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MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
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ADAPTIVE COUPLING OF BOUNDARY ELEMENTS AND FINITE ELEMENTS (*)

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Abstract. — *In this note we present an h-adaptive procedure for the symmetric coupling of boundary element and finite elements methods for two-dimensional linear and nonlinear interface problems. An a posteriori error estimate is derived which guarantees a given bound of the error in the energy norm (up to a multiplicative constant). Following the approach of Eriksson & Johnson this leads to a residual based adaptive procedure within the Galerkin discretization. Numerical examples confirm that our procedure gives good meshes leading to efficient numerical procedures.*

Subject Classifications : AMS(MOS) 65 N 35, 65 R 20, 65 D 07, 45 L 10.

Key words : adaptive finite element methods, adaptive boundary element methods, *a posteriori* estimates, symmetric coupling.

Résumé. — *Dans cette note, nous présentons un procédé adaptatif pour coupler des méthodes d'éléments finis et d'éléments frontière pour des problèmes d'interface linéaires ou non linéaires. Une estimation d'erreur* a posteriori *est obtenue. Des exemples numériques confirment que ce procédé donne de bons maillages, menant à des méthodes numériques efficaces.*

1. INTRODUCTION

Since the first mathematical justifications of the « mariage à la mode » in the later seventies by Brezzi, Johnson, Nedelec, Bielak, MacCamy and others further progress in the analysis of the coupling of finite and boundary elements concerns Lipschitz boundaries, systems of equations, and nonlinear problems (approximated by finite elements) cf. e.g. [7, 10, 11, 17, 18, 19, 29] and the literature quoted therein.

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In order to get a good convergence behavior not only asymptotically but also when we are dealing just with a few degrees of freedom, we need a good triangulation in particular when singularities appear. If the nature and the position of a singularity are known *a priori,* the mesh refinement can reflect on this. Otherwise one requires the information we may achieve from an analysis of the discrete solution and the given data. Whereas the main topics in the adaptive feedback steering of mesh refinements, usually based on the residuals, are mathematically understood for the finite element methods (we refer only to the pioneering works [1, 13] and to [20, 28] for nonlinear problems), comparably little is known for the boundary element method (cf. e.g. [4, 23, 24, 30, 31]).

In this paper an adaptive h-version of the Galerkin discretization for the symmetric coupling of the finite element method and the boundary element method is presented for linear and nonlinear interface problems. It is based on an *a posteriori* error estimate which gives a computable error estimate up to a multiplicative constant. Then, following the approach of Eriksson and Johnson elaborated for the finite element method we present an adaptive feedback algorithm for the mesh refinement of the coupling procedure.

The paper is organized as follows. For convenience of the reader we treat the interface problem and its rewritten form, problem (P), in § 2. Here, we are able to neglect the technical assumption of a Dirichlet boundary stated in the literature [11, 18]. Then, its discretization, the problem (P_h), is considered in § 3 where we conclude quasi-optimal convergence for the displacements in the H^1-norm approximated by (continuous, piecewise linear) finite elements in the domain Ω and for the tractions in the $H^{-1/2}$-norm approximated by (discontinuous piecewise constant) boundary elements on the interface Γ-these norms may be considered as natural (« energy ») norms. Then, in § 4, we state the precise assumptions and then prove an *a posteriori* error estimate. In § 5 we present and discuss the adaptive algorithm which is illustrated numerically in § 6.

We use the following notations. $H^s(\Omega)$ denotes the usual Sobolev spaces [21] with the trace spaces $H^{s-1/2}(\Gamma)$ $(s \in \mathbb{R})$ for a bounded Lipschitz domain Ω with boundary Γ. $\|\cdot\|_{H^k(\omega)}$ and $|\cdot|_{H^k(\omega)}$ denote the norm and semi-norm in $H^k(\omega)$ for $\omega \subseteq \Omega$ and an integer k.

2. THE INTERFACE PROBLEM

This section presents the interface problem and rewrites it with boundary integral operators as an equivalent problem (P) which will be treated numerically in the sequel.

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain in the plane with boundary Γ. The possibly nonlinear partial differential equation considered in Ω, the interior part of the problem, is described by the operator A defined by

$$A: L^2(\Omega;\mathbb{R}^2) \to L^2(\Omega;\mathbb{R}^2),$$

$$\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix} \mapsto \begin{pmatrix} a_{11}\left(\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix}\right)\cdot\varepsilon_1 + a_{12}\left(\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix}\right)\cdot\varepsilon_2 \\ a_{21}\left(\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix}\right)\cdot\varepsilon_1 + a_{22}\left(\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix}\right)\cdot\varepsilon_2 \end{pmatrix}.$$

The coefficients $a_{ij} = a_{ji} \in L^\infty(\Omega)$ may or may not depend on the argument $\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix} \in L^2(\Omega_i\mathbb{R}^2)$ and may vary in Ω provided that A is uniformly bounded and monotone, i.e. there exists positive constants α_0 and α_1 with

$$\alpha_0 \cdot |\delta - \varepsilon|^2 \leqslant$$

$$\leqslant (\varepsilon - \delta)^T \cdot \begin{pmatrix} a_{11}(x,\delta)\,\delta_1 + a_{12}(x,\delta)\,\delta_2 - a_{11}(x,\varepsilon)\,\varepsilon_1 - a_{12}(x,\varepsilon)\,\varepsilon_2 \\ a_{21}(x,\delta)\,\delta_1 + a_{12}(x,\delta)\,\delta_2 - a_{21}(x,\varepsilon)\,\varepsilon_1 - a_{12}(x,\varepsilon)\,\varepsilon_2 \end{pmatrix}$$

$$\leqslant \alpha_1 \cdot |\delta - \varepsilon|^2$$

for all δ, $\varepsilon \in \mathbb{R}^2$ and for a.e. $x \in \Omega$.

Example 1 : As a typical example consider $A = p \cdot I$ where I is the two-dimensional unit matrix. In the linear case $p \in L^\infty(\Omega)$ is a scalar function with $p_0 \leqslant p(x) \leqslant p_1$ for almost every $x \in \Omega$ and some global constants $p_0, p_1 > 0$:

$$(A\varepsilon)(x) = p(x)\cdot\varepsilon \quad \text{for a.e.} \quad x \in \Omega\,.$$

In particular $p = 1$ leads to the Laplacian equation, cf. (1). In the nonlinear case we consider p as a function of the argument $t := |\varepsilon|$ and may take e.g. $p(t) = 2 + \frac{1}{1+t}$ which gives

$$(A\varepsilon)(x) = p(|\varepsilon|)\cdot\varepsilon \quad \text{for a.e.} \quad x \in \Omega$$

and $\quad 2 \leqslant p(|\varepsilon|) \leqslant 3.$

For a given right hand side $f \in L^2(\Omega)$, we consider the (possibly nonlinear) partial differential equation

(1) $$-\operatorname{div}(A \operatorname{grad} u) = f \quad \text{in} \quad \Omega\,.$$

In the complement $\Omega^c := \mathbb{R}^2 \setminus \overline{\Omega}$ we consider

$$(2) \qquad -\Delta v = 0 \quad \text{in} \quad \Omega^c$$

with the radiation condition

$$(3) \qquad v(x) = \frac{b}{2\pi} \log |x| + o(1) \quad \text{as} \quad |x| \to \infty ,$$

where $b \in \mathbb{R}$ is a constant (depending on v). Both problems are coupled on the interface $\Gamma = \overline{\Omega} \cap \overline{\Omega^c}$ where we allow prescribed jumps, i.e. given $u_0 \in H^{1/2}(\Gamma)$ and $t_0 \in H^{-1/2}(\Gamma)$ we demand

$$(4) \qquad u = v + u_0, \quad (A \operatorname{grad} u) \cdot n = \frac{\partial v}{\partial n} + t_0 \quad \text{on } \Gamma$$

where $n = (n_1, n_2)$ is the unit outward normal to Γ pointing from Ω into Ω^c. We remark that $(A \operatorname{grad} u) \cdot n|_\Gamma$ and $\left|\frac{\partial v}{\partial n}\right|_\Gamma$ are defined in $H^{-1/2}(\Gamma)$ *via* Green's formula [9, Lemma 3.1].

Then, the *interface problem* (IP) of this note reads as follows where any derivative has to be interpreted in the distributional sense.

DEFINITION 1 : **(Problem (IP))** *Given* $(f, u_0, t_0) \in L^2(\Omega) \times H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ *find* $(u, v) \in H^1(\Omega) \times H^1_{\text{loc}}(\Omega^c)$ *satisfying* (1)-(4).

Remark 1 : It should be emphasized that in related works (e.g. [10, 11, 18, 19]) the constant displacements (or rigid body motions in elasticity) in the interface problem are prevented by an additional Dirichlet boundary inside of the interior domain. It is shown in this paper that this technical restriction is not necessary. Indeed, the radiation condition of the exterior problem yields positive definiteness of corresponding boundary integral operators (see Lemma 4 below) which, together with the semi-definiteness of the partial differential operators in the interior problem, avoid the constant displacements.

In order to give an equivalent formulation of problem (IP) we incorporate some boundary integral operators. Let $H^{-s}(\Gamma)$ be the dual of $H^s(\Gamma)$ $(0 \leq s \leq 1)$ (Γ is closed) where the duality $\langle \, , \rangle$ between these spaces extends the scalar product in $L^2(\Gamma)$.

Given $v \in H^{1/2}(\Gamma)$ and $\phi \in H^{-1/2}(\Gamma)$ we define for $z \in \Gamma$

$$(V\phi)(z) := -\frac{1}{\pi}\int_{\Gamma} \phi(\zeta) \log |z-\zeta| \, ds_{\zeta}$$

$$(Kv)(z) := -\frac{1}{\pi}\int_{\Gamma} v(\zeta) \frac{\partial}{\partial n_{\zeta}} \log |z-\zeta| \, ds_{\zeta}$$

$$(K'\phi)(z) := -\frac{1}{\pi}\int_{\Gamma} \phi(\zeta) \frac{\partial}{\partial n_{z}} \log |z-\zeta| \, ds_{\zeta}$$

$$(Wv)(z) := \frac{1}{\pi}\frac{\partial}{\partial n_z}\int_{\Gamma} v(\zeta) \frac{\partial}{\partial n_{\zeta}} \log |z-\zeta| \, ds_{\zeta}.$$

This defines linear and bounded boundary integral operators when mapping between the following Sobolev-spaces [8]

$$V : H^{s-1/2}(\Gamma) \to H^{s+1/2}(\Gamma)$$

$$K : H^{s+1/2}(\Gamma) \to H^{s+1/2}(\Gamma)$$

$$K' : H^{s-1/2}(\Gamma) \to H^{s-1/2}(\Gamma)$$

$$W : H^{s+1/2}(\Gamma) \to H^{s-1/2}(\Gamma)$$

where (since we allowed Γ to be a Lipschitz boundary) $s \in [-1/2, 1/2]$. Moreover, the *single layer potential* V is symmetric, the *double layer potential* K has the dual K' and the *hyper singular operator* W is symmetric. V and W are strongly elliptic in the sense that they satisfy a Gårding inequality (in the above spaces with $s = 0$) [8]. Additionally, we have definiteness, where

$$H_0^s(\Gamma) := \{v \in H^s(\Gamma) : \langle 1, v \rangle = 0\} \equiv H^s(\Gamma)/\mathbb{R}$$

with its dual $H_0^{-s}(\Gamma)$, $0 \leq s \leq 1$.

LEMMA 1 : [16, 22, 25, 26] *Provided the capacity of* Γ *is less than* 1

$$V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

is linear, bounded, symmetric and positive definite. □

Remark 2 : For a definition of *cap* (Γ), the *capacity* of Γ, we refer to [25] and only mention here that, e.g., if Ω lies in a ball with radius less than 1, then *cap* $(\Gamma) < 1$. Thus, *cap* $(\Gamma) < 1$ can always be achieved by scaling [16, 25, 26].

The relation between the single layer potential and the hypersingular integral operator is given by $W = -\frac{\partial}{\partial s} V \frac{\partial}{\partial s}$ where $\frac{\partial}{\partial s}$ denotes the derivative with respect to the arc-length (at least in the distributional sense).

LEMMA 2 : ([22]) $\langle Wv, w \rangle = \left\langle V \frac{\partial v}{\partial s}, \frac{\partial w}{\partial s} \right\rangle$ *for any* $v, w \in H^{1/2}(\Gamma)$. □

From Lemmas 1 and 2 we get directly the following known result.

LEMMA 3 : *Provided the capacity of* Γ *is less than* 1

$$W : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

is linear, bounded, symmetric and positive semi-definite. □

We are now in the position to reformulate the interface problem (IP).

DEFINITION 2 : **(Problem (P))** *Given* $(f, u_0, t_0) \in L^2(\Omega) \times H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ *find* $(u, \phi) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ *satisfying*

$$(5) \quad \int_\Omega (A \operatorname{grad} u) \cdot \operatorname{grad} \eta \, d\Omega + \frac{1}{2} \langle Wu|_\Gamma + (K' - 1)\, \phi, \eta|_\Gamma \rangle =$$

$$= \int_\Omega f \cdot \eta \, d\Omega + \left\langle t_0 + \frac{1}{2} W u_0, \eta|_\Gamma \right\rangle \quad (\eta \in H^1(\Omega))$$

$$(6) \quad \langle \psi, V\phi + (1 - K)\, u|_\Gamma \rangle = \langle \psi, (1 - K)\, u_0 \rangle \quad (\psi \in H^{-1/2}(\Gamma)) .$$

The problems (IP) and (P) are equivalent ; compare also [10, 11, 18, 19] for related results. The proof is given here for convenient reading.

THEOREM 1 : *The problems (IP) and (P) are equivalent in the following sense. If* $(u, v) \in H^1(\Omega) \times H^1_{\mathrm{loc}}(\Omega^c)$ *is a solution of (IP) then* $(u, \phi) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ *solves (P) with* $\phi := \frac{\partial v}{\partial n}\Big|_\Gamma$. *If, conversely,* (u, ϕ) *is a solution of problem (P) then* (u, v) *solves (IP) with* $v \in H^1_{\mathrm{loc}}(\Omega^c)$ *defined by*

$$(7) \qquad v(z) = \frac{1}{2\pi} \int_\Gamma \phi(\zeta) \cdot \log |z - \zeta| \, ds_\zeta$$

$$- \frac{1}{2\pi} \int_\Gamma (u - u_0)(\zeta) \cdot \frac{\partial}{\partial n_\zeta} \log |z - \zeta| \, ds_\zeta \quad (z \in \Omega^c) .$$

Proof: Assume that (u, v) solves (IP). Then, from [9, Lemma 3.5], there holds the representation formula (7) with $\phi := \left.\frac{\partial v}{\partial n}\right|_\Gamma$. Letting $z \to \Gamma$ and by using the jump conditions, one obtains that the Cauchy data $\left(v|_\Gamma, \left.\frac{\partial v}{\partial n}\right|_\Gamma\right)$ satisfy

$$H\begin{pmatrix} v|_\Gamma \\ \left.\frac{\partial v}{\partial n}\right|_\Gamma \end{pmatrix} = -\begin{pmatrix} v|_\Gamma \\ \left.\frac{\partial v}{\partial n}\right|_\Gamma \end{pmatrix}, \quad H := \begin{pmatrix} -K & V \\ W & K' \end{pmatrix}, \tag{8}$$

compare [9, Theorem 3.11] (note that $a = 0$ here, cf. (3)). Using (4) in the first component of (8) gives (6). Multiplying of (1) with a test function $\eta \in H^1(\Omega)$, integration over Ω, using Green's formula, and incorporating (4) we get

$$\int_\Omega (A \operatorname{grad} u) \cdot \operatorname{grad} \eta \, d\Omega = \int_\Omega f \cdot \eta \, d\Omega + \left\langle \left.\frac{\partial v}{\partial n}\right|_\Gamma + t_0, \eta|_\Gamma \right\rangle .$$

From the second component in (8) we have

$$|Wv|_\Gamma + (K' - 1)\left.\frac{\partial v}{\partial n}\right|_\Gamma = -2\left.\frac{\partial v}{\partial n}\right|_\Gamma .$$

The last two identities (with $\phi = \left.\frac{\partial v}{\partial n}\right|_\Gamma$) and (4) yield (5).

Conversely, let (u, ϕ) solve (P) and define v by (7). Then, according to [9], v satisfies (2), (3) and hence (8), and the jump relations yield

$$\begin{pmatrix} v|_\Gamma \\ \left.\frac{\partial v}{\partial n}\right|_\Gamma \end{pmatrix} = \frac{1}{2}(Id - H)\begin{pmatrix} u|_\Gamma - u_0 \\ \phi \end{pmatrix}. \tag{9}$$

The first component of (9) together with (6) yields $u|_\Gamma = v|_\Gamma + u_0$. From the second identity in (9) we then have

$$\left.\frac{\partial v}{\partial n}\right|_\Gamma = -\frac{1}{2}\{W(u|_\Gamma - u_0) + (K' - 1) V\phi\} .$$

Using this in (5) gives, by Green's formula again,

$$\int_\Omega (\operatorname{div}(A \operatorname{grad} u) + f)\, \eta \, d\Omega = \left\langle (A \operatorname{grad} u) \cdot n|_\Gamma - \left.\frac{\partial v}{\partial n}\right|_\Gamma - t_0, \eta|_\Gamma \right\rangle$$

for all $\eta \in H^1(\Omega)$. Choosing $\eta \in H_0^1(\Omega)$, the completion of $C_0^\infty(\Omega)$ in the H^1-norm, we conclude the weak form of (1). Hence using (1) we get (4). □

Remark 3 : We note that

$$W1 = 0 \quad \text{and} \quad K1 = -1 \tag{10}$$

with 1 being the constant function with the value one. The identities (10) follow from $H\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (cf. [9, Lemma 3.5]).

Define the continuous mapping $B : (H^1(\Omega) \times H^{-1/2}(\Gamma))^2 \to \mathbb{R}$ and the linear form $L : H^1(\Omega) \times H^{-1/2}(\Gamma) \to \mathbb{R}$ by

$$\begin{aligned} B\left(\begin{pmatrix} u \\ \phi \end{pmatrix}, \begin{pmatrix} v \\ \psi \end{pmatrix}\right) := & \int_\Omega (A \operatorname{grad} u) \cdot \operatorname{grad} v \, d\Omega \\ & + \frac{1}{2} \langle Wu|_\Gamma + (K' - 1)\phi, v|_\Gamma \rangle \\ & + \frac{1}{2} \langle \psi, V\phi + (1 - K) u|_\Gamma \rangle \\ L\begin{pmatrix} v \\ \psi \end{pmatrix} := & \int_\Omega f \cdot v \, d\Omega + \frac{1}{2} \cdot \langle \psi, (1 - K) u_0 \rangle \\ & + \left\langle t_0 + \frac{1}{2} W u_0, v|_\Gamma \right\rangle \end{aligned}$$

for any $(u, \phi), (v, \psi) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$.

COROLLARY 1 : *Problem (P) is equivalent to* $(u, \phi) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ *with*

$$B\left(\begin{pmatrix} u \\ \phi \end{pmatrix}, (\,\cdot\,)\right) = L, \tag{11}$$

i.e. for any $(v, \psi) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ *there holds* $B\left(\begin{pmatrix} u \\ \phi \end{pmatrix}, \begin{pmatrix} v \\ \psi \end{pmatrix}\right) = L\begin{pmatrix} v \\ \psi \end{pmatrix}$. □

Proof : Note that $B\left(\begin{pmatrix} u \\ \phi \end{pmatrix}, \begin{pmatrix} \cdot \\ 0 \end{pmatrix}\right) = L\begin{pmatrix} \cdot \\ 0 \end{pmatrix}$ is equivalent to (5) and $B\left(\begin{pmatrix} u \\ \phi \end{pmatrix}, \begin{pmatrix} 0 \\ \cdot \end{pmatrix}\right) = L\begin{pmatrix} 0 \\ \cdot \end{pmatrix}$ is equivalent to (6). □

We need the following result (cf. [3] for related properties in three dimensional elasticity).

LEMMA 4: *The operator* $S := W + (1 - K')\, V^{-1}(1 - K) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ *is linear, bounded, symmetric and positive definite.*

Proof: Due to the above mentioned properties of W, K, V, K', the operator S is linear, bounded, symmetric, positive semidefinite and a Fredholm operator of index zero. Thus, it suffices to prove that the kernel *ker* S is trivial in order to conclude that S is positive definite. Let $u \in \ker S$, then $0 = \langle Su, u\rangle$. On the other hand $\langle Su, u\rangle \geq \langle Wu, u\rangle \geq 0$, so that $\langle Wu, u\rangle = 0$. By Lemma 1, u is constant. Therefore $0 = \langle V^{-1}(1 - K)\, u, (1 - K)\, u\rangle$. By Lemma 1, V^{-1} is positive definite so that $(1 - K)\, u = 0$. Using (10), this implies that the constant u is equal to zero. Thus, $\ker S = \{0\}$. □

In the case that A is a linear mapping, the following result proves that the bilinear form B satisfies the Babuska-Brezzi condition.

THEOREM 2: *There exists a constant* $\beta > 0$ *such that for all* $(u, \phi), (v, \psi) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ *we have*

$$(12) \quad \beta \cdot \left\| \begin{pmatrix} u - v \\ \phi - \psi \end{pmatrix} \right\|_{H^1(\Omega) \times H^{-1/2}(\Gamma)} \cdot \left\| \begin{pmatrix} u - v \\ \eta - \delta \end{pmatrix} \right\|_{H^1(\Omega) \times H^{-1/2}(\Gamma)} \leq$$

$$\leq B\left(\begin{pmatrix} u \\ \phi \end{pmatrix}, \begin{pmatrix} u - v \\ \eta - \delta \end{pmatrix} \right) - B\left(\begin{pmatrix} v \\ \psi \end{pmatrix}, \begin{pmatrix} u - v \\ \eta - \delta \end{pmatrix} \right)$$

with

$$2\eta := \phi + V^{-1}(1 - K)\, u|_\Gamma,\; 2\delta := \psi + V^{-1}(1 - K)\, v|_\Gamma \in H^{-1/2}(\Gamma).$$

Proof: Some calculations show

$$B\left(\begin{pmatrix} u \\ \phi \end{pmatrix}, \begin{pmatrix} u - v \\ \eta - \delta \end{pmatrix} \right) - B\left(\begin{pmatrix} v \\ \psi \end{pmatrix}, \begin{pmatrix} u - v \\ \eta - \delta \end{pmatrix} \right) =$$

$$= \int_\Omega ((A \operatorname{grad} u) - (A \operatorname{grad} v)) \cdot \operatorname{grad}(u - v)\, d\Omega$$

$$+ \frac{1}{4} \langle W(u - v), u - v\rangle + \frac{1}{4} \langle S(u - v), u - v\rangle$$

$$+ \frac{1}{4} \langle V(\phi - \psi), \phi - \psi\rangle .$$

Due to Lemmas 1, 3 and 4 and since A is uniformly monotone we have that the right hand side is bounded below by

$$\alpha_0 \| \operatorname{grad} (u - v) \|^2_{L^2(\Omega)} + \frac{c_1}{4} \| (u - v)|_\Gamma \|^2_{H^{1/2}(\Gamma)} + \frac{c_2}{4} \| \phi - \psi \|^2_{H^{-1/2}(\Gamma)}$$

where c_1, $c_2 > 0$ result from the positive definiteness of V and S. Note that

$$|||u|||^2 = \| \operatorname{grad} u \|^2_{L^2(\Omega)} + \| u|_\Gamma \|^2_{H^{1/2}(\Gamma)} \quad (u \in H^1(\Omega))$$

defines a norm $||| \cdot |||$ which is equivalent to the standard norm in $H^1(\Omega)$. Thus there exists a constant $c_3 > 0$ with $|||u||| \geqslant c_3 \| u \|_{H^1(\Omega)}$. Altogether we have proved that

$$B\left(\begin{pmatrix} u \\ \phi \end{pmatrix}, \begin{pmatrix} u - v \\ \eta - \delta \end{pmatrix}\right) - B\left(\begin{pmatrix} v \\ \psi \end{pmatrix}, \begin{pmatrix} u - v \\ \eta - \delta \end{pmatrix}\right) \geqslant$$

$$\geqslant c_4 \left\| \begin{pmatrix} u - v \\ \phi - \psi \end{pmatrix} \right\|^2_{H^1(\Omega) \times H^{-1/2}(\Gamma)}$$

with

$$c_4 := \min \left\{ \frac{c_2}{4}, c_3^2 \cdot \min \left\{ \alpha, \frac{c_1}{4} \right\} \right\} > 0 .$$

On the other hand, by definition of η, δ, we have

$$\| \eta - \delta \|_{H^{-1/2}(\Gamma)} \leqslant \frac{1}{2} (\| \phi - \psi \|_{H^{-1/2}(\Gamma)} + c_5(1 + c_6) \| u - v \|_{H^{1/2}(\Gamma)})$$

$$\leqslant c_7 \left\| \begin{pmatrix} u - v \\ \phi - \psi \end{pmatrix} \right\|_{H^1(\Omega) \times H^{-1/2}(\Gamma)}$$

where $c_5 > 0$ and $c_6 > 0$ are the bounds of $V^{-1} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ and $K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$, respectively, and $c_7 = \max \{1, c_5(1 + c_6)\}$.

Combining the last two estimates we obtain (12) with $\beta := \frac{c_4}{2} \cdot \min \{1, 1/c_7\}$. □

In case that A is linear, Theorems 1 and 2 and the Lax-Milgram lemma gives existence and uniqueness of solutions of the interface problem (IP) as well as of the rewritten problem (P).

COROLLARY 2 : *The problem (IP) as well as the problem (P) have unique solutions.*

Proof: Note that (6) is equivalent to

$$\phi = - V^{-1}(1 - K)(u|_{\Gamma} - u_0) \tag{13}$$

which may be used to eliminate ϕ in (5). This leads to the problem to find $u \in H^1(\Omega)$ with

$$A'(u)(\eta) := \int_{\Omega} (A \operatorname{grad} u) \cdot \operatorname{grad} \eta \, d\Omega + \frac{1}{2} \langle Su|_{\Gamma}, \eta|_{\Gamma} \rangle \tag{14}$$

$$= L'(\eta) \quad (\eta \in H^1(\Omega)) .$$

Here, L' is some bounded linear functional. The operator A' on the left hand side maps $H^1(\Omega)$ into its dual, is continuous, bounded, uniformly monotone (cf. the arguments of the proof of Theorem 2). From the main theorem on monotone operators [32] we obtain that A' is bijective. This yields the existence of u satisfying (14). Letting ϕ as in (13) we have that (u, ϕ) solves Problem (P). Uniqueness of the solution may be concluded from the converse calculation and the bijectivity of A' or, alternatively, from Theorem 2. □

3. THE DISCRETE PROBLEM (P_h)

In this section we treat the discretization of problem (P) in the form (11).

Let $(H_h \times H_h^{-1/2} : h \in I)$ be a family of finite dimensional subspaces of $H^1(\Omega) \times H^{-1/2}(\Gamma)$. Then, the coupling of finite elements and boundary elements consists in the following Galerkin procedure.

DEFINITION 3 : **(Problem (P_h))** *For $h \in I$ find $(u_h, \phi_h) \in H_h \times H_h^{-1/2}$ such that*

$$B\left(\begin{pmatrix} u_h \\ \phi_h \end{pmatrix}, \begin{pmatrix} v_h \\ \psi_h \end{pmatrix}\right) = L\begin{pmatrix} v_h \\ \psi_h \end{pmatrix} \tag{15}$$

for all $(v_h, \psi_h) \in H_h \times H_h^{-1/2}$.

In order to prove a discret Babuska-Brezzi condition if A is linear, we need some notations and a discrete analog of Lemma 4.

ASSUMPTION 1 : *For any $h \in I$ let $H_h \times H_h^{-1/2} \subseteq H^1(\Omega) \times H^{-1/2}(\Gamma)$ where $I \subseteq (0, 1)$ with $0 \in \bar{I}$. $1 \in H_h^{-1/2}$ for any $h \in I$ where 1 denotes the constant function with value 1.*

Let $i_h : H_h \hookrightarrow H^1(\Omega)$ and $j_h : H_h^{-1/2} \hookrightarrow H^{-1/2}(\Gamma)$ denote the canonical injections with their duals $i_h^* : H^1(\Omega)^* \to H_h^*$ *and* $j_h^* : H^{1/2}(\Gamma) \to (H_h^{-1/2})^*$ being projections. Let $\gamma : H^1(\Omega) \to H^{1/2}(\Gamma)$ denote the trace operator, $\gamma u = u|_\Gamma$ for all $u \in H^1(\Omega)$, with the dual γ^*.

Then, define

$$(16) \quad V_h := j_h^* V j_h, \quad K_h := j_h^* K \gamma i_h, \quad W_h := i_h^* \gamma^* W \gamma i_h, \quad K_h' := i_h^* \gamma^* K^* j_h$$

and, since V_h is positive definite as well,

$$(17) \qquad S_h := W_h + (1_h^* - K_h') V_h^{-1}(1_h - K_h) : H_h \to H_h^*$$

with $1_h := j_h^* \gamma i_h$ and its dual 1_h^*.

LEMMA 5 : *There exist constants $c_0 > 0$ and $h_0 > 0$ such that for any $h \in I$ with $h < h_0$ we have*

$$\langle S_h u_h, u_h \rangle \geqslant c_0 \cdot \| u_h|_\Gamma \|^2_{H^{1/2}(\Gamma)} \quad \textit{for all } u_h \in H_h .$$

Proof : The proof is quite analog to that of [3, Lemma 8] so that we give only a sketch. Assume that the conclusion is false. Then one can construct a sequence of functions $(u_{h_n})_{n=1,2,3,\dots}$ in $H^1(\Omega)$ with

$$u_{h_n} \in H_{h_n}, \ \| u_{h_n}|_\Gamma \|_{H^{1/2}(\Gamma)} = 1, \langle S_{h_n} u_{h_n}, u_{h_n} \rangle \leqslant \frac{1}{n} \, (n = 1, 2, 3, \dots)$$

and $\lim_{n\to\infty} h_n = 0$. Due to the Banach-Alaoglu theorem we may assume that $(u_{h_n}|_\Gamma)_{n=1,2,3,\dots}$ converges towards some $w \in H^{1/2}(\Gamma)$ weakly in $H^{1/2}(\Gamma)$ (a subsequence at least).

Then, by definition of S_h, we firstly conclude that $\langle W u_{h_n}|_\Gamma, u_{h_n}|_\Gamma \rangle$ tends towards zero so that (by weak convexity of $\langle W \cdot , \cdot \rangle$) $\langle Ww, w \rangle = 0$, i.e. $w|_\Gamma$ is constant by Lemma 1. A decomposition of $u_{h_n}|_\Gamma = v_n + w_n$ with $v_n \in H_0^{1/2}(\Gamma)$ and $w_n \in \mathbb{R}$ shows additionally that $(v_n)_{n=1,2,3,\dots}$ tends towards zero strongly in $H^{1/2}(\Gamma)$ so that we have also strong convergence of $(u_{h_n}|_\Gamma)_{n=1,2,3,\dots}$ towards the constant $w \in \mathbb{R}$ in $H^{1/2}(\Gamma)$.

On the other hand we have $0 = \lim_{n\to\infty} \langle V z_n, z_n \rangle$ with $z_n := V_{h_n}^{-1}(\phi_n) \in H_{h_n}^{-1/2} \subseteq H^{-1/2}(\Gamma)$,

$$\phi_n := j_{h_n}^* y_n \in (H_{h_n}^{-1/2})^*, \quad y_n := u_{h_n} - K u_{h_n} \in H^{1/2}(\Gamma).$$

Thus, $0 = \lim_{n\to\infty} \| z_n \|_{H^{-1/2}(\Gamma)}$ whence $0 = \lim_{n\to\infty} \| \phi_n \|_{(H_{h_n}^{-1/2})^*}$.
Because of $(u_{h_n}|_\Gamma)_{n=1,2,3,\dots} \to w$ we get $(y_n)_{n=1,2,3,\dots} \to 2\,w$ strongly in $H^{1/2}(\Gamma)$ (by (10) and $w \in \mathbb{R}$). Hence,

$$2\,w \langle 1, 1 \rangle = \lim_{n\to\infty} \langle 1, y_n \rangle = \lim_{n\to\infty} \langle j_{h_n} 1, y_n \rangle = \lim_{n\to\infty} \langle 1, \phi_n \rangle = 0 ,$$

i.e. $w = 0$. This contradicts $\| w \|_{H^{1/2}(\Gamma)} = \lim_{n \to \infty} \| u_{h_n}|_\Gamma \|_{H^{1/2}(\Gamma)} = 1$. □

THEOREM 3 : *There exist constants* $\beta_0 > 0$ *and* $h_0 > 0$ *such that for any* $h \in I$ *with* $h < h_0$ *we have that for any* (u_h, ϕ_h), $(v_h, \psi_h) \in H_h \times H_h^{-1/2}$

$$\beta \cdot \left\| \begin{pmatrix} u_h - v_h \\ \phi_h - \psi_h \end{pmatrix} \right\|_{H^1(\Omega) \times H^{-1/2}(\Gamma)} \cdot \left\| \begin{pmatrix} u_h - v_h \\ \eta_h - \delta_h \end{pmatrix} \right\|_{H^1(\Omega) \times H^{-1/2}(\Gamma)}$$
$$\leq B\left(\begin{pmatrix} u_h \\ \phi_h \end{pmatrix}, \begin{pmatrix} u_h - v_h \\ \eta_h - \delta_h \end{pmatrix} \right) - B\left(\begin{pmatrix} v_h \\ \psi_h \end{pmatrix}, \begin{pmatrix} u_h - v_h \\ \eta_h - \delta_h \end{pmatrix} \right)$$

with

$$2\, \eta_h := \phi_h + V_h^{-1}(1_h - K_h)\, u_h, \; 2\, \delta_h := \psi_h + V_h^{-1}(1_h - K_h)\, v_h \in H_h^{-1/2}.$$

Proof: The proof is quite analog to that of Theorem 2 dealing with the discrete operators (16) and (17). All calculations in the proof of Theorem 2 can be repeated with obvious modifications. Due to Lemma 5 the constants are independent of h as well so that β_0 does not depend on $h < h_0$, h_0 chosen in Lemma 5. Hence we may omit the details. □

COROLLARY 3 : *There exist constants* $c_0 > 0$ *and* $h_0 > 0$ *such that for any* $h \in I$ *with* $h < h_0$ *the problem* (P_h) *has a unique solution* (u_h, ϕ_h) *and, if* (u, ϕ) *denotes the solution of* (P), *there holds*

$$\left\| \begin{pmatrix} u - u_h \\ \phi - \phi_h \end{pmatrix} \right\|_{H^1(\Omega) \times H^{-1/2}(\Gamma)}$$
$$\leq c_0 \cdot \inf_{\begin{pmatrix} v_h \\ \psi_h \end{pmatrix} \in H_h \times H_h^{-1/2}} \left\| \begin{pmatrix} u - v_h \\ \phi - \psi_h \end{pmatrix} \right\|_{H^1(\Omega) \times H^{-1/2}(\Gamma)}.$$

Proof: The existence and uniqueness of the discrete solutions follows as in the proof of Corollary 2. Let $(U_h, \Phi_h) \in H^h \times H_h^{-1/2}$ be the orthogonal projections onto $H^h \times H_h^{-1/2}$ of the solution (u, ϕ) of Problem (P) in $H^1(\Omega) \times H^{-1/2}(\Gamma)$. From Theorem 3 we conclude with appropriate $(\eta_h, \delta_h) \in H^h \times H_h^{-1/2}$ that

$$\beta_0 \cdot \left\| \begin{pmatrix} U_h - u_h \\ \Phi_h - \phi_h \end{pmatrix} \right\|_{H^1(\Omega) \times H^{-1/2}(\Gamma)} \cdot \left\| \begin{pmatrix} U_h - u_h \\ \eta_h - \delta_h \end{pmatrix} \right\|_{H^1(\Omega) \times H^{-1/2}(\Gamma)}$$
$$\leq B\left(\begin{pmatrix} U_h \\ \Phi_h \end{pmatrix}, \begin{pmatrix} U_h - u_h \\ \eta_h - \delta_h \end{pmatrix} \right) - B\left(\begin{pmatrix} u_h \\ \phi_h \end{pmatrix}, \begin{pmatrix} U_h - u_h \\ \eta_h - \delta_h \end{pmatrix} \right).$$

Using the Galerkin conditions and the Lipschitz continuity of B, with related constant L (which follows since A is Lipschitz continuous), we get that the right hand side is bounded by

$$L \cdot \left\| \begin{pmatrix} U_h - u_h \\ \eta_h - \delta_h \end{pmatrix} \right\|_{H^1(\Omega) \times H^{-1/2}(\Gamma)} \cdot \left\| \begin{pmatrix} U_h - u \\ \Phi_h - \phi \end{pmatrix} \right\|_{H^1(\Omega) \times H^{-1/2}(\Gamma)} .$$

Dividing the hole estimate by $\left\| \begin{pmatrix} U_h - u_h \\ \eta_h - \delta_h \end{pmatrix} \right\|_{H^1(\Omega) \times H^{-1/2}(\Gamma)}$ proves

$$\left\| \begin{pmatrix} U_h - u_h \\ \Phi_h - \phi_h \end{pmatrix} \right\|_{H^1(\Omega) \times H^{-1/2}(\Gamma)} \leqslant \frac{L}{\beta_0} \cdot \left\| \begin{pmatrix} U_h - u \\ \Phi_h - \phi \end{pmatrix} \right\|_{H^1(\Omega) \times H^{-1/2}(\Gamma)} .$$

From this, the triangle inequality yields the assertion. □

Remark 4 : If $1 \in H_h$ then $\phi - \phi_h \in H_0^{-1/2}(\Gamma)$. For a proof consider $0 = B\left(\begin{pmatrix} u - u_h \\ \phi - \phi_h \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$.

4. *A POSTERIORI* ERROR ESTIMATE

In this section we state the assumptions and the result of an *a posteriori* error estimate, proved in the following section, which is the base of our adaptive feedback procedure. For simplicity, we restrict ourselves to linear triangles as finite elements in H_h and piecewise constants $H_h^{-1/2}$.

ASSUMPTION 2 : *Let Ω be a two-dimensional domain with polygonal boundary Γ on which we consider a family $\mathcal{T} := (\mathcal{T}_h : h \in I)$ of decomposition $\mathcal{T}_h = \{\Delta_1, \dots, \Delta_N\}$ of Ω in closed triangles $\Delta_1, \dots, \Delta_N$ such that $\overline{\Omega} = \bigcup_{i=1}^{N} \Delta_i$ and two different triangles are disjoint or have a side in common or have a vertex in common. Let $\mathcal{S}_h$ denote the sides, i.e.*

$$\mathcal{S}_h = \{\partial T_i \cap \partial T_j : i \neq j \text{ with } \partial T_i \cap \partial T_j \text{ is a common side}\} ,$$

∂T_j being the boundary of T_j. Let

$$\mathcal{G}_h = \{E : E \in \mathcal{S}_h \text{ with } E \subseteq \Gamma\}$$

be the set of « boundary sides » and let

$$\mathcal{S}_h^0 = \mathcal{S}_h \backslash \mathcal{G}_h$$

be the set of « interior sides ».

We assume that all the angels of some $\Delta \in \mathcal{T}_h \in \mathcal{T}$ are $\geqslant \Theta$ for some fixed $\Theta > 0$ which does not depend on Δ or $\mathcal{T}_h$.

Then, define

$$H_h := \{\eta_h \in C(\Omega) : \eta_h|_\Delta \in P_1 \text{ for any } \Delta \in \mathcal{T}_h\}$$

$$H_h^{-1/2} := \{\eta_h \in L^\infty(\Gamma) : \eta_h|_E \in P_0 \text{ for any } E \in \mathcal{G}_h\}$$

where P_j denotes the polynomials with degree $\leqslant j$.

For fixed $\mathcal{T}_h$ let h be the piecewise constant function defined such that the constants $h|_\Delta$ and $h|_E$ equal the element sizes diam (Δ) of $\Delta \in \mathcal{T}_h$ and diam (E) of $E \in \mathcal{S}_h$.

We assume that the coefficients a_{ij} of A are piecewise smooth such that $A(\operatorname{grad} v_h) \in C^1(\Delta)$ for any $\Delta \in \mathcal{T}_h \in \mathcal{T}$ and any trial function $v_h \in H_h$. Finally, let $f \in L^2(\Omega)$, $u_0 \in H^1(\Gamma)$, and $t_0 \in L^2(\Gamma)$.

Let n be the exterior normal on Γ and on any element boundary $\partial\Delta$, let n have a fixed orientation so that $[(A \operatorname{grad} u_h) \cdot n]|_E \in L^2(E)$ denotes the jump of the discrete tractions $(A \operatorname{grad} u_h) \cdot n$ over the side $E \in \mathcal{S}_h^0$. Define

$$R_1^2 := \sum_{\Delta \in \mathcal{T}_h} \operatorname{diam}(\Delta)^2 \cdot \int_\Delta |f + \operatorname{div}(A \operatorname{grad} u_h)|^2 \, d\Omega$$

$$R_2^2 := \sum_{E \in \mathcal{S}_h^0} \operatorname{diam}(E) \cdot \int_E |\, [(A \operatorname{grad} u_h) \cdot n]|^2 \, ds$$

$$R_3 := \left\| \sqrt{h} \cdot \left(t_0 - (A \operatorname{grad} u_h) \cdot n + \frac{1}{2} W(u_0 - u_h|_\Gamma) - \frac{1}{2}(K' - 1)\,\phi_h \right) \right\|_{L^2(\Gamma)}$$

$$R_4 := \sum_{E \in \mathcal{G}_h} \operatorname{diam}(E)^{1/2} \cdot \left\| \frac{\partial}{\partial s} \{(1 - K)(u_0 - u_h|_\Gamma) - V\phi_h\} \right\|_{L^2(E)} .$$

Under the above assumptions and notations there holds the following *a posteriori* estimate where (u, ϕ) and (u_h, ϕ_h) solve problem (P) and (P_h), respectively.

THEOREM 4 : *There exists some constant $c > 0$ such that for any $h \in I$ with $h < h_0$ (h_0 from Lemma 5) we have*

$$\left\| \begin{pmatrix} u - u_h \\ \phi - \phi_h \end{pmatrix} \right\|_{H^1(\Omega) \times H^{-1/2}(\Gamma)} \leqslant c \cdot (R_1 + R_2 + R_3 + R_4) .$$

Note that $R_1, \ldots, R_4$ can be computed (at least numerically) as far as the solution (u_h, ϕ_h) of problem (P_h) is known.

The proof of Theorem 4 is divided into several lemmas. Throughout this section we adopt the notations and assumptions of Theorem 4 and let

$$e := u - u_h, \quad \varepsilon := \phi - \phi_h, \quad \delta := \frac{1}{2}(\varepsilon + V^{-1}(1-K)\, e|_\Gamma).$$

We start with a simple inequality and estimate the appearing terms in the rest of this section.

LEMMA 6 : *We have*

$$\beta \cdot \left\| \begin{pmatrix} e \\ \varepsilon \end{pmatrix} \right\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)} \cdot \left\| \begin{pmatrix} e \\ \delta \end{pmatrix} \right\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)} \leqslant T_1 + T_2 + T_3 + T_4$$

where, for any $(e_h, \delta_h) \in H_h \times H_h^{-1/2}$,

$$T_1 := \sum_{\Delta \in \mathcal{T}_h} \int_\Delta (f + \operatorname{div}(A \operatorname{grad} u_h))(e - e_h)\, d\Omega$$

$$T_2 := - \sum_{E \in \mathcal{S}_h^0} \int_E [(A \operatorname{grad} u_h) \cdot n](e - e_h)|_E\, ds$$

$$T_3 := \left\langle t_0 - (A \operatorname{grad} u_h) \cdot n + \frac{1}{2} W(u_0 - u_h|_\Gamma) - \frac{1}{2}(K' - 1)\, \phi_h, (e - e_h)|_\Gamma \right\rangle$$

$$T_4 := \frac{1}{2} \langle \delta - \delta_h, (1 - K)(u_0 - u_h|_\Gamma) - V\phi_h \rangle .$$

Proof: Due to the arguments of the proof of Theorem 2 we have

$$\beta \cdot \left\| \begin{pmatrix} e \\ \varepsilon \end{pmatrix} \right\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)} \cdot \left\| \begin{pmatrix} e \\ \delta \end{pmatrix} \right\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)}$$

$$\leqslant B\left(\begin{pmatrix} u \\ \phi \end{pmatrix}, \begin{pmatrix} e \\ \delta \end{pmatrix} \right) - B\left(\begin{pmatrix} u_h \\ \phi_h \end{pmatrix}, \begin{pmatrix} e \\ \delta \end{pmatrix} \right)$$

$$= L\begin{pmatrix} e - e_h \\ \delta - \delta_h \end{pmatrix} - B\left(\begin{pmatrix} u_h \\ \phi_h \end{pmatrix}, \begin{pmatrix} e - e_h \\ \delta - \delta_h \end{pmatrix} \right)$$

using (15) and (11). By definition of B and L, the last expression is equal to

$$\int_{\Omega} (f(e-e_h) - A \operatorname{grad} u_h \operatorname{grad}(e-e_h))\, d\Omega$$

$$+ \left\langle t_0 + \frac{1}{2} W(u_0 - u_h|_\Gamma) - \frac{1}{2}(K'-1)\,\phi_h, (e-e_h)|_\Gamma \right\rangle$$

$$+ \frac{1}{2}\left\langle \delta - \delta_h, (1-K)(u_0 - u_h|_\Gamma) - V\phi_h \right\rangle .$$

Using Green's formula on any element $\Delta \in \mathcal{T}_h$ we obtain

$$- \int_{\Omega} A \operatorname{grad} u_h \operatorname{grad}(e-e_h)\, d\Omega$$

$$= \sum_{\Delta \in \mathcal{T}_h} \int_{\Delta} \operatorname{div}(A \operatorname{grad} u_h)(e-e_h)\, d\Omega$$

$$- \sum_{E \in \mathcal{S}_h^0} \int_E [(A \operatorname{grad} u_h)\cdot n](e-e_h)|_E\, ds$$

$$- \langle (A \operatorname{grad} u_h)\cdot n, (e-e_h)|_\Gamma \rangle .$$

Combining the last two identities proves the lemma. □

We note that under the Assumption 2 the results of [6] apply here and give the following lemmas where $c > 0$ is a generic constant and depends only on $\mathcal{T}$ but not on h, Δ, N, u, etc.

LEMMA 7 : *There exists a family of interpolation operators* $(I_h : H^1(\Omega) \to H_h : h \in I)$ *and a constant* $c > 0$ *such that the following holds. For any* $\Delta \in \mathcal{T}_h \in \mathcal{T}$ *and integers* k, q *with* $0 \leqslant k \leqslant q \leqslant 2$ *and with* $N := \cup\{\Delta' \in \mathcal{T}_h : \Delta' \cap \Delta \neq \Phi\}$, *the union of all neighbor elements of* Δ, *and for all* $u \in H^q(N)$,

$$|I_h u - u|^2_{H^k(\Delta)} \leqslant c \cdot \operatorname{diam}(T)^{2(q-k)} \cdot |u|^2_{H^q(N)} .$$

Proof: The proof follows from the analysis in [6] ; compare e.g. [6, page 82, line 13] in different notations.

Remark 5 : The operator I_h is obtained in [6] locally as follows. For any knot x_j let $N_j := \cup\{\Delta : x_j \in \Delta \in \mathcal{T}_h\}$ be the support of some trial function (or « hat function ») η_j in H_h related to x_j. Let c_j be the value of the $L^2(N_j)$-projection of $u|_{N_j}$, $u \in H^1(\Omega)$, at x_j. Then, $I_h u$ is the sum of all such $c_j \cdot \eta_j$.

Remark 6 : Due to the angle condition in Assumption 2 we have that the number of neighbor elements is bounder, i.e.

$$\operatorname{card}\left\{\Delta' \in \mathcal{T}_h : \Delta' \cap \Delta \neq \Phi\right\} \leqslant 6\,\pi/\Theta .$$

Moreover, the quotient of the size of two neighboring elements is bounded, i.e. there exists $c \geqslant 1$ (depending only on $\mathcal{T}$) with

$$1/c \leqslant \operatorname{diam}(\Delta)/\operatorname{diam}(\Delta') \leqslant c \text{ if } \Delta \cap \Delta' \neq \Phi,\ \Delta,\ \Delta' \in \mathcal{T}_h \in \mathcal{T} .$$

In particular, if E is one side of $\Delta \in \mathcal{T}_h \in \mathcal{T}$, $1/c \cdot \operatorname{diam}(\Delta) \leqslant \operatorname{diam}(E) \leqslant c \cdot \operatorname{diam}(\Delta)$.

LEMMA 8 : *Choosing* $e_h := I_h e$ *we have* $T_1 \leqslant c \cdot |e|_{H^1(\Omega)} \cdot R_1$.

Proof :

$$T_1 \leqslant \sum_{\Delta \in \mathcal{T}_h} \|f + \operatorname{div}(A \operatorname{grad} u_h)\|_{L^2(\Delta)} \cdot \|e - e_h\|_{L^2(\Delta)}$$

$$\leqslant c \sum_{\Delta \in \mathcal{T}_h} \operatorname{diam}(\Delta) \cdot \|f + \operatorname{div}(A \operatorname{grad} u_h)\|_{L^2(\Delta)} \cdot |e|_{H^1(N_\Delta)} ,$$

using Lemma 7 $(k = 0, q = 1)$ with N_Δ denoting the union of all neighbors of Δ. Using Cauchy's inequality and Remark 6, this gives

$$T_1 \leqslant c \cdot R_1 \cdot \sqrt{6\,\pi/\Theta} \cdot |e|_{H^1(\Omega)}$$

which proves the lemma. □

We recall the following weighted trace inequality which can be proved using the trace inequality and equivalence of norms on the reference triangle and then by transformation on the elements.

LEMMA 9 ([6, Lemma 4]) : *There exists a constant* $c > 0$ *such that for any* E, E *is one side of* $\Delta \in \mathcal{T}_h \in \mathcal{T}$, *and any* $u \in H^1(\Delta)$ *there holds*

$$\operatorname{diam}(\Delta)\ \|u\|^2_{L^2(E)} \leqslant c \cdot \left(\|u\|^2_{L^2(\Delta)} + \operatorname{diam}(\Delta)^2 \cdot |u|^2_{H^1(\Delta)}\right) .$$ □

LEMMA 10 : *Choosing* $e_h := I_h e$ *we have* $T_2 \leqslant c \cdot |e|_{H^1(\Omega)} \cdot R_2$.

Proof: Combining Lemma 9 (with $e - I_h e$ replacing u) and Lemma 7 (with e replacing u, $k = 0$, $q = 1$ and $k = 1 = q$) we obtain for any $E \in \mathcal{S}_h^0$, $E \subseteq \Delta$, $\Delta \in \mathcal{T}_h \in \mathcal{T}$,

$$\|e - I_h e\|_{L^2(E)}^2 \leqslant c(1/\text{diam}(\Delta)\ \|e - I_h e\|_{L^2(\Delta)}^2 + \text{diam}(\Delta)\ |e - I_h e|_{H^1(\Delta)}^2) \leqslant 2c^2\ \text{diam}(\Delta)\ |e|_{H^1(N_\Delta)}^2$$

with N_Δ denoting the union of all neighbors of Δ. Therefore,

$$T_2 \leqslant \sum_{E \in \mathcal{S}_h^0} \|[(A \text{ grad } u_h)\, n]\|_{L^2(E)} \cdot \|e - I_h e\|_{L^2(E)} \leqslant \sqrt{2}\,c \sum_{E \in \mathcal{S}_h^0} \sqrt{\text{diam}(E)}\ \|[(A \text{ grad } u_h)\, n]\|_{L^2(E)} \cdot |e|_{H^1(N_\Delta)}\,.$$

Using Cauchy's inequality and Remark 6 again, this gives

$$T_2 \leqslant c\sqrt{12\,\pi/\Theta} \cdot R_2 \cdot |e|_{H^1(\Omega)}$$

which proves the lemma. □

LEMMA 11 : *Choosing* $e_h := I_h e$ *we have* $T_3 \leqslant c \cdot |e|_{H^1(\Omega)} \cdot R_3$.

Proof: Note that $t_0 \in L^2(\Gamma)$, $W(u_0, -u_h|_\Gamma) \in L^2(\Gamma)$ since $u_0 - u_h|_\Gamma \in H^1(\Gamma)$, $(K' - 1)\,\phi_h \in L^2(\Gamma)$ since $\phi_h \in L^2(\Gamma)$, and $(A \text{ grad } u_h)\, n|_\Gamma \in L^2(\Gamma)$ since grad u_h is piecewise constant a_{ij} is piecewise smooth. Thus, we may repeat the arguments of the proof of Lemma 10 in connection with $\mathcal{G}_h$. This proves the lemma. □

LEMMA 12 : *For* $\psi := (1 - K)(u_0 - u_h|_\Gamma) - V\phi_h$ *we have*

$$\|\psi\|_{H^{1/2}(\Gamma)} \leqslant c \cdot \sum_{E \in \mathcal{G}_h} \|\sqrt{h}\ \psi'\|_{L^2(E)}\,.$$

Proof: Note that $\psi \in H_0^1(\Gamma)$ has the property that $\langle \psi, \eta_h \rangle = 0$ for any piecewise constant function $\eta_h \in H_h^{-1/2}$. Then, the assertion follows from [4, Proposition 1] so that we only give a brief sketch of the proof here.

Let $\Gamma_1, \ldots, \Gamma_N$ denote the boundary elements of the considered triangulation, $\Gamma = \bigcup_{j=1}^{N} \Gamma_j$. Since $\int_{\Gamma_j} \psi\, ds = 0$ we have at least one zero y_j of the continuous function ψ in the interior of Γ_j, $j = 1, \ldots, N$. Let $\tilde{\psi}_j \in H^{1/2}(\Gamma)$ be equal to ψ on the part of Γ between y_j and y_{j+1} and equal to 0 on the remaining part of Γ. Here we set $y_0 = y_N$. Then, the triangle inequality gives

$$\|\psi\|_{H^{1/2}(\Gamma)} \leqslant \sum_{j=0}^{N-1} \|\tilde{\psi}_j\|_{H^{1/2}(\Gamma)} . \tag{18}$$

Since supp $\psi_j \subseteq \Gamma_j \cup \Gamma_{j+1}$ and by interpolation [2] we obtain

$$\begin{aligned} \|\tilde{\psi}_j\|_{H^{1/2}(\Gamma)} &\leqslant \|\tilde{\psi}_j\|^{1/2}_{H^1(\Gamma)} \cdot \|\tilde{\psi}_j\|^{1/2}_{L^2(\Gamma)} \leqslant \\ &\leqslant \|\psi\|^{1/2}_{H^1(\Gamma_j \cup \Gamma_{j+1})} \cdot \|\psi\|^{1/2}_{L^2(\Gamma_j \cup \Gamma_{j+1})} . \end{aligned} \tag{19}$$

Since ψ has at least one zero y_j in Γ_j, the main theorem of calculus shows

$$\|\psi\|_{L^2(\Gamma_j \cup \Gamma_{j+1})} \leqslant c \cdot (h_j + h_{j+1}) \cdot \|\psi'\|_{L^2(\Gamma_j \cup \Gamma_{j+1})} .$$

Here, $h_j > 0$ is the length of the boundary element Γ_j and we note that h_j / h_{j+1}, $h_{j+1} / h_j \leqslant c$ due to the angle condition (cf. Remark 6). Using this leads to

$$\|\psi\|_{H^1(\Gamma_j \cup \Gamma_{j+1})} \leqslant c \|\psi'\|_{L^2(\Gamma_j \cup \Gamma_{j+1})}$$

and (19) gives

$$\|\tilde{\psi}_j\|_{H^{1/2}(\Gamma)} \leqslant c \cdot \left\|\sqrt{h}\psi'\right\|_{L^2(\Gamma_j \cup \Gamma_{j+1})} .$$

According to (18), this proves the assertion. □

Proof of Theorem 4 : Use Lemmas 8, 10, 11 and 12 to estimate T_1, T_2, T_3 and T_4 (with $\delta_h = 0$) in Lemma 6, respectively. Then, division by $\left\|\begin{pmatrix} e \\ \delta \end{pmatrix}\right\|_{H^1(\Omega) \times H^{-1/2}(\Gamma)}$ proves the theorem. □

5. ADAPTIVE FEEDBACK PROCEDURE

For a given triangulation $\mathcal{T}_h = \{\Delta_1, \ldots, \Delta_N\}$ of Ω and the related partition $\{\Gamma_1, \ldots, \Gamma_M\} = \mathcal{G}_h$ of the boundary Γ we can consider one element

$\Delta_j \in \mathcal{T}_h$ and compute its contributions a_j, b_k to the right hand side of the *a posteriori* error estimate in Theorem 4

$$a_j^2 := \operatorname{diam}(\Delta_j)^2 \cdot \int_{\Delta_j} |f + \operatorname{div}(A \operatorname{grad} u_h)|^2 \, d\Omega$$

$$+ \sum_{E \in \mathcal{S}_h^0,\, E \subseteq \partial\Delta_j} \operatorname{diam}(E) \cdot \int_E |[(A \operatorname{grad} u_h) \cdot n]|^2 \, ds$$

$$+ \operatorname{diam}(\Gamma \cap \partial\Delta_j) \cdot \Big\| t_0 - (A \operatorname{grad} u_h) \cdot n + \frac{1}{2} W(u_0 - u_h|_\Gamma)$$

$$- \frac{1}{2}(K' - 1)\,\phi_h \Big\|^2_{L^2(\Gamma \cap \partial\Delta_j)}$$

$$b_k := \operatorname{diam}(\Gamma_k)^{1/2} \cdot \Big\| \frac{\partial}{\partial s}\{(1 - K)(u_0 - u_h|_\Gamma) - V\phi_h\} \Big\|_{L^2(\Gamma_k)} .$$

The computational details a for a_j, b_k re given in the next section. If we neglect the constant $c > 0$ in Theorem 4, the error in the energy norm is bounded by

$$\sqrt{\sum_{j=1}^{N} a_j^2} + \sum_{k=1}^{M} b_k . \tag{20}$$

This *a posteriori* error estimate is almost useless for absolute error control unless the constant $c > 0$ (or an upper bound at least) is known. But it can be used to compare the contributions to the local error.

Note that the different nature of the coefficients a_j and b_k is, in general, caused by two different discretizations : a_j is related to a finite element, b_k is related to a boundary element. Because of a simple storage organization and a simple computation of the stiffness matrices, it is convenient to use only one mesh, i.e. to take the boundary element discretization induced by the finite element triangulation. Therefore, we consider this case in the sequel. For any element Δ_j let

$$c_j := a_j + \sum_{k=1,\, \Gamma_k \subseteq \bar{\Delta}_j}^{N} b_k$$

where the sum may be zero or consists of one or two summands.

The meshes in our numerical examples are steered by the following algorithm where $0 \leqslant \theta \leqslant 1$ is a global parameter.

Algorithm (A) *Given some coarse e.g. uniform mesh refine it successively by halving some of the elements due to the following rule. For any triangulation define* $a_1, \ldots, a_N$ *as above and divide some element* Γ_j *by halving the largest side if*

$$c_j \geqslant \theta \cdot \max_{k=1,\ldots,N} c_k .$$

In a subsequent step all hanging nodes are avoided by further refinment in order to obtain a regular mesh.

Remark 7 : (i) Note that in Algorithm (A) $\theta = 0$ gives a uniform triangulation and with increasing θ the number of refined elements in the present step decreases.

(ii) By observing (20) we have some error control which, in some sense, yields a *reliable* algorithm. In particular, the relative improvement of (20) may be used as a reasonable termination criterion.

(iii) If in some step of Algorithm (A), (20) does not become smaller then we may add some uniform refinement steps ($\theta = 0$). It can be proved that in this case (20) decreases and tends towards zero. If we allow this modification we get *convergence* of the adaptive algorithm.

6. NUMERICAL EXPERIMENTS

We consider four numerical examples for the solution of linear and nonlinear interface problems related to Example 1, i.e. $A = p \cdot I$.

First, we describe the numerical implementation of the Algorithm (A).

6.1. Implementation of the Galerkin procedure

We treat the case $p(t) = 1$ and $p(t) = 2 + \frac{1}{1+t}$ yielding a linear and nonlinear operator $A = p \cdot I$, respectively, as explained in Example 1. In the sequel we explain the computation of the form in (15) where it is sufficient to describe the approximation of

$$B\left(\begin{pmatrix}\eta_j\\ \psi_k\end{pmatrix}, \begin{pmatrix}\eta_n\\ \psi_m\end{pmatrix}\right) \quad \text{and} \quad L\begin{pmatrix}\eta_n\\ \psi_m\end{pmatrix}$$

used in the numerical examples. Here $\eta_j, \eta_k \in H_h^1$ are « hat functions » and $\psi_m, \psi_n \in H_h^{-1/2}$ are constant on one boundary element Γ_m, Γ_n and vanish on the remaining part of Γ partitioned by $\Gamma_1, \ldots, \Gamma_M$.

Note that the displacements are piecewise linear such that $\operatorname{grad} u_h$ is piecewise constant. Thus, for any triangle $\Delta \in \mathcal{T}_h$, the weight p is constant on Δ. Therefore,

$$\int_\Delta p \cdot (\operatorname{grad} \eta_j \cdot \operatorname{grad} \eta_k)\, d\Omega = \text{area}\,(\Delta) \cdot p \cdot (\operatorname{grad} \eta_j \cdot \operatorname{grad} \eta_k)|_\Delta$$

can be determined explicitly. The integrals

$$I_k(x) := \int_{\Gamma_k} \log |x-y|\, ds_y \quad \text{and} \quad J_k(x) := \int_{\Gamma_k} \frac{\partial}{\partial n_y} \log |x-y|\, ds_y$$

can be calculated analytically [15]. By using the functions I_k and J_k, the outer integration of $\langle V\psi_k, \psi_m \rangle$ and $\langle (K'-1)\,\psi_k, \eta_n|_\Gamma \rangle$, respectively, is performed by a 32 point Gaussian quadrature rule on any boundary element. Since the derivative of $\eta_j|_\Gamma$ with respect to the arc-length is piecewise constant, the stiffness matrix W_h of the hypersingular integral operator can be computed using the entries of the stiffness matrix of the single layer potential due to Lemma 2.

In order to approximate the right hand side for given functions $f \in L^2(\Gamma)$, $u_0 \in H^{1/2}(\Gamma)$, and $t_0 \in H^{-1/2}(\Gamma)$ we compute $\int_\Delta f \cdot \eta_j\, d\Omega$ *via* a quadrature rule with order 19 and 73 knots on any triangle Δ as presented in [12].

The integrals $\langle \psi_k, (1-K)\, u_0 \rangle = \int_\Gamma J_k(x) \cdot u_0(x)\, ds_x$, $\int_\Gamma t_0 \cdot \eta_k\, ds$ and $\langle W u_0, \eta_j|_\Gamma \rangle = \int_\Gamma u_0'(V\eta_j')\, ds$ are computed using a 32 point Gaussian quadrature formula on any boundary element and the values of J_k, u_0, t_0, u_0' and $(V\eta_j')$. Since η_j' is piecewise constant, the values of $(V\eta_j')$ are may be calculated with I_k.

Altogether the above descriptions determine the (approximate) computation of the mappings B and L when applied to discrete functions. In the linear case ($p=1$ is a constant weight and $A=I$) this yields a linear system of equations which is solved directly *via* Gaussian elimination. In the nonlinear case we get a nonlinear system of equations which is solved *via* a Newton-Raphson method until the termination error is of the magnitude of the machine precession. Then, the second derivatives of the interior problem are calculated as above ; we refer e.g. to [5] for more details.

6.2. Calculation of norms and residuals

In the examples below the error of the displacements u and hence their gradient $\operatorname{grad} u$ and normal derivative $\phi = \frac{\partial v}{\partial n}$ (cf. Theorem 1) are known explicitly.

Hence the $L^2(\Omega)$ norm of $u - u_h$ and grad $(u - u_h)$ can be calculated *via* the 73 knot quadrature rule [12] on any triangle. This yields an approximation of the error $u - u_h$ in the $H^1(\Omega)$-norm.

The $H^{-1/2}(\Gamma)$-norm is equivalent to the « energy norm »

$$\|\psi\|_V = \sqrt{\langle V\psi, \psi\rangle}$$

which is used in the sequel. For $x \in \Gamma_j$ and $\psi = \phi - \phi_h$ we compute

$$(V\psi)(x) = -\frac{1}{\pi} \sum_{k=1}^{M} \int_{\Gamma_k} \psi(y) \log |x - y| \, ds_y \tag{21}$$

by numerical quadrature rules. For $j \neq k$ we apply a 32 point Gaussian quadrature formula. For $j = k$ we divide and transform the integral such that the « singular point » x lies at the end of the unit interval. Then, we apply a 8 point Gaussian quadrature rule with logarithmic weights [27]. This explains the approximation of the « energy norm » of $\phi - \phi_h$ we use.

The calculation of the integrals for the residuals $R_1, ..., R_4$ over the finite element Δ and the boundary element Γ_k is performed as follows : the integral

$$\int_\Delta |f + \operatorname{div} (p \operatorname{grad} u_h)|^2 \, d\Omega$$

is approximated *via* the above mentioned 73 knot quadrature rule [12]. Here, $f(x)$ is given explicitly and p grad u_h is constant on Δ (even in the nonlinear case), whence the term div $(p$ grad $u_h)$ is neglected. The jumps on the interior element boundaries in R_2 are piecewise constant and their L^2-norm is determined explicitly. The $L^2(\Gamma_k)$-norm of

$$t_0 - (A \operatorname{grad} u_h) \cdot n + \frac{1}{2} W(u_0 - u_h|_\Gamma) - \frac{1}{2}(K' - 1)\,\phi_h$$

is approximated by a 32 point Gaussian quadrature formula. Here, $t_0(x)$ is known, $(A$ grad $u_h) \cdot n$ is constant on Γ_k and determined explicitly, while the term $((K' - 1)\,\phi_h)(x)$ is computed by using the integrals $J_m(x)$ above. With $v_h := u_h|_\Gamma$ the remaining term $W(u_0 - u_h|_\Gamma)\cdot(x) = -\left(\frac{\partial}{\partial s} V(u_0 - v_h)'\right)(x)$ is computed with replacing $\frac{\partial}{\partial s}$ by a symmetric difference quotient with stepsize 10^{-5}. This requires the computation of $V(u_0 - v_h)'(y)$. Here, $Vv_h'(y)$ can be treated by using the integrals $I_m(y)$ as above while $Vu_0'(y)$ is calculated as in (21) where u_0 is differentiated analytically.

For any $x \in \Gamma_j$ we compute the first and third summand of

$$\psi(x) := (u_0 - u_h|_\Gamma)(x) - (K(u_0 - u_h|_\Gamma))(x) - (V\phi_h)(y)$$

explicitly and by using $I_k(y)$, respectively. The term $(K\eta_j|_\Gamma)(x)$ is calculated analytically giving $(Ku_h|_\Gamma)(x)$ while the integral $(Ku_0|_\Gamma)(x)$ is approximated by a 32 point Gaussian quadrature rule on any boundary element. Then, $\|\psi'\|_{L^2(\Gamma_k)}$ is approximated by a 32 point Gaussian quadrature rule on Γ_k where the value $\psi'(x_i)$ is determined for any Gaussian knot x_i as follows. For $1 < i < 32$, the values of $\psi(x_{i-1})$, $\psi(x_i)$ and $\psi(x_{i+1})$ are interpolated by a second order polynomial p_i and its derivative $p_i'(x_i)$ replaces $\psi'(x_i)$. For $i = 1$ we take $\psi(x_1)$, $\psi(x_2)$ and $\psi(x_3)$ and for $i = 32$ $\psi(x_{30})$, $\psi(x_{31})$ and $\psi(x_{32})$ to determine p_1 and p_{32}.

6.3. Numerical example on the L-shape

The domain under consideration Ω is the L-shape region with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(-1,1)$, $(-1,-1)$, $(0,-1)$. The numerical calculations are carried out as explained in the previous subsections for known displacement fields

$$(22) \qquad u = r^{2/3} \cdot \sin\left(\frac{2}{3}\alpha\right) \quad \text{and} \quad v = \frac{1}{2}\log\left(\left(x+\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2\right)$$

in polar and Cartesian coordinates (r, α) and (x, y) respectively. Even if the right hand side is smooth, the solution has a typical corner singularity such that the convergence rate of the h-verion with a uniform mesh leads not to the optimal convergence rate.

In the first example we take a linear problem with the constant weight $p = 1$ and $f = 0$. The jumps of u_0 and t_0 are given by (4). Using these data f, u_0, t_0 the Algorithm (A) generates meshes as shown in figure 1 for $\theta = 0.4$. As it is expected for a reasonable improvement, the meshes automatically refine towards the origin where we have the singularity of the solution. In view of the well-known improvement of the Galerkin procedure by using e.g. graded meshes if corner singularities appear, this is quite reasonable.

In Table 1 we have the numerical results for the uniform mesh $(\theta = 0)$ and for the meshes generated by Algorithm (A) for *theta* $= \theta = 0.2$, 0.4, 0.6, 0.8 and 1.0. Here, we show only the number of degrees of freedom N for the finite element method (chosen by the algorithm ; a new row corresponds to a new refinement step in the adaptive algorithm), and the corresponding relative error of the displacements e_N in the $H^1(\Omega)$-norm.

In order to illustrate the estimate of Theorem 3 let γ_N be the error in energy norm divided by (20). Hence, by Theorem 3, γ_N is bounded which can be observed from Table 1. Moreover, γ_N is bounded below which indicates efficiency of the estimate and hence of the adaptive scheme.

From Table 1 we compare the degrees of freedom needed to make the relative error smaller than 0.05 : the values for $\theta = 0$, 0.2, 0.4, 0.6 and 0.8

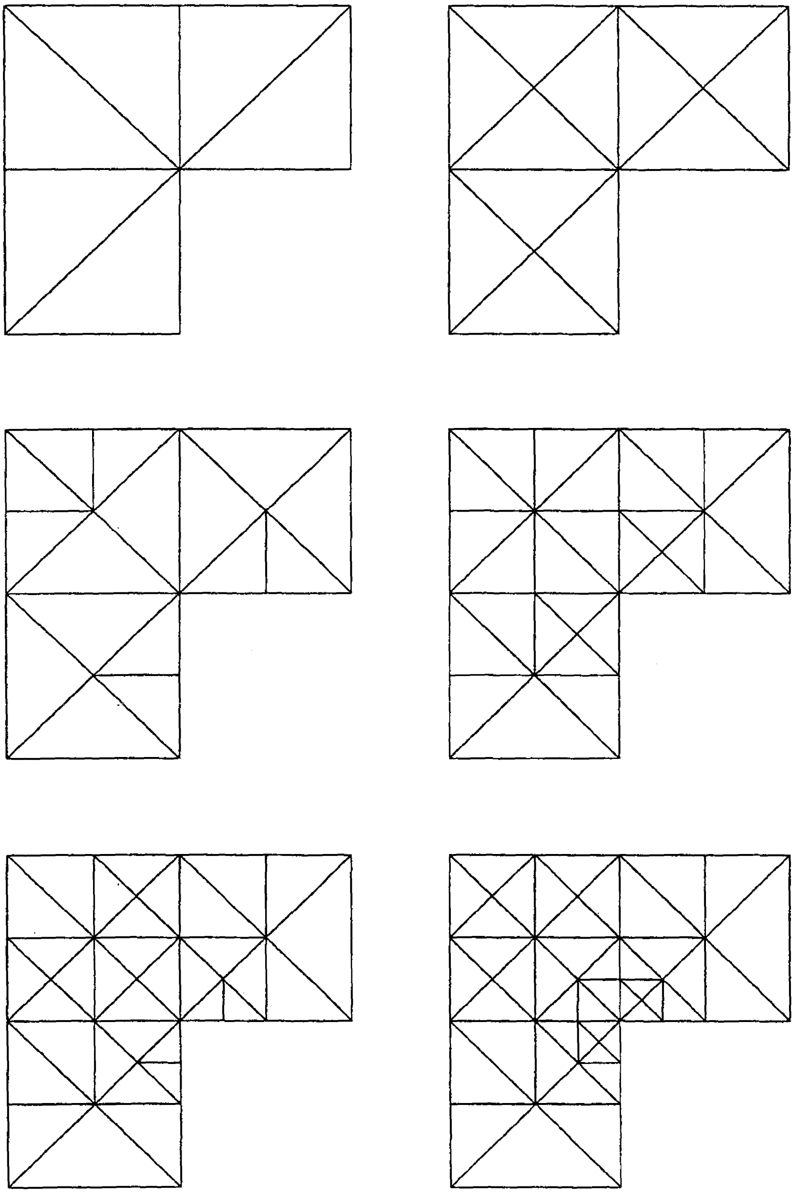


Figure 1. — Adapted meshes for the linear transmission problem.

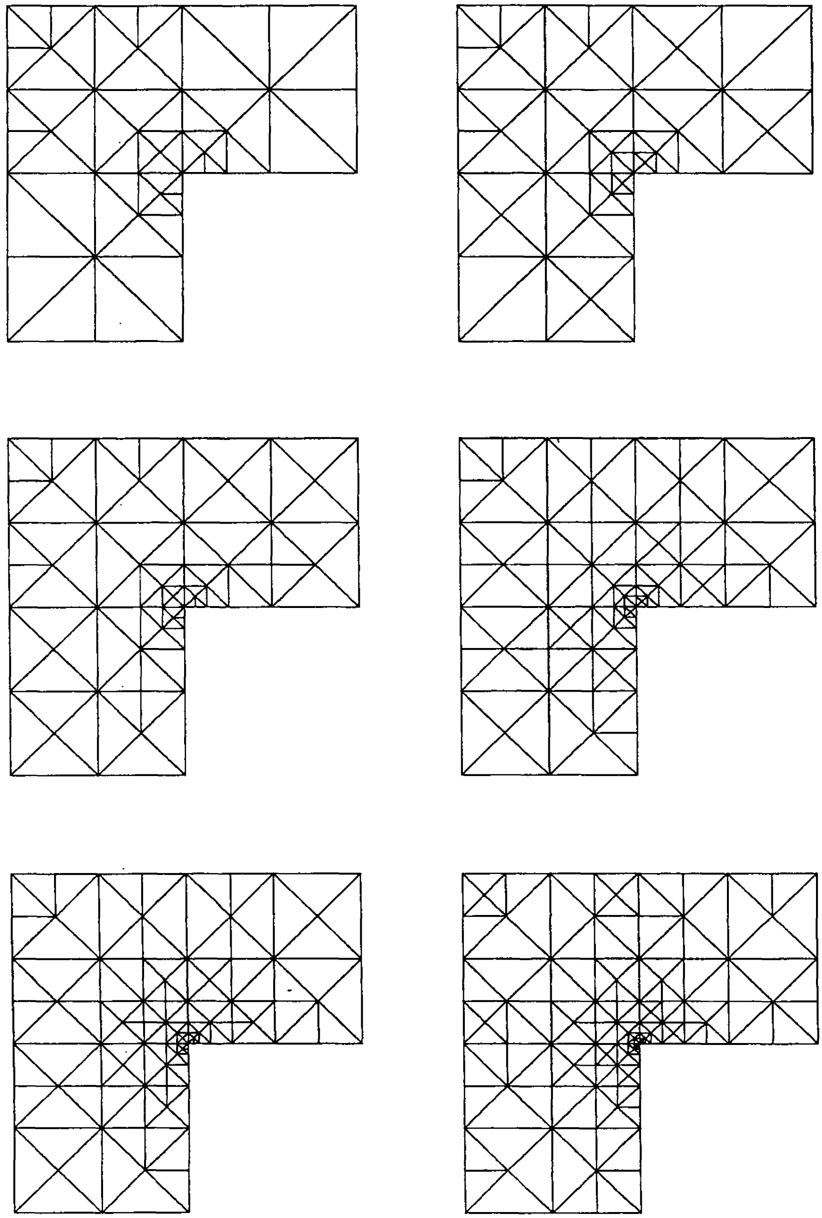


Figure 1 *(suite).*

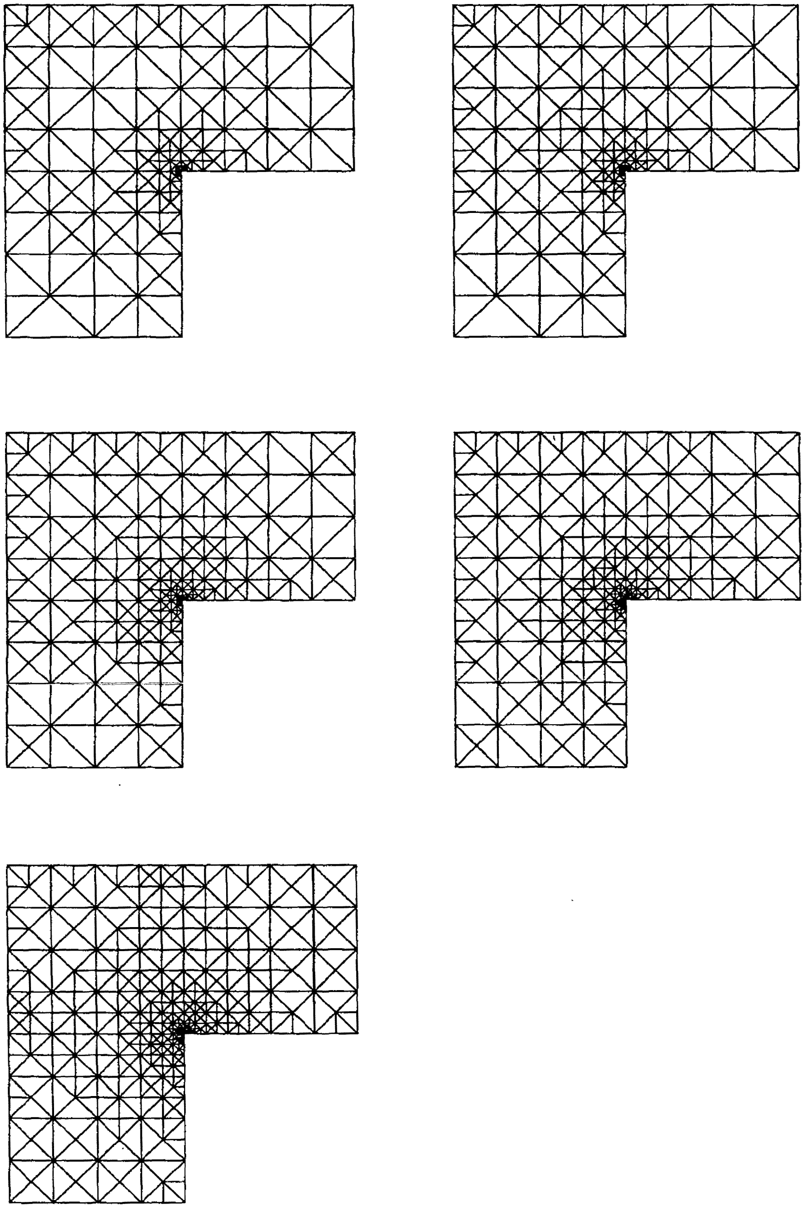


Figure 1 *(suite).*

Table 1. — Numerical results for the linear transmission problem.

Uniform mesh

N	e_N	γ_N
8	0.20434	.152
11	0.18587	.173
21	0.14485	.164
33	0.12564	.185
65	0.09563	.149
113	0.08027	.159
225	0.06230	.148

(A) for $\theta = 0.2$

N	e_N	γ_N
8	0.20434	.152
11	0.18587	.173
19	0.14621	.163
27	0.12844	.182
41	0.10297	.155
52	0.09020	.166
66	0.07554	.162
75	0.06900	.172
102	0.05947	.174
134	0.05128	.176
156	0.04646	.175
201	0.04004	.177
235	0.03604	.177

(A) for $\theta = 0.4$

N	e_N	γ_N
8	0.20434	.152
11	0.18587	.173
15	0.17074	.176
21	0.14520	.182
26	0.12197	.188
31	0.11007	.201
40	0.09420	.168
48	0.08544	.177
55	0.07824	.180
71	0.06837	.182
80	0.06260	.184
101	0.05633	.187
134	0.04959	.187
157	0.04510	.184
201	0.03904	.183
226	0.03656	.184

(A) for $\theta = 0.6$

N	e_N	γ_N
8	0.20434	.152
10	0.20467	.173
13	0.17286	.176
17	0.14848	.185
21	0.13954	.193
26	0.11594	.196
33	0.10579	.209
38	0.09402	.214
50	0.08328	.181
55	0.07744	.181
69	0.06742	.183
78	0.06448	.185
97	0.05639	.189
108	0.05448	.189
149	0.04533	.189
164	0.04367	.185
211	0.03783	.184
239	0.03562	.185

Table 1 *(suite)*.

(A) for $\theta = 0.8$		
N	e_N	γ_N
8	0.20434	.152
10	0.20467	.173
12	0.19035	.178
17	0.14848	.185
19	0.14027	.192
24	0.11730	.195
28	0.11187	.202
37	0.09378	.209
40	0.09206	.218
51	0.08049	.175
57	0.07650	.184
62	0.07401	.184
80	0.06404	.187
87	0.06071	.186
108	0.05430	.189
133	0.04934	.186
166	0.04366	.187
199	0.03927	.188
238	0.03589	.185

(A) for $\theta = 1.0$					
N	e_N	γ_N			
8	0.20434	.152	45	0.08681	.221
9	0.20451	.162	46	0.08391	.202
10	0.20467	.173	47	0.08322	.192
11	0.19777	.173	48	0.08305	.186
12	0.19035	.178	49	0.08207	.188
15	0.16267	.181	50	0.08108	.190
17	0.14848	.185	51	0.08065	.191
18	0.14462	.187	52	0.08018	.192
19	0.14027	.192	53	0.07867	.193
22	0.12391	.191	54	0.07712	.195
24	0.11730	.195	55	0.07650	.179
25	0.11651	.195	58	0.07467	.178
26	0.11459	.197	59	0.07467	.181
27	0.11246	.201	60	0.07467	.183
28	0.11187	.202	62	0.07401	.184
31	0.10368	.201	63	0.07324	.184
33	0.10067	.204	64	0.07298	.185
34	0.09884	.202	65	0.07267	.183
35	0.09698	.201	66	0.07187	.183
36	0.09545	.205	67	0.07161	.184
37	0.09378	.209	68	0.07129	.182
38	0.09285	.211	69	0.07110	.182
39	0.09181	.213	70	0.07088	.183
40	0.09206	.218	71	0.07055	.184
43	0.08820	.219	72	0.06855	.184

are > 255, 156, 134, 149 and 133. This shows that, in this example, the adapted meshes are better than a uniform triangulation. In order to determine the most efficient procedure we have to take into account the number of meshes created for this improvement. The number of corresponding required meshes (and hence the number of Galerkin equations to be solved) are > 7, 11, 13, 15 and 16. Since the values are more or less comparable, it is not clear which of the parameter leads to the most efficient procedure (the answer depends on the precise implementation and the machine we use). Conversely, we conclude that Algorithm (A) is robust concerning the parameter θ.

From Table 1 we may compute experimental convergence rates. For the uniform mesh we get experimentally a convergence of the form $O(h^{\alpha})$ with a mesh size $h = O(N^{-1/2})$ and $\alpha \approx 2/3$ as expected. In order to compress the data but compare the convergence rates, we present our numerical examples below in the form of figures where an entry corresponds to a symbol (like $\triangle$, ∇, $\diamond$ etc.) depending on the parameter *theta* $= \theta$. The entries belonging to the same parameter are connected by a straight line. The x-coordinate of a symbol is $\log(N)$ where N is the number of degrees of freedom while the y-coordinate of the symbol is $\log(e_N)$. However, the numbers shown on the axis are e_N and N.

In figure 2 we show the results for the first example where we have in the left picture the error for the displacement in relation to the number of

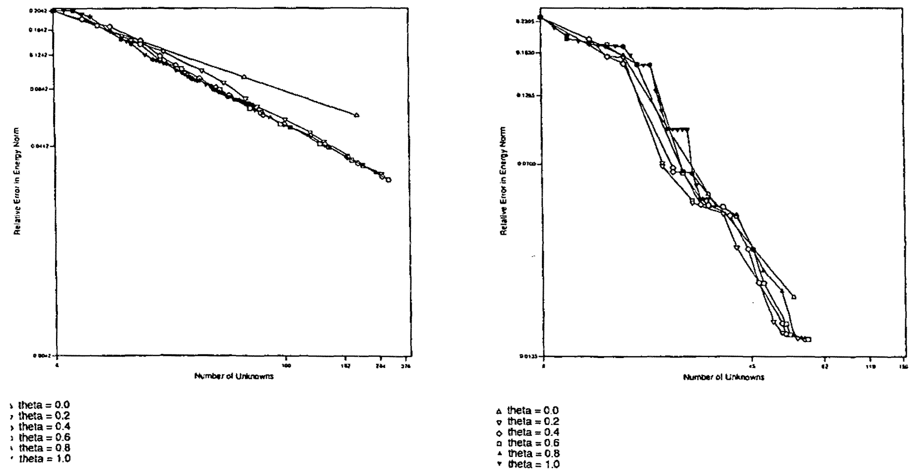


Figure 2. — Numerical results for the linear transmission problem (L-shape).

unknowns in the finite element discretization while the right picture shows the error for the tractions $\phi - \phi_h$ in relation to the number of unknowns in the boundary element discretization. The slope corresponds to the experimental convergence rates and we see an improvement of the convergence rates from 2/3 to the optimal value 1 for the displacements and an average optimal value 1.5 for the tractions.

In the second part of this example we treat the nonlinear problem where $p(t) = 2 + \frac{1}{1+t}$. We consider the same displacement fields as in (22) and obtain

$$f = -\frac{4}{27} \frac{r^{-5/3}}{\left(1 + \frac{2}{3} r^{-1/3}\right)^2} \cdot \sin\left(\frac{2}{3}\alpha\right)$$

in polar coordinates (r, α). The jumps of u_0 and t_0 are again given by (4). Using these data f, u_0, t_0 the Algorithm (A) generates meshes which refine towards the singularity as well. The related numerical output is shown in figure 3 which is quite similar to figure 2. Hence, we may conclude the same properties as above.

6.4. Numerical example on the Z-shape

The domain under consideration Ω is the Z-shape region with vertices (0, 0), (1, 0), (1, 1), $(-1, 1)$, $(-1, -1)$, $(1, -1)$. The numerical calculations

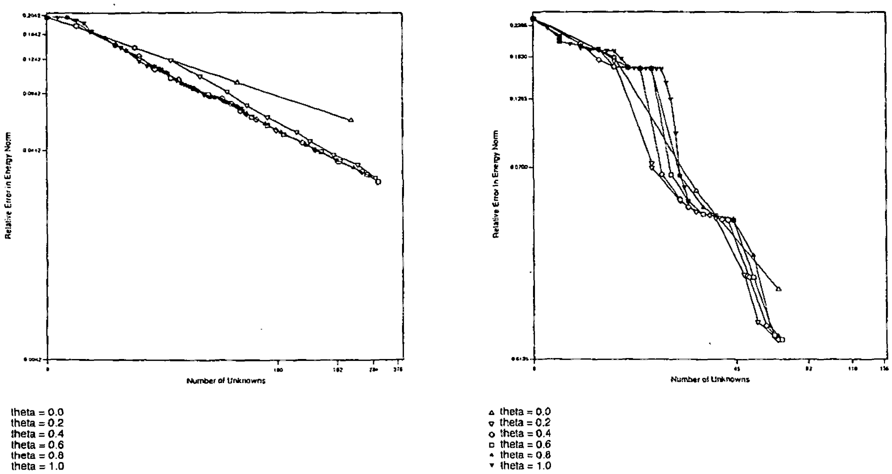


Figure 3. — Numerical results for the nonlinear transmission problem (L-shape).

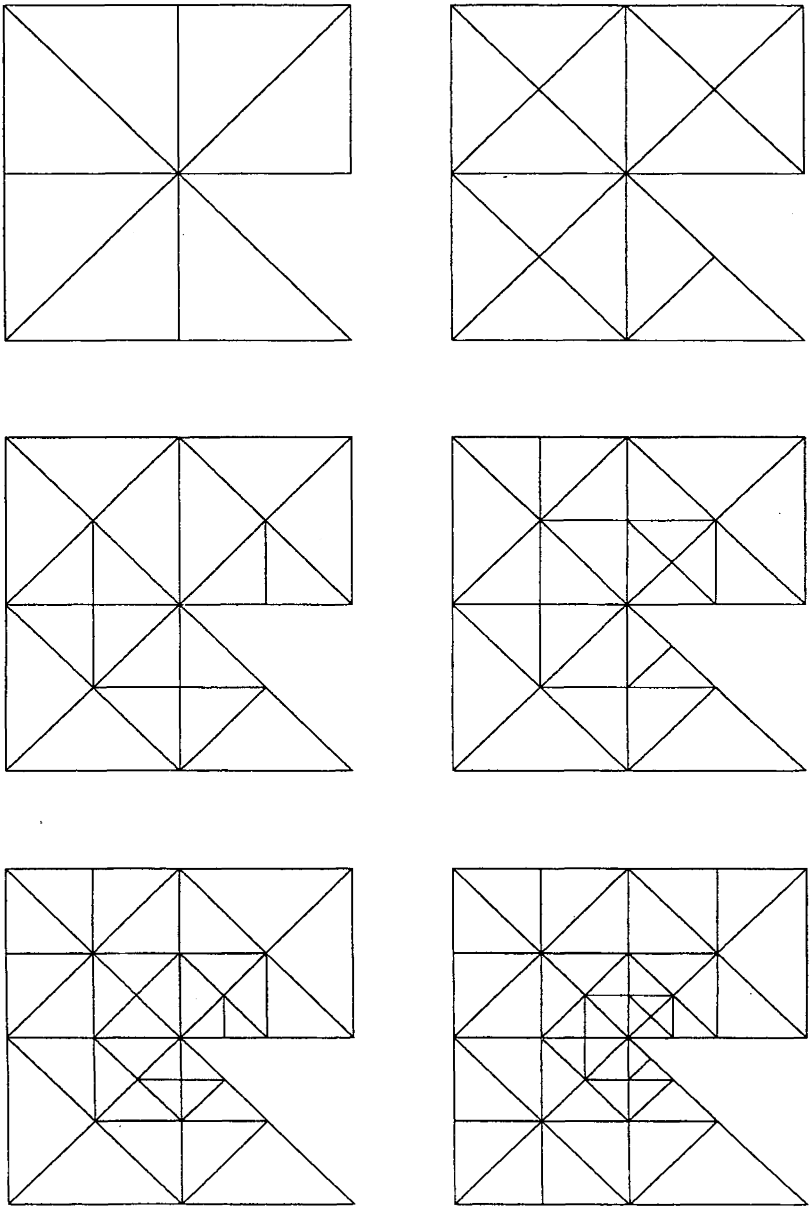


Figure 4. — Adapted meshes for the nonlinear transmission problem.

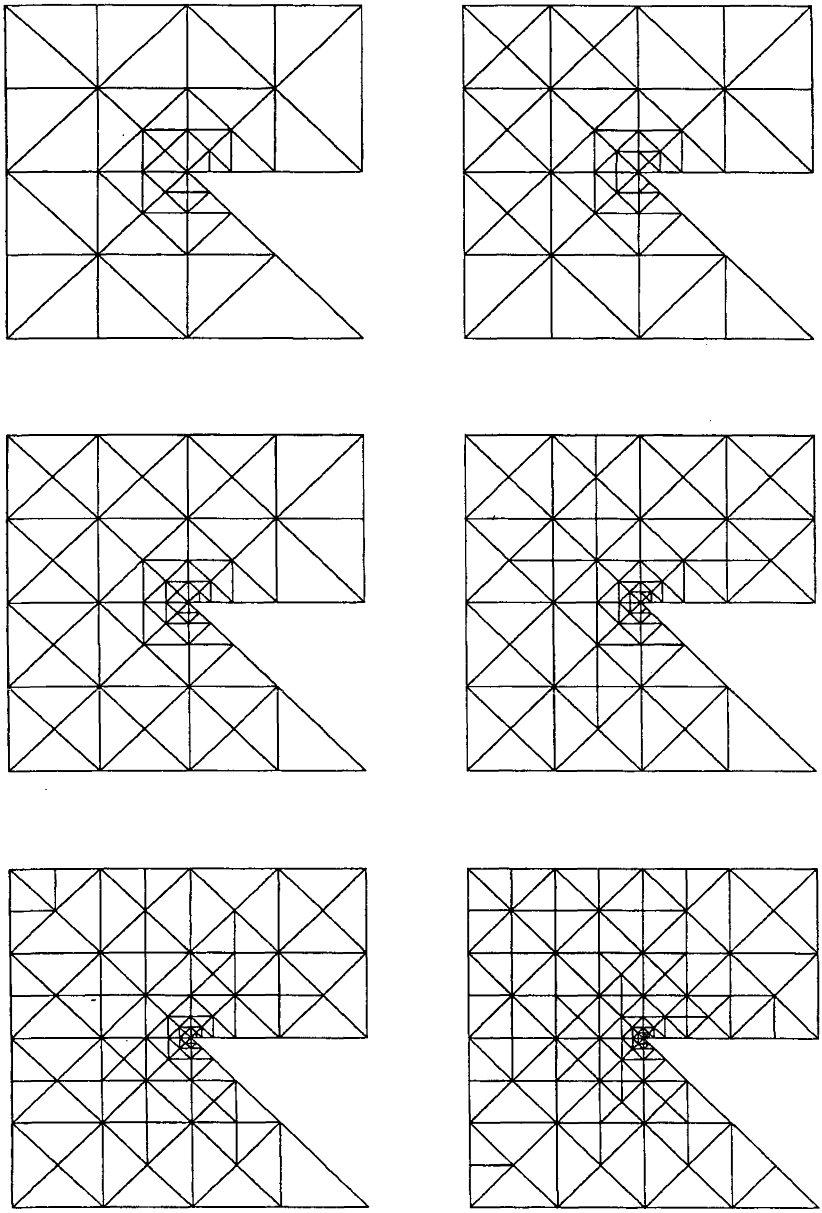


Figure 4 ***(suite).***

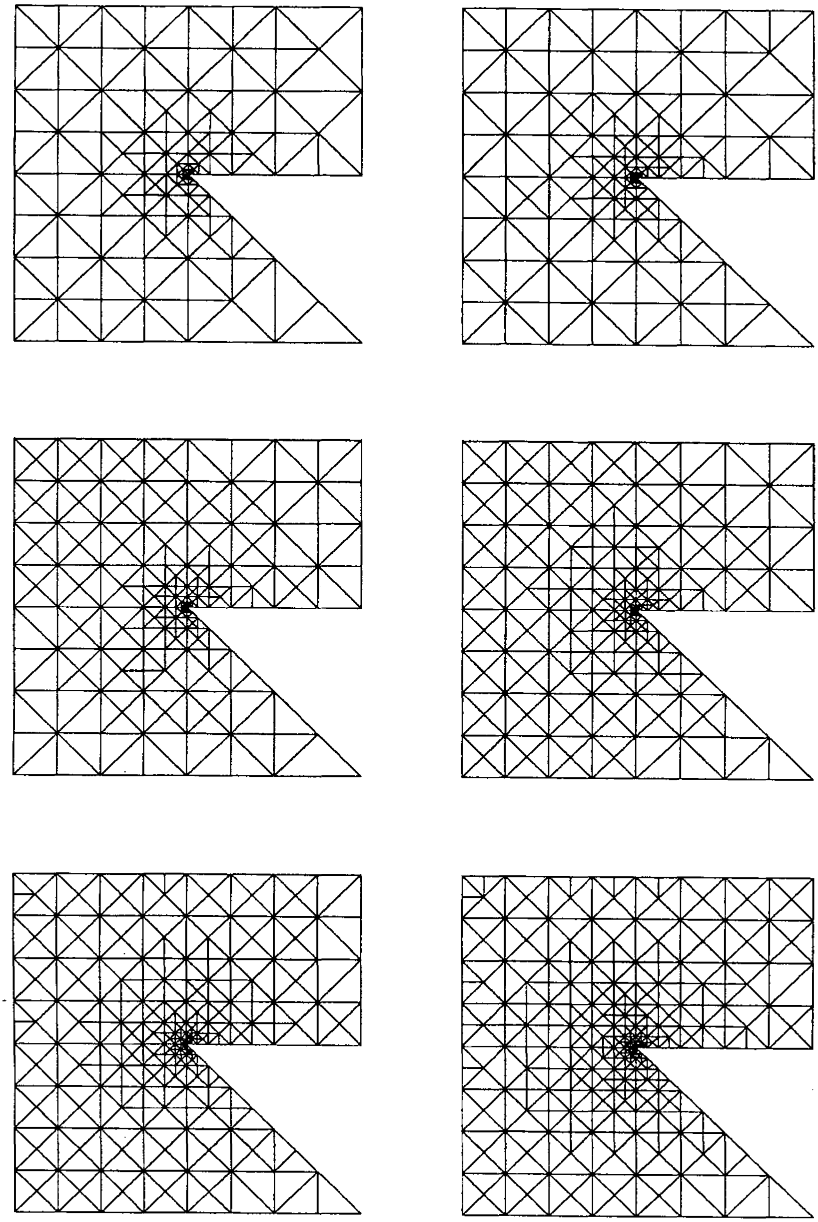


Figure 4 *(suite).*

are carried out as explained in the previous subsections for known displacement fields

$$u = r^{4/7} \cdot \sin\left(\frac{4}{7}\alpha\right) \quad \text{and} \quad v = \frac{1}{2}\log\left(\left(x+\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2\right) \tag{23}$$

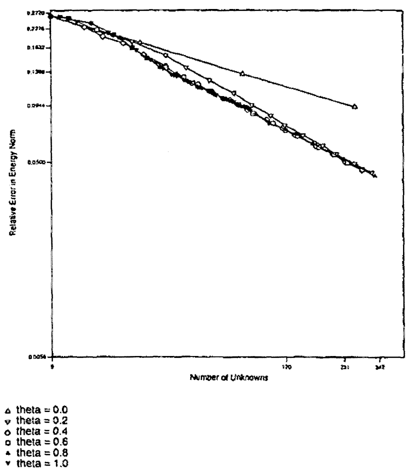
in polar and Cartesian coordinates (r, α) and (x, y) respectively. The solution has a typical corner singularity such that the convergence rate of the h-version with a uniform mesh leads not to the optimal convergence rate.

We consider the nonlinear problem where $p(t) = 2 + \frac{1}{1+t}$ with the displacement fields (23) and obtain

$$f = -\frac{48}{343}\frac{r^{-13/7}}{\left(1+\frac{4}{7}r^{-3/7}\right)^2} \cdot \sin\left(\frac{4}{7}\alpha\right)$$

in polar coordinates (r, α). The jumps of u_0 and t_0 are then given by (4). Using these data f, u_0, t_0 the Algorithm (A) generates meshes which refines towards the singularity as well. In figure 4 we show the meshes created by Algorithm (A) for $\theta = 0.4$.

The convergence rates can be seen in figure 5 which is analog to the figures of the previous examples. As in the previous examples we get an improvement of the convergence rates.



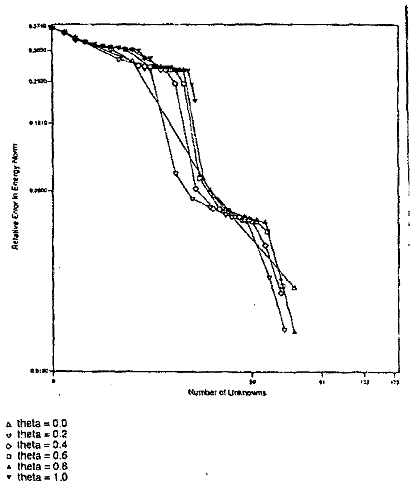


Figure 5. — Numerical results for the nonlinear transmission problem (Z-shape).

We also considered the linear problem ($p = 1$) for this example. The convergence behavior was similar as in the presented nonlinear case, hence we omit the details.

6.5. Conclusion

From the numerical experiments reported in the previous subsections, we claim that adaptive methods are important tools for an efficient numerical solution of transmission or interface problems *via* a coupling of finite elements and boundary elements. The asymptotic convergence rates are quite improved as well as the quality of the Galerkin solutions corresponding to only a few degrees of freedom. This underlines the efficiency of the adaptive algorithm as well as significance and sharpness of the *a posteriori* error estimate.

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