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## MODELING AND JUSTIFICATION OF AN EIGENVALUE PROBLEM FOR A PLATE INSERTED IN A THREE-DIMENSIONAL SUPPORT (\*)

by V. LODS (<sup>1</sup>)

Résumé. — We consider a problem in three-dimensional linearized elasticity posed over a domain consisting of a plate with thickness  $2\varepsilon$  inserted into a solid as in P. G. Ciarlet, H. Le Dret and R. Nzenywa in the static case, and F. Bourquin and P. G. Ciarlet for the eigenvalue problem. We assume that the Lamé constants of the material constituting the plate vary as  $\varepsilon^{-3}$ , the density varies as  $\varepsilon^{-1}$ , and the Lamé constants and density of the material constituting the « three-dimensional » supporting structure vary as  $\varepsilon^{-2-s}$  where  $s \in ]0, 1]$ . We prove the convergence of the eigenvalues of the three-dimensional problem as  $\varepsilon$  approaches zero either to the eigenvalues of the support, or to the eigenvalues of the plate, which then transversally vibrates at the limit. By contrast with the work of F. Bourquin and P. G. Ciarlet, these « limit » eigenvalue problems are independent. Moreover, we can prove that the eigenfunctions of the « three-dimensional » supporting structure vary as  $\varepsilon^{2+s/2}$ , while the eigenfunctions of the plate vary as  $\varepsilon^2$  for the tangential components, and as  $\varepsilon$  for the normal component. The method used here allows to improve the results of P. G. Ciarlet and H. Le Dret [5], where the displacement was proved to be of a smaller order of magnitude than  $\varepsilon^2$ .

Résumé. — On considère un problème d'élasticité linéarisée tridimensionnelle posé sur un domaine constitué d'une plaque d'épaisseur  $2\varepsilon$  insérée dans un support tridimensionnel élastique. Ce problème a déjà été à la base de travaux de P. G. Ciarlet, H. Le Dret et R. Nzenywa [7] pour le cas statique, et de F. Bourquin et P. G. Ciarlet [1] pour le problème aux valeurs propres. Nous supposons d'une part que les constantes de Lamé du matériau constituant la plaque sont en  $\varepsilon^{-3}$  et la densité en  $\varepsilon^{-1}$ , et d'autre part que les constantes de Lamé et la densité du matériau constituant le support sont en  $\varepsilon^{-2-s}$  où  $s \in ]0, 1]$ . Nous prouvons la convergence des valeurs propres du corps tridimensionnel quand  $\varepsilon$  tend vers zéro ou bien vers les valeurs propres du support, ou bien vers les valeurs propres de la plaque, qui vibre alors uniquement transversalement à la limite. Contrairement aux modèles limites obtenus par F. Bourquin et P. G. Ciarlet, les problèmes « limites » aux valeurs propres du support et de la plaque sont indépendants. De plus, nous montrons que les fonctions propres du support sont en  $\varepsilon^{2+s/2}$ , donc négligeables devant celles de la plaque qui sont en  $\varepsilon^2$  pour les composantes tangentielles et en  $\varepsilon$  pour la composante normale. La démarche utilisée ici permet d'affiner les résultats obtenus par P. G. Ciarlet et H. Le Dret [5] où il était établi que les déplacements du support sont négligeables devant  $\varepsilon^2$ .

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## INTRODUCTION

In [5], P. G. Ciarlet and H. Le Dret justified the classical boundary conditions of a clamped plate in linearized elasticity by an asymptotic analysis. More specifically, they considered a three-dimensional linearly elastic structure consisting of a « thin » plate with thickness  $2\epsilon$ , inserted into a « supporting structure » whose Lamé constants converge to  $+\infty$  as  $\epsilon^{-2-s}$  for some  $s > 0$  and the Lamé constants of the material constituting the plate are assumed to vary as  $\epsilon^{-3}$ . Then the structure supporting the plate becomes « rigid in the limit ».

Our aim here is to study the eigenvalue problem associated with the static problem considered in [5]. To this end, we make the same asymptotical hypotheses on the Lamé constants, and we introduce conditions about the densities of the materials constituting the plate and the three-dimensional supporting structure : the density of the plate varies as  $\epsilon^{-1}$ , and the density of the three-dimensional supporting structure varies as  $\epsilon^{-2-s}$ . Moreover, we have to assume that  $s \in ]0, 1]$ . Consequently, we cannot use the same material in the plate and in the « three-dimensional » supporting structure.

To study the convergence of the eigenvalues and the eigenfunctions, we use the tools developed in [1], [4], by applying the crucial idea introduced by P. G. Ciarlet and H. Le Dret in [5]. This idea consists in scaling the different parts of the full structure independently of each other, but counting the junction twice. Then, at each point in the « junction », the displacement is scaled in two ways depending on whether it is considered as a displacement of the plate or as a displacement of the three-dimensional part. As these two scalings correspond to the same displacement of the whole structure, we obtain junction conditions which depend on the choices of the scalings. Then, by arguing as in [1] for instance, we obtain both the spectrum of the plate and the spectrum of the three-dimensional part as  $\epsilon \rightarrow 0$ . Here, the two eigenvalue « limit » problems are independent, contrary to the results obtained by F. Bourquin and P. G. Ciarlet [1] where the asymptotic assumptions are different. Moreover, we prove the strong convergence of the scaled eigenfunctions. Thus, we deduce that the vibrations of the plate vary as  $\epsilon^2$  for the tangential components and as  $\epsilon$  for the normal component, while the vibrations of the three-dimensional supporting structure vary as  $\epsilon^{2+s/2}$  where  $s \in ]0, 1]$ . Consequently, the vibrations of the plate are naturally greater than the vibrations of the three-dimensional part, but the two bodies can vibrate « simultaneously » if the intersections of the two spectra is not empty.

We can make other asymptotic hypotheses on the densities. More precisely, we can choose the density of the plate as  $\epsilon^{-2-s-r}$  and the density of the three-dimensional supporting structure as  $\epsilon^{-1-r}$ . From the min-max principle, we obtain that all the eigenvalues approach 0 if  $r > 0$ , and converge to  $+\infty$  if  $r < 0$ .

1. THE THREE-DIMENSIONAL PROBLEM

Latin indices take their values in the set  $\{1, 2, 3\}$  and Greek indices take their values in the set  $\{1, 2\}$ ; the repeated index convention for summation is used. Vector-valued functions and their associated function spaces are denoted by boldface letters.

Let there be given strictly positive constants  $a_1, b_1, a_2, a_3, b_3, \beta$ , where  $\beta < b_1$ . With each  $\epsilon > 0$ , we associate the sets

$$\omega = \{(x_1, x_2) \in \mathbf{R}^2 ; 0 < x_1 < b_1, -a_2 < x_2 < a_2\}, \quad \Omega^\epsilon = \omega \times ]-\epsilon, \epsilon[ ,$$

$$\omega_\beta = \{(x_1, x_2) \in \mathbf{R}^2 ; 0 < x_1 < \beta, -a_2 < x_2 < a_2\}, \quad \Omega_\beta^\epsilon = \omega_\beta \times ]-\epsilon, \epsilon[ ,$$

$$\Gamma_0 = \{(x_1, x_2, x_3) \in \mathbf{R}^3, x_1 = -a_1, -a_2 < x_2 < a_2, -a_3 < x_3 < b_3\},$$

$$O = \{(x_1, x_2, x_3) \in \mathbf{R}^3 ; -a_1 < x_1 < \beta, -a_2 < x_2 < a_2, -a_3 < x_3 < b_3\},$$

and

$$O_\beta^\epsilon = O - \bar{\Omega}_\beta^\epsilon, \quad S^\epsilon = O \cup \Omega_\beta^\epsilon,$$

and we denote by  $x^\epsilon = (x_i^\epsilon)$  a generic point in the set  $S^\epsilon$  and by  $\partial_i^\epsilon$  the partial derivative  $\partial/\partial x_i^\epsilon$ .

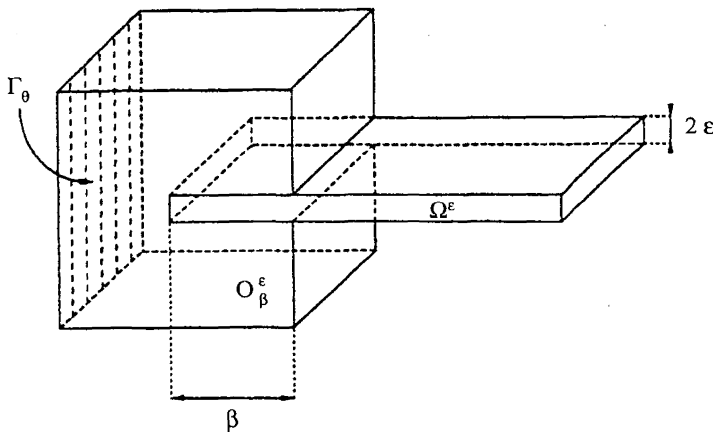


Figure 1. — The three-dimensional elastic structure.

The set  $\bar{S}^\epsilon$  is the reference configuration of an elastic structure with two parts, glued together along their common boundary : a « thin » part  $\Omega^\epsilon$  called the plate a three-dimensional part  $O_\beta^\epsilon$  called the support. The Lamé constants

$\lambda^\epsilon, \mu^\epsilon$  and the density  $\rho^\epsilon$  of the linearly elastic material constituting the plate depend on  $\epsilon$ . More precisely, we assume that there exist positive constants  $\lambda, \mu, \rho$ , which are independent of  $\epsilon$  such that

$$\lambda^\epsilon = \epsilon^{-3} \lambda, \quad \mu^\epsilon = \epsilon^{-3} \mu, \quad \rho^\epsilon = \epsilon^{-1} \rho \text{ in } \overline{\Omega^\epsilon}. \quad (1)$$

The set  $\overline{O_\beta^\epsilon}$  is the reference configuration of a linearly elastic body whose Lamé constants  $\tilde{\lambda}^\epsilon, \tilde{\mu}^\epsilon$  and the density  $\tilde{\rho}^\epsilon$  are assumed to be of the form

$$\tilde{\lambda}^\epsilon = \epsilon^{-2-s} \tilde{\lambda}, \quad \tilde{\mu}^\epsilon = \epsilon^{-2-s} \tilde{\mu}, \quad \tilde{\rho}^\epsilon = \epsilon^{-2-s} \tilde{\rho} \text{ in } \overline{O_\beta^\epsilon} \quad (2)$$

with  $s \in ]0, 1]$ ,  $\tilde{\lambda} > 0$ ,  $\tilde{\rho} > 0$  are independent of  $\epsilon$ .

The density of the three-dimensional supporting structure  $\overline{O_\beta^\epsilon}$  is a greater order of magnitude than the density of the plate  $\Omega^\epsilon$ , but we can choose exactly the same Lamé constants in the two bodies. Beside we assume here the same asymptotic hypothesis of Lamé constants as in [5].

We assume that there are no applied body forces. Thus in linearized elastodynamics, the displacement field  $\mathbf{w}^\epsilon = (w_i^\epsilon)$  of the body  $S^\epsilon$  satisfies the equations :

$$\tilde{\rho}^\epsilon \frac{\partial^2 w_i^\epsilon}{\partial t^2} = \partial_j^\epsilon \{ \tilde{\lambda}^\epsilon e_{pp}(\mathbf{w}^\epsilon) \delta_{ij} + 2 \tilde{\mu}^\epsilon e_{ij}(\mathbf{w}^\epsilon) \} \quad \text{in } O_\beta^\epsilon,$$

$$\rho^\epsilon \frac{\partial^2 w_i^\epsilon}{\partial t^2} = \partial_j^\epsilon \{ \lambda^\epsilon e_{pp}(\mathbf{w}^\epsilon) \delta_{ij} + 2 \mu^\epsilon e_{ij}(\mathbf{w}^\epsilon) \} \quad \text{in } \Omega^\epsilon,$$

with the boundary condition

$$\mathbf{w}^\epsilon(x, t) = 0 \text{ for all } x \in \Gamma_0,$$

for all time  $t \geq 0$ , where  $e_{ij}(\mathbf{w}^\epsilon) = \frac{1}{2}(\partial_i^\epsilon w_j^\epsilon + \partial_j^\epsilon w_i^\epsilon)$  are the components of the linearized strain tensor. A. Raoult [16] studied the associated problem depending on the time, with the same kind of assumptions than here. We look for stationary solutions of the form

$$\mathbf{w}^\epsilon(x^\epsilon, t) = \mathbf{u}^\epsilon(x^\epsilon) \cos(\sqrt{A^\epsilon} t)$$

$$\text{or } \mathbf{w}^\epsilon(x^\epsilon, t) = \mathbf{u}^\epsilon(x^\epsilon) \sin(\sqrt{A^\epsilon} t), \quad x^\epsilon \in S^\epsilon, t \geq 0,$$

where  $A^\epsilon$  is a positive number (cf. [9]). Thus, to find stationary solutions, we have to solve the eigenvalue problem

find  $(A^\epsilon, \mathbf{u}^\epsilon) \in \mathbf{R} \times \mathbf{V}^\epsilon$  such that

$$B^\epsilon(\mathbf{u}^\epsilon, \mathbf{v}^\epsilon) = A^\epsilon(\mathbf{u}^\epsilon, \mathbf{v}^\epsilon) \quad \text{for all } \mathbf{v}^\epsilon \in \mathbf{V}^\epsilon, \quad (3)$$

where the test space  $\mathbf{V}^\epsilon$  is given by

$$\mathbf{V}^\epsilon = \{v^\epsilon = (v_i^\epsilon) \in \mathbf{H}^1(S^\epsilon), v^\epsilon = 0 \text{ on } \Gamma_0\},$$

and

$$\begin{aligned} B^\epsilon(\mathbf{u}^\epsilon, \mathbf{v}^\epsilon) &= \int_{O_\beta^\epsilon} \{ \tilde{\lambda}^\epsilon e_{pp}(\mathbf{u}^\epsilon) e_{qq}(\mathbf{v}^\epsilon) + 2 \tilde{\mu}^\epsilon e_{ij}(\mathbf{u}^\epsilon) e_{ij}(\mathbf{v}^\epsilon) \} dx^\epsilon \\ &+ \int_{\Omega^\epsilon} \{ \lambda^\epsilon e_{pp}(\mathbf{u}^\epsilon) e_{qq}(\mathbf{v}^\epsilon) + 2 \mu^\epsilon e_{ij}(\mathbf{u}^\epsilon) e_{ij}(\mathbf{v}^\epsilon) \} dx^\epsilon, \\ (\mathbf{u}^\epsilon, \mathbf{v}^\epsilon)^\epsilon &= \int_{O_\beta^\epsilon} \tilde{\rho}^\epsilon u_i^\epsilon v_i^\epsilon dx^\epsilon + \int_{\Omega^\epsilon} \rho^\epsilon u_i^\epsilon v_i^\epsilon dx^\epsilon. \end{aligned}$$

F. Bourquin and P. G. Ciarlet ([1]) have studied the same eigenvalue problem involving different asymptotic assumptions and different boundary conditions : only the plate is clamped along the part  $\Gamma_0^\epsilon$  of its boundary where

$$\Gamma_0^\epsilon = \gamma_0 \times ] - \epsilon, \epsilon[, \quad \gamma_0 = \{(b_1, x_2) \in \mathbf{R}^2 ; |x_2| \leq a_2\}.$$

By arguing as in [1], [4], and by using the spectral theory of compact operators, we see that the eigenvalues  $A^{l,\epsilon}$ ,  $l \geq 1$  can be arranged as

$$0 < A^{1,\epsilon} \leq A^{2,\epsilon} \leq \dots A^{l,\epsilon} \leq A^{l+1,\epsilon} \leq \dots \quad \text{with } A^{l,\epsilon} \rightarrow +\infty \text{ as } l \rightarrow +\infty ;$$

and that there exists a sequence of corresponding eigenfunctions  $\mathbf{u}^{l,\epsilon} \in \mathbf{V}^\epsilon$ , which constitute a complete orthogonal set in both Hilbert spaces  $\mathbf{V}^\epsilon$  and  $L^2(\Omega^\epsilon)$ . The eigenfunctions  $\mathbf{u}^{l,\epsilon}$  satisfy the relations

$$B^\epsilon(\mathbf{u}^{l,\epsilon}, \mathbf{v}^\epsilon) = A^{l,\epsilon}(\mathbf{u}^{l,\epsilon}, \mathbf{v}^\epsilon)^\epsilon \quad \text{for all } \mathbf{v}^\epsilon \in \mathbf{V}^\epsilon, \tag{4}$$

and we assume that the orthonormalization condition

$$(\mathbf{u}^{l,\epsilon}, \mathbf{u}^{m,\epsilon})^\epsilon = \epsilon^2 \delta_{k,m} \quad 1 \leq k, m, \tag{5}$$

holds, where  $\delta_{k,m} = 1$  if  $k = m$ ,  $\delta_{k,m} = 0$  else. The associated boundary value problem is detailed in [1], with different boundary conditions.

**2. THE « SCALED » THREE-DIMENSIONAL PROBLEM OVER SETS INDEPENDENT OF  $\epsilon$**

Classically ([2], [3], [11], [10]) as in the case of a single plate, we first define the open set  $\Omega = \omega \times ]-1, 1[$  in order to deal with functions defined on sets independent of  $\epsilon$ . To avoid an overlapping over the inserted part  $\Omega_\beta^\epsilon$ , we then introduce as in [1], [5], [6] the translated set  $\tilde{\Omega} = O + t$ , where  $t$  is a vector such that  $\tilde{\Omega} \cap \Omega = \emptyset$ . Let  $x = (x_i)$  denote a generic point in the set  $\Omega$ , let  $\tilde{x} = (\tilde{x}_i)$  denote a generic point in the set  $\tilde{\Omega}$ , and let  $\partial_i = \partial/\partial x_i$ ,  $\tilde{\partial}_i = \partial/\partial \tilde{x}_i$ . On the one hand, with each point  $x^\epsilon = (x_i^\epsilon) \in \tilde{\Omega}^\epsilon$ , we associate the point  $x = (x_i) \in \Omega$  defined by

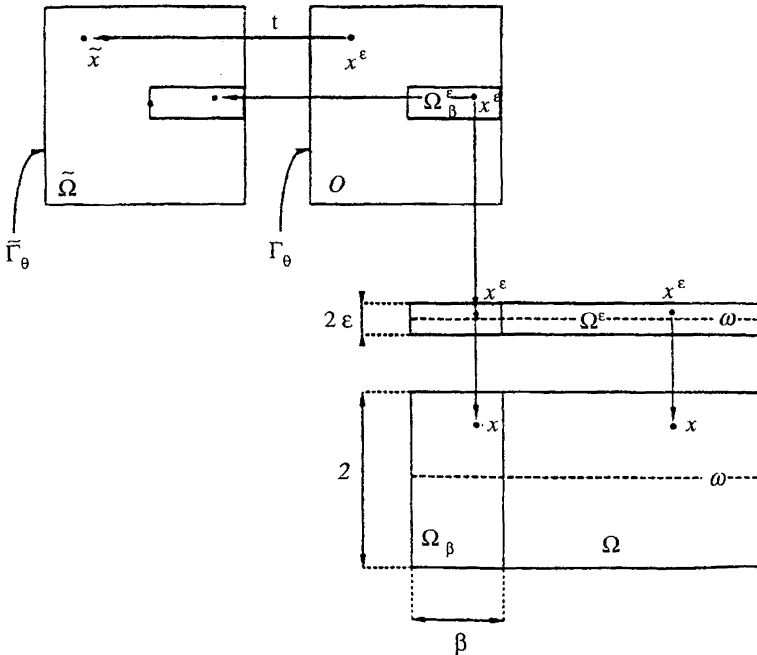


Figure 2. — The sets  $\tilde{\Omega}^\epsilon$  and  $\Omega$ , which are respectively occupied by the « thin » part and the « three-dimensional » part of the elastic structure, are mapped into two disjoint sets  $\tilde{\Omega}$  and  $\{\tilde{\Omega}\}^-$ . The « inserted » part  $\tilde{\Omega}_\beta^\epsilon$  of the thin part is thus mapped twice, once onto  $\tilde{\Omega}_\beta \subset \{\tilde{\Omega}\}^-$ .

$x_\alpha = x_\alpha^\epsilon$ ,  $x_3 = (1/\epsilon) x_3^\epsilon$ : we thus have  $\partial_\alpha^\epsilon = \partial_\alpha$  and  $\partial_3^\epsilon = \epsilon \partial_3$ . On the other hand, with each point  $x^\epsilon = (x_i^\epsilon) \in \bar{O}$ , we associate the translated point  $\bar{x} = x^\epsilon + t \in \bar{\Omega}$ : we thus have  $\bar{\partial}_i^\epsilon = \bar{\partial}_i$ .

To study the behavior of the eigenfunctions  $\mathbf{u}^{l,\epsilon}$  of problem (3), we introduce the scaled unknowns  $\mathbf{u}(\epsilon) = (u_i(\epsilon)) : \bar{\Omega} \rightarrow \mathbf{R}^3$ ,  $\tilde{\mathbf{u}}(\epsilon) = (\tilde{u}_i(\epsilon)) : \bar{\Omega} \rightarrow \mathbf{R}^3$  defined by

$$u_\alpha^\epsilon(x^\epsilon) = \epsilon^2 u_\alpha(\epsilon)(x), \quad u_3^\epsilon(x^\epsilon) = \epsilon u_3(x) \text{ for all } x^\epsilon \in \bar{\Omega}^\epsilon, \tag{6}$$

and

$$\tilde{u}_i^\epsilon(x^\epsilon) = \epsilon^{2+s/2} \tilde{u}_i(\epsilon)(\bar{x}) \text{ for all } x^\epsilon \in \bar{O}. \tag{7}$$

*Remark 1*: The scalings (6) have been introduced by P. G. Ciarlet and P. Destuynder ([2], [3], [10]) to study the behavior of the displacement of a single plate. Then P. G. Ciarlet and H. Le Dret have used formulas (6) to study the behavior of the solution of the static problem associated with the eigenvalue problem (3). But they have defined the scaled functions  $\tilde{\mathbf{u}}^\epsilon$  over the set  $\bar{O}$  by

$$u_i^\epsilon(x^\epsilon) = \epsilon^2 \tilde{u}_i(\epsilon)(\bar{x}) \text{ for all } x^\epsilon \in \bar{O}.$$

Then, under some specific assumptions on the applied forces, P. G. Ciarlet and H. Le Dret proved in [5] the convergence of the displacement  $(\tilde{\mathbf{u}}(\epsilon), \mathbf{u}(\epsilon))$  in the space  $\mathbf{H}^1(\bar{\Omega}) \times \mathbf{H}^1(\Omega)$  towards a function  $(\tilde{\mathbf{u}}, \mathbf{u})$  such that  $\tilde{\mathbf{u}} = 0$  on  $\bar{\Omega}$  and  $\mathbf{u}$  is a Kirchhoff-Love field. The proof given here allows to improve the results of P. G. Ciarlet and H. Le Dret ([5]) when  $s \in ]0, 1]$ . Indeed, by choosing the scalings (7), we obtain the strong convergence of the function  $\tilde{\mathbf{u}}(\epsilon)$ , and consequently, the displacement  $\tilde{\mathbf{u}}^\epsilon$  of the three-dimensional body  $O$  varies as  $\epsilon^{2+s/2}$ .  $\square$

The function  $(\tilde{\mathbf{u}}(\epsilon), \mathbf{u}(\epsilon))$  defined in (6)-(7), belong to the space  $\mathbf{H}^1(\bar{\Omega}) \times \mathbf{H}^1(\Omega)$  and verifies the boundary conditions  $\tilde{u}_i(\epsilon)(\bar{x}) = 0$  for all  $\bar{x} \in \bar{\Gamma}_0 = \Gamma_0 + t$ . Moreover, by (6)-(7), this function satisfies the junction conditions of the three dimensional problem

$$\tilde{u}_\alpha(\epsilon) = \epsilon^{-s/2} u_\alpha(\epsilon)(x) \tag{8}$$

$$\tilde{u}_3(\epsilon)(\bar{x}) = \epsilon^{-s/2-1} u_3(\epsilon)(x) \tag{9}$$

at each corresponding point  $\bar{x} \in \bar{\Omega}_\beta^\epsilon = \Omega_\beta^\epsilon + \mathbf{t}$  and  $x \in \Omega_\beta = \omega_\beta \times ]-1, 1[$ , i.e.  $x$  and  $\bar{x}$  correspond to the same point  $x^\epsilon \in \Omega_\beta^\epsilon$  (see fig. 2). The function  $(\tilde{\mathbf{u}}(\epsilon), \mathbf{u}(\epsilon))$  thus belongs to the space  $\mathbf{V}(\epsilon)$ , which depends on the thickness  $\epsilon$ , defined by

$$\mathbf{V}(\epsilon) = \{(\tilde{\mathbf{v}}, \mathbf{v}) \in \mathbf{H}^1(\bar{\Omega}) \times \mathbf{H}^1(\Omega); \tilde{\mathbf{v}} = 0 \text{ on } \bar{\Gamma}_0; \text{ and}$$

$$\tilde{v}_\alpha(\bar{x}) = \epsilon^{-s/2} v_\alpha(x), \quad \tilde{v}_3(\bar{x}) = \epsilon^{-s/2-1} v_3(x)$$

$$\text{at all corresponding points } x \in \Omega_\beta \text{ and } \bar{x} \in \bar{\Omega}_\beta^\epsilon\}. \tag{10}$$

To obtain the variational equations satisfied by the scaled functions  $(\tilde{\mathbf{u}}(\epsilon), \mathbf{u}(\epsilon))$ , we associate with the vector field  $\mathbf{v}^\epsilon = (v_i^\epsilon) \in V^\epsilon$ , the scaled vector fields  $\mathbf{v}(\epsilon) : \bar{\Omega} \rightarrow \mathbf{R}^3$  and  $\tilde{\mathbf{v}}(\epsilon) : \bar{\Omega} \rightarrow \mathbf{R}^3$  defined by the same formulas as in (6)-(7), i.e.,

$$v_\alpha^\epsilon(x^\epsilon) = \epsilon^2 v_\alpha(\epsilon)(x), \quad v_3^\epsilon(x^\epsilon) = \epsilon v_3(x) \text{ for all } x^\epsilon \in \bar{\Omega}^\epsilon,$$

and

$$v_i^\epsilon(x^\epsilon) = \epsilon^{2+s/2} \tilde{v}_i(\epsilon)(\bar{x}) \quad \text{for all } x^\epsilon \in \bar{O}.$$

Finally, for consistency of the notations, we also let

$$A(\epsilon) \stackrel{\text{def}}{=} A^\epsilon. \quad (11)$$

We then verify that  $(A(\epsilon), (\tilde{\mathbf{u}}(\epsilon), \mathbf{u}(\epsilon))) \in ]0, +\infty[ \times \mathbf{V}(\epsilon)$  satisfies the variational equations

$$\int \chi(\bar{O}_\beta^\epsilon) \tilde{A}e(\tilde{\mathbf{u}}(\epsilon)) : e(\tilde{\mathbf{v}}(\epsilon)) d\bar{x} + \int_\Omega A\kappa(\mathbf{u}(\epsilon)) : \kappa(\mathbf{v}(\epsilon)) dx =$$

$$A(\epsilon) \left\{ \int \chi(\bar{O}_\beta^\epsilon) \tilde{\rho} \tilde{u}_i^\epsilon \tilde{v}_i^\epsilon d\bar{x} + \int_\Omega \rho(\epsilon^2 u_\alpha(\epsilon) v_\alpha(\epsilon) + u_3(\epsilon) v_3(\epsilon)) dx \right\}, \quad (12)$$

for all  $(\tilde{\mathbf{v}}(\epsilon), \mathbf{v}(\epsilon)) \in \mathbf{V}(\epsilon)$ , where  $\chi(A)$  denotes the characteristic function of a set  $A$ ,  $\bar{O}_\beta^\epsilon = O_\beta^\epsilon + \mathbf{t}$ , and

$$\tilde{A} = (\tilde{A}_{ijkl}), \quad A = (A_{ijkl}), \quad \tilde{A}t : t = \tilde{A}_{ijkl} t_{ij} t_{kl},$$

$$At : t = A_{ijkl} t_{ij} t_{kl} \quad \text{for any tensor } t = (t_{ij}),$$

with

$$\tilde{A}_{ijkl} = \tilde{\lambda} \delta_{ij} \delta_{kl} + \tilde{\mu} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),$$

$$A_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),$$

$$e(\tilde{\mathbf{v}}) = (e_{ij}(\tilde{\mathbf{v}})), \quad \kappa(\mathbf{v}) = (\kappa_{ij}(\mathbf{v})),$$

where

$$e_{ij}(\tilde{\mathbf{v}}) = \frac{1}{2} (\tilde{\partial}_i \tilde{v}_j + \tilde{\partial}_j \tilde{v}_i),$$

$$\kappa_{\alpha\beta}(\mathbf{v}) = e_{\alpha\beta}(\mathbf{v}), \kappa_{\alpha 3}(\mathbf{v}) = \kappa_{3\alpha}(\mathbf{v}) = (1/\epsilon) e_{\alpha 3}(\mathbf{v}), \kappa_{33}(\mathbf{v}) = (1/\epsilon^2) e_{33}(\mathbf{v}),$$

with  $e_{ij}(\mathbf{v}) = \frac{1}{2} (\partial_i v_j + \partial_j v_i)$  and  $e(\mathbf{v}) = (e_{ij}(\mathbf{v}))$ .

Thus, with each eigensolution  $(A^{l,\epsilon}, \mathbf{u}^{l,\epsilon})$  of problem (3) is associated the eigensolution  $(A^l(\epsilon), (\tilde{\mathbf{u}}^l(\epsilon), \mathbf{u}^l(\epsilon)))$ , by relations (6)-(7) and (11). As the eigenfunctions  $(\mathbf{u}^{l,\epsilon})$  of problem (3) are normalized as in (5), the eigenfunction  $(\tilde{\mathbf{u}}^l(\epsilon), \mathbf{u}^l(\epsilon))$  verify

$$\int_{\tilde{\Omega}} \chi(\tilde{O}_\beta^\epsilon) \tilde{\rho} \tilde{u}_i^k(\epsilon) \tilde{u}_i^l(\epsilon) d\tilde{x} + \int_{\Omega} \rho(\epsilon^2 u_\alpha^k(\epsilon) + u_3^k(\epsilon) u_3^l(\epsilon)) dx = \delta_{kl}$$

(13)

for all  $k, l \geq 1$ .

*Remark 2:* In Lemma 3, we prove that the orthonormalized functions  $(\tilde{\mathbf{u}}^l(\epsilon), \mathbf{u}^l(\epsilon))$  are bounded independently of  $\epsilon$  in the space  $\mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega)$ . Consequently, the scalings (6)-(7) imply that the components  $u_i^\epsilon$  of the eigenfunctions of the three-dimensional supporting structure are « small » compared to the eigenfunctions of the plate, since  $s$  is assumed to be strictly positive. □

### 3. OTHER ASYMPTOTICAL ASSUMPTIONS

As we already noticed, F. Bourquin and P. G. Ciarlet ([1]) have studied the same kind of eigenvalue problem (3), with different boundary conditions and different asymptotical assumptions. Indeed in [1], the displacement  $\tilde{\mathbf{u}}^\epsilon$  is assumed to vanish along a part  $\Gamma_0^\epsilon$  of the boundary of the plate and the Lamé constants and density of the « three-dimensional » supporting structure are independent of  $\epsilon$ , which means that the body that occupies the set  $\tilde{O}_\beta^\epsilon$  is assumed to be « less rigid » and « less heavy » than here. Thus, the eigenfunctions  $\mathbf{u}_{|O_\beta^\epsilon}^\epsilon$  in the three-dimensionnal part  $\tilde{O}_\beta^\epsilon$  are greater than the displacement obtained here. Indeed, the eigenfunctions  $\mathbf{u}_{|O_\beta^\epsilon}^\epsilon$  of [1] vary as  $\epsilon$ , while the eigenfunctions of our problem vary as  $\epsilon^{2+s/2}$  (see Lemma 3), with the same orthonormalization condition.

Let us recall that the limit eigenvalue problem obtained in [1] is a coupled, pluridimensional eigenvalue problem, which is associated with the limit variational problem obtained by P. G. Ciarlet, H. Le Dret and R. Nzungwa ([7]).

We are going to use several results of F. Bourquin and P. G. Ciarlet, since they studied in [1] the eigenvalue problem (3) associated with the following test space

$$V(\epsilon) = \{(\tilde{\mathbf{v}}, \mathbf{v}) \in \mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega); \tilde{\mathbf{v}} = 0 \text{ on } \Gamma_0^\epsilon; \\ \text{and } \tilde{v}_\alpha(\tilde{x}) = \epsilon v_\alpha(x), \tilde{v}_3(\tilde{x}) = v_3(x)\} \quad (14)$$

at all corresponding points  $x \in \Omega_\beta$  and  $\tilde{x} \in \tilde{\Omega}_\beta^\epsilon$ .

Consequently, we can apply their results as soon as the boundary conditions and the transmission conditions were not used in their proofs.

4. CONVERGENCE OF  $(\mathcal{A}^l(\epsilon), (\tilde{\mathbf{u}}^l(\epsilon), \mathbf{u}(\epsilon)))$  as  $\epsilon \rightarrow 0$

Let  $\|\cdot\|_{0,A}$  and  $\|\cdot\|_{m,A}$  denote respectively the norms in  $L^2(A)$  and  $H^m(A)$ ,  $m \geq 1$ , of scalar, or vector-valued, functions defined on an open set  $A$  in  $\mathbf{R}^n$ . Strong and weak convergences are respectively denoted  $\rightarrow$  and  $\rightharpoonup$ . We establish the convergence of  $(\mathcal{A}^l(\epsilon), (\tilde{\mathbf{u}}^l(\epsilon), \mathbf{u}(\epsilon)))$  as  $\epsilon \rightarrow 0$  and we identify the limits. In particular, we prove that the eigenfunction of the plate  $\mathbf{u}^l(\epsilon)$  is a Kirchhoff-Love field.

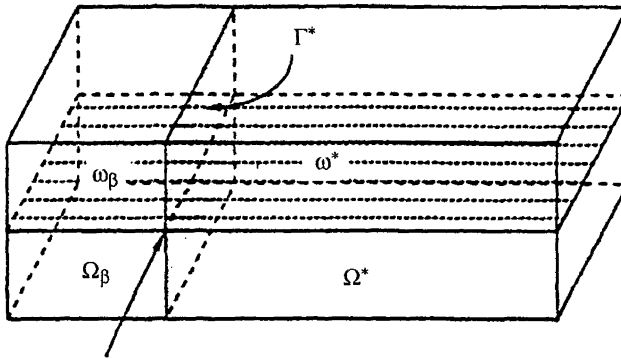


Figure 3. — Various sets found in Theorem 1 and in its proof.

THEOREM 1 : 1. Define the spaces

$$V_3(\omega^*) = \{\eta_3 \in H^2(\omega^*); \eta_3 = \partial_n \eta_3 = 0 \text{ on } \gamma^*\}, \quad (15)$$

where  $\omega^* = \omega - \bar{\omega}_\beta$ ,  $\gamma^* = \bar{\omega}_\beta \cap \bar{\omega}^*$ , and

$$\tilde{V} = \{ \tilde{v} \in \mathbf{H}^1(\tilde{\Omega}); \tilde{v} = 0 \text{ on } \tilde{\Gamma}_0 \}, \tag{16}$$

and consider the eigenvalue problems :

Find  $(\Lambda, \zeta_3) \in ]0, +\infty[ \times V_3(\omega^*)$  such that

$$\int_{\omega^*} \frac{4}{3} \left\{ \frac{\lambda \mu}{\lambda + 2\mu} \Delta \zeta_3 \delta_{\alpha\beta} + \mu \partial_{\alpha\beta} \zeta_3 \right\} \partial_{\alpha\beta} \eta_3 \, dx = 2 \Lambda \int_{\omega^*} \rho \zeta_3 \eta_3 \, dx \tag{17}$$

for all  $\eta_3 \in V_3(\omega^*)$ , and

Find  $(\tilde{\Lambda}, \tilde{u}) \in ]0, +\infty[ \times \tilde{V}$  such that

$$\int_{\tilde{\Omega}} \{ \tilde{\lambda} e_{pp}(\tilde{u}) e_{qq}(\tilde{v}) + 2 \tilde{\mu} e_{ij}(\tilde{u}) e_{ij}(\tilde{v}) \} \, d\tilde{x} = \tilde{\Lambda} \int_{\tilde{\Omega}} \tilde{\rho} \tilde{u}_i \tilde{v}_i \, d\tilde{x} \tag{18}$$

for all  $\tilde{v} \in \tilde{V}$ . The eigenvalues of two problems (17)-(18) can be arranged so as to satisfy

$$0 < \Lambda^1 \leq \Lambda^2 \leq \dots \leq \Lambda^l \leq \Lambda^{l+1} \leq \dots \text{ with } \Lambda^l \rightarrow +\infty \text{ as } l \rightarrow +\infty. \tag{19}$$

2. For each integer  $l \geq 1$ ,

$$\Lambda^l(\epsilon) \rightarrow \Lambda^l \text{ as } \epsilon \rightarrow 0. \tag{20}$$

3. We assume that  $\Lambda^l$  is a simple eigenvalue of both problems (17)-(18). Then, there exists  $\epsilon_0(l) > 0$  such that for all  $\epsilon \leq \epsilon_0(l)$ ,  $\Lambda^l(\epsilon)$  is also a simple eigenvalue of problem (12) and there exists an eigenfunction  $(\tilde{\mathbf{u}}^l(\epsilon), \mathbf{u}^l(\epsilon))$  associated with  $\Lambda^l(\epsilon)$ , normalized as in (13), that converges into the space  $\mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega)$  to a limit  $(\tilde{\mathbf{u}}^l, \mathbf{u}^l)$ , where  $\mathbf{u}^l = (u_i^l)$  can be written as follows

$$u_\alpha^l(x_1, x_2, x_3) = -x_3 \partial_\alpha \zeta_3^l(x_1, x_2) \text{ and } u_3^l(x_1, x_2, x_3) = \zeta_3^l(x_1, x_2) \tag{21}$$

for all  $x = (x_i) \in \Omega^{*\text{def}} \Omega - \bar{\Omega}_\beta$ , where the function  $\zeta_3^l$  belongs to the space  $V_3(\omega^*)$ .

Moreover  $(\tilde{\mathbf{u}}^l, \zeta_3^l)$  is an eigenfunction of both problems (17)-(18) associated with the eigenvalue  $\Lambda^l$ .

4. If  $\lambda^l$  is a simple eigenvalue of either problem (17), or problem (18), then the conclusions of part 3 are satisfied. Moreover the « limit » function  $\zeta_3^l$ , or  $\tilde{\mathbf{u}}^l$ , is an eigenfunction of problem (17), or of problem (18), associated with the eigenvalue  $\lambda^l$ , and  $\tilde{\mathbf{u}}^l = 0$ , or  $\zeta_3^l = 0$ .
5. If  $\lambda^l$  does not satisfy the conditions of part 3) or 4), there exists a subsequence of eigenfunctions associated with the eigenvalues  $\lambda^l(\epsilon)$  that satisfy either the conclusions of Part (3) if  $\lambda^l$  is an eigenvalue of both problems (17)-(18), or the conclusions of Part (4) otherwise.
6. The eigenfunctions obtained as in 3)-4) and 5) form a complete set in both spaces  $\tilde{V} \times V_3(\omega^*)$  and  $L^2(\tilde{\Omega}) \times L^2(\omega^*)$  and they satisfy

$$\int_{\tilde{\Omega}} \tilde{\rho} \tilde{u}_i^k \tilde{u}_i^l d\tilde{x} + 2 \int_{\omega^*} \rho \zeta_3^k \zeta_3^l dx = \delta_{kl} \quad k, l \geq 1. \tag{22}$$

*Remark 3 :* This theorem is still true if we assume

$$\lambda^\epsilon = \epsilon^r \lambda(\epsilon), \quad \rho^\epsilon = \epsilon^{-1-r} \rho \text{ in } \tilde{\Omega}^\epsilon, \quad \tilde{\rho}^\epsilon = \epsilon^{-2-s-r} \tilde{\rho} \text{ in } \tilde{O}_\beta^\epsilon,$$

where  $r$  is a real number independent of  $\epsilon$ , since the right-hand side of equation (12) is then not modified. Thus, the eigenvalues approach zero if the densities are « large » ( $r > 0$ ), they converge towards  $+\infty$  if the densities are « small » ( $r < 0$ ), and they converge towards strictly positive limits if  $r = 0$ . □

The proof of Theorem 1 is long and technical. So, for this reason, we proceed in several lemmas. We first establish that the eigenvalues and the eigenfunctions are bounded independently of  $\epsilon$ . Then, we identify the limits and finally we prove the strong convergences.

**LEMMA 1 :** *For each integer  $l \geq 1$ , the family  $(\lambda^l(\epsilon))_{\epsilon > 0}$  is bounded.*

*Proof :* The proof of Lemma 1 of F. Bourquin and P. G. Ciarlet ([1]) which is based on the min-max principle ([9]) allows to obtain this result. Indeed, the transmission conditions are not used by F. Bourquin and P. G. Ciarlet, and the boundary conditions have not a specific role (see Part 2). □

To prove that the eigenfunctions  $(\tilde{\mathbf{u}}^l(\epsilon), \mathbf{u}^l(\epsilon))$  are bounded in the space  $\mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega)$ , we first show that the *semi-norm*

$$(\tilde{\mathbf{v}}, \mathbf{v}) \rightarrow |(\tilde{\mathbf{v}}, \mathbf{v})| \stackrel{\text{def}}{=} \{ \|e(\tilde{\mathbf{v}})\|_{0,\tilde{\Omega}}^2 + \|e(\mathbf{v})\|_{0,\Omega}^2 \}^{1/2}$$

is a norm over the space  $\mathbf{V}(\epsilon)$  defined in (10), which in addition is uniformly equivalent (with respect to  $\epsilon$ ) to the norm

$$\|(\tilde{\mathbf{v}}, \mathbf{v})\| \stackrel{\text{def}}{=} \{ \|\tilde{\mathbf{v}}\|_{1,\tilde{\Omega}}^2 + \|\mathbf{v}\|_{1,\Omega}^2 \}^{1/2}.$$

LEMMA 2 : *There exists a constant  $C > 0$  independent of  $\epsilon$  such that*

$$\|(\tilde{\mathbf{v}}, \mathbf{v})\| \leq C |(\tilde{\mathbf{v}}, \mathbf{v})| \quad \text{for all } (\tilde{\mathbf{v}}, \mathbf{v}) \in \mathbf{V}(\epsilon). \tag{23}$$

*Proof:* The proof given here can be applied for all  $s \geq 0$ . We adapt the proof of P. G. Ciarlet and H. Le Dret ([5], Lemma 1), the transmission conditions being different here. First, we verify that the semi-norm  $(\tilde{\mathbf{v}}, \mathbf{v}) \rightarrow |(\tilde{\mathbf{v}}, \mathbf{v})|$  is a norm over the space  $\mathbf{V}(\epsilon)$ . Indeed, let  $(\tilde{\mathbf{v}}, \mathbf{v})$  be a function of the space  $\mathbf{V}(\epsilon)$  such that  $|(\tilde{\mathbf{v}}, \mathbf{v})| = 0$ . Let the « de-scaled » function  $\mathbf{v}^\epsilon = (v_i^\epsilon) \in \mathbf{H}^1(S^\epsilon)$  be defined by

$$v_\alpha^\epsilon(x^\epsilon) = \epsilon^2 v_\alpha(\epsilon)(x), \quad v_3^\epsilon(x^\epsilon) = \epsilon v_3(\epsilon)(x) \quad \text{for all } x^\epsilon \in \bar{\Omega}^\epsilon,$$

$$v_i^\epsilon(x^\epsilon) = \epsilon^{2+s/2} \tilde{v}_i(\epsilon)(\tilde{x}) \quad \text{for all } x^\epsilon \in \bar{O}.$$

Then  $|(\tilde{\mathbf{v}}, \mathbf{v})| = 0$  implies that  $e_{ij}(\mathbf{v}^\epsilon) = 0$  on  $S^\epsilon$ . Since  $\mathbf{v}^\epsilon \rightarrow |e_{ij}(\mathbf{v}^\epsilon)|_{0,S^\epsilon}$  is a norm over the space  $\mathbf{V}^\epsilon$  by Korn's inequality and the boundary condition along the part  $\Gamma_0$ , we deduce that  $\mathbf{v}^\epsilon = 0$  in  $S^\epsilon$ , and so  $(\tilde{\mathbf{v}}, \mathbf{v})$  is equal to zero. Consequently  $(\tilde{\mathbf{v}}, \mathbf{v}) \rightarrow |(\tilde{\mathbf{v}}, \mathbf{v})|$  is a norm on the space  $\mathbf{V}(\epsilon)$ .

If inequality (23) is false, there exist sequences  $\epsilon_k$  and  $(\tilde{\mathbf{v}}^k, \mathbf{v}^k) \in \mathbf{V}(\epsilon_k)$  such that

$$\epsilon_k \rightarrow 0 \quad \text{as } k \rightarrow +\infty, \tag{24}$$

$$\|(\tilde{\mathbf{v}}^k, \mathbf{v}^k)\| = 1, \tag{25}$$

$$|(\tilde{\mathbf{v}}^k, \mathbf{v}^k)| \rightarrow 0 \quad \text{as } k \rightarrow +\infty. \tag{26}$$

By (26),  $|e(\tilde{\mathbf{v}}^k)|_{0,\tilde{\Omega}} \rightarrow 0$ . Since  $\tilde{\mathbf{v}}^k = 0$  on  $\tilde{\Gamma}_0$ , Korn's inequality gives

$$\|\tilde{\mathbf{v}}^k\|_{1,\tilde{\Omega}} \rightarrow 0, \tag{27}$$

and in particular,

$$\tilde{\mathbf{v}}^k|_{\tilde{\omega}_\beta} \rightarrow 0 \quad \text{in } \mathbf{H}^{1/2}(\tilde{\omega}_\beta). \tag{28}$$

On the other hand, the convergence  $|e(\mathbf{v}^k)|_{0,\Omega} \rightarrow 0$  due to (26) implies (e.g. by involving the same kind of arguments as in Duvaut-Lions ([12], theorem 3.4, p. 117)) that there exist two vectors  $\mathbf{a}^k \in \mathbf{R}^3$  and  $\mathbf{b}^k \in \mathbf{R}^3$  and a function  $s^k \in \mathbf{H}^1(\Omega)$  such that

$$\mathbf{v}^k(x) = \mathbf{a}^k + \mathbf{b}^k \wedge \mathbf{0x} + s^k(x) \quad \text{for all } x \in \Omega \tag{29}$$

with

$$\| \mathbf{s}^k \|_{1, \Omega} \rightarrow 0. \tag{30}$$

Then by (28), the junction conditions contained in the definition of the space  $\mathbf{V}(\epsilon)$

$$\tilde{v}_\alpha^k(\tilde{x}) = \epsilon_k^{-s/2} v_\alpha^k(x), \quad v_3^k(\tilde{x}) = \epsilon_k^{-s/2-1} v_3^k(x),$$

at each point  $x \in \Omega_\beta$  corresponding to  $\tilde{x} \in \tilde{\Omega}_\beta^\epsilon$ , imply

$$\mathbf{v}_{|\omega_\beta}^k \rightarrow 0 \quad \text{in} \quad \mathbf{H}^{1/2}(\omega_\beta), \tag{31}$$

because  $s > 0$ . By (29)-(30) and (31), the vectors  $(\mathbf{a}^k + \mathbf{b}^k \wedge \mathbf{0x})_{|\omega_\beta}$  are bounded independently of  $k$  in the space  $\mathbf{H}^{1/2}(\omega_\beta)$ . Then, as the functions  $(\mathbf{a}^k + \mathbf{b}^k \wedge \mathbf{0x})_{|\omega_\beta}$  all belong to the same finite-dimensional vector space, there exists a subsequence that converges to a function of the form  $(\mathbf{a} + \mathbf{b} \wedge \mathbf{0x})_{|\omega_\beta}$ . By (29)-(30) and (31),  $(\mathbf{a} + \mathbf{b} \wedge \mathbf{0x})_{|\omega_\beta} = 0$ , so  $\mathbf{a} = \mathbf{b} = 0$ . Thus, from (29)-(30) and (27), we deduce that  $\|(\tilde{\mathbf{v}}^k, \mathbf{v}^k)\| \rightarrow 0$ , which contradicts (25).  $\square$

We can now prove that the sequence of eigenfunctions  $(\mathbf{u}^l(\epsilon), \tilde{\mathbf{u}}^l(\epsilon))$  is bounded independently of  $\epsilon$ .

LEMMA 3 : *For each integer  $l \geq 1$ , the family  $(\mathbf{u}^l(\epsilon), \tilde{\mathbf{u}}^l(\epsilon))_{\epsilon > 0}$  is bounded independently of  $\epsilon$  in the space  $\mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega)$ . Thus there exists a subsequence, still indexed by  $\epsilon$  for notational convenience, and which can be chosen to be the same for all integers  $l \geq 1$ , and there exists a number  $A^l \geq 0$  and a pair  $(\tilde{\mathbf{u}}^l, \mathbf{u}^l) \in \mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega)$  such that*

$$A^l(\epsilon) \rightarrow A^l, \tag{32}$$

$$\mathbf{u}^l(\epsilon) \rightarrow \tilde{\mathbf{u}}^l \text{ in } \mathbf{H}^1(\tilde{\Omega}) \text{ and } \tilde{\mathbf{u}}^l = 0 \text{ on } \tilde{\Gamma}_0, \tag{33}$$

$$\mathbf{u}^l(\epsilon) \rightarrow \mathbf{u}^l \text{ in } \mathbf{H}^1(\Omega), \tag{34}$$

as  $\epsilon \rightarrow 0$ .

*Proof:* We follows the ideas of P. G. Ciarlet and H. Le Dret ([5]), (see also [1], [7]). The orthonormalization condition (13) and equation (12) imply

$$A^l(\epsilon) = \int_{\tilde{\Omega}} \chi(\tilde{O}_\beta^\epsilon) \tilde{A}e(\tilde{\mathbf{u}}^l(\epsilon)) : e(\tilde{\mathbf{u}}^l(\epsilon)) \, d\tilde{x} + \int_{\Omega} A\kappa(\mathbf{u}^l(\epsilon)) : \kappa(\mathbf{u}^l(\epsilon)) \, dx. \tag{35}$$

We then write, by using the notations introduced in part 2,

$$\int_{\Omega} A\kappa(\mathbf{u}^t(\epsilon)) : \kappa(\mathbf{u}^t(\epsilon)) \, dx = \int_{\Omega} \chi(\Omega - \Omega_{\beta}) A\kappa(\mathbf{u}^t(\epsilon)) : \kappa(\mathbf{u}^t(\epsilon)) \, dx + \int_{\Omega} \chi(\Omega_{\beta}) A\kappa(\mathbf{u}^t(\epsilon)) : \kappa(\mathbf{u}^t(\epsilon)) \, dx .$$

The idea of P. G. Ciarlet, H. Le Dret consists in splitting the integral over the set  $\Omega_{\beta}$  into two equal (for definiteness) parts by using the transmission conditions (8)-(9). In this fashion, we obtain

$$\int_{\Omega} \chi(\Omega_{\beta}) A\kappa(\mathbf{u}^t(\epsilon)) : \kappa(\mathbf{u}^t(\epsilon)) \, dx = \frac{1}{2} \int_{\Omega} \chi(\Omega_{\beta}) A\kappa(\mathbf{u}^t(\epsilon)) : \kappa(\mathbf{u}^t(\epsilon)) \, dx + \frac{1}{2} \epsilon^{s-1} \int_{\tilde{\Omega}} \chi(\tilde{\Omega}_{\beta}^{\epsilon}) Ae(\tilde{\mathbf{u}}^t(\epsilon)) : e(\tilde{\mathbf{u}}^t(\epsilon)) \, d\tilde{x} .$$

Thus, equality (35) can be written as follows

$$A^t(\epsilon) = \int_{\tilde{\Omega}} \chi(\tilde{\Omega}_{\beta}^{\epsilon}) \tilde{A}e(\tilde{\mathbf{u}}^t(\epsilon)) : e(\tilde{\mathbf{u}}^t(\epsilon)) \, d\tilde{x} + \frac{1}{2} \epsilon^{s-1} \int_{\tilde{\Omega}} \chi(\tilde{\Omega}_{\beta}^{\epsilon}) Ae(\tilde{\mathbf{u}}^t(\epsilon)) e(\tilde{\mathbf{u}}^t(\epsilon)) \, d\tilde{x} + \int_{\Omega} \left\{ \chi(\Omega - \Omega_{\beta}) + \frac{1}{2} \chi(\Omega_{\beta}) \right\} A\kappa(\mathbf{u}^t(\epsilon)) : \kappa(\mathbf{u}^t(\epsilon)) \, dx .$$

Since there exists a constant  $c > 0$  which depends on the Lamé constants  $\lambda, \mu, \bar{\lambda}, \bar{\mu}$ , such that

$$\tilde{A}e : e \geq ce : e \quad \text{and} \quad Ae : e \geq ce : e , \tag{36}$$

for all symmetric tensor  $e = (e_{ij})$ , we obtain (because  $s \leq 1$  and  $\epsilon \leq 1$ )

$$\frac{c}{2} \{ |e(\tilde{\mathbf{u}}^t(\epsilon))|_{0, \tilde{\Omega}}^2 + |\kappa(\mathbf{u}^t(\epsilon))|_{0, \Omega}^2 \} \leq A^t(\epsilon) . \tag{37}$$

On the other hand, it follows from the definition of  $\kappa(\mathbf{u}^t(\epsilon))$  that

$$|e(\mathbf{u}^t(\epsilon))|_{0, \Omega} \leq |\kappa(\mathbf{u}^t(\epsilon))|_{0, \Omega} . \tag{38}$$

By (37)-(38) and by Lemma 1, we deduce that  $|(\tilde{\mathbf{u}}^l(\epsilon), \mathbf{u}^l(\epsilon))|$  is bounded independtly of  $\epsilon$ , and consequently by (23), the family  $(\tilde{\mathbf{u}}^l(\epsilon), \tilde{\mathbf{u}}^l(\epsilon))$  is bounded in the space  $\mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega)$  The other conclusions of Lemma 3 then follow from this property  $\square$

As in the case of a single plate (see for instance [2]), we next show that the weak limit found in (34) is a Kirchhoff-Love vector field over the set  $\Omega$

LEMMA 4 For each integer  $l \geq 1$ , the function  $\mathbf{u}^l$  belongs to the space

$$\mathbf{V}_{KL}(\Omega) \stackrel{\text{def}}{=} \{ \mathbf{v} \in \mathbf{H}^1(\Omega), e_{i3}(\mathbf{v}) = 0 \text{ in } \Omega \}, \tag{39}$$

which can also be defined as

$$\mathbf{V}_{KL}(\Omega) = \{ \mathbf{v} \in \mathbf{H}^1(\Omega), v_\alpha = \eta_\alpha - x_3 \partial_\alpha \eta_3, \\ v_3 = \eta_3, \text{ with } \eta_\alpha \in H^1(\omega), \eta_3 \in H^2(\omega) \}$$

*Proof* The proof of Lemma 3 given by P G Ciarlet-H Le Dret ([5]) can be used here This result comes from the fact that the sequence  $(\kappa(\mathbf{u}^l(\epsilon)))_{\epsilon > 0}$  is bounded into the space  $L^2(\Omega)$  by (37), and consequently the sequences  $e_{i3}(\mathbf{u}^l(\epsilon))$  converge to  $\mathbf{0}$  in the space  $L^2(\Omega)$  as  $\epsilon \rightarrow 0$  Thus,

$$\|e_{i3}(\mathbf{u}^l)\|_{0,\Omega} \leq \liminf \|e_{i3}(\mathbf{u}^l(\epsilon))\|_{0,\Omega} = 0,$$

hence  $\mathbf{u}^l \in \mathbf{V}_{KL}(\Omega)$  The equivalence between definitions (39) and (40) is established in [3] for instance  $\square$

We next show that the scaled « limit » displacement field vanishes in the inserted portion of the scaled plate In the following lemma, the assumption  $s > 0$  is crucial

LEMMA 5 For each integer  $l \geq 1$ , the weak limit  $\mathbf{u}^l$  satisfies

$$\mathbf{u}^l = 0 \text{ in } \Omega_\beta \tag{40}$$

*Proof* By definition of the space  $\mathbf{V}(\epsilon)$ ,

$$u_\alpha^l(\epsilon)(x) = \epsilon^{s/2} \tilde{u}_\alpha(\epsilon)(\tilde{x}) \quad \text{and} \quad u_3^l(\epsilon)(x) = \epsilon^{s/2+1} \tilde{u}_3(\epsilon)(\tilde{x})$$

at all corresponding points  $\tilde{x} \in \tilde{\Omega}_\beta^\epsilon$  and  $x \in \Omega_\beta$  Hence

$$u_{\alpha|\omega_\beta}^l(\epsilon) = \epsilon^{s/2} \tilde{u}_{\alpha|\omega_\beta}(\epsilon) \quad \text{and} \quad u_{3|\omega_\beta}^l(\epsilon) = \epsilon^{s/2+1} \tilde{u}_{3|\omega_\beta}(\epsilon), \tag{41}$$

where  $\tilde{\omega}_\beta \stackrel{\text{def}}{=} \omega_\beta + \mathbf{t}$ , these equalities being understood as holding up to a translation by the vector  $\mathbf{t}$

The trace operators from  $\mathbf{H}^1(\tilde{\Omega})$  onto  $\mathbf{H}^{1/2}(\tilde{\omega}_\beta)$  and from  $\mathbf{H}^1(\Omega)$  onto  $\mathbf{H}^{1/2}(\omega_\beta)$  being strongly continuous, they are weakly continuous. Consequently, the weakly convergence of the sequence  $(\tilde{\mathbf{u}}^l(\epsilon), \mathbf{u}^l(\epsilon))$  in the space  $\mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega)$  imply

$$\tilde{\mathbf{u}}^l(\epsilon)|_{\tilde{\omega}_\beta} \rightharpoonup \tilde{\mathbf{u}}^l|_{\tilde{\omega}_\beta} \text{ in } H^{1/2}(\tilde{\omega}_\beta) \quad \text{and} \quad \mathbf{u}^l(\epsilon)|_{\omega_\beta} \rightharpoonup \mathbf{u}^l|_{\omega_\beta} \text{ in } \mathbf{H}^{1/2}(\omega_\beta). \quad (42)$$

It then follows from (42) (because  $s > 0$ )

$$u^l_i|_{\omega_\beta} = 0. \quad (43)$$

Since the function  $\mathbf{u}^l$  belongs to the space  $V_{KL}(\Omega)$ , there exist by Lemma 4 functions  $\zeta^l_\alpha \in H^1(\Omega)$  and  $\zeta^l_3 \in H^2(\Omega)$  such that  $u^l_\alpha = \zeta^l_\alpha - x_3 \partial_\alpha \zeta^l_3$  and  $u^l_3 = \zeta^l_3$  in  $\Omega$ . Since  $\zeta^l_i = 0$  in  $\omega_\beta$  (by (44)), we deduce that  $\mathbf{u}^l = 0$  on  $\Omega_\beta = \omega_\beta \times ]-1, 1[$ .  $\square$

To identify the weak limit, we proceed exactly as in [5]. First, by (37) and by Lemma 1, there exists a subsequence, still indexed by  $\epsilon$ , of the sequence  $(\kappa(\mathbf{u}^l(\epsilon)))_{\epsilon > 0}$  and a function  $\kappa^l \in L^2(\Omega)$  such that

$$\kappa(\mathbf{u}^l(\epsilon)) \rightharpoonup \kappa^l \text{ in } L^2(\Omega) \text{ as } \epsilon \rightarrow 0.$$

By arguing as in [5] (Lemma 5), we prove that

$$\kappa_{\alpha 3}(\mathbf{u}^l(\epsilon)) \stackrel{\text{def}}{=} \frac{1}{\epsilon} e_{\alpha 3}(\mathbf{u}^l(\epsilon)) \rightharpoonup 0 \text{ in } L^2(\Omega^*), \quad (44)$$

$$\kappa_{33}(\mathbf{u}^l(\epsilon)) \stackrel{\text{def}}{=} \frac{1}{\epsilon^2} e_{33}(\mathbf{u}^l(\epsilon)) \rightharpoonup -\frac{\lambda}{\lambda + 2\mu} e_{\gamma\gamma}(\mathbf{u}^l) \text{ in } L^2(\Omega^*), \quad (45)$$

where  $\Omega^* = \Omega - \overline{\Omega}_\beta$ . We now obtain the « limit » equation satisfied by  $\mathbf{u}^l$ .

LEMMA 6: *Let  $\omega^* = \omega - \overline{\omega}_\beta$  and  $\gamma^* = \overline{\omega}_\beta \cap \overline{\omega}^*$ . By Lemmas 4 and 5, there exists functions*

$$(\zeta^l_1, \zeta^l_2) \in \mathbf{V}_{12}(\omega^*) = \{(\eta_1, \eta_2) \in H^1(\omega^*) \times H^1(\omega^*); \eta_\alpha = 0 \text{ on } \gamma^*\},$$

$$\zeta^l_3 \in \mathbf{V}_3(\omega^*) = \{\eta_3 \in H^2(\omega^*); \eta_3 = \partial_\nu \eta_3 = 0 \text{ on } \gamma^*\},$$

such that

$$u^l_\alpha = \zeta^l_\alpha - x_3 \partial_\alpha \zeta^l_3 \quad \text{and} \quad u^l_3 = \zeta^l_3.$$

Then

$$\zeta_\alpha^l = 0 \text{ on } \omega^*, \tag{46}$$

and the function  $\zeta_3^l$  solves the variational equation

$$\int_\omega \frac{4}{3} \left\{ \frac{\lambda\mu}{\lambda + 2\mu} \Delta \zeta_3^l \delta_{\alpha\beta} + \mu \partial_{\alpha\beta} \zeta_3^l \right\} \partial_{\alpha\beta} \eta_3 \, dx = 2 \mathcal{A}^l \int_\omega \rho \zeta_3^l \eta_3 \, dx \tag{47}$$

for all  $\eta_3 \in V_3(\omega^*)$ . The bilinear form of the left-hand side of equation (48) is symmetric and coercive over the space  $V_3(\omega^*)$ ; hence, each eigenvalue is strictly positive.

*Proof:* We can again use the proof of Lemma 6 of Ciarlet and Le Dret ([5]), where the functions  $f_i$  appearing in the right-hand side of equation (4.42) are replaced by

$$f_\alpha(\epsilon) = \epsilon^2 \rho \mathcal{A}^l(\epsilon) u'_\alpha(\epsilon) \quad \text{and} \quad f_3(\epsilon) = \rho \mathcal{A}^l(\epsilon) u'_3(\epsilon).$$

Thus there exist subsequences such that  $f_\alpha(\epsilon) \rightarrow 0$  and  $f_3(\epsilon) \rightarrow \rho \mathcal{A}^l u'_3$  in the space  $L^2(\Omega^*)$ . □

To find the equations satisfied by the weak limit  $\mathbf{u}^l$ , we first prove the following technical lemma.

LEMMA 7: We define the space  $\mathbf{W}_\beta$  of functions  $(\tilde{\mathbf{v}}, \mathbf{v}) \in H^1(\tilde{\Omega}) \times H^1(\Omega)$ , such that

$$\begin{cases} \tilde{\mathbf{v}}|_{\tilde{\Gamma}_0} = 0, \\ \mathbf{v} \text{ is independent of the transverse variable } x_3, \\ \tilde{\mathbf{v}}(\tilde{x}) = \mathbf{v}(x) \text{ at all corresponding points } \tilde{x} \in \tilde{\omega}_\beta \text{ and } x \in \omega_\beta, \end{cases}$$

and we define the space :

$$\begin{aligned} \mathbf{W}(\epsilon) = \{ & (\tilde{\mathbf{w}}(\epsilon), \mathbf{w}(\epsilon)) \in \mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega), \tilde{\mathbf{w}}(\epsilon) = 0 \text{ on } \tilde{\Gamma}_0, \text{ and} \\ & \tilde{\mathbf{w}}(\epsilon)(\tilde{x}) = \mathbf{w}(\epsilon)(x) \text{ at all corresponding points } \tilde{x} \in \tilde{\Omega}_\beta^\epsilon \text{ and } x \in \Omega_\beta \}. \end{aligned} \tag{48}$$

Let  $(\tilde{\mathbf{v}}, \mathbf{v})$  be a function of the space  $\mathbf{W}_\beta$  such that

$$\tilde{\mathbf{v}}|_{\tilde{\omega}} \in H^1(\tilde{\omega}). \tag{49}$$

Then there exists a subsequence  $(\tilde{\mathbf{v}}(\epsilon), \mathbf{v}(\epsilon))$  such that

$$\tilde{\mathbf{v}}(\epsilon) \rightarrow \tilde{\mathbf{v}} \text{ in } \mathbf{H}^1(\tilde{\Omega}), \text{ as } \epsilon \rightarrow 0 \tag{50}$$

$$\mathbf{v}(\epsilon) \rightarrow \mathbf{v} \text{ in } \mathbf{H}^1(\Omega) \text{ and } \partial_3 \mathbf{v}(\epsilon) = 0 \text{ in } \Omega \tag{51}$$

$$(\tilde{\mathbf{v}}(\epsilon), \mathbf{v}(\epsilon)) \in \mathbf{W}(\epsilon) \text{ for all } \epsilon > 0. \tag{52}$$

*Proof:* We use the sequence  $(\tilde{\mathbf{v}}(\epsilon), \mathbf{v}(\epsilon))$  introduced by P. G. Ciarlet, H. Le Dret and R. Nzingwa for the normal components ([7], Lemma 6), defined by :

$$\mathbf{v}(\epsilon) = \mathbf{v} \text{ in } \Omega \tag{53}$$

and

$$\tilde{\mathbf{v}}(\epsilon) = \tilde{\mathbf{v}}_{|\tilde{\omega}} \text{ in } \tilde{\Omega}^\epsilon, \tag{54}$$

$$\tilde{\mathbf{v}}(\epsilon) = \frac{2\epsilon - |x_3|}{\epsilon} \tilde{\mathbf{v}}_{|\tilde{\omega}} + \frac{|x_3| - \epsilon}{\epsilon} \tilde{\mathbf{v}} \text{ in } \tilde{\Omega}^{2\epsilon} - \overline{\tilde{\Omega}}^\epsilon \tag{55}$$

$$\tilde{\mathbf{v}}(\epsilon) = \tilde{\mathbf{v}} \text{ in } \overline{\tilde{\Omega}}^{2\epsilon}, \tag{56}$$

where  $\tilde{\Omega}^\epsilon = \tilde{\omega} \times ]-\epsilon, \epsilon[$  and  $\tilde{\Omega}^{2\epsilon} = \tilde{\omega} \times ]-\epsilon, \epsilon[ - 2\epsilon, 2\epsilon[$ . Since the function  $\tilde{\mathbf{v}}_{|\tilde{\omega}}$  is assumed to belong to the space  $\mathbf{H}^1(\tilde{\omega})$  by (50), the function  $\tilde{\mathbf{v}}(\epsilon)$  constructed in (55)-(57) belongs to the space  $\mathbf{H}^1(\tilde{\Omega})$ . Besides, the function  $\mathbf{v}$  being independent of the transverse variable  $x_3$ , so is the function  $\mathbf{v}(\epsilon)$ , and thus we deduce from the assumptions made on the functions  $\mathbf{v}$  and  $\tilde{\mathbf{v}}$ , that

$$\tilde{\mathbf{v}}(\epsilon)(\tilde{x}) = \mathbf{v}(\epsilon)(x)$$

at all corresponding points  $\tilde{x} \in \tilde{\Omega}^\epsilon$  and  $x \in \Omega_\beta$ . The boundary condition  $\tilde{\mathbf{v}}_{|\tilde{\Gamma}_0} = 0$  being satisfied, the pair  $(\tilde{\mathbf{v}}(\epsilon), \mathbf{v}(\epsilon))$  belongs to the space  $\mathbf{W}(\epsilon)$ . The convergence of the sequence  $(\tilde{\mathbf{v}}(\epsilon), \mathbf{v}(\epsilon))$  is established by P. G. Ciarlet, H. Le Dret, R. Nzingwa ([6], [7]). □

We now identify the « limit » problem satisfied by the three-dimensional displacement  $\tilde{u}^l$

LEMMA 8 *The weak limit  $\tilde{\mathbf{u}}^l$  belongs to the space*

$$\tilde{\mathbf{V}} = \{ \tilde{\mathbf{v}} \in \mathbf{H}^1(\tilde{\Omega}), \tilde{\mathbf{v}}|_{\Gamma_0} = 0 \}, \tag{57}$$

and solves the equation

$$\int_{\Omega} \bar{A}e(\tilde{\mathbf{u}}^l) : e(\tilde{\mathbf{v}}) \, d\tilde{x} = A^l \int_{\Omega} \bar{\rho} \tilde{u}_i^l \tilde{v}_i \, d\tilde{x} \tag{58}$$

for all  $\tilde{\mathbf{v}} \in \tilde{\mathbf{V}}$  Besides, the following normalization condition is satisfied

$$\int_{\Omega} \bar{\rho} \tilde{u}_i^l \tilde{u}_i^k \, d\tilde{x} + 2 \int_{\omega} \rho \zeta_3^l \zeta_3^k \, dx = \delta_{kl} \text{ for all } k, l \geq 1 \tag{59}$$

Let us notice that the bilinear form of the left-hand side of equation (59) is coercive over the space  $\tilde{\mathbf{V}}$

*Proof* We proceed in three steps

*First step* we search variational equations satisfied by  $(\tilde{\mathbf{u}}^l(\epsilon), \mathbf{u}^l(\epsilon))$  and associated with the test space  $\mathbf{W}(\epsilon)$

Let  $(\tilde{\mathbf{v}}(\epsilon), \mathbf{v}(\epsilon))$  be a function in the space  $\mathbf{V}(\epsilon)$  To apply Lemma 7, we define the function  $(\tilde{\mathbf{w}}(\epsilon), \mathbf{w}(\epsilon))$  by

$$w_{\alpha}(\epsilon) = \epsilon^{-s/2} v_{\alpha}(\epsilon) \quad \text{and} \quad w_3(\epsilon) = \epsilon^{-s/2-1} v_3(\epsilon) \text{ in } \Omega, \tag{60}$$

and

$$\tilde{\mathbf{w}}(\epsilon) = \tilde{\mathbf{v}}(\epsilon) \text{ in } \tilde{\Omega} \tag{61}$$

The function  $(\tilde{\mathbf{w}}(\epsilon), \mathbf{w}(\epsilon))$  belongs to the space  $\mathbf{W}(\epsilon)$ . Thus, equation (12) can be written as follows :

$$\begin{aligned} & \int_{\tilde{\Omega}} \chi(\tilde{O}_\beta^\epsilon) \tilde{A}e(\tilde{\mathbf{u}}^l(\epsilon)) : e(\tilde{\mathbf{w}}(\epsilon)) \, d\tilde{x} \\ & + \epsilon^{s/2} \int_{\Omega} \{ \lambda e_{\alpha\alpha}(u^l(\epsilon)) e_{\beta\beta}(\mathbf{w}(\epsilon)) + 2 \mu e_{\alpha\beta}(\mathbf{u}^l(\epsilon)) e_{\alpha\beta}(\mathbf{w}(\epsilon)) \} \, dx \\ & + \epsilon^{s/2-1} \int_{\Omega} \lambda e_{\alpha\alpha}(\mathbf{u}^l(\epsilon)) e_{33}(\mathbf{w}(\epsilon)) \, dx \\ & + \epsilon^{s/2-2} \left\{ \int_{\Omega} \lambda e_{33}(\mathbf{u}^l(\epsilon)) e_{\alpha\alpha}(\mathbf{w}(\epsilon)) \, dx \right. \\ & \qquad \qquad \qquad \left. + \int_{\Omega} \mu e_{\alpha 3}(\mathbf{u}^l(\epsilon)) (\partial_3 w_\alpha(\epsilon) + \epsilon \partial_\alpha w_3(\epsilon)) \, dx \right\} \\ & = A^l(\epsilon) \left\{ \int_{\tilde{\Omega}} \tilde{\rho} \tilde{u}_i^l(\epsilon) \tilde{w}_i^l(\epsilon) \, d\tilde{x} + \epsilon^{s/2+1} \int_{\Omega} \rho(\epsilon u_\alpha^l(\epsilon) w_\alpha(\epsilon) \right. \\ & \qquad \qquad \qquad \left. + u_3^l(\epsilon) w_3(\epsilon) \, dx \right\} \quad (62) \end{aligned}$$

for all  $(\tilde{\mathbf{w}}(\epsilon), \mathbf{w}(\epsilon)) \in \mathbf{W}(\epsilon)$ .

*Second step :* we obtain the variational equations that the weak limite  $\tilde{\mathbf{u}}^l$  should satisfy when the test function  $\tilde{\mathbf{v}}$  is subjected to the same restrictions as in Lemma 7.

Let  $\tilde{\mathbf{v}}$  be a function of the space  $\tilde{\mathbf{V}}$  such that

$$\tilde{\mathbf{v}}|_{\tilde{\omega}} \in \mathbf{H}^1(\tilde{\omega}). \tag{63}$$

With such a function  $\tilde{\mathbf{v}}$  is first associated the function  $\mathbf{v}^* \in \mathbf{H}^1(\omega_\beta)$  defined by

$$\mathbf{v}^*(x) = \tilde{\mathbf{v}}|_{\tilde{\omega}}(\tilde{x})$$

at all corresponding points  $x \in \omega_\beta$  and  $\bar{x} \in \bar{\omega}_\beta = \omega_\beta + \mathbf{t}$ . The assumption  $\tilde{\mathbf{v}}|_{\omega_\beta}$  is thus crucially used here. The function  $\mathbf{v}^* \in \mathbf{H}^1(\omega_\beta)$  can be extended to a function still denoted  $\mathbf{v}^* \in \mathbf{H}^1(\omega)$ , because the set  $\omega$  has a Lipschitz-continuous boundary (see e.g. [14], p. 80). We then define the function  $\mathbf{v} \in \mathbf{H}^1(\Omega)$  as follows

$$\mathbf{v}(x) = \mathbf{v}^*(x_1, x_2) \quad \text{for all } x = (x_i) \in \Omega = \omega x] - 1, 1[$$

Thus, the function  $\mathbf{v}$  is independent of the transverse variable  $x_3$  and besides

$$\tilde{\mathbf{v}}(\bar{x}) = \mathbf{v}(x) \quad \text{for all corresponding points } x \in \omega_\beta \text{ and } \bar{x} \in \bar{\omega}_\beta$$

The function  $(\tilde{\mathbf{v}}, \mathbf{v})$  then belongs to the space  $\mathbf{W}_\beta$  defined in (49). Consequently, by (64) and by Lemma 7, there exists a subsequence  $\{\tilde{\mathbf{v}}(\epsilon), \mathbf{v}(\epsilon)\}_{\epsilon > 0}$  of the space  $\mathbf{W}(\epsilon)$  verifying conditions (51)-(52). With these functions  $(\tilde{\mathbf{v}}(\epsilon), \mathbf{v}(\epsilon)) \in \mathbf{W}(\epsilon)$ , equation (63) implies (because  $\partial_3 \mathbf{v}(\epsilon) = 0$  in  $\Omega$ )

$$\begin{aligned} & \int_{\Omega} \chi(\bar{O}_\beta^\epsilon) \bar{A}e(\bar{\mathbf{u}}^l(\epsilon)) \cdot e(\tilde{\mathbf{v}}(\epsilon)) \, d\bar{x} \\ & + \epsilon^{1/2} \int_{\Omega} \{ \lambda e_{\alpha\alpha}(\mathbf{u}^l(\epsilon)) e_{\beta\beta}(\mathbf{v}(\epsilon)) + 2\mu e_{\alpha\beta}(\mathbf{u}^l(\epsilon)) e_{\alpha\beta}(\mathbf{v}(\epsilon)) \} \, dx \\ & + \epsilon^{1/2} \left\{ \int_{\Omega} \lambda \kappa_{33}(\mathbf{u}^l(\epsilon)) e_{\alpha\alpha}(\mathbf{v}(\epsilon)) \, dx + \int_{\Omega} \mu \kappa_{\alpha 3}(\mathbf{u}^l(\epsilon)) \partial_\alpha v_3(\epsilon) \, dx \right\} \\ & = A^l(\epsilon) \left\{ \int_{\Omega} \bar{\rho} \bar{u}_i^l(\epsilon) \bar{v}_i^l(\epsilon) \, d\bar{x} + \epsilon^{1/2+1} \int_{\Omega} \rho(\epsilon u_\alpha^l(\epsilon) v_\alpha(\epsilon) \right. \\ & \qquad \qquad \qquad \left. + u_3^l(\epsilon) v_3(\epsilon)) \, dx \right\} \quad (64) \end{aligned}$$

Let us recall the following properties (cf. (34), (37) and (52))

$$\begin{cases} \mathbf{u}^l(\epsilon) \text{ is bounded independently of } \epsilon \text{ in the space } \mathbf{H}^1(\Omega), \\ \kappa_{i3}(\mathbf{u}^l(\epsilon)) \text{ is bounded independently of } \epsilon \text{ in the space } \mathbf{L}^2(\Omega), \\ \mathbf{v}(\epsilon) \rightarrow \mathbf{v} \text{ in the space } \mathbf{H}^1(\Omega) \text{ as } \epsilon \rightarrow 0. \end{cases}$$

We then deduce that all terms in equation (65) in which integrals over the open set  $\Omega$  appear, converge towards 0, because  $s > 0$ . We now establish the convergence

$$\begin{aligned} I(\epsilon) &\stackrel{\text{def}}{=} \int_{\tilde{\Omega}} \chi(\tilde{\Omega}_\beta^\epsilon) \tilde{A}e(\tilde{\mathbf{u}}^l(\epsilon)) : e(\tilde{\mathbf{v}}(\epsilon)) \, d\tilde{x} \\ &\rightarrow \int_{\tilde{\Omega}} \tilde{A}e(\tilde{\mathbf{u}})^l : e(\tilde{\mathbf{v}}) \, d\tilde{x} \text{ as } \epsilon \rightarrow 0. \end{aligned} \quad (65)$$

To this end, we write  $I(\epsilon)$  as

$$I(\epsilon) = \int_{\tilde{\Omega}} \tilde{A}e(\tilde{\mathbf{u}}^l(\epsilon)) : e(\tilde{\mathbf{v}}(\epsilon)) \, d\tilde{x} - \int_{\tilde{\Omega}} \chi(\tilde{\Omega}_\beta^\epsilon) \tilde{A}e(\tilde{\mathbf{u}}^l(\epsilon)) : e(\tilde{\mathbf{v}}(\epsilon)) \, d\tilde{x}.$$

First,

$$\int_{\tilde{\Omega}} \chi(\tilde{\Omega}_\beta^\epsilon) \tilde{A}e(\tilde{\mathbf{u}}^l(\epsilon)) : e(\tilde{\mathbf{v}}(\epsilon)) \, d\tilde{x} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0,$$

since the family  $\{\chi(\tilde{\Omega}_\beta^\epsilon) \tilde{A}e(\tilde{\mathbf{v}}(\epsilon))\}_{\epsilon > 0}$  converges strongly to  $\mathbf{0}$  in the space  $L^2(\tilde{\Omega})$ , and by (33). Furthermore

$$\int_{\tilde{\Omega}} \tilde{A}e(\tilde{\mathbf{u}}^l(\epsilon)) : e(\tilde{\mathbf{v}}(\epsilon)) \, d\tilde{x} \rightarrow \int_{\tilde{\Omega}} \tilde{A}e(\tilde{\mathbf{u}}^l) : e(\tilde{\mathbf{v}}) \, d\tilde{x} \quad \text{as } \epsilon \rightarrow 0,$$

since the family  $(\tilde{\mathbf{u}}^l(\epsilon))_{\epsilon > 0}$  weakly converges to  $\tilde{\mathbf{u}}^l$  and the family  $(\tilde{\mathbf{v}}(\epsilon))_{\epsilon > 0}$  strongly converges to  $\tilde{\mathbf{v}}$  in the space  $\mathbf{H}^1(\tilde{\Omega})$ . Hence (66) is proved. In the same way, we can establish

$$\int_{\tilde{\Omega}} \tilde{\rho} \tilde{u}_i^l(\epsilon) \tilde{v}_i^l(\epsilon) \, d\tilde{x} \rightarrow \int_{\tilde{\Omega}} \tilde{\rho} \tilde{u}_i^l \tilde{v}_i^l \, d\tilde{x}.$$

Consequently equation (65) implies

$$\int_{\tilde{\Omega}} \tilde{A}e(\tilde{\mathbf{u}}^l) : e(\tilde{\mathbf{v}}) \, d\tilde{x} = A^l \int_{\tilde{\Omega}} \tilde{\rho} \tilde{u}_i^l \tilde{v}_i^l \, d\tilde{x} \quad (66)$$

for all functions  $\tilde{v} \in \tilde{V}$  subjected to restriction (64) Thus, equation (65) is satisfied as soon as the function  $\tilde{v}$  belongs to the space

$$\tilde{V}_0 \stackrel{\text{def}}{=} \{ \tilde{v} \mid_{\bar{\Omega}}, \tilde{v} \in \mathcal{C}_c^\infty(\bar{\Omega}), \tilde{v}|_{\Gamma_0} = 0 \}$$

By density of the space  $\tilde{V}_0$  into the space  $\tilde{V}$  with respect to the norm of  $\mathbf{H}^1(\bar{\Omega})$ , equation (65) is satisfied for all functions of the space  $\tilde{V}$

*Third step* The normalization condition follows from the strong convergences of family  $(\tilde{\mathbf{u}}^l(\epsilon), \mathbf{u}^l(\epsilon))_{\epsilon > 0}$  towards  $(\tilde{\mathbf{u}}^l, \mathbf{u}^l)$  into the space  $L^2(\bar{\Omega}) \times L^2(\Omega)$ , by Rellich-Kondrasov's theorem and by (13)  $\square$

We now establish the *strong* convergence in the space  $\mathbf{H}^1(\bar{\Omega}) \times \mathbf{H}^1(\Omega)$  of the subsequence  $(\tilde{\mathbf{u}}^l(\epsilon), \mathbf{u}^l(\epsilon))$

LEMMA 9 *By Lemma 3 and by (45)-(46), there exists at least one subsequence verifying the following properties*

$$A^l(\epsilon) \rightarrow A^l, \tag{67}$$

$$(\tilde{\mathbf{u}}^l(\epsilon), \mathbf{u}^l(\epsilon)) \rightharpoonup (\tilde{\mathbf{u}}^l, \mathbf{u}^l) \text{ in } \mathbf{H}^1(\bar{\Omega}) \times \mathbf{H}^1(\Omega), \tag{68}$$

$$\kappa_{\alpha 3}(\epsilon) \stackrel{\text{def}}{=} \frac{1}{\epsilon} e_{\alpha 3}(\mathbf{u}^l(\epsilon)) \rightarrow \mathbf{0} \text{ in } L^2(\Omega^*), \tag{69}$$

$$\kappa_{33}(\epsilon) \stackrel{\text{def}}{=} \frac{1}{\epsilon^2} e_{33}(\mathbf{u}^l(\epsilon)) \rightarrow -\frac{\lambda}{\lambda + 2\mu} e_{\gamma\gamma}(\mathbf{u}^l) \text{ in } L^2(\Omega^*) \tag{70}$$

as  $\epsilon \rightarrow 0$ , for all integers  $l \geq 1$  All convergences (69) (71) are strong

*Proof* We proceed as in [5] (Lemma 10) Let  $l \geq 1$  be a given integer By Lemma 3, the subsequence  $(\tilde{\mathbf{u}}^l(\epsilon), \mathbf{u}^l(\epsilon))$  converges weakly to  $(\tilde{\mathbf{u}}^l, \mathbf{u}^l)$  in the space  $\mathbf{H}^1(\bar{\Omega}) \times \mathbf{H}^1(\Omega)$ , and strongly to  $(\tilde{\mathbf{u}}^l, \mathbf{u}^l)$  in the space  $L^2(\bar{\Omega}) \times L^2(\Omega)$  by the Rellich-Kondrasov's theorem It thus suffices to show that family  $(e(\tilde{\mathbf{u}}^l(\epsilon)), e(\mathbf{u}^l(\epsilon)))_{\epsilon > 0}$  strongly converges in the space  $L^2(\bar{\Omega}) \times L^2(\Omega)$ , as the conclusion will then follow from Korn's inequality applied in the spaces  $\mathbf{H}^1(\bar{\Omega})$  and  $\mathbf{H}^1(\Omega)$

Let  $\kappa^l = (\kappa_{ij}^l)$  denote the weak limit of the sequence  $(\kappa(\mathbf{u}^l(\epsilon)))_{\epsilon > 0}$  in the space  $L^2(\Omega^*)$  (see Lemma 3 and (45)-(46)), which is given by

$$\kappa_{\alpha\beta}^l = e_{\alpha\beta}(\mathbf{u}^l), \quad \kappa_{\alpha 3}^l = 0, \quad \kappa_{33}^l = -\frac{\lambda}{\lambda + 2\mu} e_{\gamma\gamma}(\mathbf{u}^l) \text{ in } \Omega^* \tag{71}$$

By inequalities (36), there exists a constant  $c > 0$  such that

$$\begin{aligned}
 & c \{ |e(\tilde{u}^l(\epsilon)) - e(\tilde{u}^l)|_0^2_{\Omega} + |\kappa(u^l(\epsilon)) - \kappa^l|_0^2_{\Omega'} + |\kappa(u^l(\epsilon))|_0^2_{\Omega_\beta} \} \\
 & \leq \int_{\Omega} \chi(\tilde{\Omega}_\beta^\epsilon) A(e(\tilde{\mathbf{u}}^l(\epsilon)) - e(\tilde{\mathbf{u}}^l)) (e(\tilde{\mathbf{u}}^l(\epsilon)) - e(\tilde{\mathbf{u}}^l)) d\tilde{x} \\
 & + \int_{\Omega} \chi(\tilde{O}_\beta^\epsilon) \tilde{A}(e(\tilde{\mathbf{u}}^l(\epsilon)) - e(\tilde{\mathbf{u}}^l)) (e(\tilde{\mathbf{u}}^l(\epsilon)) - e(\tilde{\mathbf{u}}^l)) d\tilde{x} \\
 & + \int_{\Omega'} A(\kappa(\mathbf{u}^l(\epsilon)) - \kappa^l) (\kappa(\mathbf{u}^l(\epsilon)) - \kappa^l) dx + \\
 & + \int_{\Omega_\beta} A(\kappa(\mathbf{u}^l(\epsilon))) \kappa(\mathbf{u}^l(\epsilon)) dx \quad (72)
 \end{aligned}$$

The aim is to prove that the right-hand side of the above inequality converges towards 0. Let us study the first term of the right hand side of inequality (73)

$$\begin{aligned}
 & \int_{\Omega} \chi(\tilde{\Omega}_\beta^\epsilon) A(e(\tilde{\mathbf{u}}^l(\epsilon)) - e(\tilde{u}^l)) (e(\tilde{\mathbf{u}}^l(\epsilon)) - e(\tilde{u}^l)) d\tilde{x} \\
 & \leq 2 \int_{\Omega} \chi(\tilde{\Omega}_\beta^\epsilon) Ae(\tilde{\mathbf{u}}^l) (e(\tilde{\mathbf{u}}^l) - e(\tilde{\mathbf{u}}^l(\epsilon))) d\tilde{x} + \\
 & + \int_{\Omega} \chi(\tilde{\Omega}_\beta^\epsilon) Ae(\tilde{\mathbf{u}}^l(\epsilon)) e(\tilde{\mathbf{u}}^l(\epsilon)) d\tilde{x} \quad (73)
 \end{aligned}$$

The transmission conditions (8)-(9) for the three-dimensional problem imply

$$\begin{aligned}
 & \int_{\Omega} \chi(\tilde{\Omega}_\beta^\epsilon) Ae(\tilde{\mathbf{u}}^l(\epsilon)) e(\tilde{\mathbf{u}}^l(\epsilon)) d\tilde{x} \\
 & = \epsilon^{1-s} \int_{\Omega} \chi(\Omega_\beta) A\kappa(\mathbf{u}^l(\epsilon)) \kappa(\mathbf{u}^l(\epsilon)) dx
 \end{aligned}$$

Hence, since  $\epsilon$  may be chosen  $\leq 1$ , and since  $s \leq 1$  by assumption, we deduce

$$\int_{\Omega} \chi(\tilde{\Omega}_\beta^\epsilon) Ae(\tilde{\mathbf{u}}^l(\epsilon)) e(\tilde{\mathbf{u}}^l(\epsilon)) d\tilde{x} \leq \int_{\Omega} \chi(\Omega_\beta) A\kappa(\mathbf{u}^l(\epsilon)) \kappa(\mathbf{u}^l(\epsilon)) dx \quad (74)$$

Inequalities (73)-(74)-(75) then imply that there exists a constant  $d > 0$  such that

$$\begin{aligned}
 & d\{ |e(\tilde{\mathbf{u}}^l(\epsilon)) - e(\tilde{\mathbf{u}}^l)'|_{0, \tilde{\Omega}}^2 + |\kappa(\mathbf{u}^l(\epsilon)) - \kappa^l|_{0, \Omega^*}^2 + |\kappa(\mathbf{u}^l(\epsilon))|_{0, \tilde{\Omega}_\beta}^2 \} \\
 & \leq \int_{\tilde{\Omega}} \chi(\tilde{\Omega}_\beta^\epsilon) A e(\tilde{\mathbf{u}}^l) : (e(\tilde{\mathbf{u}}^l) - e(\tilde{\mathbf{u}}^l(\epsilon))) \, d\tilde{x} \\
 & \quad + \int_{\tilde{\Omega}} \chi(\tilde{O}_\beta^\epsilon) \bar{A}(e(\tilde{\mathbf{u}}^l(\epsilon)) - e(\tilde{\mathbf{u}}^l)) : (e(\tilde{\mathbf{u}}^l(\epsilon)) - e(\tilde{\mathbf{u}}^l)) \, d\tilde{x} \\
 & \quad + \int_{\Omega^*} A(\kappa(\mathbf{u}^l(\epsilon)) - \kappa^l) : (\kappa(\mathbf{u}^l(\epsilon)) - \kappa^l) \, dx \\
 & \quad + \int_{\Omega_\beta} A\kappa(\mathbf{u}^l(\epsilon)) : \kappa(\mathbf{u}^l(\epsilon)) \, dx . \tag{75}
 \end{aligned}$$

We now show that the right-hand side of (76) converges to 0 as  $\epsilon$  approaches 0.

First,

$$\int_{\tilde{\Omega}} \chi(\tilde{\Omega}_\beta^\epsilon) A e(\tilde{\mathbf{u}}^l) : (e(\tilde{\mathbf{u}}^l) - e(\tilde{\mathbf{u}}^l(\epsilon))) \, d\tilde{x} \rightarrow 0 ,$$

since the family  $\{\chi(\tilde{\Omega}_\beta^\epsilon) A e(\tilde{\mathbf{u}}^l)\}_{\epsilon > 0}$  strongly converges to  $\mathbf{0}$  in the space  $\mathbf{L}^2(\tilde{\Omega})$  and the family  $\{e(\tilde{\mathbf{u}}^l) - e(\tilde{\mathbf{u}}^l(\epsilon))\}_{\epsilon > 0}$  weakly converges to  $\mathbf{0}$  in the space  $\mathbf{L}^2(\tilde{\Omega})$ . We then have to prove that the following quantities  $J(\epsilon)$  converge to 0 :

$$\begin{aligned}
 J(\epsilon) & \stackrel{\text{def}}{=} \int_{\tilde{\Omega}} \chi(\tilde{O}_\beta^\epsilon) \bar{A}(e(\tilde{\mathbf{u}}^l(\epsilon)) - e(\tilde{\mathbf{u}}^l)) : (e(\tilde{\mathbf{u}}^l(\epsilon)) - e(\tilde{\mathbf{u}}^l)) \, d\tilde{x} \\
 & \quad + \int_{\Omega^*} A(\kappa(\mathbf{u}^l(\epsilon)) - \kappa^l) : (\kappa(\mathbf{u}^l(\epsilon)) - \kappa^l) \, dx \\
 & \quad + \int_{\Omega_\beta} A\kappa(\mathbf{u}^l(\epsilon)) : \kappa(\mathbf{u}^l(\epsilon)) \, dx .
 \end{aligned}$$

First, we split  $J(\epsilon)$  as

$$J(\epsilon) = J_1(\epsilon) + J_2(\epsilon) ,$$

with

$$J_1(\epsilon) = \int_{\tilde{\Omega}} \chi(\tilde{O}_\beta^\epsilon) \bar{A} e(\tilde{\mathbf{u}}^l(\epsilon)) : e(\tilde{\mathbf{u}}^l(\epsilon)) \, d\tilde{x} + \int_{\Omega} A\kappa(\mathbf{u}^l(\epsilon)) : \kappa(\mathbf{u}^l(\epsilon)) \, dx$$

and

$$J_2(\epsilon) = \int_{\tilde{\Omega}} \chi(\tilde{O}_\beta^\epsilon) \tilde{A}e(\tilde{\mathbf{u}}^l) : (e(\tilde{\mathbf{u}}^l) - 2e(\tilde{\mathbf{u}}^l(\epsilon))) d\tilde{x} + \int_{\Omega^*} A\kappa^l : (\kappa^l - 2k(\mathbf{u}^l(\epsilon))) dx .$$

Since the variational equations (12) are satisfied in particular by the pair  $(\tilde{\mathbf{u}}^l(\epsilon), \mathbf{u}^l(\epsilon))$  in the space  $\mathbf{V}(\epsilon)$ , we find that

$$J_1(\epsilon) = A^l(\epsilon) .$$

Moreover we prove that, as before,

$$J_2(\epsilon) \rightarrow J_2 \stackrel{\text{def}}{=} - \int_{\tilde{\Omega}} \tilde{A}e(\tilde{\mathbf{u}}^l) : e(\tilde{\mathbf{u}}^l) d\tilde{x} - \int_{\Omega^*} A\kappa^l : \kappa^l dx .$$

Indeed the family  $(\kappa(\mathbf{u}^l(\epsilon)))_{\epsilon > 0}$  converges weakly to  $\kappa^l$  in  $\mathbf{L}^2(\Omega^*)$  and

$$\begin{aligned} & \int_{\tilde{\Omega}} \chi(\tilde{O}_\beta^\epsilon) \tilde{A}e(\tilde{\mathbf{u}}^l) : (e(\tilde{\mathbf{u}}^l) - 2e(\tilde{\mathbf{u}}^l(\epsilon))) d\tilde{x} \\ &= \int_{\tilde{\Omega}} \tilde{A}e(\tilde{\mathbf{u}}^l) : (e(\tilde{\mathbf{u}}^l) - 2e(\tilde{\mathbf{u}}^l(\epsilon))) d\tilde{x} \\ & - \int_{\tilde{\Omega}} \chi(\tilde{\Omega}_\beta^\epsilon) \tilde{A}e(\tilde{\mathbf{u}}^l) : (e(\tilde{\mathbf{u}}^l) - 2e(\tilde{\mathbf{u}}^l(\epsilon))) d\tilde{x} , \end{aligned}$$

with

$$\int_{\tilde{\Omega}} \chi(\tilde{\Omega}_\beta^\epsilon) \tilde{A}e(\tilde{\mathbf{u}}^l) : (e(\tilde{\mathbf{u}}^l) - 2e(\tilde{\mathbf{u}}^l(\epsilon))) d\tilde{x} \rightarrow 0 ,$$

since the family  $(\chi(\tilde{\Omega}_\beta^\epsilon) e(\tilde{\mathbf{u}}))_{\epsilon > 0}$  strongly converges to  $\mathbf{0}$  in the space  $\mathbf{L}^2(\tilde{\Omega})$ . Now, it suffices to show that  $J_2 = -A^l$ , as the conclusion will then follow from the convergence of  $J(\epsilon)$  towards 0. By Lemma 8,

$$\int_{\tilde{\Omega}} \tilde{A}e(\tilde{\mathbf{u}}^l) : e(\tilde{\mathbf{u}}^l) d\tilde{x} = A^l \int_{\tilde{\Omega}} \tilde{\rho} \tilde{u}_i^l \tilde{u}_i^l d\tilde{x} . \tag{76}$$

Besides, a simple computation based on the definition of  $\kappa^l$  and on equation (48) satisfied by the function  $\zeta_3^l$  shows that

$$\int_{\Omega^*} A\kappa^l : \kappa^l dx = A^l \int_{\Omega^*} \rho \{u_3^l\}^2 dx ,$$

and consequently

$$J_2 = - A^l \left\{ \int_{\Omega} \bar{\rho} \bar{u}_i^l \bar{u}_i^l d\bar{x} + \int_{\Omega} \rho \{u_3^l\}^2 dx \right\}$$

Equality (60) then implies that  $J_2 = - A^l$  □

We have shown the strong convergence of each subsequence  $(A^l(\epsilon), (\bar{\mathbf{u}}^l(\epsilon), \mathbf{u}^l(\epsilon)))$  in the space  $]0, +\infty[ \times \mathbf{H}^1(\bar{\Omega}) \times \mathbf{H}^1(\Omega)$  to a solution  $(A^l, (\bar{\mathbf{u}}^l, \mathbf{u}^l))$  of the « limit » eigenvalue problem (48)-(59), which satisfies the orthonormalization condition (60) This « limit » eigenvalue problem can be written as

$$B((\bar{\mathbf{u}}^l, \zeta_3^l), (\bar{\mathbf{v}}, \eta_3^l)) = A^l \left\{ \int_{\Omega} \bar{\rho} \bar{u}_i^l \bar{v}_i d\bar{x} + 2 \int_{\omega} \rho \zeta_3^l \eta_3 dx \right\} \tag{77}$$

for all  $(\bar{\mathbf{v}}, \eta_3^l) \in \bar{\mathbf{V}} \times V_3(\omega^*)$ , where

$$B((\bar{\mathbf{u}}^l, \zeta_3^l), (\bar{\mathbf{v}}, \eta_3^l)) \stackrel{\text{def}}{=} \int_{\omega} \frac{4}{3} \left\{ \frac{\lambda \mu}{\lambda + 2 \mu} \Delta \zeta_3 \delta_{\alpha\beta} + \mu \partial_{\alpha\beta} \zeta_3 \right\} \partial_{\alpha\beta} \eta_3 dx + \int_{\Omega} \bar{A} e(\bar{\mathbf{u}}^l) e(\bar{\mathbf{v}}) dx$$

Let us recall that F Bourquin and P G Ciarlet have studied the same « initial » three-dimensional problem, but under other asymptotical assumptions The « limit » problem then obtained is a coupled eigenvalue problem All the preceding lemmas are an adaptation of Lemmas 1 to 10 of F Bourquin and P G Ciarlet ([1]) Consequently we do not prove the following propositions which are consequences of preceding lemmas, since their demonstrations are detailed in [1] We can apply the results of F Bourquin and P G Ciarlet ([1], Lemmas 11 and 12) because the bilinear form  $B$  defined in (78) is coercive over the space  $\bar{\mathbf{V}} \times V_3(\omega^*)$

**PROPOSITION 1** *Let  $(A^l, (\bar{\mathbf{u}}^l, \mathbf{u}^l))$ ,  $l \geq 1$  be the eigensolutions of problem (78) found as limits of the subsequence  $(A^l(\epsilon), (\bar{\mathbf{u}}^l(\epsilon), \mathbf{u}^l(\epsilon)))_{\epsilon > 0}$ ,  $l \geq 1$ , orthonormalized as in (13) of Problem (12) Then the sequence  $(\bar{\mathbf{u}}^l, \zeta_3^l)$  of eigenfunctions orthonormalized as in (60) associated with the eigenvalues  $(A^l)_{l \geq 1}$ , counting their multiplicites, of problem (78) form a complete set of the space  $\mathbf{H}^1(\bar{\Omega}) \times V_3(\omega^*)$*

**PROPOSITION 2** *For each integer  $l \geq 1$ , the whole family  $(A^l(\epsilon))_{\epsilon > 0}$  converges as  $\epsilon \rightarrow 0$  If, for a given integer  $l \geq 1$ , the eigenvalue  $A^l$  of*

problem (78) is simple, there exists  $\epsilon_0(l) > 0$  such that for all  $\epsilon \leq \epsilon_0(l)$ , the eigenvalue  $\Lambda^l(\epsilon)$  of problem (12) is simple and there exists for all  $\epsilon \leq \epsilon_0(l)$ , an eigenfunction  $(\tilde{\mathbf{u}}(\epsilon), \mathbf{u}(\epsilon))$  associated with  $\Lambda^l(\epsilon)$ , satisfying the orthonormalization condition (13), such that the whole family  $(\tilde{\mathbf{u}}(\epsilon), \mathbf{u}(\epsilon))$  converges into the space  $\mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega)$  as  $\epsilon \rightarrow 0$ .

5. CONCLUSIONS AND COMMENTS

1. As in [1], [5], we describe the « limit » eigenvalue problem satisfied by the limit functions  $(\tilde{\mathbf{u}}^l, \mathbf{u}^l)$  of the family  $(\tilde{\mathbf{u}}^l(\epsilon), \mathbf{u}^l(\epsilon))_{\epsilon > 0}$  for each integer  $l \geq 1$ . This limit problem is, at least formally, that associated with the variational equations (17)-(18). By Theorem 1, there exists a function  $\zeta_3^l \in H^2(\omega)$  such that  $u_\alpha^l = -x_3 \partial_\alpha \zeta_3^l$  and  $u_3^l = \zeta_3^l$  in the set  $\Omega$ , which solves the following eigenvalue problem, under some regularity assumptions about the function  $\zeta_3$ , (see for example [2])

$$\begin{cases} -\partial_{\alpha\beta} m_{\alpha\beta}(\zeta_3^l) & = 2 \Lambda^l \rho \zeta_3^l \text{ in } \omega^*, \\ \zeta_3^l = \partial_\nu \zeta_3^l & = 0 \text{ on } \gamma^* \\ m_{\alpha\beta}(\zeta_3^l) \nu_\alpha \nu_\beta & = 0 \text{ on } \gamma_1^* \\ \partial_\tau (m_{\alpha\beta}(\zeta_3^l) \nu_\alpha \tau_\beta) + \{\partial_\alpha m_{\alpha\beta}(\zeta_3^l)\} \nu_\beta & = 0 \text{ on } \gamma_1^*, \end{cases}$$

where  $\gamma_1^* = (\partial\omega^* - \gamma^*)$  and  $m_{\alpha\beta}(\zeta_3^l)$  is defined by

$$m_{\alpha\beta}(\zeta_3^l) = -\frac{4\mu}{3} \left\{ \partial_{\alpha\beta} \zeta_3^l + \frac{\lambda}{\lambda + 2\mu} \Delta \zeta_3^l \delta_{\alpha\beta} \right\},$$

and  $(\nu_\alpha), (\tau_\alpha)$  denote the unit outer normal and unit tangential vectors along  $\partial\omega$ ,  $\partial\tau$  denotes the tangential derivative operator along  $\partial\omega$ . In the same way, we can verify that a smooth enough solution  $\tilde{\mathbf{u}}^l$  of the variational equations (18) solves the following equations in the set  $\tilde{\Omega}$

$$\begin{cases} -\bar{\partial}_j \bar{\sigma}_j(\tilde{\mathbf{u}}^l) = \Lambda^l \bar{\rho} \tilde{u}_i & \text{in } \tilde{\Omega}, \\ \bar{\sigma}_j(\tilde{\mathbf{u}}^l) \bar{n}_j = 0 & \text{on } \partial\tilde{\Omega} - \tilde{\Gamma}_0, \\ \tilde{\mathbf{u}}^l = 0 & \text{on } \tilde{\Gamma}_0, \end{cases}$$

where

$$\bar{\sigma}_j(\tilde{\mathbf{u}}^l) = \bar{\lambda} e_{pp}(\tilde{\mathbf{u}}^l) + 2 \bar{\mu} e_j(\tilde{\mathbf{u}}^l),$$

and  $\bar{\mathbf{n}} = (\bar{n}_i)$  is the unit outer normal vector along  $\partial\tilde{\Omega}$ .

2. As in [5], we « de-scale » equations of Part 1 to obtain the equations satisfy by the « de-scaled » eigenfunctions  $(\tilde{\mathbf{u}}^{l,\epsilon}, \zeta^{l,\epsilon})_{\epsilon>0}$  attached to the physical structure. Thus, with the « limit » eigenfunctions  $\tilde{\mathbf{u}}^l: \bar{\Omega} \rightarrow \mathbf{R}^3$  we associate the limit vectors fields  $\tilde{\mathbf{u}}^{l,\epsilon} = (\tilde{\mathbf{u}}_i^{l,\epsilon}): \bar{O} \rightarrow \mathbf{R}^3$  by letting, in view of (7),

$$\tilde{u}_i^{l,\epsilon}(\tilde{x}^\epsilon) = \epsilon^{2+s/2} \tilde{u}_i^l(\tilde{x}) \tag{78}$$

at all corresponding points  $x^\epsilon \in \bar{O}$  and  $\tilde{x} \in \bar{\Omega}$ ; and with the « limit » eigenfunctions  $\zeta^l = (\zeta_i^l): \bar{\omega} \rightarrow \mathbf{R}^3$ , we associate the limit functions  $\tilde{\zeta}_3^{l,\epsilon} = (\tilde{\zeta}_{3,i}^{l,\epsilon}): \bar{\omega} \rightarrow \mathbf{R}$  by letting, in view of (6),

$$\zeta_\alpha^{l,\epsilon}(x_1, x_2) = \epsilon^2 \zeta_\alpha^l(x_1, x_2) \quad \text{for all } (x_1, x_2) \in \bar{\omega}. \tag{79}$$

$$\tilde{\zeta}_3^{l,\epsilon}(x_1, x_2) = \epsilon \zeta_\alpha^l(x_1, x_2) \quad \text{for all } (x_1, x_2) \in \bar{\omega}. \tag{80}$$

Then, we obtain the following equations satisfied by the « de-scaled » eigenfunctions  $(\tilde{\mathbf{u}}^{l,\epsilon}, \zeta^{l,\epsilon})_{\epsilon>0}$ ,

$$\begin{cases} \zeta_1^{l,\epsilon} = \zeta_2^{l,\epsilon} & = 0 & \text{in } \omega^*, \\ -\epsilon^2 \partial_{\alpha\beta} m_{\alpha\beta}^\epsilon(\zeta_3^{l,\epsilon}) & = 2 A^{l,\epsilon} \rho^\epsilon \zeta_3^{l,\epsilon} & \text{in } \omega^*, \\ \zeta_3^{l,\epsilon} = \partial_\nu \zeta_3^{l,\epsilon} & = 0 & \text{on } \gamma^* \\ m_{\alpha\beta}^\epsilon(\zeta_3^{l,\epsilon}) \nu_\alpha \nu_\beta & = 0 & \text{on } \gamma_1^* \\ \partial_\tau (m_{\alpha\beta}^\epsilon(\zeta_3^{l,\epsilon}) \nu_\alpha \tau_\beta) + \{ \partial_\alpha m_{\alpha\beta}^\epsilon(\zeta_3^{l,\epsilon}) \} \nu_\beta & = 0 & \text{on } \gamma_1^*, \end{cases}$$

where  $A^{l,\epsilon} = A^l(\epsilon)$ ,  $m_{\alpha\beta}^\epsilon(\zeta_3^{l,\epsilon})$  is defined by

$$m_{\alpha\beta}^\epsilon(\zeta_3^{l,\epsilon}) = -\frac{4\mu^\epsilon}{3} \left\{ \partial_{\alpha\beta\tau_3} \zeta_3^{l,\epsilon} + \frac{\lambda^\epsilon}{\lambda^\epsilon + 2\mu^\epsilon} \Delta \zeta_3^{l,\epsilon} \delta_{\alpha\beta} \right\},$$

and

$$\begin{cases} -\tilde{\partial}_j^\epsilon \tilde{\sigma}_{ij}(\tilde{\mathbf{u}}^\epsilon) = A^{l,\epsilon} \tilde{\rho}^\epsilon \tilde{u}_i^\epsilon & \text{in } O, \\ \tilde{\sigma}_{ij}(\tilde{\mathbf{u}}^\epsilon) \tilde{n}_j & = 0 & \text{on } \partial\tilde{\Omega} - \tilde{\Gamma}_O, \\ \tilde{\mathbf{u}} & = O & \text{on } \tilde{\Gamma}_O, \end{cases}$$

where  $\tilde{\partial}_j^\epsilon = \partial/\partial\tilde{x}_j^\epsilon$  and

$$\tilde{\sigma}_{ij}(\tilde{\mathbf{u}}^\epsilon) = \tilde{\lambda}^\epsilon e_{pp}(\tilde{\mathbf{u}}^\epsilon) + 2\mu^\epsilon e_{ij}(\tilde{\mathbf{u}}^\epsilon).$$

3. We first recall the conclusions derived by P. G. Ciarlet and H. Le Dret ([5]) about the « static » problem associated with the eigenvalue problem (3) : The three-dimensional body becomes rigid in the limit ; the inserted portion plate also becomes rigid in the limit, and the « limit » normal displacement  $\zeta_3^{l,\epsilon}$  solves the classical two-dimensional plate eigenvalue equations in  $\omega^*$ .

4. The spectrum of problem (78) is constituted of the eigenvalues of problems (17)-(18). If the intersection of the spectrums of problems (17) and (18) is empty, then, for all integer  $l \geq 1$ , either  $\mathbf{u}^l = \mathbf{0}$ , or  $\zeta_3^l = 0$ . If there exists an eigenvalue that belongs to the intersection of both spectrums associated with problem (17) and (18), then the three-dimensional supporting structure and the plate « vibrate simultaneously » as  $\epsilon \rightarrow 0$ .

In both cases, the displacements of the plate are larger than the displacements of the three-dimensionnal part  $O$ . Indeed by (6)-(7), the tangential components of the « limit » displacements of the plate vary as  $\epsilon^2$ , and the normal component varies as  $\epsilon$ , while the displacement of the three-dimensional supporting structure varies as  $\epsilon^{2+s/2}$ , with  $s > 0$ .

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