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M. VANMAELE
R. VAN KEER

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AN OPERATOR METHOD FOR A NUMERICAL QUADRATURE FINITE ELEMENT APPROXIMATION FOR A CLASS OF SECOND-ORDER ELLIPTIC EIGENVALUE PROBLEMS IN COMPOSITE STRUCTURES (*)

M. VANMAELE [1] and R. VAN KEER [1]

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Abstract. — *We consider a second-order elliptic eigenvalue problem on a convex polygonal domain, divided in M non-overlapping subdomains. The conormal derivative of the unknown function is continuous on the interfaces, while the function itself is discontinuous. In this paper, we study the finite element approximation without and with numerical quadrature of this eigenvalue problem by means of the perturbation theory for linear, compact, self-adjoint operators, see [13, § IV.2-IV.3, § V.4.3] and [9]. We refine the method, developed in [18], by incorporating some basic ideas of [4] and [9]. This improved method is then extended to the underlying multi-component structure with discontinuities at the interfaces. Furthermore, in contrast to [3] and [4], which are dealing with a Dirichlet eigenvalue problem on a one-component domain, discretized by a triangular mesh, we allow for a rectangular mesh, for mixed Dirichlet-Robin boundary conditions and for a more general second-order differential operator. Finally, in contrast to [18], we also consider finite elements of higher degree (quadratic, biquadratic,...).*

Crucial to the finite element analysis is a non-standard variational formulation to the eigenvalue problem, similar to the one in [11] for some classes of parabolic problems. The emphasis of this paper is on the error analysis of the approximate eigenpairs.

AMS classification : 65N25, 65N30, 65D30, 65N15.

Key words : eigenvalue problem, multi-component domain, operator method, numerical integration, finite element approximation.

Résumé. — *Nous considérons un problème elliptique spectral du second ordre sur un domaine convexe polygonal, divisé en M sous-domaines disjoints. La dérivée conormale de la fonction inconnue est continue sur les frontières intérieures, tandis que la fonction même y est discontinue. Cet article est consacré à l'étude de l'approximation de ce problème spectral par des méthodes aux éléments finis sans et avec intégration numérique, en employant la théorie de perturbation pour des opérateurs linéaires, compacts et auto-adjoints, voir [13, § IV.2-IV.3, § V.4.3] et [9]. La méthode, développée dans [18], est raffinée par l'incorporation des idées fondamentales de [4] et [9]. Puis, cette méthode ainsi améliorée est étendue à la structure multi-composante avec des discontinuités aux frontièress intérieures. Par opposition à [3] et [4], où un problème spectral*

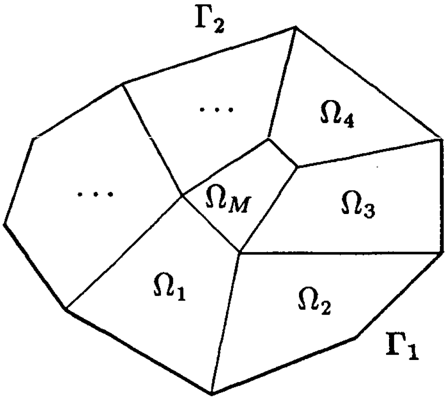
(*) Received May 9, 1994.
[1] Department of Numerical Analysis, University Gent, Galglaan 2, 9000 Gent, Belgium.

de Dirichlet sur un domaine single-composant est discrétisé par un maillage triangulaire, nous tenons compte en outre d'un maillage rectangulaire, des conditions aux limites mêlées de type Dirichlet-Robin et d'un opérateur différentiel du second ordre plus général. Finalement, contrairement à [18], nous considérons aussi des éléments finis de degré plus élevé (quadratique, biquadratique, ...).

La formulation variationnelle non standard du problème spectral, similaire à celle de [11] pour des classes de problèmes paraboliques, est essentielle pour les méthodes d'éléments finis. L'accent est mis sur l'analyse d'erreur des couples de valeurs et de fonctions propres approchées.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain with boundary $\partial\Omega$. Assume that $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Lambda_1$, where Γ_1 and Γ_2 are open and consist of an integer number of sides, $\Gamma_1 \cap \Gamma_2 = \Phi$ and $\text{meas}_1 \Lambda_1 = 0$. Further, let Ω be divided in M non-overlapping, open, convex, polygonal domains $\Omega_i \subset \Omega$, $1 \leqslant i \leqslant M$. We denote by $\mathcal{N}_i \subset \{1, \dots, M\}$ the set of integers σ for which $\text{meas}_1(\partial\Omega_i \cap \partial\Omega_\sigma) > 0$. We set $\Gamma_{i,\sigma} = \partial\Omega_i \cap \partial\Omega_\sigma$ for $\sigma \in \mathcal{N}_i$, $1 \leqslant i \leqslant M$. We assume that $\Gamma_{i,\sigma} \cap \Gamma_{j,\rho} = \Phi$ when $\{i, \sigma\} \neq \{j, \rho\}$.



Then we may write

$$\Omega = \bigcup_{i=1}^{M} \Omega_i \bigcup_{i=1}^{M} \left(\bigcup_{\substack{\sigma \in \mathcal{N}_i \\ \sigma > i}} \Gamma_{i,\sigma} \right) \cup \Lambda_2\,, \quad \text{meas}_1 \Lambda_2 = 0\,.$$

Finally, denote $\mathcal{N}(\Gamma_j) = \{i : 1 \leqslant i \leqslant M$ and $\{\text{meas}_1(\Gamma_j \cap \partial\Omega_i) > 0\}$, $j = 1, 2$, and $\Gamma_j^i = \Gamma_j \cap \partial\Omega_i$, $i \in \mathcal{N}(\Gamma_j)$, $j = 1, 2$. In what follows, when writing Γ_j^i, we will sometimes delete the restriction $i \in \mathcal{N}(\Gamma_j)$, of course taking $\Gamma_j^i = \Phi$ when $i \notin \mathcal{N}(\Gamma_j)$.

This paper deals with the eigenvalue problem (EVP) of determining the real numbers λ and the corresponding functions $u^i : \Omega_i \to \mathbb{R}$, $1 \leqslant i \leqslant M$, which obey, in a weak sense, the following coupled system : the second-order differential equations on the respective domains Ω_i

$$-\sum_{\ell, m=1}^{2} \frac{\partial}{\partial x_\ell}\left(a^i_{\ell m} \frac{\partial u^i}{\partial x_m}\right) + a^i_0 u^i = \lambda u^i \quad \text{in} \quad \Omega_i \tag{1.1}$$

the transmission conditions (TCs) on the interfaces

$$\left.\begin{aligned} - a^i \partial_\nu u^i &= h^{i,\sigma}(u^i - u^\sigma) \quad \text{on } \Gamma_{i,\sigma}, \qquad (1.2)\\ a^i \partial_\nu u^i &= a^\sigma \partial_\nu u^\sigma \quad \text{on } \Gamma_{i,\sigma}, \qquad (1.3) \end{aligned}\right\} \forall \sigma \in \mathcal{N}_i$$

and the homogeneous boundary conditions (BCs) on parts of the boundary $\partial\Omega$

$$u^i = 0 \ \text{ on } \Gamma^i_1, \tag{1.4}$$

$$a^i \partial_\nu u^i + a^i_1 u^i = 0 \ \text{ on } \Gamma^i_2. \tag{1.5}$$

The conormal derivative in (1.5) is given by

$$a^i \partial_\nu u^i = \sum_{\ell, m=1}^{2} a^i_{\ell m} \frac{\partial u^i}{\partial x_m} \nu_\ell$$

with ν_ℓ the ℓ-th component of the unit outward normal vector $\bar{\nu}$ to Γ^i_2. The conditions (1.2)-(1.3) have to be understood similarly, $\bar{\nu}$ being the unit normal vector to $\Gamma_{i,\sigma}$ and pointing from Ω_i to Ω_σ.

In this problem $a^i_{\ell m}$, a^i_0, $h^{i,\sigma}$ and a^i_1, $1 \leqslant \ell, m \leqslant 2$, $\sigma \in \mathcal{N}_i$, $1 \leqslant i \leqslant M$, are given space dependent functions, which are sufficiently regular, as specified below,. Note that on account of (1.2), the TC (1.3) is equivalent with the symmetry condition $h^{i,\sigma} = h^{\sigma,i}$ on $\Gamma_{i,\sigma} = \Gamma_{\sigma,i}$.

In this paper we study the approximation of the EVP (1.1)-(1.5). We leave from a non-standard formally equivalent variational formulation in an abstract setting, similar to the one in [11] for some classes of parabolic problems. We lean upon the perturbation theory for linear, compact, self-adjoint operators, see [13, § IV.2-IV.3, § V.4.3] and [9].

EVPs of the type above have practical relevance, e.g. in the context of heat transfer problems in multi-component domains (« non-perfect thermal contact problems »), see for instance [15].

An outline of the paper is now in order. The precise variational and operator formulation of the EVP (1.1)-(1.5) are stated in Section 2, together with some preliminary results concerning the involved function spaces and the bilinear form. In Section 3 we first introduce suitable approximation spaces and next the elliptic projector. The consistent mass finite element method (FEM) is dealt with in Section 4, while the numerical quadrature FEM is discussed in considerable detail in Section 5. Finally, in Section 6 we formulate some conclusions.

2. VARIATIONAL AND OPERATOR EIGENVALUE PROBLEM

2.1. Notations and assumptions (see also [19])

Let $H^1(\Omega_i)$ be the usual first order Sobolev space on Ω_i with norm $\|\cdot\|_{1,\Omega_i}$, $1 \leqslant i \leqslant M$, and let $V^i = \{w \in H^1(\Omega_i) : w = 0 \text{ on } \Gamma_1^i\}$. We recall that $\Gamma_1^i = \Phi$ when $i \notin \mathcal{N}(\Gamma_1)$. Then, we introduce the product space

$$V = \{v = (v^1, \dots, v^M) : v^i \in V^i, 1 \leqslant i \leqslant M\}$$

and we identify $v \in V$ with a scalar function $v : \Omega \to \mathbb{R}$ for which $v|_{\Omega_i} = v^i$ on Ω_i, $1 \leqslant i \leqslant M$. Similarly, we introduce the product space $H = L_2(\Omega_1) \times \cdots \times L_2(\Omega_M)$ with inner-product (.,.) and associated norm $|\cdot|$ given by

$$(v, w) = \sum_{i=1}^{M} \int_{\Omega_i} v^i w^i \, dx, \quad |v| = \sqrt{(v, v)}, \quad \forall v, w \in H. \tag{2.1}$$

Further, we denote

$$\mathcal{A}(v, w) = \sum_{i=1}^{M} \int_{\Omega_i} \left(\sum_{\ell, m=1}^{2} a_{\ell m}^i \frac{\partial v^i}{\partial x_\ell} \frac{\partial w^i}{\partial x_m} + a_0^i v^i w^i \right) dx,$$

$$\mathcal{B}(v, w) = \frac{1}{2} \sum_{i=1}^{M} \sum_{\sigma \in \mathcal{N}_i} \int_{\Gamma_{i,\sigma}} h^{i,\sigma}(v^i - v^\sigma)(w^i - w^\sigma) \, ds,$$

$$\mathcal{C}(v, w) = \sum_{i \in \mathcal{N}(\Gamma_2)} \int_{\Gamma_2^i} a_1^i v^i w^i \, ds,$$

and

$$a(v, w) = \mathcal{A}(v, w) + \mathcal{B}(v, w) + \mathcal{C}(v, w), \quad \forall v, w \in V. \tag{2.2}$$

We will also use the product space $\hat{H}^m(\Omega) = H^m(\Omega_1) \times \cdots \times H^m(\Omega_M)$, $m \in \mathbb{N}_0$, and its (product) norm $\| \cdot \|_{\hat{H}^m}$ and (product) semi-norm $| \cdot |_{\hat{H}^m(\Omega)}$, both defined in the natural way. For $m = 1$, the product norm is simply denoted by $\| \cdot \|$.

Throughout this paper, the data are assumed to fulfill the hypotheses (H1)-(H2) :

$$(\mathrm{H1})\quad (1)\ a^i_{\ell m}, a^i_0 \in L_\infty(\Omega_i);\ a^i_{\ell m} = a^i_{m\ell} \text{ a.e. in } \Omega_i \qquad \ell, m = 1, 2,\ 1 \leqslant i \leqslant M\,;$$
$$(2)\ h^{i,\sigma} \in L_\infty(\Gamma_{i,\sigma})\,;\ 0 < h^{i,\sigma} = h^{\sigma,i} \text{ a.e. on } \Gamma_{i,\sigma}, \quad \sigma \in \mathcal{N}_i,\ 1 \leqslant i \leqslant M\,;$$
$$(3)\ a^i_1 \in L_\infty(\Gamma^i_2)\,;\ \exists \alpha_1 \geqslant 0 : a^i_1 \geqslant \alpha_1 \text{ a.e. on } \Gamma^i_2 \quad i \in \mathcal{N}(\Gamma_2)\,.$$

(H2) (1) The matrices $a^i = (a^i_{\ell m})$, $1 \leqslant i \leqslant M$, are positive definite, i.e.

$$\exists \alpha > 0 : \forall \xi \in \mathbb{R}^2,\ \sum_{\ell, m = 1}^{2} a^i_{\ell m}(x)\, \xi_\ell\, \xi_m \geqslant \alpha |\xi|^2 \text{ a.e. in } \Omega_i\,,\quad 1 \leqslant i \leqslant M\,;$$

$$(2) \qquad \exists \alpha_0 > 0 : a^i_0 \geqslant \alpha_0 \quad \text{a.e. in } \Omega_i\,, \qquad 1 \leqslant i \leqslant M\,.$$

These hypotheses guaranty the ellipticity of the EVP (1.1)-(1.5).

2.2. Variational and operator formulation

The weak or variational EVP associated with (1.1)-(1.5) reads :

$$\text{Find } (\lambda, u) \in \mathbb{R} \times V,\ u \neq 0 : a(u, v) = \lambda(u, v)\,, \quad \forall v \in V\,, \tag{2.3}$$

where (.,.) and a(.,.) are defined by (2.1) and (2.2) respectively.

The data H, V and a(.,.) have the following properties, cfr. [19],

PROPOSITION 2.1 : (1) *The spaces H and V are Hilbert spaces, V being compactly and densely imbedded in H.* (2) *The bilinear form a(.,.) : $V \times V \to \mathbb{R}$, (2.2), is symmetric, bounded and strongly coercive.*

Hence, the exact EVP fits into the general abstract framework of [14].

With these data we define the exact solution operator T by

$$T : H \to V,\ a(Tf, v) = (f, v) \quad \forall f \in H, \forall v \in V\,. \tag{2.4}$$

From the strong coercivity of a(.,.), T is easily seen to be bounded.

The corresponding exact EVP reads :

$$\text{Find } \mu \in \mathbb{R},\ u \in V : Tu = \mu u\,. \tag{2.5}$$

This is the operator formulation of the variational EVP (2.3) with $\mu = \lambda^{-1}$.

We recall, see e.g. [14],

PROPOSITION 2.2 : *Let $i : V \to H$ be the imbedding operator. Then the operator $T_r = T|_V = T \circ i : V \to V$ is positive definite and self-adjoint w.r.t. $a(.,.)$. As i is compact, also T_r is.*

Hence, the spectrum $sp(T_r)$ consists of an infinite sequence of eigenvalues, all being strictly positive and having finite multiplicity, with zero as accumulation point.

We arrange the eigenvalues as $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n \geqslant \cdots \to 0$, where each eigenvalue occurs as many times as given by its multiplicity.

In what follows we often write T instead of T_r when the meaning is clear from the context.

3. PRELIMINARY RESULTS

We consider a regular family of triangulations $(\mathcal{T}^i_{h_i})_{h_i}$, see e.g. [6], of each component $\overline{\Omega}_i$, $1 \leqslant i \leqslant M$, consisting of either triangular or, when Ω_i is a rectangle, of rectangular elements. Hereby we assume that the triangulations in each pair of adjacent components $\overline{\Omega}_i$, $\overline{\Omega}_\sigma$ match in the sense that the interface $\Gamma_{i,\sigma}$ consists of non-overlapping segments, each of them figuring as a common side of an element $K \in \mathcal{T}^i_{h_i}$ and an element $K' \in \mathcal{T}^\sigma_{h_\sigma}$.

With a triangulation $\mathcal{T}^i_{h_i}$, we associate the finite-dimensional subspace $X^i_{h_i}$ of $H^1(\Omega_i)$, $X^i_{h_i} = \{v^i \in \mathscr{C}^0(\overline{\Omega}_i) : v^i|_K \in P(K),\ \forall K \in \mathcal{T}^i_i\}$, where h_i is the mesh parameter and $P(K)$ is given by

$$P(K) = \begin{cases} P_k(K) & \text{if } K \text{ is triangular,} \\ Q_k(K) & \text{if } K \text{ is rectangular,} \end{cases} \qquad k \in \mathbb{N}_0 .$$

P_k stands for the set of polynomials of degree $\leqslant k$ and Q_k for the set of polynomials of degree $\leqslant k$ in each variable.

Finally, we consider the product spaces

$$X_h = \{v = (v^1, \ldots, v^M) : v^i \in X^i_{h_i},\ 1 \leqslant i \leqslant M\} \subset \hat{H}^1(\Omega),$$

$$V_h = \{v \in X_h : v^i = 0 \text{ on } \Gamma^i_1,\ 1 \leqslant i \leqslant M\} \subset V,$$

where $h = \max_{1 \leqslant i \leqslant M} h_i$ is the overall mesh parameter, taken to be sufficiently small in what follows.

We also introduce the following notation for $r \geqslant 1$,

$$|v|_{r,\mathcal{T}_h} = \left(\sum_{i=1}^{M} \sum_{K \in \mathcal{T}^i_{h_i}} |v^i|^2_{r,K} \right)^{1/2}, \ \|v\|_{r,\mathcal{T}_h} = \left(\sum_{s=0}^{r} |v|^2_{s,\mathcal{T}_h} \right)^{1/2}, \quad v \in X_h . \quad (3.1)$$

(Trivially, $\|v\|_{1,\mathcal{T}_h} = \|v\|$ for $v \in X_h$.)

In what follows C denotes a generic constant not depending on h (even when it is not explicitly stated).

3.1. Inverse inequality

From now on, we assume that each regular family of triangulations $(\mathcal{T}^i_{h_i})_{h_i}$, $1 \leqslant i \leqslant M$, is « quasi-uniform » in the sense of [6, (3.2.28)]. Then, [6, Theorem 3.2.6] implies for $m \in \mathbb{N}_0$

$$|w|_{m,K} \leqslant C h_i^{-s} |w|_{m-s,K}, \quad 0 \leqslant s \leqslant m, \forall w \in P(K), \forall K \in \mathcal{T}^i_{h_i}, \quad (3.2)$$

$$\|w\|_{m,K} \leqslant C h_i^{-s} \|w\|_{m-s,K}, \quad 0 \leqslant s \leqslant m, \forall w \in P(K), \forall K \in \mathcal{T}^i_{h_i}, \quad (3.3)$$

where C is a constant independent of $K \in \mathcal{T}^i_{h_i}$ and of h_i.

Further, we assume that the families $(\mathcal{T}^i_{h_i})_{h_i}$ are mutually « quasi-uniform » too, i.e.

$$\exists \nu \geqslant 0 : h \leqslant \nu h_i, \quad \forall h_i, 1 \leqslant i \leqslant M. \quad (3.4)$$

3.2. Density and approximation property

PROPOSITION 3.1 : [19, Proposition 3.1]

(1) *$\hat{H}^2(\Omega) \cap V$ is dense in V.*

(2) *The finite element space $V_h \subset V$ satisfies the approximation property*

$$inf\{|v - v_h| + h|v - v_h|_{\hat{H}^1(\Omega)} \ ; v_h \in V_h\} \leqslant C h^{r+1} \|v\|_{\hat{H}^{r+1}(\Omega)},$$

$$\forall v \in \hat{H}^{r+1}(\Omega) \cap V, 1 \leqslant r \leqslant k . \quad (3.5)$$

3.3. The elliptic projector

We define the elliptic projector $P : V \rightarrow V_h$ by

$$a(v - Pv, w) = 0, \quad \forall v \in V, \forall w \in V_h . \quad (3.6)$$

From (3.5), the elliptic projector P satisfies the following property, cfr. [19, § 3.4] :

$$\| v - Pv \| \leqslant Ch^r \| v \|_{\hat{H}^{r+1}(\Omega)} \quad \forall v \in \hat{H}^{r+1}(\Omega) \cap V, 1 \leqslant r \leqslant k . \quad (3.7)$$

PROPOSITION 3.2 : *Consider regular families of triangulations* $(\mathcal{T}^i_{h_i})_{h_i}$ *of* $\overline{\Omega}_i$, $1 \leqslant i \leqslant M$, *satisfying the mutual quasi-uniformity (3.4). Then*

$$\| Pv \|_{r, \mathcal{T}_h} \leqslant C \| v \|_{\hat{H}^r(\Omega)}, \quad \forall v \in \hat{H}^r(\Omega) \cap V, r = 1, \dots, k+1 . \quad (3.8)$$

Proof: The case $r = 1$ directly follows from (3.6) and the properties of $a(.,.)$. For $r \geqslant 2$, we first consider the standard Lagrange interpolator $\Pi^i_{h_i} : \mathcal{C}^0(\overline{\Omega}_i) \to X^i_{h_i}$. This operator has the well known properties, see e.g. [6, Theorem 3.2.1],

$$\sum_{K \in \mathcal{T}^i_{h_i}} | w - \Pi^i_{h_i} w |^2_{r,K} \leqslant C | w |^2_{r, \Omega_i}, \sum_{K \in \mathcal{T}^i_{h_i}} | w - \Pi^i_{h_i} w |^2_{1,K} \leqslant C h_i^{2(r-1)} | w |^2_{r, \Omega_i}$$

$$\forall w \in H^r(\Omega_i) . \quad (3.9)$$

From $(3.9)_1$ and the triangle inequality, we get

$$| Pv |^2_{r, \mathcal{T}_h} \leqslant C \left[\sum_{i=1}^{M} \sum_{K \in \mathcal{T}^i_{h_i}} | (Pv)^i - \Pi^i_{h_i} v^i |^2_{r,K} + | v |^2_{\hat{H}^r(\Omega)} \right] .$$

First applying (3.2), (3.4) and the triangle inequality and next invoking (3.7) and $(3.9)_2$, we have

$$| Pv |^2_{r, \mathcal{T}_h} \leqslant C \left[h^{-2(r-1)} \sum_{i=1}^{M} \sum_{K \in \mathcal{T}^i_{h_i}} \left(| (Pv)^i - v^i |^2_{1,K} + \right.\right.$$

$$\left.\left. + | v^i - \Pi^i_{h_i} v^i |^2_{1,K} \right) + | v |^2_{\hat{H}^r(\Omega)} \right]$$

$$\leqslant C \| v \|^2_{\hat{H}^r(\Omega)}, \quad \forall v \in \hat{H}^r(\Omega) \cap V .$$

From $(3.1)_2$ this estimate implies (3.8). □

Remark 3.1 : In the case of a triangular mesh on Ω_i, $\|p\|_{k+1,K} = \|p\|_{k,K}$, for all $p \in P_k(K)$, for all $K \in \mathcal{T}^i_{h_i}$.

4. CONSISTENT MASS FINITE ELEMENT METHOD

We consider the operator T^h, defined by

$$T^h : V \to V_h , a(T^h f, v) = (f, v) \quad \forall f \in V, \forall v \in V_h , \tag{4.1}$$

as an approximation to the operator T_r. Then, from the definition of the elliptic projector P, (3.6), it is easily seen that

$$T^h = P \circ T_r . \tag{4.2}$$

The associated EVP reads :

$$\text{Find } \mu_h \in \mathbb{R}, u_h \in V_h : T^h u_h = \mu_h u_h . \tag{4.3}$$

This is the operator formulation of the variational consistent mass EVP

$$\text{Find } \lambda_h(= \mu_h^{-1}) \in \mathbb{R} , u_h \in V_h : a(u_h, v) = \lambda_h(u_h, v) \quad \forall v \in V_h .$$

From the formal equivalence of the variational formulation of the EVP on a multi-component domain and of an EVP on a one-component domain and from the properties of the approximation space V_h and the elliptic projector, we may expect similar results for the convergence and error estimates for the eigenpairs of the consistent mass EVP (4.3) as in [2, Chap. II], [10] and [14, Chap. 9].

For further reference, we quote explicity some of the auxiliary results without proof. As the proof of Theorem 4.1 below or of its analogon for an EVP on a one-component domain appears not to have been given before in its present form, we deal in some detail with this proof.

Before studying (the rate of) convergence of the approximate operator T^h to T_r, we define the BVP (2.4) to be *regular* when the exact solution operator T, (2.4), has the properties

$$T : H \to V \cap \hat{H}^2(\Omega) \text{ and } \exists C > 0 : \|Tg\|_{\hat{H}^2(\Omega)} \leqslant C|g| , \quad \forall g \in H . \tag{4.4}$$

PROPOSITION 4.1 : *For the operators T, (2.4), and T^h, (4.1), we have* (1)

$$\|T_r - T^h\| \equiv \sup \{ \|(T - T^h) v\| : v \in V, \|v\| \leqslant 1\} \to 0 \text{ as } h \to 0 . \tag{4.5}$$

(2) *Under the condition (4.4) we have*

$$\| T_r - T^h \| \leqslant Ch . \tag{4.6}$$

Let μ be an $(L+1)$ –fold exact eigenvalue of (2.5). Further, let d be its isolation distance, i.e.

$$d = d(\mu) = \min \{ |\mu - \nu| \; ; \nu \in \operatorname{sp}(T), \nu \neq \mu \} , \operatorname{sp}(T) = \text{spectrum of } T ,$$

and set

$$\partial B = \left\{ z \in \mathbb{C} : |z - \mu| = \frac{d}{2} \right\} \subset \operatorname{res}(T), \operatorname{res}(T) = \text{resolvent set of } T . \tag{4.7}$$

From (4.5) and [14, § 8.5] follows

LEMMA 4.1 : *There are $(L+1)$ eigenvalues $\mu_{t,h}$, $0 \leqslant t \leqslant L$, of T^h, (4.1), for which*

$$|\mu - \mu_{t,h}| \to 0 \quad \text{as} \quad h \to 0 , \quad 0 \leqslant t \leqslant L . \tag{4.8}$$

Further, we have $\partial B \subset \operatorname{res}(T^h)$. Hence we may define the spectral projector $\mathscr{P}^h : V \to V_h$ by

$$\mathscr{P}^h = - \frac{1}{2 \pi i} \int_{\partial B} R_z(T^h) \, dz$$

with

$$R_z(T^h) = (T^h - z)^{-1} , \quad z \in \operatorname{res}(T^h) .$$

Let u_r, $0 \leqslant r \leqslant L$, be eigenfunctions of T, (2.4), corresponding to μ and being orthonormal in H. The space spanned by these eigenfunctions is denoted by $\mathscr{E}$.

Further, let $u_{r,h}$, $0 \leqslant r \leqslant L$, be eigenfunctions of T^h, (4.1), associated with $\mu_{r,h}$, $0 \leqslant r \leqslant L$, also being orthonormal in H.

Set $\mathscr{E}^h = \operatorname{span}(u_{0,h}, \ldots, u_{L,h})$, then we have, see [14, § 8.5], [13, § IV 3.4-3.5], $\mathscr{P}^h(V) = \mathscr{E}^h$.

LEMMA 4.2 : *For $w \in \mathscr{E}$ we have,*

$$\| w - \mathscr{P}^h w \| \leqslant C \| (T - T^h) w \| . \tag{4.9}$$

Moreover, let $\mathscr{E} \subset \hat{H}^{k+1}(\Omega)$. Then it holds that

$$\| w - \mathscr{P}^h w \| \leqslant Ch^k \| w \| . \tag{4.10}$$

COROLLARY 4.1 : *We have*

$$\delta(\mathscr{E}, \mathscr{E}^h) \equiv \sup \{ d(w, \mathscr{E}^h) \; ; w \in \mathscr{E}, \|w\| \leq 1 \} \to 0 \quad \text{as} \quad h \to 0 , \tag{4.11}$$

where

$$d(w, \mathscr{E}^h) = \inf \{ \|w - v\| \; ; v \in \mathscr{E}^h \} . \tag{4.12}$$

Moreover, let $\mathscr{E} \subset \hat{H}^{k+1}(\Omega)$, *then*

$$\delta(\mathscr{E}, \mathscr{E}^h) \leq Ch^k . \tag{4.13}$$

THEOREM 4.1 : *Let* $u_{t,h}$, $0 \leq t < L$, *be eigenfunctions of (4.3) orthonormalized in H and associated with* $\mu_{t,h}$, *which satisfy (4.8),* $0 \leq t < L$. *Then, there exists a set* $(W_t(h))_{0 \leq t \leq L}$ *of exact eigenfunctions of (2.5) corresponding to* μ, *such that*

$$\| W_t(h) - u_{t,h} \| \to 0 \quad \text{as} \quad h \to 0, 0 \leq t \leq L. \tag{4.14}$$

Moreover, when $\mathscr{E} \subset \hat{H}^{k+1}(\Omega)$, *we have*

$$\| W_t(h) - u_{t,h} \| \leq Ch^k , \quad 0 \leq t \leq L . \tag{4.15}$$

Proof: Defining $\delta(\mathscr{E}^h, \mathscr{E})$ analogously to (4.11)-(4.12), we have for $u_{t,h} \in \mathscr{E}^h$, $0 \leq t \leq L$, that

$$d(u_{t,h}, \mathscr{E}) \equiv \inf \{ \|u_{t,h} - w\| \; ; w \in \mathscr{E} \} \leq \delta(\mathscr{E}^h, \mathscr{E}) \|u_{t,h}\| . \tag{4.16}$$

Combining (4.1) and (4.3) with the strong coercivity of $a(.,.)$ and next using (4.8) and $|u_{t,h}| = 1$, we easily see that

$$\|u_{t,h}\| \leq C , \quad 0 \leq t \leq L .$$

Since $\mathscr{E}$ is finite-dimensional and hence the infimum in (4.16) is attained, there exists an exact eigenfunction $W_t(h) \in \mathscr{E}$, $0 \leq t \leq L$, which satisfies

$$\| W_t(h) - u_{t,h} \| \leq C\delta(\mathscr{E}^h, \mathscr{E}) .$$

Finally, we recall that, cfr. [14, § 8.5, Remark 1], $\delta(\mathscr{E}^h, \mathscr{E}) = \delta(\mathscr{E}, \mathscr{E}^h)$ and invoke Corollary 4.1. □

5. NUMERICAL QUADRATURE FINITE ELEMENT METHOD

5.1. Preliminaries (cfr. [6, § 4.1], [8, § 2.7, § 5.6], [17, § 2.1] and [19, § 5.1])

Consider the (affine invertible) mapping

$$F_K : \hat{K} \,(\text{reference element}) \to K : \hat{x} \mapsto x = F_K(\hat{x}) = B_K \hat{x} + b_K , \quad \det B_K > 0 , \tag{5.1}$$

with $B_K \in \mathbb{R}^{2\times 2}$ and $b_K \in \mathbb{R}^{2\times 1}$, $K \in \mathcal{T}^i_{h_i}$.

Next, introduce a quadrature formula on $\hat{K}$

$$I^X_{\hat{K}}(\hat{\varphi}) = \sum_{r=1}^{N(X)} \hat{\omega}^X_r \ \hat{\varphi}(\hat{b}^X_r) \simeq \int_{\hat{K}} \hat{\varphi}(\hat{x})\,\mathrm{d}\hat{x} \quad \forall \hat{\varphi} \in \mathcal{C}^0(\hat{K}), X = L, G\,,$$

where $\hat{b}^X_r$ and $\hat{\omega}^X_r > 0$, $r = 1, \dots, N(X)$, are the quadrature nodes and weights respectively, and where $X = L$ or $X = G$ refers to the quadrature formulas having degree of precision $2k - 1$ or $2k + 1$ respectively. The quadrature error is

$$E^X_{\hat{K}}(\hat{\varphi}) \equiv \int_{\hat{K}} \hat{\varphi}(\hat{x})\,d\hat{x} - I^X_{\hat{K}}(\hat{\varphi})\,.$$

Putting $\varphi(x) = \hat{\varphi}(\hat{x})$ whenever $x = F_K(\hat{x})$, $\hat{x} \in \hat{K}$, we define the corresponding quadrature formulas on K by

$$I^X_K(\varphi) = (\det B_K)\, I^X_{\hat{K}}(\hat{\varphi}) \simeq \int_K \varphi(x)\,dx \quad X = L, G\,. \tag{5.2}$$

In a similar way we approximate the line integrals on a side $\partial_t\hat{K}$ of $\hat{K}$, using a one-dimensional Lobatto $(X = L)$ or Gauss-Legendre $(X = G)$ quadrature formula

$$I^X_{\partial_t\hat{K}}(\hat{\varphi}) = \sum_{r=1}^{k+1} \hat{\omega}^X_r\, \hat{\varphi}(\hat{g}^{\,X}_{\,r}) \simeq \int_{\partial_t\hat{K}} \hat{\varphi}(\hat{s})\,d\hat{s} \quad \forall \hat{\varphi} \in \mathcal{C}^0(\partial_t\hat{K})\,,$$

where the quadrature point $\hat{g}^{\,X}_{\,r}$ (characterized by their arc length) correspond with the Lobatto or Gauss points in the interval $[-1, 1]$ respectively. This quadrature formula has precision $2k - 1$ or $2k + 1$ respectively, see [8, § 2.7].

According to (5.1) we put $\varphi(s) = \hat{\varphi}(\hat{s})$ whenever $s = F_K|_{\partial_t\hat{K}}(\hat{s})$, $\hat{s} \in \partial_t\hat{K}$. The quadrature formula on a side $\partial_t K$ of an element K is defined by

$$I^X_{\partial_t K}(\varphi) = \frac{\text{meas}\ \partial_t K}{\text{meas}\ \partial_t\hat{K}}\, I^X_{\partial_t\hat{K}}(\hat{\varphi})\,. \tag{5.3}$$

We refer to [1, § 3] and to [6, pp. 181-184], [7, § XII 1.5, p. 780], [12, pp. 2.100-2.104] for examples of quadrature formulas on a rectangular and a triangular element K respectively.

5.2. Approximation to (.,.) and to a (.,.)

We define the discrete H-inner-product and the associated norm in X_h by

$$(v, w)_h = \sum_{i=1}^{M} \sum_{K \in \mathcal{T}^i_{h_i}} I^L_K(v^i w^i), \ |v|_h = \sqrt{(v, v)_h}, \qquad \forall v, w \in X_h . \tag{5.4}$$

PROPOSITION 5.1 : [19, Proposition 5.1] *The equivalence of the norms* $|\cdot|_h$, *(5.4), and* $|\cdot|$, *(2.1), on the space* X_h *is uniform w.r.t. h.*

Introducing the notations

$$\begin{aligned}
\mathcal{N}(\Gamma^i_2) &= \{j : 1 \leq j \leq N(\Omega_i) \text{ and } \partial_j \Omega_i \text{ (side of } \Omega_i) \subset \Gamma^i_2\}, \\
&\qquad (N(\Omega_i) \text{ number of sides of } \Omega_i) \\
\mathcal{B}_{j, h_i} &= \{K \in \mathcal{T}^i_{h_i} : \exists t : \partial_t K \subset \partial_j \Omega_i\}, j \in \mathcal{N}(\Gamma^i_2) \\
\mathcal{B}^i_{\sigma, h_i} &= \{K \in \mathcal{T}^i_{h_i} : \exists t : \partial_t K \subset \Gamma_{i,\sigma}\}, \sigma \in \mathcal{N}_i,
\end{aligned}$$

with $t \in \{1, 2, 3, 4\}$ when K is rectangular and $t \in \{1, 2, 3,\}$ when K is triangular, we may decompose a line integral on Γ^i_2 and on $\Gamma_{i,\sigma}$ into line integrals on suitable sides $\partial_t K$, $K \in \mathcal{T}^i_{h_i}$, similarly as in [19, § 5.3].

In what follows, we assume that $a^i_{\ell m}$, $a^i_0 \in \mathcal{C}^0(\overline{\Omega}_i)$, $\ell, m = 1, 2$, $h^{i,\sigma} \in \mathcal{C}^0(\overline{\Gamma}_{i,\sigma})$, $\sigma \in \mathcal{N}_i$ and $a^i_1 \in \mathcal{C}^0(\overline{\Gamma}^i_2)$, $1 \leq i \leq M$. The discrete analogon of the bilinear from $a(.,.)$, (2.2), on $X_h \times X_h$ is defined by

$$a_h(v, w) = \mathcal{A}_h(v, w) + \mathcal{B}_h(v, w) + \mathcal{C}_h(v, w), \quad \forall v, w \in X_h, \tag{5.5}$$

where

$$\mathcal{A}_h(v, w) = \sum_{i=1}^{M} \sum_{K \in \mathcal{T}^i_{h_i}} I^X_K \left(\sum_{\ell, m=1}^{2} a^i_{\ell m} \partial_\ell v^i \partial_m w^i + a^i_0 v^i w^i \right), \tag{5.6}$$

$$\mathcal{B}_h(v, w) = \frac{1}{2} \sum_{i=1}^{M} \sum_{\sigma \in \mathcal{N}_i} \sum_{K \in \mathcal{B}^i_{\sigma, h_i}} I^K_{\partial_t K}(h^{i,\sigma}(v^i - v^\sigma)(w^i - w^\sigma)), \tag{5.7}$$

$$\mathcal{C}_h(v, w) = \sum_{i \in \mathcal{N}(\Gamma_2)} \sum_{j \in \mathcal{N}(\Gamma^i_2)} \sum_{K \in \mathcal{B}_{j, h_i}} I^X_{\partial_t K}(a^i_1 v^i w^i). \tag{5.8}$$

PROPOSITION 5.2 : [19, Proposition 5.2] *$a_h(.,.)$ (clearly) retains the symmetry property of $a(.,.)$. Further, $a_h(.,.)$ is uniformly bounded and strongly coercive w.r.t. h in the $\hat{H}^1(\Omega)$ –norm.*

5.3. Estimate of the total errors of quadrature

The approximate inner-product (5.4) and bilinear from (5.5) induce the respective quadrature errors

$$E^L(v, w) \equiv (v, w) = \sum_{i=1}^{M} \sum_{K \in \mathcal{T}_{h_i}^i} E_K^L(v^i w^i) , \quad \forall v, w \in X_h . \tag{5.9}$$

$$E_a^X(v, w) \equiv a(v, w) - a_h(v, w)$$

$$= E_{\mathcal{A}}^X(v, w) + E_{\mathcal{B}}^X(v, w) + E_{\mathcal{C}}^X(v, w) , \quad \forall v, w \in X_h \tag{5.10}$$

where $E_{\mathcal{A}}^X$, $E_{\mathcal{B}}^X$ and $E_{\mathcal{C}}^X$ are given by expressions similar to (5.6), (5.7) and (5.8) respectively, in terms of E_K^X and $E_{\partial, K}^X$, i.e. the errors of quadrature on an element K and on a side of K.

These last (local) errors are estimated in [19, Propositions 5.3, 5.4, 5.5]. We add here the analogon of [19, (5.14) and (5.17)] for a rectangular element K, which is proved in a similar way as [16, Theorem 2.1].

PROPOSITION 5.3 : *Let K be a rectangle. For the quadrature formula (5.2), $X = L$, we have for $2 \leq r \leq k$,*

$$|E_K^L(dpq)| \leq C h_K^r \|d\|_{r,\infty,K} \|p\|_{r,K} |q|_{0,K} ,$$

$$\forall d \in W^{r,\infty}(K), \forall p, q \in Q_k(K) . \tag{5.11}$$

For $k = 1$ we have

$$|E_K^L(dpq)| \leq C h_K \|d\|_{1,\infty K} \|p\|_{1,K} |q|_{0,K} ,$$

$$\forall d \in W^{1,\infty}(K), \forall p, q \in Q_1(K) . \tag{5.12}$$

When $d \in W^{2k,\infty}(K)$, $k \geq 1$, we have, with C independent of d,

$$|E_K^L(dpq)| \leq C h_K^{2k} \|d\|_{2k,\infty,K} \|p\|_{k,K} \|q\|_{k,K}, \quad \forall p, q \in Q_k(K) . \tag{5.13}$$

In these estimates the constant $C > 0$ is independent of K, while h_K stands for the diameter of K.

These estimates for the local errors lead to the estimates of the total quadrature errors (5.9) and (5.10).

THEOREM 5.1 : [19, Theorem 5.1] *For the total error of quadrature* $E^L(.,.)$, *(5.9), we have*

$$|E^L(v, w)| \leqslant Ch^2 |v|_{\tilde{H}^1(\Omega)} |w|_{\tilde{H}^1(\Omega)}, \quad \forall v, w \in X_h . \tag{5.14}$$

THEOREM 5.2 : (cfr. [19, Theorem 5.2])
(1) *Let* $a^i_{\ell m} \in W^{k+s, \infty}(\Omega_i)$, $\ell, m = 1, 2$, $a^i_0 \in W^{k, \infty}(\Omega_i)$ *and* $h^{i, \sigma} \in W^{k+s, \infty}(\Gamma_{i, \sigma})$, $\sigma \in \mathcal{N}_i$, $1 \leqslant i \leqslant M$. *Further, let* $a^i_1 \in W^{k+s, \infty}(\Gamma^i_2)$, $i \in \mathcal{N}(\Gamma_2)$. *Here,* $s = 1$ *when dealing with a rectangular mesh and* $s = 0$ *for a triangular mesh. Then,*

$$|E^G_a(v, w)| \leqslant Ch \|v\| \, \|w\| , \quad \forall v, w \in X_h . \tag{5.15}$$

(2) *Let* $a^i_{\ell m}$, $a^i_0 \in W^{k, \infty}(\Omega_i)$, $\ell, m = 1, 2$, $h^{i, \sigma} \in W^{k, \infty}(\Gamma_{i, \sigma})$, $\sigma \in \mathcal{N}_i$, $1 \leqslant i \leqslant M$, *and let* $a^i_1 \in W^{k, \infty}(\Gamma^i_2)$, $i \in \mathcal{N}(\Gamma_2)$. *Then, for the case of a triangular mesh, we have*

$$|E^L_a(v, w)| \leqslant Ch \|v\| \, \|w\| , \quad \forall v, w \in X_h . \tag{5.16}$$

Remark 5.1 : For the case of a rectangular mesh, Proposition 5.3 will not lead to the analogon of estimate (5.16).

5.4. Approximate eigenvalue problem with numerical quadrature

With the operator T_r we associate the approximate operator $\hat{T}^h$ defined by

$$\hat{T}^h : V_h \to V_h, a_h(\hat{T}^h f, v) = (f, v)_h, \quad \forall f, v \in V_h , \tag{5.17}$$

where $a_h(.,.)$ and $(.,.)_h$ are respectively given by (5.5) and (5.4).
The corresponding EVP then reads :

$$\text{Find } (\hat{\mu}_h, \hat{u}_h) \in \mathbb{R} \times V_h : \hat{T}^h \hat{u}_h = \hat{\mu}_h \hat{u}_h . \tag{5.18}$$

This is the operator formulation of the variational EVP

$$\text{Find } (\hat{\lambda}_h, \hat{u}_h) \in \mathbb{R} \times V_h : a_h(\hat{u}_h, v) = \hat{\lambda}_h(\hat{u}_h, v)_h ,$$

$$\forall v \in V_h , \quad (\hat{\mu}_h = \hat{\lambda}_h^{-1}) . \tag{5.19}$$

From Propositions 5.1 and 5.2 it readily follows that

PROPOSITION 5.4 : *The linear operator $\hat{T}^h$, (5.17), is positive definite and self-adjoint (w.r.t. to $a_h(.,.)$) in V_h. This operator is bounded uniformly w.r.t. h.*

We want to prove the convergence and to estimate the error for the approximate eigenpairs, i.e. the eigenpairs of (5.18). Hereto we study in the next section the (rate of the) uniform convergence of $\hat{T}^h$ to T.

5.5. Uniform convergence of $\hat{T}^h$ to T

First, we compare $\hat{T}^h$ with the operator T^h, (4.1).

LEMMA 5.1 :

(1) *Assume that the coefficients $a^i_{\ell m}$, a^i_0, a^i_1 and $h^{i,\sigma}$ satisfy the conditions underlying estimate (5.15). Then, for $X = G$ and, when dealing with a triangular mesh on Ω_i, $1 \leqslant i \leqslant M$, also for $X = L$, we have*

$$\| T^h - \hat{T}^h \|_{1, V_h} \equiv \sup \{ \| (T^h - \hat{T}^h) v \| \; ; v \in V_h, \| v \| \leqslant 1 \} \leqslant Ch . \tag{5.20}$$

(2) *In the case of a triangular mesh or, when $k = 1$, also in the case of a rectangular mesh on Ω_i, $1 \leqslant i \leqslant M$, assume that $a^i_{\ell m}$, $a^i_0 \in W^{1,\infty}(\Omega_i)$, $\ell, m = 1, 2$, $1 \leqslant i \leqslant M$. In the remaining case, i.e. the case of a rectangular mesh with $k \geqslant 2$, assume that $a^i_{\ell m}$, $a^i_0 \in W^{2,\infty}(\Omega_i)$, $\ell, m = 1, 2$, $1 \leqslant i \leqslant M$.*

Further, let, for both situations, $h^{i,\sigma} \in W^{1,\infty}(\Gamma_{i,\sigma})$, $\sigma \in \mathcal{N}_i$, $1 \leqslant i \leqslant M$ and let $a^i_1 \in W^{1,\infty}(\Gamma^i_2)$, $i \in \mathcal{N}(\Gamma_2)$.

Then, for $X = L$ estimate (5.20) holds under the regularity condition (4.4).

Proof : Set $w = (T^h - \hat{T}^h) v$.

Using the uniform strong coercivity of $a_h(.,.)$, invoking the definitions (4.1) and (5.17) of T^h and $\hat{T}^h$ respectively, as well as those of E^L and E^X_a, see (5.9) and (5.10) respectively, we have

$$C \| (T^h - \hat{T}^h) v \|^2 \leqslant a_h((T^h - \hat{T}^h) v, w) = E^L(v, w) - \\ - E^X_a(T^h v, w) , \forall v \in V_h . \tag{5.21}$$

(1) Invoke Theorems 5.1, 5.2 and use the uniform boundedness of T^h, see (4.2).

(2) We successively estimate the three terms appearing in $E^L_a(T^h v, w)$, (5.10).

1. For the terms constituting $E^L_{\mathcal{A}}$ we distinguish between a rectangular and a triangular mesh.

Rectangular mesh $(P(K) = Q_k(K))$

Case $k \geq 2$: Using (5.11) (with $r = 2$), the notation (3.1) and (3.3) we find

$$|E^L_{\mathscr{A}}(T^h v, w)| \leq Ch \| T^h v \|_{2, \mathscr{T}_h} \| w \| , \quad \forall v \in V_h \quad .$$

Combination of (4.2), (3.8) (with $r = 2$) and (4.4) leads to

$$\| T^h v \|_{2, \mathscr{T}_h} \leq C \| Tv \|_{\hat{H}^2(\Omega)} \leq C \| v \| , \quad \forall v \in V .$$

Hence, we have

$$|E^L_{\mathscr{A}}(T^h v, w)| \leq Ch \| v \| \, \| w \| , \quad \forall v \in V_h . \tag{5.22}$$

Case $\mathbf{k = 1}$: Now (5.22) readily follows from (5.12).

Triangular mesh $(P(K) = P_k(K))$.

In this case (5.22) may be derived in an analogous way by using [4, Lemma 3.2 (with $i = 1$)].

2. For the terms constituting $E^L_{\mathscr{C}}$ we may proceed in a similar way, when invoking the estimate [17, (2.23) (with $r = 1$)] as well as the trace inequality :

$$\begin{aligned} |E^L_{\mathscr{C}}(T^h v, w)| &\leq C \sum_{i \in \mathscr{N}(\Gamma_2)} \sum_{j \in \mathscr{N}(\Gamma^i_2)} \sum_{K \in \mathscr{B}_{j, hi}} h_K \| (T^h v)^i \|_{1, \partial_i K} | w^i |_{0, \partial_i K} \\ &\leq C \sum_{i=1}^{M} \sum_{K \in \mathscr{T}^i_{hi}} h_K \| (T^h v)^i \|_{2, K} \| w^i \|_{1, K} \\ &\leq Ch \| v \| \, \| w \| , \quad \forall v \in V_h . \end{aligned}$$

3. Recalling that $h^{i, \sigma} = h^{\sigma, i}$, using similar arguments to those for $E^L_{\mathscr{C}}$ and noting also that

$$\sum_{K \in \mathscr{B}^i_{\sigma, hi}} | v^\sigma |^2_{0, \partial_i K} = \sum_{K' \in \mathscr{B}^\sigma_{i, h\sigma}} | v^\sigma |^2_{0, \partial_{i'} K'} ,$$

which follows from the convention introduced in Section 3 on the triangulations of two adjacent domains Ω_i and Ω_σ, we get

$$\begin{aligned} |E^L_{\mathscr{B}}(T^h v, w)| &\leq C \sum_{i=1}^{M} \sum_{\sigma \in \mathscr{N}_i} h_i \sum_{K \in \mathscr{B}^i_{\sigma, hi}} \| (T^h v)^i \|_{2, K} (\| w^i \|_{1, K} + | w^\sigma |_{0, \partial_i K}) \\ &\leq Ch \| v \| \, \| w \| , \quad \forall v \in V_h , \end{aligned}$$

From these three estimates just obtained, (5.10) gives

$$|E_a^L(T^h v, w)| \leqslant Ch\|v\| \, \|w\| \, .$$

Substitution of this estimate and of (5.14) into (5.21) yields (5.20). □

Next, combining the lemma above with Proposition 4.1, we obtain.

THEOREM 5.3 :

(1) *Under the conditions of Lemma 5.1(1) we have*

$$\|T - \hat{T}^h\|_{1, V_h} \equiv \sup\{\|(T - \hat{T}^h) v\| \; ; v \in V_h, \|v\| \leqslant 1\} \to 0 \quad \text{as} \quad h \to 0 \, .$$

(2) *When moreover the regularity condition (4.4) holds, we have*

$$\|T - \hat{T}^h\|_{1, V_h} \leqslant Ch \, .$$

(3) *This estimate also holds under the assumptions of Lemma 5.1 (2), [including (4.4)].*

Now, we deal with an alternative for the estimate of $(T^h - \hat{T}^h) v, v \in V_h$, obtained in Lemma 5.1.

LEMMA 5.2 : *Let $a_{\ell m}^i, a_0^i \in W^{k,\infty}(\Omega_i)$, $\ell, m = 1, 2$, and $h^{i,\sigma} \in W^{k,\infty}(\Gamma_{i,\sigma})$, $\sigma \in \mathcal{N}_i$, $1 \leqslant i \leqslant M$. Further, let $a_1^i \in W^{k,\infty}(\Gamma_2^i)$, $i \in \mathcal{N}(\Gamma_2)$. Using quadrature formulas with $X = L$ or $X = G$, we have for both a rectangular and a triangular mesh*

$$\|(T^h - \hat{T}^h) v\| \leqslant C(h^{k+1}|v|_{k, \mathcal{T}_h} + h^k \|T^h v\|_{k+1, \mathcal{T}_h}) \, , \quad \forall v \in V_h \, . \tag{5.23}$$

Proof. We again leave from (5.21).

Applying [17, Theorem 2.4] (with $s = 0$, $t = k - 1$) to (5.9), we find for both a rectangular and triangular mesh,

$$|E^L(v, w)| \leqslant C \sum_{i=1}^{M} h_i^{k+1} \sum_{K \in \mathcal{T}_{h_i}^i} |v^i|_{k, K} |w^i|_{1, K}$$

$$\leqslant Ch^{k+1} |v|_{k, \mathcal{T}_h} \|w\| \, , \quad \forall v \in V_h \, . \tag{5.24}$$

We next deal with the second term at the right hand side of (5.21).

1. **Case X = L**

Rectangular mesh

Combining (5.11) (with $r = k$) for $k \geq 2$ or (5.12) for $k = 1$ we obtain for the first term of (5.10),

$$|E^L_{\mathscr{A}}(T^h v, w)| \leq C \sum_{i=1}^{M} \sum_{K \in \mathcal{T}^i_{hi}} h^k_K \|(T^h v)^i\|_{k+1,K} \; \|w^i\|_{1,K}$$

$$\leq C h^k \|T^h v\|_{k+1,\mathcal{T}_h} \|w\|, \quad \forall v \in V_h . \tag{5.25}$$

Using [17, (2.23) (with $r = k$)], we may proceed similarly as in the proof of Lemma 5.1 (2), to arrive at

$$|E^L_{\mathscr{B}}(T^h v, w)| + |E^L_{\mathscr{C}}(T^h v, w)| \leq C h^k \|T^h v\|_{k+1,\mathcal{T}_h} \|w\| .$$

As this estimate obviously is the same as the one in (5.25), $|E^L_a(T^h v, w)|$ is found to obey this estimate too, on account of the definition (5.10).

Triangular mesh

The same estimate for $|E^L_a(T^h v, w)|$ is obtained by applying [4, Lemma 3.2] and its analogon for $E^L_{\partial_t K}$.

Thus for both a rectangular and triangular mesh, this estimate for $|E^L_a(T^h v, w)|$ and the one for $|E^L(v, w)|$, (5.24), imply (5.23) on account of (5.21).

2. **Case X = G**

We may proceed in a similar way, now leaning upon [17, (2.19) and (2.22) (with $r = k - 2$)] for $k \geq 2$ and upon [17, (2.21) and (2.25)] for $k = 1$. □

5.6. Convergence of the approximate eigenvalues

From the canonical convergence $V_h \to V$, implied by Proposition 3.1, and from Theorem 5.3, some results, obtained in [9] and [13] and summarized in [4], remain valid for the type of EVPs under consideration. For further reference, we quote some of these results without proof.

LEMMA 5.3 : *Retain the conditions of Theorem 5.3, (1) or (3). Let* $F \subset \operatorname{res}(T)$, *be a closed subset of the resolvent set of T, (2.4). Then,* $F \subset \operatorname{res}(\hat{T}^h)$, *where* $\hat{T}^h$ *is defined by (5.17). Moreover,* $R_z(\hat{T}^h) = (\hat{T}^h - z)^{-1}$, $\forall z \in \operatorname{res}(\hat{T}^h)$, *is a bounded operator in* V_h, *uniformly w.r.t.* $z \in F$ *and h.*

As, in particular, for ∂B, (4.7), we have $\partial B \subset \operatorname{res}(\hat{T}^h)$, we may define the spectral projector $\hat{\mathcal{P}}^h : V_h \to V_h$, associated with $\hat{T}^h$, by

$$\hat{\mathcal{P}}^h = -\frac{1}{2\pi i}\int_{\partial B} R_z(\hat{T}^h)\, dz . \tag{5.26}$$

LEMMA 5.4 : *Retain the assumptions of Theorem 5.3, (1) or (3).*

(1) *There are* $(L+1)$ *eigenvalues* $\hat{\mu}_{t,h}$, $0 \leq t \leq L$, *of* $\hat{T}^h$, *(5.17), for which*

$$|\mu - \hat{\mu}_{t,h}| \to 0 \quad \text{as} \quad h \to 0\,, \quad 0 \leq t \leq L\,. \tag{5.27}$$

(2) *Denoting by* $\hat{u}_{t,h}$, $0 \leq t \leq L$, *eigenfunctions of* $\hat{T}^h$ *corresponding with* $\hat{\mu}_{t,h}$, $0 \leq t \leq L$, *we set* $\hat{\mathcal{E}}^h = \operatorname{span}(\hat{u}_{0,h}, \dots, \hat{u}_{L,h})$. *It holds that* $\hat{\mathcal{P}}^h(V_h) = \hat{\mathcal{E}}^h$.

5.7. Convergence and error estimate of the approximate eigenfunctions

Let us start with some auxiliary results.

LEMMA 5.5 : *Let* $w \in \mathcal{E}$. *Retaining the conditions of Theorem 5.3, (1) or (3), we have*

$$\| w - \hat{\mathcal{P}}^h \mathcal{P}^h w \| \leq C[\, \| T - T^h \|_{1,\mathcal{E}} + \| T^h - \hat{T}^h \|_{1,\mathcal{E}^h}]\, \| w \| \,, \tag{5.28}$$

with

$$\| T - T^h \|_{1,\mathcal{E}} = \sup \{ \| (T - T^h)\, v \| \; ; v \in \mathcal{E},\ \| v \| \leq 1 \} \,,$$

$$\| T^h - \hat{T}^h \|_{1,\mathcal{E}^h} = \sup \{ \| (T^h - \hat{T}^h)\, v \| \; ; v \in \mathcal{E}^h,\ \| v \| \leq 1 \} \,.$$

Proof: Choose $w \in \mathcal{E}$ arbitrarily. We leave from

$$\| w - \hat{\mathcal{P}}^h \mathcal{P}^h w \| \leq \| w - \mathcal{P}^h w \| + \| \mathcal{P}^h w - \hat{\mathcal{P}}^h \mathcal{P}^h w \| \,. \tag{5.29}$$

To estimate the second term, we note that, cfr. [14, § 8.5, Lemma 1],

$$\mathcal{P}^h w - \hat{\mathcal{P}}^h \mathcal{P}^h w = (\mathcal{P}^h - \hat{\mathcal{P}}^h)\, \mathcal{P}^h w =$$

$$= -\frac{1}{2\pi i}\int_{\partial B} R_z(\hat{T}^h)(\hat{T}^h - T^h)\, R_z(T^h)\, \mathcal{P}^h w \, dz \,.$$

Subsequently using the boundedness of $R_z(\hat{T}^h)$ uniformly w.r.t. h and $z \in \partial B$, see Lemma 5.3, the fact that $R_z(T^h)\,\mathscr{P}^h w \in \mathscr{E}^h$, the boundedness of $R_z(T^h|_{V_h})$ in V_h uniformly w.r.t. h and $z \in \partial B$, see the analogon of Lemma 5.3 for T^h, implied by (4.5), and the boundedness of $\mathscr{P}^h$ uniformly w.r.t. h, as projection operator, we may find

$$\begin{aligned}\|\mathscr{P}^h w - \hat{\mathscr{P}}^h \mathscr{P}^h w\| &\leqslant C \int_{\partial B} \|(\hat{T}^h - T^h)\, R_z(T^h)\, \mathscr{P}^h w\|\, dz \\ &\leqslant C \|\hat{T}^h - T^h\|_{1,\mathscr{E}^h} \int_{\partial B} \|R_z(T^h)\, \mathscr{P}^h w\|\, dz \\ &\leqslant C \|\hat{T}^h - T^h\|_{1,\mathscr{E}^h} \|\mathscr{P}^h w\| \\ &\leqslant C \|\hat{T}^h - T^h\|_{1,\mathscr{E}^h} \|w\|\,, \quad \forall w \in \mathscr{E}\,.\end{aligned}$$

From this inequality and (4.9), (5.29) leads to (5.28). □

LEMMA 5.6 :

(1) *Under the assumptions of Theorem 5.3, (1) or (3), we have for* $\delta(\mathscr{E}, \hat{\mathscr{E}}^h)$, *defined analogously to (4.11)-(1.12) ;*

$$\delta(\mathscr{E}, \hat{\mathscr{E}}^h) \leqslant C[\, \|T - T^h\|_{1,\mathscr{E}} + \|T^h - \hat{T}^h\|_{1,\mathscr{E}^h}] \to 0 \quad \text{as} \quad h \to 0\,. \tag{5.30}$$

(2) *For* $X = G$ *in (5.5) or, when dealing with a triangular mesh on* Ω_i, $1 \leqslant i \leqslant M$, *for* $X = L$, *retain the conditions of Theorem 5.3 (1).*

For the case $X = L$, *with a rectangular mesh, assume that* T *satisfies (4.4) and retain the assumptions of Lemma 5.2 [so that the conditions of Theorem 5.3 (3) are implied]. Moreover, let for both situations* $\mathscr{E} \subset \hat{H}^{k+1}(\Omega)$. *Then, we have*

$$\delta(\mathscr{E}, \hat{\mathscr{E}}^h) \leqslant Ch^k\,. \tag{5.31}$$

Proof: (1) The estimate (5.30) directly follows from the definition of $\delta(\mathscr{E}, \hat{\mathscr{E}}^h)$ and (5.28), while the convergence is implied by (4.5) and Lemma 5.1.

(2) To proof (5.31), we estimate the operator norms in (5.30) separately. Let $\mathscr{E} \subset \hat{H}^{k+1}(\Omega)$. Using (4.2), (2.5) and (3.7) (with $r = k$), we find

$$\|Tw - T^h w\| = \|Tw - PTw\| \leqslant Ch^k \|w\|_{\hat{H}^{k+1}(\Omega)}\,, \quad \forall w \in \mathscr{E}\,,$$

such that, from the equivalence of norms on $\mathscr{E}$,

$$\| T - T^h \|_{1,\mathscr{E}} \leqslant Ch^k \sup \{ \| v \|_{\hat{H}^{k+1}(\Omega)} ; v \in \mathscr{E}, \| v \| \leqslant 1 \} \leqslant Ch^k .$$

On the other hand, Lemma 5.2 implies

$$\| T^h - \hat{T}^h \|_{1,\mathscr{E}^h} \leqslant Ch^k \sup \{ | v |_{k,\mathscr{T}_h} + \| T^h v \|_{k+1,\mathscr{T}_h} ; v \in \mathscr{E}^h, \| v \| \leqslant 1 \} . \tag{5.32}$$

It remains to show that this supremum is bounded independently of h. We retain the notations of Theorem 4.1 and leave from

$$\| u_{t,h} \|_{k+1,\mathscr{T}_h} \leqslant C[\| PW_t(h) \|_{k+1,\mathscr{T}_h} + \| PW_t(h) - W_t(h) \|_{k+1,\mathscr{T}_h} + \\ + \| W_t(h) - u_{t,h} \|_{k+1,\mathscr{T}_h}] .$$

For the first term we invoke (3.8) (with $r = k + 1$) and the equivalence of norms of $\mathscr{E}$. For the second term we apply (3.3) (with $s = k$), (3.7) (with $r = k$) and the equivalence of norms on $\mathscr{E}$. For the last term we use also (3.3) (with $s = k$) and next (4.15). Thus we get

$$\| u_{t,h} \|_{k+1,\mathscr{T}_h} \leqslant C[\| W_t(h) \| + 1] \leqslant C , \quad 0 \leqslant t \leqslant L , \tag{5.33}$$

where in the last step we combined (4.15) and (4.17).
For $v = \sum_{t=0}^{L} \alpha_t u_{t,h} \in \mathscr{E}^h$, with $\| v \| \leqslant 1$, we have $\sum_{t=0}^{L} \alpha_t^2 = | v |^2 \leqslant 1$. Hence, from (5.33) we have

$$| v |_{k,\mathscr{T}_h}^2 \leqslant C \sum_{t=0}^{L} \alpha_t^2 | u_{t,h} |_{k,\mathscr{T}_h}^2 \leqslant C \sum_{t=0}^{L} \alpha_t^2 \leqslant C ,$$

and, when also using (4.3) and (4.8),

$$\| T^h v \|_{k+1,\mathscr{T}_h}^2 \leqslant C \sum_{t=0}^{L} \alpha_t^2 \mu_{t,h}^2 \| u_{t,h} \|_{k+1,\mathscr{T}_h}^2 \leqslant C .$$

These two estimates show the supremum entering (5.32) to be bounded independently of h. □

Remark 5.2 : By applying the results of [5, § 2.5.1] and those of [13, Theorem I.6.34] to the present context, it is readily seen that $\delta(\mathscr{E}, \hat{\mathscr{E}}^h) = \delta(\hat{\mathscr{E}}^h, \mathscr{E})$.

THEOREM 5.4 :

(1) *Retain the assumptions of Theorem 5.3, (1) or (3). Then, there exists a set* $(U_{t,*}(h))_{0 \leqslant t \leqslant L}$ *of exact eigenfunctions, corresponding to* μ *and being orthonormal in* H, *such that for* $\hat{u}_{t,h}$, $0 \leqslant t \leqslant L$, *orthonormal w.r.t.* $(.,.)_h$, *we have*

$$\| U_{t,*}(h) - \hat{u}_{t,h} \| \to 0 \quad \text{as} \quad h \to 0, \quad 0 \leqslant t \leqslant L . \tag{5.34}$$

(2) *Under the assumptions of Lemma 5.6 (2), there exists a set* $(U_{t,*}(h))_{0 \leqslant t \leqslant L}$ *of exact eigenfunctions, corresponding to* μ *and being orthonormal in* H, *such that for* $\hat{u}_{t,h}$, $0 \leqslant t \leqslant L$, *orthonormal w.r.t.* $(.,.)_h$, *we have*

$$\| U_{t,*}(h) - \hat{u}_{t,h} \| \leqslant Ch^k . \tag{5.35}$$

Proof: Combining the uniform strong coercivity of $a_h(.,.)$ with (5.17), (5.18) and $|\hat{u}_{t,h}|_h = 1$, we arrive at

$$\| \hat{u}_{t,h} \| \leqslant C\hat{\mu}_{t,h}^{-1/2} \leqslant C, \quad 0 \leqslant t \leqslant L, \tag{5.36}$$

where in the last step (5.27) is used.
Then, analogously as in the proof of Theorem 4.1, from the definition of $\delta(\hat{\mathscr{E}}^h, \mathscr{E})$ and (5.36) we derive

$$\inf\{ \| \hat{u}_{t,h} - w \| ; w \in \mathscr{E} \} \leqslant C\delta(\hat{\mathscr{E}}^h, \mathscr{E}) .$$

Hence, the exact eigenfunction $U_t(h) \in \mathscr{E}$, which realizes this infimum, satisfies

$$\| U_t(h) - \hat{u}_{t,h} \| \leqslant C\delta(\hat{\mathscr{E}}^h, \mathscr{E}) . \tag{5.37}$$

From Lemma 5.6, Remark 5.2 and (5.36), this estimate leads to

$$\| U_t(h) \| \leqslant C . \tag{5.38}$$

Moreover, as

$$| |U_t(h)| - 1 | \leqslant |U_t(h) - \hat{u}_{t,h}| + | |\hat{u}_{t,h}|^2 - 1 | ,$$

we infer from (5.37) and (5.9)

$$| |U_t(h)| - 1 | \leqslant C\delta(\hat{\mathscr{E}}^h, \mathscr{E}) + |E^L(\hat{u}_{t,h}, \hat{u}_{t,h})| .$$

For the second term we first apply [17, Theorem 2.4 (with $s = 0$, $t = k$)], use (5.36) [or Proposition 5.1] and next proceed similarly as in the proof of (5.33) [now using (5.37)] to find

$$|E^L(\hat{u}_{t,h}, \hat{u}_{t,h})| \leqslant Ch^k |\hat{u}_{t,h}|_{k,\mathcal{T}_h} |\hat{u}_{t,h}| \leqslant Ch^k [\|U_t(h)\| + 1] \leqslant Ch^k, \tag{5.39}$$

where in the last step (5.38) is used. Hence,

$$||U_t(h)| - 1| \leqslant C[\delta(\hat{\mathscr{E}}^h, \mathscr{E}) + h^k]. \tag{5.40}$$

Furthermore, from (5.36), (5.37), (5.38) and the analogon of (5.39), we obtain for $t \neq s$

$$|(U_t(h), U_s(h))| = |(U_t(h) - \hat{u}_{t,h}, U_s(h)) +$$

$$+ (\hat{u}_{t,h}, U_s(h) - \hat{u}_{s,h}) + E^L(\hat{u}_{t,h}, \hat{u}_{s,h})|$$

$$\leqslant C[\delta(\hat{\mathscr{E}}^h, \mathscr{E}) + h^k].$$

Finally, by the Gram-Schmidt procedure we construct out of the set $(U_t(h))_{0 \leqslant t \leqslant L}$ a new set $(U_{t^*}(h))_{0 \leqslant t \leqslant L}$ of $(L+1)$ exact eigenfunctions, corresponding to μ and being orthonormal in H. For this new set we may readily (show (5.34)-(5.35) to hold by a proof by induction on $t = 0, 1, \ldots, L$. Here we use (5.37), (5.40) and (5.41), and finally invoke Lemma 5.6. □

Remark 5.3 : When dealing with a triangular mesh, following the arguments of [3], we may decrease the degree of precision of the involved quadrature formulas by one unit to $2k - 2$ and retain the optimal rate of convergence $O(h^k)$ for the eigenfunctions. This improvement may rest upon [6, Theorem 4.1.5]. However, the analogon of this auxiliary result for a rectangular mesh, see [6, Exercice 4.1.7 (ii)], only holds for a quadrature formula with degree of precision $2k - 1$. Thus, the technique in [3] may not be extended to the rectangular case.

5.8. Error estimate of the approximate eigenvalues

Let λ be an $(L+1)$-fold exact eigenvalue of (2.3) and let $\hat{\lambda}_{t,h}$, $0 \leqslant t \leqslant L$, be the corresponding eigenvalues of (5.19). We may obtain an optimal estimate for $|\lambda - \hat{\lambda}_{t,h}|$. First, we give an auxiliary result.

LEMMA 5.7 : *Let (λ, u) be an eigenpair of (2.3), with $|u| = 1$. Further, let $v \in V_h$ with $|v|_h = 1$. Then, we have*

$$a_h(v, v) - \lambda = a(u - v, u - v) - \lambda|u - v|^2 + \lambda E^L(v, v) - E_a^X(v, v) \tag{5.42}$$

Proof : From the definitions of E^L and E_a^X, we have

$$a_h(v, v) - \lambda = a(v, v) - E_a^X(v, v) - \lambda|v|^2 + \lambda E^L(v, v),$$

which may be rewritten in the form (5.42) on account of (2.3). □

THEOREM 5.5 : *Let $a_{\ell m}^i$, $a_0^i \in W^{2k,\infty}(\Omega_i)$, $\ell, m = 1, 2$, $h^{i,\sigma} \in W^{2k,\infty}(\Gamma_{i,\sigma})$, $\sigma \in \mathcal{N}_i$, $1 \leq i \leq M$, and let $a_1^i \in W^{2k,\infty}(\Gamma_2^i)$, $i \in \mathcal{N}(\Gamma_2)$. Further, let $\mathcal{E} \subset \hat{H}^{k+1}(\Omega)$. For $X = L$ in (5.5), when using a rectangular mesh, we assume in addition that T satisfies condition (4.4). Then, we have*

$$|\lambda - \hat{\lambda}_{t,h}| \leq Ch^{2k}, \quad 0 \leq t \leq L. \tag{5.43}$$

Proof : Let $\hat{u}_{t,h}$, $0 \leq t \leq L$, be eigenfunctions of (5.19) corresponding to $\hat{\lambda}_{t,h}$, $0 \leq t \leq L$, and being orthonormal w.r.t. $(.,.)_h$. Let $U_{t,*}$ be as in (5.34). Then, Lemma 5.7 implies

$$|\lambda - \hat{\lambda}_{t,h}| = |\lambda - a_h(\hat{u}_{t,h}, \hat{u}_{t,h})| \leq C\|U_{t,*} - \hat{u}_{t,h}\|^2 + \lambda|E^L(\hat{u}_{t,h}, \hat{u}_{t,h})| + |E_a^X(\hat{u}_{t,h}, \hat{u}_{t,h})|. \tag{5.44}$$

To estimate the second term, we now invoke [17, Theorem 2.4 (with $s = t = 0$)] to obtain, analogously as in (5.39),

$$|E^L(\hat{u}_{t,h}, \hat{u}_{t,h})| \leq Ch^{2k}|\hat{u}_{t,h}|_{k,\mathcal{T}_h}^2 \leq Ch^{2k}.$$

On the other hand, combining [17, (2.20)] with [17, (2.24)] when $X = G$ and [17, (2.18)] or (5.13) with [17, (2.24)] when $X = L$, we find

$$|E_a^X(\hat{u}_{t,h}, \hat{u}_{t,h})| \leq Ch^{2k}\|\hat{u}_{t,h}\|_{k+1,\mathcal{T}_h}^2 \leq Ch^{2k},$$

where in the last step we used the uniform boundedness of $\|\hat{u}_{t,h}\|_{k+1,\mathcal{T}_h}$, which follows along similar lines as (5.33).

Substitution of these two estimates and of (5.35) into (5.44) yields (5.43).□

Adapting the arguments in [20, Theorems 3.5 and 3.7], using (5.37), (5.40), (5.41) combined with Lemma 5.6 (1) and Lemma 5.4 (1), or with Lemma 5.6 (2) and Theorem 5.5, we may arrive at

THEOREM 5.6 :

(1) *Under the same conditions as in Theorem 5.4 (1) there exist a set $(\overline{U}_{t,*})_{0 \leq t \leq L}$ of fixed eigenfunctions of T, correspondinng to μ and being orthonormal in H, and a sequence h_j with $h_j \to 0$, such that*

$$\|\overline{U}_{t,*} - \hat{u}_{t,h_j}\| \to 0 \quad \text{as} \quad h_j \to 0, \quad 0 \leq t \leq L.$$

(2) *Under the same conditions as in Theorem 5.5 there exist a set* $(\overline{U}_{t,*})_{0 \leq t \leq L}$ *of fixed eigenfunctions of T, corresponding to* μ *and being orthonormal in H, a sequence* h_j *with* $h_j \to 0$, *and a number* m, $0 < m \leq k$, *such that*

$$\|\overline{U}_{t,*} - \hat{u}_{t,h_j}\| \leq C h_j^m , \quad 0 \leq t \leq L .$$

6. CONCLUSIONS

The method of [4] has been extended in four respects : we consider an EVP in a composite medium ; we allow for a more general self-adjoint 2nd-order differential operator as well as for mixed Dirichlet-Robin BCs ; finally, we also deal with the case of rectangular meshes.

In comparison to the variational approach in [19] the most important features of the present method are :

- By the present operator method we obtain optimal estimates for the approximate eigenpairs.
- For the case of a triangular mesh, this is even true when we use a quadrature formula with degree of precision $2k-1$ instead of $2k+1$ to approximate the bilinear form, without having to increase the smoothness assumptions on the coefficients in the bilinear form.
- For a rectangular mesh, the lower degree of precision, viz. $2k-1$, requires the additional regularity condition (4.4) for the BVP (2.4) in order to retain the same rate of convergence for the approximate eigenpairs.

REFERENCES

[1] A. B. ANDREEV, V. A. KASCIEVA & M. VANMAELE, Some results in lumped, mass finite-element approximation of eigenvalue problems using numerical quadrature, *J. Comp. Appl. Math.*, **43**, 1992, 291-311.

[2] I. BABUŠKA & J. E. OSBORN, Eigenvalue Problems. In : *Handbook of Numerical Analysis, Vol. II, Finite Element Methods (Part 1)* (P. G. Ciarlet, J. L. Lions, eds.). Amsterdam : North-Holland, 1991, 641-787.

[3] U. BANERJEE, A note on the effect of numerical quadrature in finite element eigenvalue approximation, *Numer. Math.*, **61**, 1992, 145-152.

[4] U. BANERJEE & J. E. OSBORN, Estimation of the Effect of Numerical Integration in Finite Element Eigenvalue Approximation, *Numer. Math.*, **56**, 1990, 735-762.

[5] F. CHATELIN, *Spectral Approximation of Linear Operators,* New York : Academic Press, 1983.

[6] P. G. CIARLET, *The Finite Element Method for Elliptic Problems,* Amsterdam : North-Holland, 1978.

[7] R. DAUTRAY & & J.-L. LIONS, *Analyse Mathématique et Calcul Numérique pour les Sciences et les Techniques, Tome 2.* Paris : Masson, 1985.

[8] P. J. DAVIS & P. RABINOWITZ, *Methods of numerical integration,* New York : Academic Press, 1975.

[9] J. DESCLOUX, N. NASSIF & J. RAPPAZ, On Spectral Approximation. Part 1. The Problem of Convergence, *R.A.I.R.O. Numerical Analysis,* **12,** 1978, 97-112.

[10] G. J. FIX, Eigenvalue Approximation by the Finite Element Method, *Advances in Mathematics,* **10,** 1973, 300-316.

[11] J. KAČUR & R. VAN KEER, On the Numerical Solution of some Heat Transfer Problems in Multi-component Structures with Non-perfect Thermal Contacts. In : *Numerical Methods for Thermal Problems VII* (R. W. Lewis, ed.). Swansea : Pineridge Press, 1991, 1378-1388.

[12] H. KARDESTUNCER & D. H. NORRIE, *Finite Element Handbook,* New York : McGraw-Hill Book Comp, 1987.

[13] T. KATO, *Perturbation Theory for Linear Operators,* Berlin : Springer-Verlag, 1976.

[14] B. MERCIER, *Lectures on Topics in Finite Element Solution of Elliptic Problems,* Berlin : Springer-Verlag, 1976.

[15] M. N. ÖZISIK, *Heat Conduction,* New York : John Wiley & Sons, 1980.

[16] M. VANMAELE, On optimal and nearly optimal error estimates of a numerical quadrature finite element method for 2nd-order eigenvalue problems with Dirichlet boundary conditions, *Simon Stevin,* **67,** 1992, 121-132.

[17] M. VANMAELE, A numerical quadrature finite element method for 2nd-order eigenvalue problems with Dirichlet-Robin boundary conditions. *Proceedings ISNA'92.* Prague, 1994, 269-292.

[18] M. VANMAELE & R. VAN KEER, Error estimates for a finite element method with numerical quadrature for a class of elliptic eigenvalue problems. In : *Numerical Methods* (D. Greenspan, P. Rósza, eds.). *Colloq. Math. Soc. János Bolyai,* **59,** 1990, Amsterdam, North-Holland, 267-282.

[19] M. VANMAELE & R. VAN KEER, On a numerical quadrature finite element method for a class of elliptic eigenvalue problems in composite structures, *Math. Comp.* (submitted).

[20] M. VANMAELE & A. ŽENÍŠEK, External finite element approximations of eigenfunctions in case of multiple eigenvalues. *J. Comp. Appl. Math.* **50** (to appear).