.2 again, we conclude that

$$\|P^{\varepsilon} K^{\varepsilon} f^{\varepsilon}\|_{H^1(\Omega)} \leq C . \tag{3.54}$$

Consequently, there exists a subsequence of $\{P^{\varepsilon} K^{\varepsilon} f^{\varepsilon}\}$ (denoted again by $\{P^{\varepsilon} K^{\varepsilon} f^{\varepsilon}\}$) and a function $h \in H^1(\Omega)$, such that

$$P^{\varepsilon} K^{\varepsilon} f^{\varepsilon} \to h \quad \text{weakly in} \quad H^1(\Omega), \tag{3.55}$$

$$\text{strongly in} \quad L^2(\Omega) \quad \text{and} \tag{3.56}$$

$$\text{strongly in} \quad L^2(\Gamma_1). \tag{3.57}$$

From (3.3), (3.57) and the equality

$$\int_{\Gamma_1} \chi^{\varepsilon} \cdot (P^{\varepsilon} K^{\varepsilon} f^{\varepsilon})\, dx_1\, dx_2 = 0 , \tag{3.58}$$

we obtain $h \in V$. Let $w = \gamma h$; then it holds

$$\|A^{\varepsilon} f^{\varepsilon} - R^{\varepsilon} w\|_{H^{\varepsilon}} \leq$$

$$\leq \|A^{\varepsilon} f^{\varepsilon} - w\|_{L^2(\Gamma_1^{\varepsilon})} + \frac{1}{|\Gamma^{\varepsilon}|} \left| \int_{\Gamma_1^{\varepsilon}} w\, dx_1\, dx_2 \right|$$

$$\leq \|P^{\varepsilon} K^{\varepsilon} f^{\varepsilon} - w\|_{L^2(\Gamma_1)} + \frac{1}{|\Gamma^{\varepsilon}|} \left| \int_{\Gamma_1} w\, dx_1\, dx_2 \right| . \tag{3.59}$$

Taking into account (3.5), (3.6) and (3.57), we obtain (3.51).

4. PROOF OF THEOREM 2.1

The conclusions of Theorem 2.1 follow immediately from Lemmas 3.1, 3.3, 3.8 and 3.9, which appear to be the verifications of assumptions of the mentioned abstract result [4].

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