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EXISTENCE RESULTS FOR A NONLINEAR PROBLEM MODELING THE DISPLACEMENT OF A SOLID IN A TRANSVERSE FLOW (*)

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Abstract — *In this paper, we study a simplified mathematical model that describes the stationary displacements of a solid body immersed in a transverse flow This model involves the Laplace equation with a non-homogeneous Neumann boundary condition in a domain whose geometry depends on the displacement of the solid under the action of the fluid The solution of the equation and the displacement are related by a nonlinear condition The nonlinear character of the model is also present in the dependence of the domain on the solution We give here an existence result for this case and for a more general situation*

Résumé — *Dans cet article, on étudie un modèle mathématique simplifié qui décrit, en régime stationnaire, les déplacements d'un solide immergé dans un écoulement transversal Ce modèle fait intervenir l'équation de Laplace avec une condition aux limites non homogène de type Neumann, dans un domaine dont la géométrie dépend du déplacement du solide sous l'action du fluide La solution a ce probleme aux limites et le déplacement du solide sont liés par une relation non linéaire Le caractere non linéaire du modèle est aussi présent par le fait que le domaine dépend de la solution On démontre ici un résultat d'existence pour le cas décrit ci-dessus, ainsi que pour une situation plus générale*

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INTRODUCTION

Let Ω be a bounded open set of $\mathbb{R}^N$ and T_0 a smooth region of Ω. Consider the translated sets T_s of T_0 by vectors $s \in \mathbb{R}^N$ such that the displaced set $\bar{T}_s$ remains included in Ω (i.e., $T_s = s + T_0$ and $\bar{T}_s \subset \Omega$; see *fig.* 1).

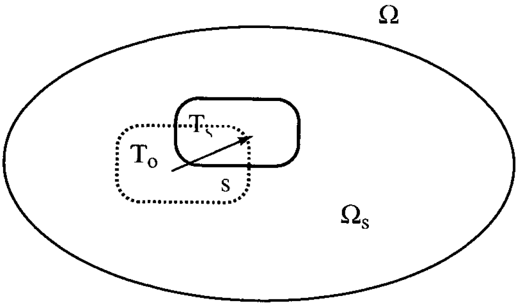

Figure 1. — The region Ω_s

Let s be such a displacement of T_0 and φ_s be the solution of the Laplace equation in $\Omega_s = \Omega \backslash \bar{T}_s$, with a homogeneous Neumann condition on the external boundary $\partial\Omega$ and with a given non-homogeneous Neumann condition on the boundary of T_s.

We are interested in finding s such that (s, φ_s) verifies

$$s = \alpha \int_{\partial T_s} |\nabla \varphi_s + \dot{u}_s|^2 \, n \, d\sigma \, , \tag{0.1}$$

where $\alpha \in R$ and $u_s \in L^2(\partial T_s)$ are given and n denotes the outward unit normal on the boundary of Ω_s. Condition (0.1) might seems rather strange at first glance and there is no doubt that some comments and further explanations are needed. This will be done in § 2, where we present a complete physical interpretation of this mathematical problem and a detailed derivation of (0.1) is consequently also included.

This is a nonlinear boundary-value problem, since equation (0.1) is a nonlinear condition on the couple (s, φ_s) and φ_s solves a boundary-value problem in a domain which also depends on s.

The purpose of this paper is to prove that for α sufficiently small (say, for all α, $|\alpha| \leq \alpha_0$), this problem has at least one solution. This solution is also small in the sense that the displacement s belongs to a small neighborhood of the origin which depends on α_0.

Our approach consists of reducing the original problem to that of finding a fixed point of a nonlinear function F acting on $\mathbb{R}^N$. As usual, the existence of the fixed point is done in two steps : the first is to prove that F is continuous and the next is to prove that F maps a ball of $\mathbb{R}^N$ into itself. The continuity of F is based on the fact that the family of couples $\{(s, \nabla\varphi_s(x-s))\}$, with s admissible, is relatively compact in $\mathbb{R}^N \times L^2(\partial T_0)$. To prove this compactness, we use a local a priori estimate of the H^2-norm of the solution φ_s and a uniform Poincaré's inequality, which provides a global H^1-estimate. The cluster points of this family are easily identified by a simple limit process.

From an engineering point of view, this type of nonlinear boundary-value problem arises in the description of some kinds of fluid-solid interactions. For example, if $\Omega \backslash \bar{T}_0$ is regarded as the bidimensional section of a region occupied by a fluid (or liquid) and T_0 as a transversal section of a solid structure immersed in this fluid, then the problem with which we are concerned is nothing other than a simple mathematical model (see D. J. Gorman & J. Planchard [1988] or J. Planchard & B. Thomas [1991]) describing the stationary displacement of the solid by the action of the fluid. In § 2, following the above authors, we give a complete description of this model as well as the mathematical derivation of its equations.

As we shall see, this model is really very simple, since it assume the fluid to be perfect, incompressible, and irrotational. Of course, for practical purposes this is not realistic except in the case of a very slowly flowing fluid. It corresponds to a first approximation to the viscous case which is our intention to consider in a forthcoming paper. Our goal in studying this simpler model is to establish some elementary properties of its solutions and to introduce a general mathematical framework where this kind of nonlinear problems can be set out. The present study can also serve as a guide to envisage new methods for tackling the case where the fluid motion is governed by the Navier-Stokes equations.

To conclude this Introduction let us discuss the content of the remaining sections. In § 1 we give a precise formulation of our boundary-value problem and we state the main Existence Theorem. Section 3 is devoted to proving this result. Finally, in § 4 we include a complementary existence result for the case where the solid T_0 has several connected components which can move independently.

1. STATEMENT OF THE PROBLEM AND THE MAIN RESULT

Let Ω be a bounded open set of $\mathbb{R}^N$ with boundary $\delta\Omega$ locally Lipschitz, and consider an open subset T_0 of Ω (not necessarily connected) such that

$\bar{T}_0 \subset \Omega$ We assume that the boundary γ_0 of T_0 is of class $\mathscr{C}^2$ For any $s \in \mathbb{R}^N$, we define the *translate* sets of T_0 and γ_0 by the vector s, as follows

$$\begin{aligned} T_s &= s + T_0 \\ \gamma_s &= \partial T_s = s + \gamma_0 \end{aligned}$$

Let $\delta > 0$ be a given constant We shall call *ring around T_0 of thickness δ* the set $C(T_0, \delta)$ defined by

$$C(T_0, \delta) = \left\{ x \in (\mathbb{R}^N \backslash \bar{T}_0) \mid \operatorname{dist}(x, \gamma_0) < \delta \right\}$$

Analogously, we denote by $C(T_s, \delta)$ (or simply by C_s) the ring of thickness δ around T_s, that is,

$$C(T_s, \delta) = s + C(T_0, \delta)$$

We define the set S_δ of the admissible displacements of T_0 by

$$S_\delta = \left\{ s \in \mathbb{R}^N \mid \bar{C}_s \subset \Omega \right\}$$

Observe that S_δ is an open bounded subset of $\mathbb{R}^N$, and assume that

$$S_\delta \neq \emptyset , \tag{1 1}$$

which holds for δ small enough since 0 belongs to S_δ for all $\delta < \operatorname{dist}(T_0, \partial\Omega)$ For each $s \in S_\delta$ we denote by Ω_s the region of Ω defined as follows (see *fig* 1)

$$\Omega_s = \Omega \backslash \bar{T}_s$$

Let $\alpha \in \mathbb{R}$, $u \in L^2(\gamma_0)^N$ and $g \in H^{1/2}(\gamma_0)$ be given with $\int_{\gamma_0} g \, d\sigma = 0$ Our aim is to prove an existence result for the following problem

Find $s \in S_\delta$ and $\varphi_s \in H^1(\Omega_s) \cap H^2(C_s)$ such that

$$\left\{ \begin{aligned} \Delta \varphi_s &= 0 \quad \text{in} \quad \Omega_s \\ \frac{\partial \varphi_s}{\partial n} &= 0 \quad \text{on} \quad \partial\Omega \\ \frac{\partial \varphi_s}{\partial n} &= g_s \quad \text{on} \quad \gamma_s \\ s &= \alpha \int_{\gamma_s} |\nabla \varphi_s + u_s|^2 \, n \, d\sigma , \end{aligned} \right. \tag{1 2}$$

where $u_s(x) = u(x - s)$, $g_s(x) = g(x - s)$ for $x \in \gamma_s$, and n denotes the outward unit normal on the boundary of Ω_s, $\partial/\partial n$ denotes the derivation along that normal

Our existence theorem shows that (1 2) has at least one (small) solution if α is sufficiently small More precisely, we prove the following result

THEOREM 1 1 *Assume that* (1 1) *holds Define* r_0 *by*

$$r_0 = \max \{ r > 0 \mid B(0, r) \subset S_\delta \} ,$$

where $B(0, r)$ *denotes the ball of radius* r, *centered at the origin Then there exists a strictly positive constant* α_0 *such that for all* α, $|\alpha| < \alpha_0$, *problem* (1 2) *has a solution* (s, φ_s) *in* $B(0, r_0) \times (H^1(\Omega_s) \cap H^2(C_s))$

Problem (1 2) allows us to take into account the case where T_0 is connected and can be translated in any direction as well as the case where T_0 is multi-connected, but all its connected components can only be translated simultaneously (that is, the case in which all the components of T_0 are treated as one rigid body)

In § 4 we shall consider a more general problem than (1 2) which will allow us to consider the case where any component of T_0 can move independently

2 PHYSICAL MOTIVATION

In order to provide an engineering justification for the study of problem (1 2), let us present a mathematical model which describes the interaction between a solid and a fluid, and which gives rise to this kind of boundary-value problems To this end, let us imagine a homogeneous fluid (or liquid) contained in a three-dimensional rectangular box (with edges parallel to the coordinate axes) Denote by Ω any section of this box, perpendicular to the OX_3-axis

Within this box there is a solid structure, immersed in the fluid, whose projection on Ω is also a constant region which is represented by T_0 To fix ideas, let us simply consider a cylindrical tube with a circular section, whose generating line is parallel to the OX_3-axis We assume that the fluid flows *transversally* to the tube More precisely, it enters by one of the faces of the box, perpendicular to the OX_2 axis, with a constant velocity $Ue_2(U > 0,\ e_2 = (0, 1, 0))$, and leaves by the opposite face, also with the constant velocity Ue_2 The other faces of the box are assumed to be rigid walls

We are interested in modeling the interaction of this fluid-solid structure on the following assumptions (i) The tube is rigid (that is, its section is not deformed by the action of the fluid), (ii) it is long enough for three-dimensional effects to be neglected and for the problem to be studied in Ω, and (iii) the ends of the tube are joined to two opposite faces of the cavity, in such a way that the tube can be likened to a long elastic bar (of section T_0) which can move transversally, but which does not allow movement perpendicular to its section *Concerning the section Ω, this means that* T_0 can move in any direction inside Ω Of course, we assume that T_0 is far away from the boundary of Ω (see *fig* 2)

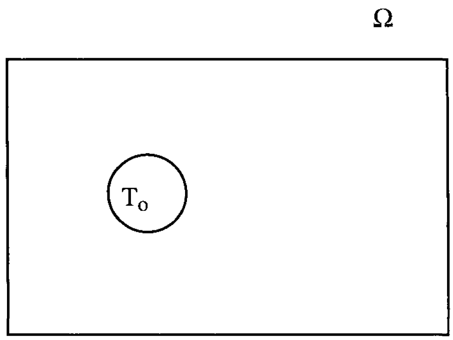

Figure 2 — The region Ω and the section T_0

In such conditions, engineers are interested in studying the transverse displacement of this solid under the action of the fluid's motion Many models have been proposed by them to deal with this physical problem Each corresponds to different types of assumptions about the fluid and its movement Here, we describe a mathematical model which has been proposed by D J Gorman & J Planchard [1988] For an overall understanding of the problem, we shall go into detail on how this model derives from this physical situation

In this model, the fluid is assumed to be perfect and incompressible, and it only considers small oscillations of the fluid around a state of rest The movement of the fluid is thus *irrotational* and its velocity therefore derives from a *potential function* $\psi(x, t)$ Under the action of the fluid, the section T_0 of the tube moves Denote by $s(t)$ the *transverse displacement vector* of T_0 in Ω As in § 1, let T_s and γ_s be the translate sets by s of T_0 and ∂T_0, respectively Define $\Omega_s = \Omega \backslash \bar{T}_s$, and let us split up the boundary of Ω into three parts Γ_1, Γ_2 and Γ_3 , Γ_1 and Γ_2 denote the in flow and outflow sections of $\partial\Omega$, respectively , Γ_3 is the remaining part of $\partial\Omega$, it represents the rigid walls (see *fig* 3)

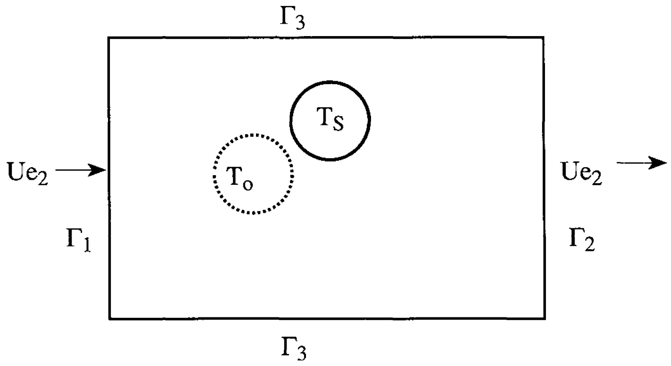

Figure 3. — The in-flow and out-flow sections and the translate tube T_s

Since the fluid is incompressible, the equation of its movement is simply the Laplace equation in Ω_s,

$$\Delta\psi = 0 \tag{2.1}$$

for all t On Γ_3 the fluid satisfies the classical conditions of not penetrating outside (that is, zero normal velocity), on Γ_1 and Γ_2 the normal velocity is given (with a constant absolute value U), and on γ_s, the normal velocity of the fluid coincides with the normal component of the tube-section's velocity. Thus, for every t, it holds that

$$\begin{cases} \dfrac{\partial\psi}{\partial n} = 0 & \text{on } \Gamma_3 \\ \dfrac{\partial\psi}{\partial n} = \begin{cases} -U & \text{on } \Gamma_1 \\ +U & \text{on } \Gamma_2 \end{cases} \\ \dfrac{\partial\psi}{\partial n} = \dfrac{ds}{dt}\cdot n & \text{on } \gamma_s \end{cases} \tag{2.2}$$

for any t Since the movement of the fluid was assumed to be irrotational, the *pressure* $p(x, t)$ of the fluid is given by the Bernoulli equation

$$p = -\rho\left(\frac{\partial\psi}{\partial t} + \frac{1}{2}|\nabla\psi|^2\right),$$

where $\rho > 0$ is the *density* of the fluid (it is a constant independent of x and t, since the fluid is homogeneous and incompressible)

Due to assumption (i), s only depends on t Since it was assumed that there is no interaction between T_0 and $\partial\Omega$, the motion of T_0 obeys a simple harmonic oscillation with a forced term, generated by its interaction with the fluid Thus, the dynamic equation for s is

$$m \frac{d^2 s}{dt^2} + ks = \int_{\gamma_s} pn \, d\sigma , \tag{2 4}$$

where m is the *mass* of the tube and k denotes its *stiffness* constant

Now, let us look for a steady-state of this system Using (2 3) and (2 4) this consists of finding a constant vector $\bar{s} \in \mathbb{R}^2$ (which represents the stationary displacement vector of T_0), a solution of the following nonlinear equation

$$\bar{s} = - \frac{\rho}{2k} \int_{\gamma_s} |\nabla \bar{\psi}|^2 \, n \, d\sigma , \tag{2 5}$$

where $\nabla \bar{\psi}(x)$ is the stationary velocity of the fluid

Therefore, if we define φ_s by

$$U \varphi_s(x) = \bar{\psi}(x) - U x_1 \quad \text{for} \quad x \in \Omega_s ,$$

it follows from (2 1) and (2 2) that φ_s is a solution of the following boundary-value problem in $\Omega_{\bar{s}}$

$$\begin{cases} \Delta \varphi_s = 0 & \text{in} \quad \Omega_{\bar{s}} \\ \dfrac{\partial \varphi_s}{\partial n} = 0 & \text{on} \quad \partial \Omega \\ \dfrac{\partial \varphi_s}{\partial n} = - n_1 & \text{on} \quad \gamma_s , \end{cases} \tag{2 6}$$

where $n_1 = n \, . \, e_1$, and equation (2 5) becomes,

$$\bar{s} = - \frac{\rho U^2}{2k} \int_{\gamma_s} |\nabla \varphi_s + e_1|^2 \, n \, d\sigma \tag{2 7}$$

These equations define φ_s up to an additive constant, which we fix imposing the following complementary condition

$$\int_{\Omega_{\bar{s}}} \varphi_{\bar{s}} \, dx = 0 \tag{2 8}$$

Observe that (2 6), together with (2 7) and (2 8) is a nonlinear boundary-value problem of the same type as problem (1 2) In this case, $\alpha = - \dfrac{\rho U^2}{2k}$, $u = e_1$ and $g(x) = n(x)$ for $x \in \gamma_0$

Theorem 1.1 shows that if the in-flow velocity of the fluid is sufficiently small, then there exists at least one stationary displacement vector $\bar{s}$, and a stationary potential $\varphi_{\bar{s}}$ such that the couple $(\bar{s}, \varphi_{\bar{s}})$ is a solution of (2.6), (2.7), (2.8). This existence result provides a mathematical justification of Gorman and Planchard's model.

A similar mathematical model has been proposed by the same authors to deal with the case where a tube bundle (with a finite number of tubes) is immersed in the fluid. Of course, in this more general case, each tube of the bundle can move independently. Therefore, Theorem 1.1 does not apply to this case ; but we shall see in § 4 how this result can be generalized in order to be applied to this other situation.

3. PROOF OF THE EXISTENCE RESULT

In this section we prove Theorem 1.1. To this end, let us introduce the function $F : S_\delta \to \mathbb{R}^N$ defined by

$$F(s) = \alpha \int_{\gamma_s} |\nabla \varphi_s + u_s|^2 \, n \, d\sigma ,$$

where φ_s is the (unique) solution of the following non-homogeneous Neumann problem in Ω_s :

$$\begin{cases} \Delta \varphi_s = 0 & \text{in} \quad \Omega_s \\ \dfrac{\partial \varphi_s}{\partial n} = 0 & \text{on} \quad \partial \Omega \\ \dfrac{\partial \varphi_s}{\partial n} = g_s & \text{on} \quad \gamma_s \\ \displaystyle\int_{\Omega_s} \varphi_s \, dx = 0 . \end{cases} \tag{3.1}$$

Since $g_s \in H^{1/2}(\gamma_s)$, it is well-known that (3.1) has one and only one (weak) solution φ_s in $H^1(\Omega_s)$. Furthermore, since ∂T_0 was assumed to be of class $\mathscr{C}^2$, classical regularity results for elliptic problems show that φ_s is of class H^2, except near the external boundary $\partial \Omega$ of Ω_s. In particular, we have that φ_s belongs to $H^2(C_s)$. This implies that $\nabla \varphi_s \in L^2(\gamma_s)^N$ and that F is therefore well-defined. The proof of Theorem 1.1 consists of showing that there exists α_0 such that for all α, $|\alpha| \leq \alpha_0$, the map F has a unique fixed point in the ball $B(0, r_0)$.

First, we prove that F is continuous, and next, we show that for α sufficiently small F maps $B(0, r_0)$ into itself. In order to prove the continuity of F, without loss of generality, we shall prove that F is continuous at the origin The proof is completely analogous in any other point $s \in S_\delta$ To prove continuity at the origin, the idea is to prove that the sequence of solutions $\{\varphi_s\}$ of problem (3 1) converges, as $s \to 0$, to the solution φ_0 of problem (3 1) with $s = 0$, and that this convergence holds in such a way that

$$\nabla \psi_s \to \nabla \varphi_0 \quad \text{in} \quad L^2(\gamma_0)^N\text{-strongly},$$

as $s \to 0$, where $\psi_s(x) = \varphi_s(x+s)$, $x \in \gamma_0$

The foregoing result is based on a local H^2-estimate of φ_s in a neighborhood of γ_s and on the identification of its cluster points in H^1 This is done using a uniform family of extension operators

Let us begin by establishing some notations If $\mathcal{O}$ is any measurable subset of $\mathbb{R}^N$, by $|\mathcal{O}|$ we will mean the Lebesgue measure of $\mathcal{O}$ and $\chi_{\mathcal{O}}$ will stand for the characteristic function of the set $\mathcal{O}$ Besides that, if $\mathcal{O}$ denotes any open bounded subset of $\mathbb{R}^N$, we will write

$$\|\cdot\|_{0\ \mathcal{O}} = \|\cdot\|_{L^2(\mathcal{O})^n}$$

and

$$\|\cdot\|_{t\ \mathcal{O}} = \|\cdot\|_{H^t(\mathcal{O})^m},$$

$m \in \mathbb{N}$, and for all $t > 0$ For each $s \in S_\delta$, we define the space V_s by

$$V_s = \left\{ v \in H^1(\Omega_s) \mid \int_{\Omega_s} v\, dx = 0 \right\}$$

From the generalized Poincaré's inequality in Ω we can easily deduce that the map $v \to \|\nabla v\|_{0\ \Omega_s}$ is a norm in V_s, equivalent to the standard $H^1(\Omega_s)$-norm

Now, let us introduce the variational formulation of (3 1), which is

$$\left\{ \begin{array}{l} \text{Find } \varphi_s \in V_s \text{ such that} \\ \displaystyle\int_{\Omega_s} \nabla\varphi_s \cdot \nabla\varphi\, dx = \int_{\gamma_s} g_s\, \varphi\, d\sigma \quad \forall \varphi \in V_s \end{array} \right. \tag{3 2}$$

In order to prove that the norm of the solution φ_s of (3 2) is bounded in $H^1(\Omega_s)$, independently of s in S_δ, we will use the following lemma

LEMMA 3 1 *There exist an extension operator $P_s \in \mathcal{L}(H^1(\Omega_s), H^1(\Omega))$ and a constant $C = C(\Omega, T_0)$, which is independent of s in S_δ, such that*

$$\left\{ \begin{array}{ll} a) & P_s v(x) = v(x) \quad \forall x \in \Omega_s \\ b) & \|\nabla P_s v\|_{0\ \Omega} \leq C\, \|\nabla v\|_{0\ \Omega_s} \\ c) & \|P_s\|_{\mathcal{L}(H^1(\Omega_s)\ H^1(\Omega))} \leq C \end{array} \right. \tag{3 3}$$

Proof: Let $Q_0 \in \mathscr{L}(H^1(C_0), H^1(C_0 \cup \bar{T}_0))$ be any operator with the following properties :

$$\begin{cases} a) & Q_0 v(x) = v(x) \quad \forall x \in C_0 \\ b) & \|\nabla Q_0 v\|_{0, C_0 \cup \bar{T}_0} \leq C_2 \|\nabla v\|_{0, C_0}, \end{cases} \tag{3.4}$$

where $C_2 = C_2(T_0, \delta)$. A proof of the existence of such an extension operator can be found, for example, in D. Cioranescu & J. Saint Jean-Paulin [1979].

Now, we define P_s as follows :

$$P_s v(x) = \begin{cases} v(x) & \text{if} \quad x \in \Omega_s \backslash \bar{C}_s \\ Q_0(v(x-s)) & \text{if} \quad x \in \bar{C}_s . \end{cases}$$

Obviously, $P_s \in \mathscr{L}(H^1(\Omega_s), H^1(\Omega))$ and verifies (3.3) a. On the other hand, using its definition we have

$$\|\nabla(P_s v)\|^2_{0, \Omega} = \|\nabla v\|^2_{0, \Omega_s \backslash \bar{C}_s} + \|\nabla(Q_0 v_s)\|^2_{0, C_0 \cup \bar{T}_0},$$

where $v_s(x) = v(x-s)$, $x \in C_0$. Thus, using (3.4) b we obtain

$$\|\nabla(P_s v)\|^2_{0, \Omega} \leq \|\nabla v\|^2_{0, \Omega_s \backslash \bar{C}_s} + C_2 \|\nabla v_s\|^2_{0, C_0},$$

which implies (3.3) b with $C^2 = \max\{1, C_2\}$. To prove (3.3) c it suffices to use (3.3) b and the following inequality

$$\|P_s v\|_{0, \Omega} \leq C \|v\|_{1, \Omega_s}$$

which is true because

$$\begin{aligned} \|P_s v\|^2_{0, \Omega} &= \|v\|^2_{0, \Omega_s \backslash \bar{C}_s} + \|Q_0 v_s\|^2_{0, C_0 \cup \bar{T}_0} \leq \\ &\leq \|v\|^2_{1, \Omega_s}, \end{aligned}$$

since $Q_0 \in \mathscr{L}(H^1(C_0), H^1(C_0 \cup \bar{T}_0))$. This completes the proof of Lemma 3.1. ∎

LEMMA 3.2 : *Let $\{\Omega_s\}_s$ be any family of open bounded subsets of Ω for which there exists a family $\{P_s\}_s$ of extension-operators verifying* (3.3). *There then exists a positive constant $C_1 = C_1(\Omega, T_0)$, independent of s, such that*

$$\left\| v - \left(\frac{1}{|\Omega_s|} \int_{\Omega_s} v \, dx \right) \right\|_{0, \Omega_s} \leq C_1 \|\nabla v\|_{0, \Omega_s} \quad \forall v \in H^1(\Omega_s) . \tag{3.5}$$

Proof It is clear that for all $v \in H^1(\Omega_s)$, we have

$$\inf_{c \in \mathbb{R}} \|v + c\|_{0\,\Omega_s}^2 \leq \inf_{c \in \mathbb{R}} \|P_s(v + c)\|_{0\,\Omega}^2 = $$
$$= \inf_{c \in \mathbb{R}} \|P_s v + c\|_{0\,\Omega}^2 ,$$

since (3 3) b implies that $P_s c = c$ for all $c \in \mathbb{R}$ But it is easy to check that for any open bounded set $\mathcal{O} \subset \mathbb{R}^N$ the following identity holds

$$\inf_{c \in \mathbb{R}} \|v + c\|_{0\,\mathcal{O}}^2 = \|v - \mathcal{M}_{\mathcal{O}}(v)\|_{0\,\mathcal{O}}^2 \quad \forall v \in L^2(\mathcal{O}),$$

where $\mathcal{M}_{\mathcal{O}}(v) = \dfrac{1}{|\mathcal{O}|}\displaystyle\int_{\mathcal{O}} v\,dx$ Therefore,

$$\|v - \mathcal{M}_{\Omega_s}(v)\|_{0\,\Omega_s}^2 \leq \|P_s v - \mathcal{M}_{\Omega}(P_s v)\|_{0\,\Omega}^2 \quad \forall v \in H^1(\Omega_s) \tag{3 6}$$

On the other hand, from the generalized Poincare's inequality in Ω, we know that there exists $C_3 = C_3(\Omega)$ such that

$$\|P_s v - \mathcal{M}_{\Omega}(P_s v)\|_{0\,\Omega} \leq C_3 \|\nabla P_s v\|_{0\,\Omega} \tag{3 7}$$

Combining (3 6) with (3 7) and (3 3) b we obtain (3 5), with $C_1 = CC_3$ This completes the proof of Lemma 3 2 ■

It is worth observing from (3 5) that the equivalence between $\|\nabla v\|_{0\,\Omega_s}$ and $\|v\|_{1\,\Omega_s}$ is uniform in S_δ in the following sense

$$\|v\|_{1\,\Omega_s} \leq (1 + C_1^2)^{1/2} \|\nabla v\|_{0\,\Omega_s}, \tag{3 8}$$

for all $v \in H^1(\Omega_s)$, where C_1 is defined by (3 5)

Now, to prove the bound for the $H^1(\Omega_s)$-norm of φ_s we choose $\varphi = \varphi_s$ as test function in (3 2) Using Cauchy Schwarz's inequality we obtain

$$\|\nabla\varphi_s\|_{0\,\Omega_s}^2 \leq \|g_s\|_{0\,\gamma_s} \|\varphi_s\|_{0\,\gamma_s} \leq$$
$$\leq C\|g_s\|_{0\,\gamma_s} \|\varphi_s\|_{1\,C_s} \leq C\|g_s\|_{0\,\gamma_s} \|\varphi_s\|_{1\,\Omega_s},$$

where C is the norm of the continuous embedding of $H^1(C_0)$ into $L^2(\gamma_0)$ But $\|g_s\|_{0\,\gamma_s} = \|g\|_{0\,\gamma_0}$, then using (3 8) we deduce the following estimate

$$\|\nabla\varphi_s\|_{0\,\Omega_s} \leq C\|g\|_{0\,\gamma_0} \quad \forall s \in S_\delta, \tag{3 9}$$

where here, and in what follows, C will denote different constants, independent of s in S_δ Estimate (3 9) allows us to prove the following uniform regularity result

LEMMA 3 3 *There exists a constant* $C = C(\Omega, T_0, \delta)$ *such that*

$$\|\varphi_s\|_{2\, C_s \frac{1}{2}} \leqslant C\, \|g\|_{\frac{1}{2}\, \gamma_0} \tag{3 10}$$

where $C_{s\,\frac{1}{2}} = s + C(T_0, \delta/2)$

Proof We use a localization method Let $\xi \in \mathscr{D}(C_0 \cup \bar{T}_0)$ be a smooth cut-off function such that

$$0 \leqslant \xi \leqslant 1 \quad \text{in} \quad C_0 \cup \bar{T}_0 \quad \text{and} \quad \xi = 1 \quad \text{in} \quad C\left(T_0, \frac{\delta}{2}\right)$$

and set

$$\psi_s(x) = \varphi_s(x + s) \quad x \in C_0$$

Since φ_s solves (3 1), a brief computation shows that $(\psi_s\, \xi)$ verifies

$$\begin{cases} -\Delta(\psi_s\, \xi) = -\psi_s\, \Delta\xi - 2\, \nabla\psi_s \,.\, \nabla\xi & \text{in} \quad C_0 \\ \psi_s\, \xi = 0 & \text{on} \quad (\partial C_0 \backslash \gamma_0) \\ \dfrac{\partial(\psi_s\, \xi)}{\partial n} = g & \text{on} \quad \gamma_0 \end{cases}$$

Since γ_0 is of class $\mathscr{C}^2$, from standard regularity results for elliptic problems we know that $(\psi_s\, \xi) \in H^2(C_0)$ and there exists a constant $C = C(C_0)$ such that

$$\|\psi_s\, \xi\|_{2\, C_0} \leqslant C\, \|g\|_{\frac{1}{2}\, \gamma_0}$$

Then (3 10) follows immediately from the properties of the function ξ ∎

The foregoing lemma implies that there exists a sequence $\{s_n\} \subset S_\delta(s_n \to 0)$ such that

$$\psi_s \rightharpoonup \psi^* \quad \text{in} \quad H^2(C_{0\,\frac{1}{2}}) \text{ weakly},$$

as $s_n \to 0$ Since the canonical embedding of $H^1(C_{0\,\frac{1}{2}})$ into $H^{1-\eta}(C_{0\,\frac{1}{2}})$ is compact for all η such that $0 < \eta < 1$, it can be assumed that the subsequence has been chosen so that

$$\psi_s \to \psi^* \quad \text{in} \quad H^{2-\eta}(C_{0\,\frac{1}{2}})\text{-strongly}, \tag{3 11}$$

as $s_n \to 0$, and since the trace operator is continuous from $H^{1-\eta}(C_{0\,\frac{1}{2}})$ into $L^2(\gamma_0)$, also verifying

$$\nabla \psi_{s_n} \to \nabla \psi^* \quad \text{in} \quad L^2(\gamma_0)\text{-strongly}\,, \tag{3.12}$$

as $s_n \to 0$.

Our next step consists in proving that $\psi^* = \varphi_0$ in $C_{0,\frac{1}{2}}$. The fact that φ_0 is independent of the subsequence $\{s_n\}$ proves that the whole family $\{\psi_s\}$ converges to φ_0 in $H^2(C_{0\,\frac{1}{2}})$ weakly as $s \to 0$ In particular, from (3.12) we deduce that

$$\nabla \psi_s \to \nabla \varphi_0 \quad \text{in} \quad L^2(\gamma_0)\text{-strongly}\,, \tag{3.13}$$

as $s \to 0$, which clearly implies that F is continuous at the origin. It only remains to identify ψ^*.

LEMMA 3.4 : *Let $\{P_s\}_s$ be the family of extension-operators given by Lemma* 3 1 *Then,*

$$P_s\,\varphi_s\big|_{\Omega_0} \to \varphi_0 \quad in \quad H^1(\Omega_0)\text{-}weakly\,,$$

as $s \to 0$, where φ_0 is the solution of (3.1) *for $s = 0$*

Proof From (3 3) c, (3.8) and (3.9), it follows that

$$\{P_s\,\varphi_s\}_{s \in S_\delta} \quad \text{is bounded in} \quad H^1(\Omega) \tag{3.14}$$

Let $s \to 0$ First, note that

$$\chi_{\Omega_s} \to \chi_{\Omega_0} \quad \text{in} \quad L^2(\Omega)\text{-strongly}\,. \tag{3 15}$$

Second, from the boundedness of $\{P_s\,\varphi_s\}$ there exists a sequence $\{s_n\}_n$ $(s_n \to 0)$ and a function $\Phi \in H^1(\Omega)$ such that

$$P_{s_n}\,\varphi_{s_n} \rightharpoonup \Phi \quad \text{in} \quad H^1(\Omega)\text{-weakly}\,. \tag{3.16}$$

Let $\varphi \in \mathscr{C}^\infty(\bar{\Omega})$ be given Choosing $\varphi - \mathscr{M}_{\Omega_{s_n}}(\varphi)$ as test function in the variational formulation of (3 1) (see (3.2)) we have

$$\int_\Omega \chi_{\Omega_{s_n}} \nabla (P_{s_n}\,\varphi_{s_n}) \,.\, \nabla\varphi\, dx = \int_{\gamma_{s_n}} g_{s_n}\,\varphi\, d\sigma\,. \tag{3 17}$$

It is clear that

$$\lim_n \int_{\gamma_{s_n}} g_{s_n}\,\varphi\, d\sigma = \lim_n \int_{\gamma_0} g(x)\,\varphi(x + s_n)\, d\sigma = \int_{\gamma_0} g\varphi\, d\sigma\,. \tag{3 18}$$

Using (3 15) and (3 16) we can pass to the limit in the left hand side term of (3 17) Then we obtain

$$\int_{\Omega_0} \nabla \Phi \cdot \nabla \varphi \, dx = \int_{\gamma_0} g \varphi \, d\sigma \quad \forall \varphi \in H^1(\Omega), \tag{3 19}$$

since $\mathscr{C}^\infty(\bar{\Omega})$ is dense in $H^1(\Omega)$

On the other hand, we have

$$\int_{\Omega} \chi_{\Omega_{s_n}} P_{s_n} \varphi_{s_n} \, dx = 0,$$

and passing to the limit as $n \to \infty$,

$$\int_{\Omega_0} \Phi \, dx = 0 \tag{3 20}$$

Thus, (3 19) and (3 20) prove that $\Phi|_{\Omega_0}$ is the (unique) solution of (3 1) with $s = 0$, i e, $\Phi|_{\Omega_0} = \varphi_0$ Since φ_0 is independent of the subsequence s_n, we can finally conclude that all the family $\{P_s \varphi_s|_{\Omega_0}\}$ converges to φ_0 in $H^1(\Omega_0)$ This completes the proof ■

Now, let us bring together the above results to conclude that $\psi^* = \varphi_0$ in $C_{0\frac{1}{2}}$ First, by the triangle's inequality we have

$$\begin{aligned} \|\varphi_0 - \psi^*\|_{0\, C_{0\frac{1}{2}}} \leq \|\varphi_0 - P_s \varphi_{s_n}\|_{0\, C_{0\frac{1}{2}}} + \|P_s \varphi_{s_n} - \psi_{s_n}\|_{0\, C_{0\frac{1}{2}}} + \\ + \|\psi_{s_n} - \psi^*\|_{0\, C_{0\frac{1}{2}}}, \end{aligned} \tag{3 21}$$

where $\{s_n\}$ is the sequence in S_δ for which (3 11) holds

Using (3 11) and Lemma 3 4 we observe that the first and the third term in the right hand side of (3 21) go to zero as $s_n \to 0$ In order to pass to the limit in the second term, let us rewrite it as follows

$$\|P_{s_n} \varphi_{s_n} - \psi_{s_n}\|^2_{0\, C_{0\frac{1}{2}}} = \int_{C_{0\frac{1}{2}}} |P_s \varphi_{s_n}(x) - P_{s_1} \varphi_{s_1}(x + s_n)|^2 \, dx$$

Denote by $\mathscr{F}$ the family $\{P_s \varphi_s\}_{s \in S_\delta}$ From (3 14) we know that $\mathscr{F}$ is relatively compact in $L^2(\Omega)$ Then by the Kolmogorov Compactness Theorem (see e g , J Nečas [1967] Theorem 2 1 3) the family $\mathscr{F}$ is equicontinuous in mean, that is,

$\forall \varepsilon > 0 \quad \exists \eta > 0 \quad$ with $\quad \eta < \text{dist}\,(C_{0\frac{1}{2}}, \partial\Omega)$ such that

$$\int_{C_{0\frac{1}{2}}} |P_s \varphi_s(x) - P_s \varphi_s(x + h)|^2 \, dx < \varepsilon$$

for all $|h| < \delta$ and $s \in S_\delta$. In particular, we conclude that

$$\lim_{s_n \to 0} \int_{C_{0,\frac{1}{2}}} \left|P_{s_n} \varphi_{s_n}(x) - P_{s_n} \varphi_{s_n}(x + s_n)\right|^2 dx = 0 .$$

Therefore, all the right hand side terms of (3.21) go to zero as $s_n \to 0$, which allows us to conclude that

$$\psi^* = \varphi_0 \quad \text{in} \quad C_{0,\frac{1}{2}} .$$

This completes the proof of the continuity of F.

Now, let us prove that there exists $\alpha_0 > 0$ such that for all α, $|\alpha| < \alpha_0$, $F(B(0, r_0)) \subset B(0, r_0)$. Using the definition of F and Lemma 3.3, we get

$$|F(s)| \leqslant |\alpha| \int_{\gamma_s} |\nabla\varphi_s + u_s|^2 \, d\sigma = |\alpha| \int_{\gamma_0} |\nabla\psi_s + u|^2 \, d\sigma \leqslant$$

$$\leqslant C|\alpha| (\|\psi_s\|^2_{2, C_0} + \|u\|^2_{0, \gamma_0}) \leqslant C|\alpha| (\|g\|^2_{\frac{1}{2}, \gamma_0} + \|u\|^2_{0, \gamma_0}) ,$$

where C depends only on Ω, T_0 and δ.

Therefore, if we define α_0 by

$$\alpha_0 = \frac{r_0}{C(\|g\|^2_{\frac{1}{2}, \gamma_0} + \|u\|^2_{0, \gamma_0})} ,$$

the map F has the desired property. Using the classical Brouwer Fixed Point Theorem we conclude that F has at least one fixed point s in $B(0, r_0)$. The couple (s, φ_s) verifies (1.2). ∎

4. AN EXISTENCE RESULT FOR A MORE GENERAL CASE

In this section we consider the case where T_0 has K connected components, and each of them can move independently in Ω, Ω being defined as in § 1. Let us denote as $T_{01}, \ldots, T_{0K}$ the K connected components of T_0. As before, we assume that $\bar{T}_0 \subset \Omega$ and that the boundary γ_{0i} of T_{0i} is of class $\mathscr{C}^2$, for $i = 1, \ldots, K$. Any translation of the whole structure T_0 will be now represented by a vector $\mathrm{s} = (s_1, \ldots, s_K)$, with $s_i \in \mathbb{R}^n$. For all $i = 1, \ldots, K$, set

$$\begin{aligned} T_{s_i} &= s_i + T_{0i} \\ \gamma_{s_i} &= \partial T_{s_i} \\ C_{s_i} &= s_i + \left\{x \in (\mathbb{R}^N \backslash \bar{T}_{0i}) \,|\, \mathrm{dist}\,(x, \gamma_{0i}) < \delta\right\} \\ C_{\mathrm{s}} &= \bigcup_{i=1}^{K} C_{s_i} . \end{aligned}$$

We define the set of admissible displacements by

$$\mathbf{S}_\delta = \left\{ \mathrm{s} \in \mathbb{R}^{KN} \,|\, \bar{C}_{\mathrm{s}} \subset \Omega \quad \text{and} \quad \bar{C}_{s_i} \cap \bar{C}_{s_j} = \emptyset \quad \text{for} \quad i \neq j \right\} .$$

Observe that $\mathbf{S}_\delta$ in an open bounded set of $\mathbb{R}^{KN}$ and assume that δ has been chosen sufficiently small so that

$$\mathbf{S}_\delta \neq \emptyset . \tag{4.1}$$

For each $\mathrm{s} \in \mathbf{S}_\delta$, we denote by Ω_{s} the subset of Ω defined as follows (see *fig.* 4) :

$$\Omega_{\mathrm{s}} = \Omega \setminus \bigcup_{i=1}^{K} \bar{T}_{s_i} .$$

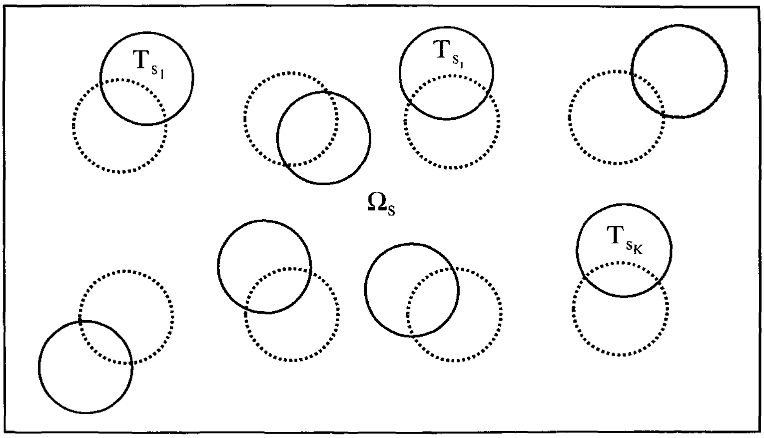

Figure 4. — The region Ω_{s}

Let $\underset{\sim}{\alpha} = (\alpha_1, \ldots, \alpha_K) \in \mathbb{R}^K$, $\mathbf{u} = (u_1, \ldots, u_K) \in L^2(\gamma_{01})^N \times \cdots \times L^2(\gamma_{0K})^N$, $\mathbf{g} = (g_1, \ldots, g_K) \in H^{1/2}(\gamma_{01}) \times \cdots \times H^{1/2}(\gamma_{0K})$ be given with $\int_{\gamma_{0i}} g_i \, d\sigma = 0$, for all $i = 1, \ldots, K$. In this context, we can state an analogous problem to (1.2) as follows :

Find $\mathrm{s} \in \mathbf{S}_\delta$ and $\varphi_{\mathrm{s}} \in H^1(\Omega_{\mathrm{s}}) \cap H^2(C_{\mathrm{s}})$ such that

$$\left\{ \begin{aligned} \Delta \varphi_{\mathrm{s}} &= 0 \quad \text{in} \quad \Omega_{\mathrm{s}} \\ \frac{\partial \varphi_{\mathrm{s}}}{\partial n} &= 0 \quad \text{on} \quad \partial \Omega \\ \frac{\partial \varphi_{\mathrm{s}}}{\partial n} &= g_{s_i} \quad \text{on} \quad \gamma_{s_i} \quad \forall i = 1, \ldots, K \\ s_i &= \alpha_i \int_{\gamma_{s_i}} \left| \nabla \varphi_{\mathrm{s}} + u_{s_i} \right|^2 n \, d\sigma \quad \forall i = 1, \ldots, K , \end{aligned} \right. \tag{4.2}$$

where $g_{s_i}(x) = g_i(x - s_i)$ and $u_{s_i}(x) = u_i(x - s_i)$ for $x \in \gamma_{s_i}$ and n denotes the outward unit normal on the boundary of Ω_s.

THEOREM 4.1 : *Assume that* (4.1) *holds. Define* r_0 *by*

$$r_0 = \max \{r > 0 \,|\, B(\mathbf{O}, r) \subset \mathbf{S}_\delta\}$$

where $B(\mathbf{O}, r)$ *denotes the ball of radius* r, *centered at the origin* $\mathbf{O}$ *of* $\mathbb{R}^{KN}$. *Then there exists a strictly positive constant* α_0 *such that for all* $\underline{\alpha} \in \mathbb{R}^{KN}$, $|\underline{\alpha}| < \alpha_0$, *problem* (4.2) *has a solution* $(\mathbf{O}, \varphi_s)$ *in* $B(\mathbf{O}, r_0) \times (H^1(\Omega_s) \cap H^2(C_s))$.

Proof : It is almost identical to the proof of Theorem 1.2. It is sufficient to follow step by step the proof of the above theorem, but replacing F by $\mathbf{F} = (F_1, \dots, F_K)$ defined by

$$\mathbf{F} : \mathbf{s} \in \mathbf{S}_\delta \to \mathbb{R}^{KN}$$

$$F_i(\mathbf{s}) = \alpha_i \int_{\gamma_{s_i}} |\nabla \varphi_s + u_{s_i}|^2 \, n \, d\sigma$$

and replacing C_s by C_s. It can also be easily checked that Lemma 3.1 and Lemma 3.2 remain valid if we replace Ω_s by Ω_s as well as convergence (3.15). This provides all the elements which allow us to complete the proof of Theorem 4.1. ∎

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