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FINITE ELEMENT APPROXIMATIONS OF LANDAU-GINZBURG'S EQUATION MODEL FOR STRUCTURAL PHASE TRANSITIONS IN SHAPE MEMORY ALLOYS (*)

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Abstract. — *This paper deals with finite element approximations of the Landau-Ginzburg model for structural phase transitions in shape memory alloys. The non-linear evolutionary system of partial differential equations is discretized in time by finite differences and in space by very simple finite elements, that is, the linear element for the absolute temperature and the Hermite cubic element for the displacement. Thus both the displacement and the strain are obtained directly. Error estimates for the fully discrete scheme are derived.*

Résumé. — *Dans cet article on présente des approximations par éléments finis d'un modèle de Ginzburg-Landau pour les transitions de phases dans des alliages à mémoire de forme. Le système non linéaire est discritisé en temps par une méthode de différences finies et en espace par des éléments finis très simples, linéaires pour la température, cubiques de type Hermite pour le déplacement. On obtient des estimations d'erreur pour le schéma discrétisé.*

1. INTRODUCTION

Recently much attention has been paid to mathematical models for thermomechanical phase transitions in shape memory alloys. For the survey of physical backgrounds and theoretical investigations on these models, we refer to two detailed introductory papers [2, 7]. There have been in the literature a great deal of theoretical results on the well-posedness and the optimal controls of mathematical models for the description of the phenomenology of shape memory alloys, but only a few references which deal with their numerical simulations. [1, 12] have made many numerical experiments, but no theoreti-

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cal analyses to their numerical schemes were given there. In [6] a discrete approximation to the Landau-Ginzburg model was constructed by the Galerkin method, and convergence was proved. The finite dimensional subspace for the approximation to the displacement was constructed in [6] by using the eigenvalue functions of the 4th order ordinary differential equation, and the functions of the resulting discrete subspace were then infinitely smooth. Therefore the techniques for getting *a priori* estimates in continuous cases could be repeated for their discrete system, and finally the compactness arguments led to the convergence of the discrete problem. Recently, the authors were notified by Prof. Sprekels that the error estimate was obtained in a recent work [5] for the discrete scheme proposed in [6].

The more effective and practical discretizations for these problems are obviously finite element methods. In our present paper, we approach the Landau-Ginzburg model by a very simple finite element, thus very practical for the applications. With our simple element the discrete subspace possesses only a very low smoothness. Not so many *a priori* estimates as for the original continuous problems, or as in [5, 6], could be obtained in the present case. Nevertheless, these *a priori* estimates are enough for us to attain error estimates for the fully discrete finite element approximation. To our knowledge it is the first time to obtain error estimates for the finite element approximations to such highly nonlinear shape memory alloy models.

The paper is arranged as follows. In Section 2 the Landau-Ginzburg mathematical model is introduced and their finite element problem is constructed in Section 3. Section 4 is devoted to *a priori* estimates, the uniqueness and existence of solutions of the discrete system. In Section 5 we derive error estimates for the finite element approximation.

2. LANDAU-GINZBURG MODEL

In this paper we consider the following Landau-Ginzburg model arising from modelling the dynamics of solid-state phase transitions in shape memory alloys :

$$\rho u_{tt} - (\alpha_1(\theta - \theta_1)\, u_x - \alpha_2\, u_x^3 + \alpha_3\, u_x^5)_x + \gamma u_{xxxx} = f(x, t), \text{ in } Q_T, \qquad (2.1a)$$

$$c_0\, \theta_t - \kappa\theta_{xx} - \alpha_1\, \theta u_x\, u_{xt} = g(x, t), \text{ in } Q_T \qquad (2.1b)$$

with the boundary conditions

$$u(0, t) = u_{xx}(0, t) = u_{xx}(1, t) = u(1, t) = 0\,, \qquad (2.2a)$$

$$\theta_x(0, t) = \theta_x(1, t) = 0 \qquad (2.2b)$$

and the initial conditions

$$u(x,0)=u_0(x)\,,\quad u_t(x,0)=u_1(x)\,, \tag{2.3a}$$

$$\theta(x,0)=\theta_0(x) \tag{2.3b}$$

where $Q_t=\Omega\times(0,t)$, $0\leqslant t\leqslant T$, $\Omega=(0,1)$. Throughout the paper, we let

$$p(\theta,\varepsilon)=\psi_\varepsilon(\theta,\varepsilon,\varepsilon_x)=\alpha_1(\theta-\theta_1)\,\varepsilon-\alpha_2\,\varepsilon^3+\alpha_3\,\varepsilon^5$$

with $\varepsilon=u_x$ denoting the shear strain. The unknown functions u and θ in (2.1a, b) represent the displacement and the absolute temperature, and ψ denotes Helmholtz free energy which is assumed in the Landau-Ginzburg form

$$\psi=\psi(\theta,\varepsilon,\varepsilon_x)=\psi_0(\theta)+E^0(\theta,\varepsilon)+\frac{\gamma}{2}\,\varepsilon_x^2 \tag{2.4}$$

with E^0 and ψ_0 expressed by (the solution θ is a positive function, as we see later)

$$E^0(\theta,\varepsilon)=\psi_1(\theta)\,\psi_2(\varepsilon)+\psi_3(\varepsilon)\,,\quad \psi_0(\theta)=c_0(\theta-\theta\log\theta/\theta_2)\,, \tag{2.5a}$$

$$\psi_1(\theta)=\frac{1}{2}\,\alpha_1\,\theta\,,\ \psi_2(\varepsilon)=\varepsilon^2,\ \psi_3(\varepsilon)=-\frac{1}{2}\,\alpha_1\,\theta_1\,\varepsilon^2-\frac{1}{4}\,\alpha_2\,\varepsilon^4+\frac{1}{6}\,\alpha_3\,\varepsilon^6 \tag{2.5b}$$

which are capable of reproducing the developments observed in real materials under thermomechanical activations. Equations (2.1a, b) represent the balance laws of linear momentum and energy, respectively. The material is assumed as a wire of unit length, simply supported at both ends, and thermally isolated at both ends (only for simplicity. For more general nonhomogeneous conditions, e.g., as in [6], our results hold with little modification). In our context, the quantities appearing in (2.1a, b) have the physical meanings : ρ-mass density, f-volumetric load, c_0-specific heat (per volume), κ-heat conductivity, g-rate of distributed energy sources. The coefficients κ, α_1, α_2, α_3, θ_1, θ_2 and γ are assumed to be positive constants. For the physical background and a derivation of equations (2.1a, b), we refer to [8] and the references therein.

Under the appropriate regularity assumptions on the given functions f, g, u_0, u_1 and θ_0, e.g., the following ones (2.8a, b, c, d), the system (2.1)-(2.3) has a unique classical solution (u,θ) with θ being always positive, see [8]. For our later finite element error estimates, we make the following assumptions on the regularities of the solution (u, θ), which are a little stronger than the regularity results stated in Theorem 2.1 in [8] and those derived from their proofs :

$$u \in H^1(0,T;H^4(\Omega)) \cap H^2(0,T;H^2(\Omega)) \cap H^3(0,T;L^2(\Omega)), \quad (2.7a)$$

$$u \in L^\infty(0,T;H^5(\Omega)), \quad u_{tt} \in L^\infty(0,T;H^1(\Omega)), \quad (2.7b)$$

$$\theta \in L^\infty(0,T;H^3(\Omega)) \cap H^1(0,T;H^2(\Omega)) \quad (2.7c)$$

and on the given data

$$f, g \in H^1(0,T;H^1(\Omega)), f_{tt}, g_{xx} \in L^2(0,T;L^2(\Omega)), \quad (2.8a)$$

$$u_1 \in H^3_E(\Omega) = \{u \in H^3(\Omega)\,;\, u(0) = u''(0) = 0 = u(1) = u''(1)\}, \quad (2.8b)$$

$$u_0 \in H^5(\Omega) \cap H^3_E(\Omega), \quad (2.8c)$$

$$\theta_0 \in H^3(\Omega), \quad \theta_0(x) > 0 \text{ on } \overline{\Omega} \quad \text{and} \quad g(x,t) \geqslant 0 \text{ on } \overline{Q}_T. \quad (2.8d)$$

Furthermore, we can assume, by Sobolev extension theory [9], that the solution u defined on $\overline{\Omega} \times [0, T]$ can be extended for some constant $\tau_0 \leqslant T$ such that u is well-defined also on $\overline{\Omega} \times [-\tau_0, T]$ and

$$u_{ttx} \in L^\infty(-\tau_0, T; L^2(\Omega)), \quad u_{ttxx} \quad \text{and} \quad u_{ttt} \in L^2(-\tau_0, T; L^2(\Omega)). \quad (2.9)$$

Since these extensions can be constructed such that they depend only on the initial conditions (2.3a), we can even get extensions with higher regularity on $(-\tau_0, T]$ provided the initial conditions are smoother.

Throughout the paper we utilise $|\cdot|_{m,p}$ and $\|\cdot\|_{m,p}$ to denote the seminorm and the norm of the usual Sobolev space $W^{m,p}(\Omega)$. But we write $|\cdot|_m = |\cdot|_{m,p}$, $\|\cdot\|_m = \|\cdot\|_{m,p}$ and $H^m(\Omega) = W^{m,p}(\Omega)$, if $p = 2$; $\|\cdot\| = \|\cdot\|_0$ if $m = 0$. Constants C are independent of mesh size h and time step τ.

3. A FULLY DISCRETE APPROACH TO THE SYSTEM (2.1)-(2.3)

In this section we propose a fully discrete finite element scheme to Landau-Ginzburg model (2.1)-(2.3). To avoid the non-essential technicalities, we take $\theta_1 = 0$, and all constants c_0, κ, α_1, α_2, α_3, γ in (2.1)-(2.5) are normalized to unity, i.e. our equations can be rewritten as

$$u_{tt} - (\theta u_x - u_x^3 + u_x^5)_x + u_{xxxx} = f(x,t), \text{ in } Q_T,$$

$$\theta_t - \theta_{xx} - \theta u_x u_{xt} = g(x,t), \text{ in } Q_T.$$

We shall use the difference scheme to discretize the system in time. Let $\tau = T/M$ be time step size with M a positive integer. For any $n = 1, 2, \dots, M$, we denote $t^n = n\tau$ and $I^n = (t^{n-1}, t^n]$. For a given sequence $\{u^n\}_{n=0}^M \subset L^2(\Omega)$, we define

$$\partial_\tau u^n = \frac{u^n - u^{n-1}}{\tau}, \quad u^{n-\frac{1}{2}} = \frac{1}{2}(u^n + u^{n-1}),$$

$$\partial_\tau^2 u^n = \frac{\partial_\tau u^n - \partial_\tau u^{n-1}}{\tau}, \quad n = 1, 2, \dots, M.$$

For a continuous mapping $u : [0, T] \to L^2(\Omega)$, we define $u^n = u(\,\cdot\,, n\tau)$, $0 \leqslant n \leqslant M$.

In space we utilize the linear finite element approximation to the absolute temperature θ, and the Hermite finite element to the displacement u. Suppose for $h \to 0$ we have been given a family of quasiuniform partitions $= \mathcal{T}_h$ of $\Omega = [0, 1]$:

$$\mathcal{T}_h : 0 = x_0^h < x_1^h < \dots < x_N^h = 1 .$$

Let $\Delta_h^i = (x_{i-1}^h, x_i^h]$, $h = \max_{1 \leqslant i \leqslant N} h_i$ with $h_i = x_i^h - x_{i-1}^h$. By quasiuniform we mean that there is a constant σ such that $h/h_i \leqslant \sigma$, for any $1 \leqslant i \leqslant N$.

Define the finite element spaces

$$\Theta_h = \{\theta \in C(\overline{\Omega})\,;\, \theta|_{\Delta_h^i} \text{ is linear, for all } \Delta_h^i \in \mathcal{T}_h\},$$

$$V_h = \{v \in C^1(\overline{\Omega})\,;\, v|_{\Delta_h^i} \text{ is a polynomial of degree} \leqslant 4, \text{ for all } \Delta_h^i \in \mathcal{T}_h\}$$

$$V_h^0 = V_h \cap H_0^1(\Omega) .$$

Here we may choose as the degrees of freedom the nodal value, the first-order derivative at each node and the midpoint value. In the subsequent sections, we always use u_π and θ_I to denote the standard interpolations of any function $u \in C^1(\overline{\Omega})$ and $\theta \in C(\overline{\Omega})$ or $u : [0, T] \to C^1(\overline{\Omega})$ and $\theta : [0, T] \to C(\overline{\Omega})$ related to subspaces V_h^0 and Θ_h, respectively, see [10, 11].

For simplicity, we let

$$P(\theta, \varepsilon_1, \varepsilon_2) = \frac{E^0(\theta, \varepsilon_1) - E^0(\theta, \varepsilon_2)}{\varepsilon_1 - \varepsilon_2}. \tag{3.1}$$

It is easy to check that

$$p(\theta, \varepsilon) = \psi_\varepsilon(\theta, \varepsilon, \varepsilon_x) = E^0_\varepsilon(\theta, \varepsilon), \tag{3.2a}$$

$$P(\theta, \varepsilon_1, \varepsilon_2) = \frac{1}{2}\theta(\varepsilon_1 + \varepsilon_2) + Q(\varepsilon_1, \varepsilon_2) \tag{3.2b}$$

with

$$Q(\varepsilon_1, \varepsilon_2) = -\frac{1}{4}(\varepsilon_1^2 + \varepsilon_2^2)(\varepsilon_1 + \varepsilon_2) + \frac{1}{6}(\varepsilon_1^3 + \varepsilon_2^3)(\varepsilon_1^2 + \varepsilon_1 \varepsilon_2 + \varepsilon_2^2). \tag{3.3}$$

Now we give a fully discrete finite element scheme for the system (2.1)-(2.3) :

(FEP) : For $n = 1, 2, \dots, M$, find $(u_h^n, \theta_h^n) \in V_h^0 \times \Theta_h$ such that

$$\int_\Omega \partial_\tau^2 u_h^n \, v \, dx + \int_\Omega \partial_\varepsilon E^0(\theta_h^{n-1}, \varepsilon_h^n) \, v_x \, dx + \\ + \int_\Omega (u_h^n)_{xx} \, v_{xx} \, dx = \int_\Omega \bar{f}^n \, v \, dx, \quad \forall v \in V_h^0, \tag{3.4a}$$

$$\int_\Omega \partial_\tau \theta_h^n \, \eta \, dx + \int_\Omega (\theta_h^n)_x \, \eta_x \, dx - \\ - \int_\Omega \theta_h^{n-1} \, \varepsilon_h^{n-1/2} \, \partial_\tau \varepsilon_h^n \, \eta \, dx = \int_\Omega \bar{g}^{\,n} \eta dx, \quad \forall \eta \in \Theta_h, \tag{3.4b}$$

$$u_h^0 = (u_0)_\Pi, \quad u_h^0 - u_h^{-1} = \tau(u_1)_\Pi, \quad \theta_h^0 = (\theta_0)_{\mathrm{I}} \tag{3.4c}$$

with

$$\varepsilon_h^n = (u_h^n)_x, \quad \bar{f}^n(x) = \frac{1}{\tau}\int_{I^n} f(x,t)\, dt, \quad \bar{g}^{\,n}(x) = \frac{1}{\tau}\int_{I^n} g(x,t)\, dt, \tag{3.5}$$

and $\partial_\varepsilon E^0(\theta_h^{n-1}, \varepsilon_h^n)$ denoting the finite difference

$$\partial_\varepsilon E^0(\theta_h^{n-1}, \varepsilon_h^n) = \frac{E^0(\theta_h^{n-1}, \varepsilon_h^n) - E^0(\theta_h^{n-1}, \varepsilon_h^{n-1})}{\varepsilon_h^n - \varepsilon_h^{n-1}} = P(\theta_h^{n-1}, \varepsilon_h^n, \varepsilon_h^{n-1}). \tag{3.6}$$

4. *A PRIORI* ESTIMATES, EXISTENCE AND UNIQUENESS OF THE SOLUTIONS TO THE DISCRETE SYSTEM (3.4*a*, *b*, *c*)

Except for the uniqueness of solutions which was not given in [6], our way of getting *a priori* estimates and the existence of solutions to (3.4*a*, *b*, *c*) in this section is almost the same as the one used in [6].

THEOREM 4.1 : *Under the assumptions* (2.8*a*, *b*, *c*, *d*), *there exist positive constants* C_1 *and* C_2 *independent of* h *and* τ *such that if* τ *and* h *fulfill the condition*

$$h^2/6 < \tau \leqslant C_1 , \tag{4.1}$$

then for each n, $1 \leqslant n \leqslant M$, *the discrete problem (FEP) has a unique solution* (u_h^n, θ_h^n) *which satisfies the a priori estimates*

$$\theta_h^n(x) \geqslant 0 , \quad 0 \leqslant x \leqslant 1 , \quad 0 \leqslant n \leqslant M , \tag{4.2a}$$

$$\max_{0 \leqslant n \leqslant M} (\| \partial_\tau u_h^n \|^2 + |u_h^n|_2^2 + \| u_h^n \|_{1,\infty}) \leqslant C_2 , \tag{4.2b}$$

$$\max_{0 \leqslant n \leqslant M} \| \theta_h^n \|^2 + \sum_{n=1}^{M} \tau |\theta_h^n|_1^2 + \sum_{n=1}^{M} \tau \| \theta_h^n \|_{0,\infty}^2 \leqslant C_2 . \tag{4.2c}$$

As in [6] we prove Theorem 4.1 by induction. By the definition of θ_h^0, $\theta_h^0 \geqslant 0$ on $\bar{\Omega}$. Suppose that solutions $\{(u_h^n, \theta_h^n)\}_{n=0}^k$ of the problem (FEP) have been constructed for some $k \in \{0, 1, ..., M\}$, and $\theta_h^n(x) \geqslant 0$ on $\bar{\Omega}$, $0 \leqslant n \leqslant k$. We prove the required results by the following three steps, from which Theorem 4.1 follows immediately :

a) There exists a constant $C_2 > 0$, independent of h, τ and k such that the estimates (4.2*a*, *b*, *c*) hold with M replaced by k ;

b) For $h^2/6 < \tau$, (3.4*a*, *b*, *c*) has a unique solution $(u_h^{k+1}, \theta_h^{k+1})$ for $n = k + 1$;

c) There exists a constant $C_1 > 0$, independent of h, τ, and k such that $\theta_h^{k+1} \geqslant 0$ on $\bar{\Omega}$, provided that $\tau \leqslant C_1$.

To prove *a*), we first notice that by the standard interpolation theory of finite element method we have

$$\| \partial_\tau u_h^0 \| = \| (u_1)_\Pi \| \leqslant C , \quad |u_h^0|_2 = |(u_0)_\Pi|_2 \leqslant C , \tag{4.3}$$

then with (4.3), part *a*) can be proved in the same way as in proving Lemma 3.2 and Lemma 3.3 in [6].

Now we prove *b*). Rewrite (3.4*a*) with $n = k + 1$ as

$$\langle F(u_h^{k+1}), v \rangle = \int_\Omega \tilde{f}^n v \, dx , \quad \forall v \in V_h^0 \tag{4.4}$$

where $F(u_h^{k+1}) \in (V_h^0)^* =$ dual space of V_h^0. The dual pairing of V_h^0 and $(V_h^0)^*$ is $\langle \cdot , \cdot \rangle$. Then $F : V_h^0 \to (V_h^0)^*$ is defined by

$$\langle F(u), v \rangle = \int_\Omega \left(\frac{u - 2 u_h^k + u_h^{k-1}}{\tau^2} v + u_{xx} v_{xx} + \right.$$

$$\left. + \frac{E^0(\theta_h^k, \varepsilon) - E^0(\theta_h^k, \varepsilon_h^k)}{\varepsilon - \varepsilon_h^k} v_x \right) dx \quad (4.5)$$

with $\varepsilon = u_x$. We check by using $\theta_h^k \geqslant 0$ and the proved result *a*) and Young's inequality that

$$\langle F(u), u - u_h^k \rangle \geqslant \frac{1}{2\tau^2} \| u - u_h^k \|^2 - \frac{1}{2} \| \partial_\tau u_h^k \|^2 + \frac{1}{2} |u|_2^2 - \frac{1}{2} |u_h^k|_2$$

$$+ \frac{1}{2} \int_\Omega \theta_h^k (u_x^2 - (u_h^k)_x^2) \, dx - \frac{1}{4} \int_\Omega (u_x^4 - (u_h^k)_x^4) \, dx$$

$$+ \frac{1}{6} \int_\Omega (u_x^6 - (u_h^k)_x^6) \, dx$$

$$\geqslant \frac{1}{2\tau^2} \| u - u_h^k \|^2 + \frac{1}{2} |u|_2^2 - \frac{1}{2} \int_\Omega \theta_h^k (u_h^k)_x^2 \, dx + \frac{1}{12} \int_\Omega u_x^6 \, dx - C$$

$$\geqslant \frac{1}{2\tau^2} \| u - u_h^k \|^2 - C . \quad (4.6)$$

Therefore

$$\frac{\langle F(u) - F(u_h^k), u - u_h^k \rangle}{\| u - u_h^k \|} \to + \infty , \quad \text{as } \| u - u_h^k \| \to + \infty , \quad \forall u \in V_h^0 ,$$

i.e. F is coercive with respect to u_h^k. Moreover, F is continuous on V_h^0. Thus by standard theory, see [4], we know that (4.4) has a solution u_h^{k+1}. And more, from (4.4) and (4.6) we have

$$\| \partial_\tau u_h^{k+1} \|^2 + |u_h^{k+1}|_2^2 + \int_\Omega \theta_h^k (u_h^{k+1})_x^2 \, dx + \int_\Omega (u_h^{k+1})_x^6 \, dx \leqslant C . \quad (4.7)$$

To prove uniqueness, it suffices to show the uniqueness of the solution u_h^{k+1} to (3.4a), since obviously the solution θ_h^{k+1} of equation (3.4b) exists uniquely. Let u_h^{k+1} and $\bar{u}_h^{k+1}$ be the solutions of (3.4a) and $u = u_h^{k+1} - \bar{u}_h^{k+1}$. From (3.4$a$) it follows that $u \in V_h^0$ satisfies

$$\tau^{-2} \int_\Omega uv \, dx + \int_\Omega u_{xx} v_{xx} \, dx = -\frac{1}{2} \int_\Omega \theta_h^k u_x v_x \, dx +$$

$$+ \int_\Omega (Q(\bar{\varepsilon}_h^{k+1}, \varepsilon_h^k) - Q(\varepsilon_h^{k+1}, \varepsilon_h^k)) v_x \, dx, \quad \forall v \in V_h^0 \tag{4.8}$$

with $Q(q_1, q_2)$ defined by (3.3). Taking $v = u$ in (4.8) and noticing $\theta_h^k \geq 0$, we obtain that

$$\tau^{-2} \|u\|^2 + |u|_2^2 \leq \int_\Omega (Q(\bar{\varepsilon}_h^{k+1}, \varepsilon_h^k) - Q(\varepsilon_h^{k+1}, \varepsilon_h^k)) u_x \, dx. \tag{4.9}$$

But

$$|Q(\bar{\varepsilon}_h^{k+1}, \varepsilon_h^k) - Q(\varepsilon_h^{k+1}, \varepsilon_h^k)| = \left| - \int_0^1 Q_{q_1}(\varepsilon_h^{k+1} - \lambda u_x, \varepsilon_h^k) \, u_x \, d\lambda \right|,$$

and from (4.2b, c) and (4.7) one gets

$$\|\varepsilon_h^{k+1} - \lambda u_x\|_{0,\infty}^2 \leq C, \quad \|\varepsilon_h^k\|_{0,\infty}^2 \leq C,$$

Thus it follows from (4.9) and above that

$$\tau^{-2} \|u\|^2 + |u|_2^2 \leq C \|u_x\|_{0,\infty}^2 .$$

By Nirenberg's inequality [8] we get

$$\tau^{-2} \|u\|^2 + |u|_2^2 \leq C(\|u_{xx}\|^{\frac{3}{4}} \|u\|^{\frac{1}{4}} + \|u\|)^2 \leq C(\|u\|^2 + |u|_2^{\frac{3}{2}} \|u\|^{\frac{1}{2}})$$

$$\leq \frac{1}{4} |u|_2^2 + C \|u\|^2 + C |u|_2 \|u\| \leq \frac{1}{2} |u|_2^2 + C \|u\|^2,$$

so if τ is small enough, we have $u = 0$, i.e., (3.4a) has a unique solution u_h^{k+1}.

Now the same as in [6], we can show that there is a positive constant C_1 independent of h, τ and k such that if $h^2/6 < \tau \leqslant C_1$, then $\theta_h^{k+1}(x) \geqslant 0$ on $\overline{\Omega}$. That completes the proof of Theorem 4.1.

5. ERROR ESTIMATES FOR THE FULLY DISCRETE SCHEME (3.4*a*, *b*, *c*)

In this section we derive error estimates for the fully discrete finite element approximation (3.4*a*, *b*, *c*) to (2.1)-(2.3). Our main results are stated in the following theorem :

THEOREM 5.1 : *Suppose that* (u_h^n, θ_h^n) *is the solution of (3.4a, b, c) and* (u, θ) *the solution of (2.1)-(2.3). Then there is a constant independent of h and* τ *such that*

$$\max_{1 \leqslant n \leqslant M} \left(\|\theta^n - \theta_h^n\|^2 + |u^n - u_h^n| + \|u_t^n - \partial_\tau u_h^n\|^2 \right) + \\ + \sum_{n=1}^{M} \tau |\theta^n - \theta_h^n|_1^2 \leqslant C(h^2 + \tau^2).$$

Let $\xi_h^n = \theta_h^n - \theta_I^n$ and $\rho_h^n = u_h^n - u_\Pi^n$, $1 \leqslant n \leqslant M$. Here $\theta_I^n = (\theta(\,.\,, n\tau))_I$ and $u_\Pi^n = (u(\,.\,, n\tau))_\Pi$. Before proving Theorem 5.1 we first introduce the *a priori* bound of the solution (u, θ) of (2.1) :

$$\sup_{0 \leqslant t \leqslant T} \left(\|u\|_5 + \|u_t\|_3 + \|u_{tt}\|_1 + \|\theta\|_3 + \|\theta_t\|_1 \right) \leqslant C. \tag{5.1}$$

This constant depends only on the given data (2.8*a*, *b*, *c*, *d*), see [8]. And also we cite some standard finite element interpolation results, see [10, 11] :

$$\|w - w_\Pi\| \leqslant Ch^2 |w|_2, \quad \forall w \in H^2(\Omega), \tag{5.2a}$$

$$|w - w_\Pi|_1 + h|w - w_\Pi|_2 \leqslant Ch^{m-1}|w|_m, \forall w \in H^m(\Omega), m = 3, 4, 5, \tag{5.2b}$$

$$\|w - w_I\| + h|w - w_I|_1 \leqslant Ch^2 |w|_2, \quad \forall w \in H^2(\Omega). \tag{5.2c}$$

Furthermore, we give here a few *a priori* bounds and inequalities needed later. For $n = 1, 2, ..., M$, there exists a constant C such that

$$\|\theta_I^n\|_1 \leqslant C, \quad \|u_\Pi^n\|_2 \leqslant C, \quad \|\partial_\tau \rho_h^n\| \leqslant C, \tag{5.3a}$$

$$\|\xi_h^n\|_{0,\infty} \leqslant C(\|\xi_h^n\| + |\xi_h^n|_1^{\frac{1}{2}}), \quad \|\xi_h^n\|_{0,\infty} \leqslant \|\xi_h^n\| + |\xi_h^n|_1 \tag{5.3b}$$

$$\|\theta_h^n\|_{0,\infty} \leqslant C(1 + |\theta_h^n|_1^{1/2}), \quad \|\theta_h^n\|_{0,\infty} \leqslant \|\theta_h^n\| + |\theta_h^n|_1 . \tag{5.3c}$$

(5.3a, b, c) follow from (5.2a, b, c), (4.2b, c), Nirenberg's inequality [8] and the following inequality :

$$\max_{x \in \Omega} |w(x)| \leqslant \int_\Omega |w(x)|\, dx + \int_\Omega |w_x(x)|\, dx\,, \quad \forall w \in H^1(\Omega)\,.$$

To start the proof of Theorem 5.1, we first multiply (2.1a) by function $\partial_\tau \rho_h^n \in V_h^0$, integrate it then over $\Omega \times I^n$ by parts, and by means of conditions (2.2a), (3.5) and the definition of $p(\theta, \varepsilon)$, we derive that

$$\int_\Omega (\partial_\tau u_t^n - \partial_\tau^2 u^n)\, v\, dx + \int_\Omega \partial_\tau^2 u^n\, v\, dx + \frac{1}{\tau}\int_{I^n}\int_\Omega p(\theta, u_x)\, v_x\, dx\, dt$$
$$+ \frac{1}{\tau}\int_{I^n}\int_\Omega u_{xx}\, v_{xx}\, dx\, dt = \int_\Omega \bar{f}^n\, v\, dx$$

with $v = \rho_h^n - \rho_h^{n-1}$ above. Subtracting this equation from (3.4a) and using (3.6), we obtain

$$\int_\Omega \partial_\tau^2 \rho_h^n\, v\, dx + \int_\Omega (\rho_h^n)_{xx}\, v_{xx}\, dx = \int_\Omega (\partial_\tau^2 u^n - \partial_\tau^2 u_\Pi^n)\, v\, dx + \int_\Omega (\partial_\tau u_t^n - \partial_\tau^2 u^n)\, v\, dx$$
$$+ \left[\frac{1}{\tau}\int_{I^n}\int_\Omega u_{xx}\, v_{xx}\, dx\, dt - \int_\Omega (u_\Pi^n)_{xx}\, v_{xx}\, dx\right]$$
$$+ \left[\frac{1}{\tau}\int_{I^n}\int_\Omega p(\theta, u_x)\, v_x\, dx\, dt - \int_\Omega P(\theta_h^{n-1}, \varepsilon_h^n, \varepsilon_h^{n-1})\, v_x\, dx\right]$$
$$=: \sum_{i=1}^{4} (I)_i\,. \tag{5.4}$$

Note that in (5.4) $u^{-1} = u(\,.\,, -\tau)$, $u_\Pi^{-1} = (u(\,.\,, -\tau))_\Pi$ if $n = 1$, so we have utilized the extension of u onto $[-\tau, 0]$.

By the fact that $ab \geqslant a^2/2 - b^2/2$, for any real a and b, we get from (5.4)

$$\frac{1}{2}\,\|\partial_\tau \rho_h^n\|^2 - \frac{1}{2}\,\|\partial_\tau \rho_h^{n-1}\|^2 + \frac{1}{2}\,|\rho_h^n|_2^2 - \frac{1}{2}\,|\rho_h^{n-1}|_2^2 \leqslant \sum_{i=1}^{4} (I)_i\,. \tag{5.5}$$

Similarly, multiplying (2.1b) by function $\tau\xi_h^n \in \Theta_h$, integrating it then over $\Omega \times I^n$ and subtracting the resultant equation from (3.4b) we have

$$\int_\Omega \partial_\tau \xi_h^n \eta \, dx + \int_\Omega (\xi_h^n)_x \eta_x \, dx = \int_\Omega (\partial_\tau \theta^n - \partial_\tau \theta_{\mathrm{I}}^n) \eta \, dx$$
$$+ \left[\frac{1}{\tau} \int_{I^n} \int_\Omega \theta_x \eta_x \, dx - \int_\Omega (\theta_{\mathrm{I}}^n)_x \eta_x \, dx \right]$$
$$+ \left[\int_\Omega \theta_h^{n-1} \varepsilon_h^{n-\frac{1}{2}} \partial_\tau \varepsilon_h^n \eta \, dx - \frac{1}{\tau} \int_{I^n} \int_\Omega \theta u_x u_{xt} \eta \, dx \, dt \right] =: \sum_{i=1}^{3} (II)_i \tag{5.6}$$

with $\eta = \tau \xi_h^n$ above. Again by the fact that $ab \geqslant a^2/2 - b^2/2$, for any real a and b, we get

$$\frac{1}{2} \| \xi_h^n \|^2 - \frac{1}{2} \| \xi_h^{n-1} \|^2 + \tau | \xi_h^n |_1^2 \leqslant \sum_{i=1}^{3} (II)_i \,. \tag{5.7}$$

Thus, for error estimates, it suffices to estimate the terms $(I)_i$ and $(II)_i$ in (5.5) and (5.7) which will be done in a number of following lemmas.

LEMMA 5.2 : *We have*

$$|(I)_1| \leqslant \frac{1}{2} \tau \| \partial_\tau \rho_h^n \|^2 + C h^4 \int_{t^{n-2}}^{t^n} |u_{tt}|_2^2 \, dt \,, \tag{5.8}$$

$$|(I)_2| \leqslant \frac{1}{2} \tau \| \partial_\tau \rho_h^n \|^2 + \frac{1}{2} \tau^2 \int_{t^{n-2}}^{t^n} \int_\Omega u_{ttt}^2 \, dx \, dt \,, \tag{5.9}$$

$$|(I)_3| \leqslant \tau \| \partial_\tau \rho_h^n \|^2 + C \tau h^2 + \frac{1}{2} \tau^2 \int_{I^n} \int_\Omega u_{xxxxt}^2 \, dx \, dt \,. \tag{5.10}$$

Proof: Recall the previous notation $v = \rho_h^n - \rho_h^{n-1}$, then by (5.2$a$) and the standard arguments it follows that

$$|(I)_1| \leqslant \tau \| \partial_\tau \rho_h^n \| \, \| \partial_\tau^2 (u^n - u_\Pi^n) \|$$
$$\leqslant \frac{1}{2} \tau \| \partial_\tau \rho_h^n \|^2 + \frac{1}{2} \int_{t^{n-2}}^{t^n} \| u_{tt} - (u_{tt})_\Pi \|^2 \, dt$$
$$\leqslant \frac{1}{2} \tau \| \partial_\tau \rho_h^n \|^2 + C h^4 \int_{t^{n-2}}^{t^n} |u_{tt}|_2^2 \, dt \,,$$

and

$$|(I)_2| \leqslant \tau \| \partial_\tau \rho_h^n \| \ \| \partial_\tau (u_t^n - \partial_\tau u^n) \| \leqslant \tau \sqrt{\tau} \ \| \partial_\tau \rho_h^n \| \left(\int_{t^{n-2}}^{t^n} \int_\Omega u_{ttt}^2 \, dx \, dt \right)^{1/2}$$

$$\leqslant \frac{1}{2} \tau \| \partial_\tau \rho_h^n \|^2 + \frac{1}{2} \tau^2 \int_{t^{n-2}}^{t^n} \int_\Omega u_{ttt}^2 \, dx \, dt .$$

To evaluate $(I)_3$, we rewrite

$$(I)_3 = \int_{I^n} \int_\Omega (\partial_\tau \rho_h^n)_{xx} (u - u^n)_{xx} \, dx \, dt + \int_{I^n} \int_\Omega (\partial_\tau \rho_h^n)_{xx} (u^n - u_{\mathrm{II}}^n)_{xx} \, dx \, dt$$

$$=: R_3^1 + R_3^2 .$$

By Green's formula and boundary conditions (2.2*a*),

$$|R_3^1| = \left| \int_{I^n} \int_\Omega (\partial_\tau \rho_h^n) \, (u - u^n)_{xxxx} \, dx \, dt \right| \leqslant \| \partial_\tau \rho_h^n \| \int_{I^n} |u - u^n|_4 \, dt$$

$$\leqslant \frac{1}{2} \tau \| \partial_\tau \rho_h^n \|^2 + \frac{1}{2} \tau^2 \int_{I^n} \int_\Omega u_{xxxxt}^2 \, dx \, dt ,$$

while by (5.2*a*) and the inverse inequality of FE theory we get

$$|R_3^2| \leqslant \tau | \partial_\tau \rho_h^n |_2 | u^n - u_{\Pi}^n |_2 \leqslant C \tau (h^{-2} \| \partial_\tau \rho_h^n \|) (h^3 | u^n |_5)$$

$$\leqslant \frac{1}{2} \tau \| \partial_\tau \rho_h^n \|^2 + C \tau h^2 | u^n |_5^2 ,$$

so (5.10) follows from above.

LEMMA 5.3 : *We have*

$$|(I)_4| \leqslant \frac{1}{4} \tau | \xi_h^{n-1} |_1^2 + \frac{1}{2} \tau \| \xi_h^{n-1} \|^2 + C \Big(\tau \| \partial_\tau \rho_h^n \|^2 + \tau h^2$$

$$+ \tau | \rho_h^n |_2^2 + \tau | \rho_h^{n-1} |_2^2 + (\tau | \theta_h^{n-1} |_1^2) \, h^2 \Big)$$

$$+ \frac{1}{2} \tau^2 \int_{I^n} \int_\Omega (\theta_t^2 + \theta_{xt}^2) \, dx \, dt + C \tau^2 \int_{I^n} \int_\Omega u_{xxt}^2 \, dx \, dt . \qquad (5.11)$$

Proof: We rewrite $(I)_4$ as

$$(I)_4 = \int_{I^n}\int_\Omega (\partial_\tau \rho_h^n)_x \, E_\varepsilon^0(\theta, \varepsilon)\, dx - \int_{I^n}\int_\Omega (\partial_\tau \rho_h^n)_x \, \partial_\varepsilon E^0(\theta_h^{n-1}, \varepsilon_h^n)\, dx$$

$$= \int_{I^n}\int_\Omega (\partial_\tau \rho_h^n)_x (E_\varepsilon^0(\theta, \varepsilon) - E_\varepsilon^0(\theta_h^{n-1}, \varepsilon))\, dx\, dt$$

$$+ \int_{I^n}\int_\Omega (\partial_\tau \rho_h^n)_x (E_\varepsilon^0(\theta_h^{n-1}, \varepsilon) - E_\varepsilon^0(\theta_h^{n-1}, \varepsilon^n))\, dx\, dt$$

$$+ \int_{I^n}\int_\Omega (\partial_\tau \rho_h^n)_x (E_\varepsilon^0(\theta_h^{n-1}, \varepsilon^n) - \partial_\varepsilon E^0(\theta_h^{n-1}, \varepsilon^n))\, dx\, dt$$

$$+ \int_{I^n}\int_\Omega (\partial_\tau \rho_h^n)_x (\partial_\varepsilon E^0(\theta_h^{n-1}, \varepsilon^n) - \partial_\varepsilon E^0(\theta_h^{n-1}, \varepsilon_h^n))\, dx\, dt$$

$$=: \sum_{i=1}^{4} R_4^i\,. \tag{5.12}$$

We remark that by $\partial_\varepsilon E^0(\cdot\,,\cdot)$ we denote the difference quotient with respect to the second variable, like (3.6), e.g.,

$$\partial_\varepsilon E^0(\theta_h^{n-1}, \varepsilon^n) = \frac{E^0(\theta_h^{n-1}, \varepsilon^n) - E^0(\theta_h^{n-1}, \varepsilon^{n-1})}{\varepsilon^n - \varepsilon^{n-1}}\,.$$

From (2.5*a*) we know $E_\varepsilon^0(\theta, \varepsilon) - E_\varepsilon^0(\theta^{n-1}, \varepsilon) = \varepsilon(\theta - \theta_h^{n-1})$, thus by Green's formula we have

$$|R_4^1| = \left|\int_{I^n}\int_\Omega (\partial_\tau \rho_h^n)\,(E_\varepsilon^0(\theta, \varepsilon) - E_\varepsilon^0(\theta_h^{n-1}, \varepsilon))_x\, dx\, dt\right|$$

$$\leqslant \int_{I^n}\int_\Omega |\partial_\tau \rho_h^n (u_{xx}(\theta - \theta_h^{n-1}) + u_x(\theta - \theta_h^{n-1})_x)|\, dx\, dt$$

$$\leqslant \|\partial_\tau \rho_h^n\| \max_{t\in[0,T]} \|u\|_{2,\infty} \int_{I^n} (\|\theta - \theta_h^{n-1}\| + |\theta - \theta_h^{n-1}|_1)\, dt\,. \tag{5.13}$$

Combining the triangle inequality with (5.2c) gives

$$\int_{I^n} \| \theta - \theta_h^{n-1} \| \, dt \leqslant \tau \sqrt{\tau} \left(\int_{I^n} \int_{\Omega} \theta_t^2 \, dx \, dt \right)^{\frac{1}{2}} + $$

$$+ C\tau h^2 |\theta^{n-1}|_2 + \tau \| \xi_h^{n-1} \| . \quad (5.14)$$

Analogously, we get

$$\int_{I^n} |\theta - \theta_h^{n-1}|_1 \, dt \leqslant \tau \sqrt{\tau} \left(\int_{I^n} \int_{\Omega} \theta_{xt}^2 \, dx \, dt \right)^{\frac{1}{2}} + C\tau h \, |\theta^{n-1}|_2 + \tau |\xi_h^{n-1}|_1 . \quad (5.15)$$

Therefore from (5.13)-(5.15) and Young's inequality, we derive

$$|R_4^1| \leqslant \frac{1}{16} \tau |\xi_h^{n-1}|_1^2 + \frac{1}{2} \tau \| \xi_h^{n-1} \|^2 + C\tau \| \partial_\tau \rho_h^n \|^2$$

$$+ C\tau h^2 + \frac{1}{2} \tau^2 \int_{I^n} \int_{\Omega} (\theta_t^2 + \theta_{xt}^2) \, dx \, dt . \quad (5.16)$$

To treat R_4^2, by (2.5a) and Taylor's formula one can express

$$E_\varepsilon^0(\theta_h^{n-1}, \varepsilon^n) - E_\varepsilon^0(\theta_h^{n-1}, \varepsilon) = \int_0^1 E_{\varepsilon\varepsilon}^0(\theta_h^{n-1}, \bar{\varepsilon}_n) (\varepsilon^n - \varepsilon) \, d\alpha$$

$$= (\varepsilon^n - \varepsilon) \int_0^1 (\theta_h^{n-1} - 3 \bar{\varepsilon}_n^2 + 5 \bar{\varepsilon}_n^4) \, d\alpha \quad (5.17)$$

with $\bar{\varepsilon}_n = \varepsilon + \alpha(\varepsilon^n - \varepsilon)$, then by Green's formula we obtain

$$R_4^2 = - \int_{I^n} \int_{\Omega} \partial_\tau \rho_h^n \left[(\varepsilon^n - \varepsilon)_x \int_0^1 (\theta_h^{n-1} - 3 \bar{\varepsilon}_n^2 + 5 \bar{\varepsilon}_n^4) \, d\alpha \right.$$

$$\left. + (\varepsilon^n - \varepsilon) \int_0^1 (\theta_h^{n-1} - 3 \bar{\varepsilon}_n^2 + 5 \bar{\varepsilon}_n^4)_x \, d\alpha \right] dx \, dt . \quad (5.18)$$

Thus from (4.10*b*) and the *a priori* estimates in Theorem 4.1 it follows that

$$|R_4^2| \leqslant \int_{I^n}\int_\Omega |\partial_\tau \rho_h^n| \left(|\theta_h^{n-1}(u^n - u)_{xx}| + C|(u^n - u)_{xx}| \right.$$

$$\left. + |(\theta_h^{n-1})_x (u^n - u)_x| + C|(u^n - u)_x| \right) dx\, dt$$

$$\leqslant \| \partial_\tau \rho_h^n \| \int_{I^n} \left(\| \theta_h^{n-1} \|_{0,\infty} |u - u^n|_2 + C|u - u^n|_2 \right.$$

$$\left. + |u - u^n|_{1,\infty} |\theta_h^{n-1}|_1 + C|u - u^n|_1 \right) dt$$

$$\leqslant \| \partial_\tau \rho_h^n \| \int_{I^n} |u - u^n|_2 (C + 2\, |\theta_h^{n-1}|_1)\, dt\,. \tag{5.19}$$

By using $|\theta_h^{n-1}|_1 \leqslant |\xi_h^{n-1}|_1 + C$ and Young's inequality, one deduces

$$|R_4^2| \leqslant C\tau \sqrt{\tau}\, \| \partial_\tau \rho_h^n \| \, (1 + |\xi_h^{n-1}|_1) \left(\int_{I^n}\int_\Omega u_{xxt}^2\, dx\, dt \right)^{\frac{1}{2}}$$

$$\leqslant \frac{1}{16} \tau |\xi_h^{n-1}|_1^2 + C\tau^2 \int_{I^n}\int_\Omega u_{xxt}^2\, dx\, dt + \tau \| \partial_\tau \rho_h^n \|^2\,. \tag{5.20}$$

Similar to (5.17), to evaluate the term R_4^3 in (5.12), we first use Taylor's expansion to get (with $(\bar{\varepsilon}_n = \varepsilon^{n-1} + \alpha(\varepsilon^n - \varepsilon^{n-1}))$

$$E_\varepsilon^0(\theta_h^{n-1}, \varepsilon^n) - \partial_\varepsilon E^0(\theta_h^{n-1}, \varepsilon^n) =$$

$$= (\varepsilon^n - \varepsilon^{n-1}) \int_0^1 (1 - \alpha)\, (\theta_h^{n-1} - 3\, \bar{\varepsilon}_n^2 + 5\, \bar{\varepsilon}_n^4)\, d\alpha\,,$$

the same way as in the derivation of (5.18)-(5.20) shows

$$|R_4^3| \leqslant \frac{1}{16} \tau |\xi_h^{n-1}|_1^2 + C\tau^2 \int_{I^n}\int_\Omega u_{xxt}^2\, dx\, dt + \tau \| \partial_\tau \rho_h^n \|^2\,. \tag{5.21}$$

Now we come to the key estimate R_4^4 for $(I)_4$. Again from the definition (2.5a) of $E^0(\theta, \varepsilon)$ we can write

$$\partial_\varepsilon E^0(\theta_h^{n-1}, \varepsilon^n) - \partial_\varepsilon E^0(\theta_h^{n-1}, \varepsilon_h^n) = \frac{1}{2}\theta_h^{n-1}(\varepsilon^n + \varepsilon^{n-1} - \varepsilon_h^n - \varepsilon_h^{n-1})$$

$$+ \Big(Q(\varepsilon^n, \varepsilon^{n-1}) - Q(\varepsilon_h^n, \varepsilon^{n-1})\Big) + \Big(Q(\varepsilon_h^n, \varepsilon^{n-1}) - Q(\varepsilon_h^n, \varepsilon_h^{n-1})\Big)$$

$$=: r_1 + r_2 + r_3 \tag{5.22}$$

with $Q(\,.\,,\,.\,)$ defined by (3.3b). Substituting (5.22) into R_4^4 and using Green's formula, we obtain

$$R_4^4 = -\tau \int_\Omega (\partial_\tau \rho_h^n)\,(r_1 + r_2 + r_3)_x\,dx\,. \tag{5.23}$$

Note that

$$(r_1)_x = \frac{1}{2}(\theta_h^{n-1})_x(\varepsilon^n - \varepsilon_h^n) + \frac{1}{2}(\theta_h^{n-1})_x(\varepsilon^{n-1} - \varepsilon_h^{n-1})$$

$$+ \frac{1}{2}\theta_h^{n-1}(u^n - u_h^n)_{xx} + \frac{1}{2}\theta_h^{n-1}(u^{n-1} - u_h^{n-1})_{xx}\,. \tag{5.24}$$

Using (5.2b), (5.3a) and the fact $|u^n - u_h^n|_{1,\infty} \leqslant |u^n - u_h^n|_2$ one comes to

$$\left|\tau\int_\Omega \partial_\tau\rho_h^n(\theta_h^{n-1})_x(\varepsilon^n - \varepsilon_h^n)\,dx\right| \leqslant \tau\|\partial_\tau\rho_h^n\|\;|\theta_h^{n-1}|_1|u^n - u_h^n|_2$$

$$\leqslant \tau\|\partial_\tau\rho_h^n\|\;|\theta_h^{n-1}|_1(|u^n - u_\Pi^n|_2 + |\rho_h^n|_2)$$

$$\leqslant C\tau h\|\partial_\tau\rho_h^n\|\;|\theta_h^{n-1}|_1|u^n|_3 + \tau\|\partial_\tau\rho_h^n\|\;|\xi_h^{n-1}|_1|\rho_h^n|_2$$

$$+ \tau\|\partial_\tau\rho_h^n\|\;|\theta_I^{n-1}|_1|\rho_h^n|_2$$

$$\leqslant \tau\|\partial_\tau\rho_h^n\|^2 + C\tau h^2|u^n|_3^2|\theta_h^{n-1}|_1^2 + C\tau|\rho_h^n|_2^2$$

$$+ 16\,\tau|\rho_h^n|_2^2\|\partial_\tau\rho_h^n\|^2 + \frac{1}{64}\tau|\xi_h^{n-1}|_1^2$$

$$\leqslant \frac{1}{64}\tau|\xi_h^{n-1}|_1^2 + \tau\|\partial_\tau\rho_h^n\|^2 + C\tau|\rho_h^n|_2^2 + C(\tau|\theta_h^{n-1}|_1^2)\,h^2\,. \tag{5.25}$$

The same as deriving (5.25), we get

$$\left|\tau\int_{\Omega}\partial_\tau\rho_h^n(\theta_h^{n-1})_x(\varepsilon^{n-1}-\varepsilon_h^{n-1})\right| \leqslant \frac{1}{64}\tau|\xi_h^{n-1}|_1^2+\tau\|\partial_\tau\rho_h^n\|^2+$$

$$+C\tau|\rho_h^{n-1}|_2^2+C(\tau|\theta_h^{n-1}|_1^2)\,h^2\,. \quad (5.26)$$

Furthermore, from (5.3), (4.2*c*), (5.2*b*) and the inverse inequality $|\theta_h^{n-1}|_1 \leqslant Ch^{-1}\|\theta_h^{n-1}\|$ it follows that

$$\left|\tau\int_{\Omega}\theta_h^{n-1}(u^n-u_h^n)_{xx}\,\partial_\tau\rho_h^n\,dx\right| \leqslant \tau\|\theta_h^{n-1}\|_{0,\infty}|u^n-u_h^n|_2\|\partial_\tau\rho_h^n\|$$

$$\leqslant \tau(\|\theta_h^{n-1}\|+|\theta_h^{n-1}|_1)\,|u^n-u_\Pi^n|_2\|\partial_\tau\rho_h^n\|$$

$$+\tau(\|\theta_h^{n-1}\|+|\theta_h^{n-1}|_1)\,|\rho_h^n|_2\|\partial_\tau\rho_h^n\|$$

$$\leqslant C\tau h\|\partial_\tau\rho_h^n\|\;|u^n|_4+\tau|\rho_h^n|_2\|\partial_\tau\rho_h^n\|\;(C+|\xi_h^{n-1}|_1)$$

$$\leqslant \frac{1}{64}\tau|\xi_h^{n-1}|_1^2+C(\tau\|\partial_\tau\rho_h^n\|^2+\tau h^2+\tau|\rho_h^n|_2^2)\,. \quad (5.27)$$

The same derivation as to (5.27) gives

$$\left|\tau\int_{\Omega}\theta_h^{n-1}(u^{n-1}-u_h^{n-1})_{xx}\,\partial_\tau\rho_h^n\,dx\right|$$

$$\leqslant \frac{1}{64}\tau|\xi_h^{n-1}|_1^2+C(\tau\|\partial_\tau\rho_h^n\|^2+\tau h^2+\tau|\rho_h^{n-1}|_2^2)\,. \quad (5.28)$$

Thus the sum of (5.25)-(5.28) implies

$$\left|\tau\int_{\Omega}\partial_\tau\rho_h^n(r_1)_x\,dx\right| \leqslant \frac{1}{16}\tau|\xi_h^{n-1}|_1^2+$$

$$+C(\tau\|\partial_\tau\rho_h^n\|^2+\tau|\rho_h^n|_2^2+\tau|\rho_h^{n-1}|_2^2+\tau h^2+(\tau|\theta_h^{n-1}|_1^2)\,h^2)\,. \quad (5.29)$$

To analyse $(r_2)_x$, by the definition of $Q(\,.\,,\,.\,)$ and Taylor's formula one gets $(\bar{\varepsilon}_n = \varepsilon_h^n + \alpha(\varepsilon^n - \varepsilon_h^n))$

$$(r_2)_x = \left[(\varepsilon^n - \varepsilon_h^n) \int_0^1 Q_{q_1}(\bar{\varepsilon}_n, \varepsilon^{n-1})\, d\alpha \right]_x$$

$$= (\varepsilon^n - \varepsilon_h^n)_x \int_0^1 Q_{q_1}(\bar{\varepsilon}_n, \varepsilon^{n-1})\, d\alpha$$

$$+ (\varepsilon^n - \varepsilon_h^n) \int_0^1 (Q_{q_1 q_1}(\bar{\varepsilon}_n, \varepsilon^{n-1})(\bar{\varepsilon}_n)_x + Q_{q_1 q_2}(\bar{\varepsilon}_n, \varepsilon^{n-1})\, \varepsilon_x^{n-1})\, d\alpha\,,$$

then the *a priori* estimates for u_h^n and u lead to

$$\|(r_2)_2\| \leqslant C(|u^n - u_h^n|_2 + |u^n - u_h^n|_1) \leqslant C|u^n - u_h^n|_2$$

$$\leqslant C(|u^n - u_\Pi^n|_2 + |\rho_h^n|)\,, \tag{5.30}$$

so (5.2*b*) and Schwarz's inequality imply

$$\left| \tau \int_\Omega \partial_\tau \rho_h^n (r_2)_x\, dx \right| \leqslant \frac{1}{2} \tau |\rho_h^n|_2^2 + C\tau h^2 + C\tau \| \partial_\tau \rho_h^n \|^2\,. \tag{5.31}$$

The same arguments show that

$$\left| \tau \int_\Omega \partial_\tau \rho_h^n (r_3)_x\, dx \right| \leqslant \frac{1}{2} \tau |\rho_h^{n-1}|_2^2 + C\tau h^2 + C\tau \| \partial_\tau \rho_h^n \|^2\,. \tag{5.32}$$

Therefore it follows from (5.29), (5.31) and (5.32) that

$$|R_4^4| \leqslant \frac{1}{16} \tau |\xi_h^{n-1}|_1^2 + C(\tau \| \partial_\tau \rho_h^n \|^2 + \tau |\rho_h^n|_2^2 + \tau |\rho_h^{n-1}|_2^2 +$$

$$+ \tau h^2 + (\tau |\theta_h^{n-1}|_1^2)\, h^2)\,. \tag{5.33}$$

Now Lemma 5.3 is a consequence of (5.16), (5.20), (5.21) and (5.33).

In the remainder of the section we turn to the estimation of all three terms $(\Pi)_1$, $(\Pi)_2$ and $(\Pi)_3$ in (5.6) and (5.7).

LEMMA 5.4 : *We have*

$$|(II)_1| \leqslant \frac{1}{2}\tau\|\xi_h^n\|^2 + Ch^4\int_{I^n}\int_\Omega \theta_{xxt}^2\,dx\,dt\,, \tag{5.34}$$

$$|(II)_2| \leqslant \frac{1}{16}\tau\|\xi_h^n\|_1^2 + 8\,\tau^2\int_{I^n}\int_\Omega \theta_{xt}^2\,dx\,dt + C\tau h^2\,. \tag{5.35}$$

Proof: It follows immediately by (5.2c) and Young's inequality that

$$|(II)_1| \leqslant \tau\|\xi_h^n\|\;\|\partial_\tau(\theta^n - \theta_I^n)\| \leqslant C\sqrt{\tau}\;\|\xi_h^n\|\;h^2\left(\int_{I^n}\int_\Omega \theta_{xxt}^2\,dx\,dt\right)^{\frac{1}{2}}$$

$$\leqslant \frac{1}{2}\tau\|\xi_h^n\|^2 + Ch^4\int_{I^n}\int_\Omega \theta_{xxt}^2\,dx\,dt\,,$$

and

$$|(II)_2| \leqslant |\xi_h^n|_1\int_{I^n}(|\theta - \theta^n|_1 + |\theta^n - \theta_I^n|_1)\,dt$$

$$\leqslant \frac{1}{16}\tau|\xi_h^n|_1^2 + 8\,\tau^2\int_{I^n}\int_\Omega \theta_{xt}^2\,dx\,dt + C\tau h^2\,,$$

that completes the proof Lemma 5.4.

The key estimate for the right-hand side of (5.7) is the term $(II)_3$. To estimate $(II)_3$, we first rewrite it as follows

$$\begin{aligned}(II)_3 &= -\int_{I^n}\int_\Omega \xi_h^n(\theta\varepsilon\varepsilon_t - \theta_h^{n-1}\,\varepsilon_h^{n-\frac{1}{2}}\,\partial_\tau\,\varepsilon_h^n)\,dx\,dt\\
&= -\int_{I^n}\int_\Omega \xi_h^n\,\varepsilon\varepsilon_t(\theta - \theta_h^{n-1})\,dx\,dt\\
&\quad - \int_{I^n}\int_\Omega \xi_h^n\,\theta_h^{n-1}(\varepsilon\varepsilon_t - \varepsilon^{n-\frac{1}{2}}\,\partial_\tau\,\varepsilon^n)\,dx\,dt\\
&\quad - \tau\int_\Omega \xi_h^n\,\theta_h^{n-1}\,\partial_\tau\,\varepsilon^n(\varepsilon^{n-\frac{1}{2}} - \varepsilon_h^{n-\frac{1}{2}})\,dx\\
&\quad - \tau\int_\Omega \xi_h^n\,\theta_h^{n-1}\,\varepsilon_h^{n-\frac{1}{2}}(\partial_\tau\,\varepsilon^n - \partial_\tau\,\varepsilon_h^n)\,dx\\
&=: \beta_1 + \beta_2 + \beta_3 + \beta_4\,.\end{aligned} \tag{5.36}$$

Now we estimate all four terms β_i, one by one.

LEMMA 5.5 : *We have*

$$|\beta_1| \leqslant C\tau \| \xi_h^n \|^2 + \frac{1}{2} \tau \| \xi_h^{n-1} \|^2 + C\tau h^4 + \frac{1}{2} \tau^2 \int_{I^n} \int_{\Omega} \theta_t^2 \, dx \, dt . \tag{5.37}$$

$$|\beta_2| \leqslant \frac{1}{16} \tau | \xi_h^n |_1^2 + 2 \tau \| \xi_h^n \|^2 + C\tau^2 \int_{I^n} \int_{\Omega} (u_{xxt}^2 + u_{xtt}^2) \, dx \, dt . \tag{5.38}$$

$$|\beta_3| \leqslant \frac{1}{16} \tau | \xi_h^n |_1^2 + C(\tau \| \xi_h^n \|^2 + \tau | \rho_h^n |_2^2 + \tau | \rho_h^{n-1} |_2^2 + \tau h^2) . \tag{5.39}$$

$$|\beta_4| \leqslant \frac{1}{8} \tau | \xi_h^n |_1^2 + \frac{1}{8} \tau | \xi_h^{n-1} |_1^2 + C(\tau \| \partial_\tau \rho_h^n \|^2 + \tau \| \xi_h^n \|^2 + + \tau h^4 + (\tau | \theta_h^{n-1} |_1^2) \, h^4) . \tag{5.40}$$

Proof: It is easy to see from (5.36) and (5.1) that

$$|\beta_1| \leqslant C \| \xi_h^n \| \int_{I^n} \| \theta - \theta_h^{n-1} \| \, dt ,$$

so (5.37) follows from (5.14) and Schwarz's inequality.

To analyse β_2, we rewrite it into two parts

$$\beta_2 = - \int_{I^n} \int_{\Omega} \xi_h^n \, \theta_h^{n-1} \, \varepsilon_t (\varepsilon - \varepsilon^{n-\frac{1}{2}}) \, dx \, dt - \int_{I^n} \int_{\Omega} \xi_h^n \, \theta_h^{n-1} \, \varepsilon^{n-\frac{1}{2}} (\varepsilon_t - \partial_\tau \varepsilon^n) \, dx \, dt .$$

The first part of β_2 is easily treated by (4.2*c*) and the fact that $|u - u^{n-1/2}|_{1,\infty} \leqslant |u - u^{n-1/2}|_2$,

$$\begin{aligned} \left| \int_{I^n} \int_{\Omega} \xi_h^n \, \theta_h^{n-1} \, \varepsilon_t (\varepsilon - \varepsilon^{n-\frac{1}{2}}) \, dx \, dt \right| &\leqslant C \| \xi_h^n \| \int_{I^n} | u - u^{n-\frac{1}{2}} |_2 \, dt \\ &\leqslant C \| \xi_h^n \| \, \tau \sqrt{\tau} \left(\int_{I^n} \int_{\Omega} u_{xxt}^2 \, dx \, dt \right)^{\frac{1}{2}} \\ &\leqslant \frac{1}{2} \tau \| \xi_h^n \|^2 + C\tau^2 \int_{I^n} \int_{\Omega} u_{xxt}^2 \, dx \, dt . \end{aligned}$$

The estimate of the second part of β_2 follows from (5.3) and Taylor's expansion :

$$\left|\int_{I^n}\int_\Omega \varepsilon_h^n\, \theta_h^{n-1}\, \varepsilon^{n-\frac{1}{2}}(\varepsilon_t - \partial_\tau \varepsilon^n)\, dx\, dt\right|$$

$$\leqslant C\|\xi_h^n\|_{0,\infty}\int_{I^n}(\|\varepsilon_t - \varepsilon_t^{n-1}\| + \|\varepsilon_t^{n-1} - \partial_\tau \varepsilon^n\|)\, dt$$

$$\leqslant C(\|\xi_h^n\| + |\xi_h^n|_1)\,\tau\sqrt{\tau}\left(\int_{I^n}\int_\Omega u_{xtt}^2\, dx\, dt\right)^{\frac{1}{2}}$$

$$\leqslant \frac{1}{16}\tau|\xi_h^n|_1^2 + \tau\|\xi_h^n\|^2 + C\tau^2\int_{I^n}\int_\Omega u_{xtt}^2\, dx\, dt\,,$$

we have proved (5.38).
Now we consider β_3. (4.2c) and (5.3) yield

$$|\beta_3| \leqslant \tau\|\xi_h^n\|_{0,\infty}\|\theta_h^{n-1}\|\ \|\partial_\tau \varepsilon^n\|_{0,\infty}(\|\varepsilon^{n-\frac{1}{2}} - \varepsilon_\Pi^{n-\frac{1}{2}}\| + \|\varepsilon_\Pi^{n-\frac{1}{2}} - \varepsilon_h^{n-\frac{1}{2}}\|)$$

$$\leqslant C\tau(\|\xi_h^n\| + |\xi_h^n|_1)\,|\partial_\tau \varepsilon^n|_1(h^2|u^{n-\frac{1}{2}}|_3 + |\rho_h^{n-\frac{1}{2}}|)$$

$$\leqslant C\tau(\|\xi_h^n\| + |\xi_h^n|_1)\,(h^2 + |\rho^{n-\frac{1}{2}}|_2)$$

$$\leqslant \frac{1}{16}\tau|\xi_h^n|_1^2 + C(\tau\|\xi_h^n\|^2 + \tau|\rho_h^n|_2^2 + \tau|\rho_h^{n-1}|_2^2 + \tau h^2)\,. \tag{5.41}$$

Finally we estimate β_4 in (5.36). By Green's formula we rewrite β_4 as

$$\beta_4 = -\tau\int_\Omega \xi_h^n\, \theta_h^{n-1}\, \varepsilon_h^{n-\frac{1}{2}}(\partial_\tau u^n - \partial_\tau u_h^n)_x\, dx$$

$$= \tau\int_\Omega \partial_\tau(u^n - u_\Pi^n)\,(\xi_h^n\, \theta_h^{n-1}\, \varepsilon_h^{n-\frac{1}{2}})_x\, dx$$

$$- \tau\int_\Omega \partial_\tau \rho_h^n(\xi_h^n\, \theta_h^{n-1}\, \varepsilon_h^{n-\frac{1}{2}})_x\, dx$$

$$=: \beta_4^1 + \beta_4^2\,. \tag{5.42}$$

By (5.2a) and (4.2b) we derive

$$|\beta_4^1| \leqslant \tau \| \partial_\tau (u^n - u_\Pi^n) \| \ \| (\xi_h^n \theta_h^{n-1} \varepsilon^{n-\frac{1}{2}})_x \|$$

$$\leqslant C\tau h^2 (\| \theta_h^{n-1} \|_{0,\infty} |\xi_h^n|_1 |u_h^{n-\frac{1}{2}}|_{1,\infty} + \| \xi_h^n \|_{0,\infty} |\theta_h^{n-1}|_1 |u_h^{n-\frac{1}{2}}|_{1,\infty}$$

$$+ \| \xi_h^n \|_{0,\infty} \| \theta_h^{n-1} \|_{0,\infty} |u_h^{n-\frac{1}{2}}|_2)$$

$$\leqslant C\tau h^2 (\| \theta_h^{n-1} \|_{0,\infty} |\xi_h^n|_1 + \| \xi_h^n \|_{0,\infty} |\theta_h^{n-1}|_1 + \| \xi_h^n \|_{0,\infty} |\theta_h^{n-1}|_{0,\infty}) .$$

From (5.3c) and (4.2c) one gets

$$C\tau h^2 \| \theta_h^{n-1} \|_{0,\infty} |\xi_h^n|_1 \leqslant C\tau h^2 (1 + |\theta_h^{n-1}|_1) \, |\xi_h^n|_1$$

$$\leqslant \frac{1}{48} \tau |\xi_h^n|_1^2 + C(\tau h^4 + (\tau |\theta_h^{n-1}|_1^2) \, h^4)$$

and

$$\tau h^2 \| \xi_h^n \|_{0,\infty} |\theta_h^{n-1}|_1 \leqslant C\tau h^2 |\theta_1^{n-1}|_1 \| \xi_h^n \|_{0,\infty} + C\tau h^2 |\xi_h^{n-1}|_1 \| \xi_h^n \|_{0,\infty}$$

$$C \leqslant \tau h^2 (\| \xi_h^n \| + |\xi_h^n|_1) + \tau h^2 |\xi_h^{n-1}|_1 (1 + |\xi_h^n|_1^{\frac{1}{2}})$$

$$\leqslant C(\tau \| \xi_h^n \|^2 + \tau h^4) + \frac{1}{48} \tau |\xi_h^n|_1^2 + \frac{1}{48} \tau |\xi_h^{n-1}|_1^2$$

where for the term $\tau h^2 |\xi_h^{n-1}|_1 |\xi_h^n|_1^{1/2}$ we have used Young's inequality twice. And analogously, we deduce

$$C\tau h^2 \| \xi_h^n \|_{0,\infty} \| \theta_h^{n-1} \|_{0,\infty} \leqslant C\tau h^2 \| \xi_h^n \|_{0,\infty} (\| \theta_h^{n-1} \| + |\theta_h^{n-1}|_1)$$

$$\leqslant C\tau h^2 (\| \xi_h^n \| + |\xi_h^n|_1) + C\tau h^2 \| \xi_h^n \|_{0,\infty} |\xi_h^{n-1}|_1$$

$$\leqslant C(\tau \| \xi_h^n \|^2 + \tau h^4) + \frac{1}{48} \tau |\xi_h^n|_1^2 + \frac{1}{48} \tau |\xi_h^{n-1}|_1^2 ,$$

$$|\beta_4^1| \leqslant \frac{1}{16} \tau |\xi_h^n|_1^2 + \frac{1}{16} \tau |\xi_h^{n-1}|_1^2 + C(\tau \| \xi_h^n \|^2 + \tau h^4 + (\tau |\theta_h^{n-1}|_1^2) \, h^4) . \quad (5.43)$$

As above we obtain

$$|\beta_4^2| \leqslant \tau \| \partial_\tau \rho_h^n \| \; \| (\xi_h^n \theta_h^{n-1} \varepsilon_h^{n-\frac{1}{2}})_x \|$$

$$\leqslant C\tau \| \partial_\tau \rho_h^n \| \times$$

$$\times (\| \theta_h^{n-1} \|_{0,\infty} | \xi_h^n |_1 + \| \xi_h^n \|_{0,\infty} | \theta_h^{n-1} |_1 + \| \xi_h^n \|_{0,\infty} \| \theta_h^{n-1} \|_{0,\infty}) . \quad (5.44)$$

From (5.3c) one comes to

$$C\tau \| \partial_\tau \rho_h^n \| \; \| \theta_h^{n-1} \|_{0,\infty} | \xi_h^n |_1 \leqslant C\tau \| \partial_\tau \rho_h^n \| \; | \xi_h^n |_1 (1 + | \theta_h^{n-1} |_1^{1/2})$$

$$\leqslant C\tau \| \partial_\tau \rho_h^n \| \; | \xi_h^n |_1 (1 + | \xi_h^{n-1} |_1^{\frac{1}{2}})$$

$$\leqslant \frac{1}{48} \tau | \xi_h^n |_1^2 + C\tau \| \partial_\tau \rho_h^n \|^2 + \frac{1}{48} \tau | \xi_h^{n-1} |_1^2$$

where for the term $\tau \| \partial_\tau \rho_h^n \| \; | \xi_h^n |_1 | \xi_h^{n-1} |_1^{1/2}$ we have used Young's inequality twice and (5.3). Furthermore

$$C\tau \| \partial_\tau \rho_h^n \| \; | \theta_h^{n-1} |_1 \| \xi_h^n \|_{0,\infty}$$

$$\leqslant C\tau \| \partial_\tau \rho_h^n \| \; \| \xi_h^n \|_{0,\infty} + C\tau \| \partial_\tau \rho_h^n \| \; | \xi_h^{n-1} |_1 \| \xi_h^n \|_{0,\infty}$$

$$\leqslant C\tau \| \partial_\tau \rho_h^n \| (\| \xi_h^n \| + | \xi_h^n |_1) + C\tau \| \partial_\tau \rho_h^n \| \; | \xi_h^{n-1} |_1 (1 + | \xi_h^n |_1^{\frac{1}{2}})$$

$$\leqslant \frac{1}{48} \tau | \xi_h^n |_1^2 + \frac{1}{48} \tau | \xi_h^{n-1} |_1^2 + C(\tau \| \xi_h^n \|^2 + \tau \| \partial_\tau \rho_h^n \|^2) ,$$

and

$$C\tau \| \partial_\tau \rho_h^n \| \; \| \xi_h^n \|_{0,\infty} \| \theta_h^{n-1} \|_{0,\infty} \leqslant \tau \| \partial_\tau \rho_h^n \| \; \| \xi_h^n \|_{0,\infty} (C + | \xi_h^{n-1} |_{0,\infty})$$

$$\leqslant C\tau \| \partial_\tau \rho_h^n \| (\| \xi_h^n \| + | \xi_h^n |_1) + C\tau \| \partial_\tau \rho_h^n \| \; \| \xi_h^n \|_{0,\infty} (1 + | \xi_h^{n-1} |_1^{\frac{1}{2}})$$

$$\leqslant C\tau \| \partial_\tau \rho_h^n \| (\| \xi_h^n \| + | \xi_h^n |_1) + C\tau \| \partial_\tau \rho_h^n \| (\| \xi_h^n \| + | \xi_h^n |_1^{\frac{1}{2}}) \; | \xi_h^{n-1} |_1^{\frac{1}{2}}$$

$$\leqslant \frac{1}{48} \tau | \xi_h^n |_1^2 + \frac{1}{48} \tau | \xi_h^{n-1} |_1^2 + C(\tau \| \partial_\tau \rho_h^n \|^2 + \tau \| \xi_h^n \|^2)$$

where we have utilized (5.3) and

$$\tau \| \partial_\tau \rho_h^n \| \; |\xi_h^n|_1^{\frac{1}{2}} |\xi_h^{n-1}|_1^{\frac{1}{2}} \leqslant \frac{1}{2} \tau |\xi_h^n|_1 \| \partial_\tau \rho_h^n \|^2 + \frac{1}{2} \tau |\xi_h^{n-1}|_1 \| \partial_\tau \rho_h^n \|^2$$

$$\leqslant \frac{1}{96} \tau |\xi_h^n|_1^2 + \frac{1}{96} \tau |\xi_h^{n-1}|_1^2 + C\tau \| \partial_\tau \rho_h^n \|^2 .$$

From (5.44) and above we obtain

$$|\beta_4^2| \leqslant \frac{1}{16} \tau |\xi_h^n|_1^2 + \frac{1}{16} \tau |\xi_h^{n-1}|_1^2 + C(\tau \| \partial_\tau \rho_h^n \|^2 + \tau \| \xi_h^n \|^2) . \qquad (5.45)$$

Now (5.40) follows from (5.43) and (5.45).

So far we have finished all the estimations of $(I)_i$, $i = 1, 2, 3, 4$ and $(II)_i$, $i = 1, 2, 3$ in (5.5) and (5.7). We can now prove Theorem 5.1.

The proof of Theorem 5.1 : Summation from $n = 1$ to $k \leqslant M$ in (5.5) and using Lemma 5.2, Lemma 5.3, (4.2*c*) implies

$$\frac{1}{2} \| \partial_\tau \rho_h^k \|^2 + \frac{1}{2} |\rho_h^k|_2^2 \leqslant Ch^2 + C(h^2 + \tau^2) \, B_1(u, \theta) + \frac{1}{4} \sum_{n=1}^{k} |\xi_h^n|_1^2$$

$$+ C\left(\tau \sum_{n=1}^{k} \| \partial_\tau \rho_h^n \|^2 + \sum_{n=1}^{k} \tau |\rho_h^n|_2^2 \right) + \frac{1}{2} \sum_{n=1}^{k} \tau \| \xi_h^n \|^2 \qquad (5.46)$$

with

$$B_1(u, \theta) = \int_0^T \int_\Omega (u_{xxt}^2 + \theta_t^2 + \theta_{xt}^2 + u_{xxxxt}^2 + u_{tt}^2 + u_{ttt}^2 + u_{ttxx}^2) \, dx \, dt$$

$$+ \int_{-\tau}^0 \int_\Omega (u_{ttt}^2 + u_{ttxx}^2) \, dx \, dt + \sup_{t \in (-\tau, 0)} \| u_{ttx} \|^2 .$$

Here in (5.46) we have used

$$\rho_h^0 = 0 \,, \quad \xi_h^0 = 0 \,, \quad \| \partial_\tau \rho_h^0 \| \leqslant C\tau \sup_{t \in (-\tau, 0)} \| u_{ttx} \|$$

where the last inequality is just a consequence of Taylor's formula and FE interpolation results for the extension of u on $(-\tau, 0)$, see (2.9).

By taking the sum of (5.7) from $n = 1$ to $k \leqslant M$ and using Lemma 5.4 and Lemma 5.5 we obtain

$$\frac{1}{2}\|\xi_h^k\|^2 + \sum_{n=1}^{k} \tau |\xi_h^n|_1^2 \leqslant C\left(\sum_{n=1}^{k} \tau \|\xi_h^n\|^2 + \sum_{n=1}^{k} \tau |\rho_h^n|_2^2 + \sum_{n=1}^{k} \tau \|\partial_\tau \rho_h^n\|^2\right)$$

$$+ Ch^2 + C(\tau^2 + h^4) B_2(u, \theta) + \frac{7}{16} \sum_{n=1}^{k} \tau |\xi_h^n|_1^2 \tag{5.47}$$

with

$$B_2(u, \theta) = \int_0^T \int_\Omega (\theta_{xxt}^2 + \theta_{xt}^2 + \theta_t^2 + u_{xxt}^2 + u_{xtt}^2)\, dx\, dt .$$

Now noticing the assumptions (2.7*a*, *b*, *c*) and (2.9), from the sum of (5.46) and (5.47) and the discrete Gronwall's inequality it follows that

$$\max_{1 \leqslant n \leqslant M} (\|\partial_\tau \rho_h^n\|^2 + |\rho_h^n|_2^2 + \|\xi_h^n\|^2) + \sum_{n=1}^{M} \tau |\xi_h^n|_1^2 \leqslant C(h^2 + \tau^2) .$$

Combining this with the triangle inequality and (5.2*a*, *b*, *c*) and the expression

$$u_t^n - \partial_\tau u_h^n = (u_t^n - \partial_\tau u^n) + (\partial_\tau u^n - \partial_\tau u_\Pi^n) - \partial_\tau \rho_h^n$$

implies the conclusions of Theorem 5.1.

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