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A REGULARITY RESULT FOR A LINEAR MEMBRANE SHELL PROBLEM (*)

by K. GENEVEY (1)

Abstract. — *We consider the membrane shell equations of a linearly elastic shell, clamped along its entire boundary and whose middle surface is uniformly elliptic. The fact that an associated reduced problem is amenable to the theory of S. Agmon, A. Douglis and L. Nirenberg allows us to prove a regularity result for the corresponding solution.*

Résumé. — *On considère, en élasticité linéarisée, le problème membranaire bi-dimensionnel d'une coque encastrée dont la surface moyenne est uniformément elliptique. La mise en évidence, pour un problème réduit associé, de certaines propriétés de la théorie de S. Agmon, A. Douglis et L. Nirenberg, permet d'établir un résultat de régularité pour la solution correspondante.*

INTRODUCTION

The *linear membrane shell model* is established through an *asymptotic analysis,* as the thickness goes to 0, of the solution of the equations of three-dimensional elasticity. The method, introduced by Ciarlet & Destuynder [1979] for plates, is as follows : Passage to a fixed domain, scalings of the components of the displacement, assumptions on the data (see Ciarlet [1996]). However, for shells, it is not possible to find *simultaneously* the *membrane* model and the *bending* model. These two problems, posed on the middle surface $S = \varphi(\bar{\omega})$, are obtained separately, according as to whether or not a certain *space of inextensional displacements* $V_0(\omega)$ reduces to $\{0\}$ (see Destuynder [1980, 1985], Sanchez-Palencia [1990], Miara & Sanchez-Palencia [1996], Ciarlet & Lods [1994a] and Ciarlet, Lods & Miara [1994]). In other words, it is the kinematic conditions and the geometry of the shell that induce the limit behavior of the three-dimensional unknown. In particular, if

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the shell is uniformly elliptic and clamped along its entire boundary, then $V_0(\omega) = \{0\}$ (and an appropriate equivalence of norms holds ; *cf.* Ciarlet & Lods [1994a]), and the covariant components ζ_i of the limit displacement solve a *two-dimensional membrane shell problem.*

By contrast, *W. T. Koiter's model* (see Koiter [1970]), commonly used in engineering, is *not* a limit model when the thickness ε of the shell goes to 0. It posseses this distinctive characteristic that the left-hand side of the variational equation is precisely the sum of the left-hand sides of the membrane and bending problems. For a mathematical justification of Koiter's model, see Ciarlet & Lods [1994b] where it is proved that in a certain sense, its solution approaches the solution of the three-dimensional model as ε goes to 0.

Existence and uniqueness for *W. T. Koiter's model* were established by Bernadou & Ciarlet [1976]. Then another proof was given in Ciarlet & Miara [1992] ; see also Bernadou, Ciarlet & Miara [1994]. The ellipticity of the bilinear form of the *bending problem* is then a simple corollary of this result, while the situation is more delicate for the *membrane problem.* Indeed, while the variational formulation of Koiter's model is set over the space $H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)$ and the variational formulation of the bending shell model is set over a closed subspace of $H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)$, in the membrane-dominated case the third unknown ζ_3 is sought in the space $L^2(\omega)$, which makes difficult the proof of ellipticity of the bilinear form. Ciarlet & Sanchez-Palencia [1996] have established this result under assumptions of regularity on φ and on the boundary γ of ω, provided that the shell is clamped and uniformly elliptic. The proof makes use of a reduced problem posed in terms of the tangential components of the displacement, and which is proven to have a unique solution. Another proof was given by Ciarlet & Lods [1996], which is more similar, as regards to its principle, to other proofs of existence in linearized elasticity.

The present article is organized as follows : After recalling the variational formulations of the linear membrane shell problem and of the reduced problem, we show that the latter is *uniformly* and *strongly elliptic* in the sense of Agmon, Douglis and Nirenberg [1964] (Theorem 2). Then, a *regularity* result for the solution of the membrane problem (Theorem 3) is obtained as a consequence of theorems of Nečas [1967] and Geymonat [1965].

1. THE MEMBRANE PROBLEM FOR A LINEARLY ELASTIC SHELL ; THE REDUCED PROBLEM

We consider only linearized elasticity.

In what follows, Greek indices and exponents take their values in the set $\{1, 2\}$, Latin indices and exponents take their values in the set $\{1, 2, 3\}$, and we use the repeated index and exponent convention for summation.

Let ω be an open, bounded, connected subset of $\mathbf{R}^2$; we assume that the boundary γ of ω is at least of class $\mathscr{C}^1$, the set ω being locally on one side of γ. Let $y = (y^1, y^2)$ denote a generic point of the set $\bar{\omega}$, and let $\partial_\alpha = \partial/\partial y^\alpha$.

Let $\boldsymbol{\varphi} = \varphi^i \mathbf{e}_i : \bar{\omega} \to \mathbf{R}^3$ be a given injective mapping, at least of class $\mathscr{C}^2$, the vectors $(\mathbf{e}_i)$ forming an orthonormal basis of the Euclidean space, henceforth identified with $\mathbf{R}^3$. We also assume that the two vectors

$$\mathbf{a}_\alpha = \partial_\alpha \boldsymbol{\varphi} = (\partial_\alpha \varphi^i)\, \mathbf{e}_i \tag{1.1}$$

are linearly independant at each point $y \in \bar{\omega}$. Hence these two vectors $(\mathbf{a}_\alpha)$ span the tangent plane to $S = \boldsymbol{\varphi}(\bar{\omega})$ at the point $\boldsymbol{\varphi}(y)$.

At each point $y \in \bar{\omega}$, we define the vectors $\mathbf{a}^\alpha(y)$ of the tangent plane to S at the point $\boldsymbol{\varphi}(y)$ by the relations

$$\mathbf{a}^\alpha \cdot \mathbf{a}_\beta = \delta^\alpha_\beta , \tag{1.2}$$

where . denotes the Euclidean inner product in $\mathbf{R}^3$ and δ^α_β is the Kronecker's delta. We also define the vector

$$\mathbf{a}_3 = \mathbf{a}^3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|} , \tag{1.3}$$

where $\times$ denotes the vector product and $|\,.\,|$ denotes the Euclidean norm. The three vectors $\mathbf{a}_i$ form the *covariant basis,* and the three vectors $\mathbf{a}^i$ form the *contravariant basis,* at each point of S.

The *metric tensor,* or *first fundamental form,* of the surface S is defined by

$$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta . \tag{1.4}$$

Since this symmetric tensor is definite positive at all points of $\bar{\omega}$, there exists a constant a_0 such that :

$$a(y) \overset{\text{def}}{=} \det(a_{\alpha\beta}(y)) \geq a_0 > 0 \quad \text{for all } y \in \bar{\omega} . \tag{1.5}$$

The contravariant components of the metric tensor are :

$$a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta , \tag{1.6}$$

so that the matrix $(a^{\alpha\beta})$ is the inverse of the matrix $(a_{\alpha\beta})$ defined in (1.4).

The *second fundamental form* $(b_{\alpha\beta})$ of the surface S is the symmetric tensor defined by :

$$b_{\alpha\beta} = \partial_\alpha \mathbf{a}_\beta \cdot \mathbf{a}_3 = -\, \mathbf{a}_\alpha \cdot \partial_\beta \mathbf{a}_3 . \tag{1.7}$$

Finally, the Christoffel symbols $\Gamma^{\rho}_{\alpha\beta}$ of the surface S are defined by

$$\Gamma^{\rho}_{\alpha\beta} = \partial_{\alpha} \mathbf{a}_{\beta} \cdot \mathbf{a}^{\rho} \,. \tag{1.8}$$

We have the following symmetry properties :

$$a_{\alpha\beta} = a_{\beta\alpha} \,, \quad a^{\alpha\beta} = a^{\beta\alpha} \,, \quad b_{\alpha\beta} = b_{\beta\alpha} \,, \quad \Gamma^{\rho}_{\alpha\beta} = \Gamma^{\rho}_{\beta\alpha} \,.$$

Since we assumed that the mapping $\boldsymbol{\varphi} : \bar{\omega} \to \mathbf{R}^3$ is at least of class $\mathscr{C}^2$, the functions $a_{\alpha\beta}$, $a_{\beta\alpha}$ are at least in $\mathscr{C}^1(\bar{\omega})$, and the functions $b_{\alpha\beta}$, $\Gamma^{\rho}_{\alpha\beta}$ are at least in $\mathscr{C}^0(\bar{\omega})$.

Let the mapping $\boldsymbol{\Phi} : \bar{\omega} \times [-\varepsilon, \varepsilon] \mapsto \mathbf{R}^3$ be defined by :

$$\boldsymbol{\Phi}(y, y^3) = \boldsymbol{\varphi}(y) + y^3 \, \mathbf{a}_3(y) \,.$$

The set $\boldsymbol{\Phi}(\bar{\omega} \times [-\varepsilon, \varepsilon])$ is the *reference configuration* of an *elastic shell,* with *middle surface* S and *thickness* $2\varepsilon > 0$. We assume that the elastic material constituting the shell is *homogeneous* and *isotropic,* and that the reference configuration is a *natural state ;* the shell is then completely characterized by its two *Lamé constants* λ and μ, with $\lambda > 0$ and $\mu > 0$.

We consider a linearly elastic shell with middle surface S and thickness 2ε, *clamped along its entire boundary.*

The covariant components $\zeta_i : \bar{\omega} \to \mathbf{R}$ of the displacement $\zeta_i \, \mathbf{a}^i$ of the points of S are the unknowns of the *two-dimensional membrane problem,* which can be written in the following variational form :

$$\boldsymbol{\zeta} \in \mathbf{V} \quad \text{and} \quad B(\boldsymbol{\zeta}, \boldsymbol{\eta}) = L(\boldsymbol{\eta}) \quad \text{for all } \boldsymbol{\eta} \in \mathbf{V} \,, \tag{1.9}$$

where the space $\mathbf{V}$ is defined as :

$$\mathbf{V} = \{ \boldsymbol{\eta} = (\eta_i) \,; \eta_{\alpha} \in H^1_0(\omega), \eta_3 \in L^2(\omega) \} = H^1_0(\omega) \times H^1_0(\omega) \times L^2(\omega) \,, \tag{1.10}$$

the symmetric billinear form B is defined by :

$$B(\boldsymbol{\zeta}, \boldsymbol{\eta}) = \int_{\omega} \varepsilon a^{\alpha\beta\rho\sigma} \gamma_{\rho\sigma}(\boldsymbol{\zeta}) \, \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a} \, dy \,, \tag{1.11}$$

with

$$a^{\alpha\beta\rho\sigma} = \frac{4 \lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\rho\sigma} + 2\mu(a^{\alpha\rho} a^{\beta\sigma} + a^{\alpha\sigma} a^{\beta\rho}) \,, \tag{1.12}$$

and

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - \Gamma^\rho_{\alpha\beta}\eta_\rho - b_{\alpha\beta}\eta_3 . \tag{1.13}$$

The linear form $L : \mathbf{V} \to \mathbf{R}$ can be written as :

$$L(\boldsymbol{\eta}) = \int_\omega p^i \eta_i \sqrt{a}\, dy \quad \text{for all } \boldsymbol{\eta} \in \mathbf{V} , \tag{1.14}$$

where we shall assume that $p^i \in L^2(\omega)$.

The fourth-order tensor $(a^{\alpha\beta\rho\sigma})$ defined in (1.12) satisfies the following property (*cf.* Bernadou, Ciarlet & Miara [1994]) : There exists a constant c such that

$$c > 0 \quad \text{and} \quad a^{\alpha\beta\rho\sigma} t_{\rho\sigma} t_{\alpha\beta} \geq c \sum_{\alpha,\beta} |t_{\alpha\beta}|^2 , \tag{1.15}$$

for all $y \in \bar{\omega}$, and for all *symmetric* tensor $(t_{\alpha\beta})$.

In this paper, we shall be concerned with *uniformly elliptic* shells, i.e., those whose middle surface S is *uniformly elliptic* according to the following definition : There exists a constant b such that

$$b > 0 \quad \text{and} \quad b_{\alpha\beta}\xi^\alpha \xi^\beta \geq b|\boldsymbol{\xi}|^2 , \quad \text{for all } \boldsymbol{\xi} = (\xi^\alpha) \in \mathbf{R}^2 . \tag{1.16}$$

This means that there exists a constant $\rho > 0$ such that the two *principal radii of curvature* $R_1(y)$ and $R_2(y)$ are of the *same* sign for all $y \in \bar{\omega}$ and that they satisfy

$$\rho^{-1} \leq |R_\alpha(y)| \leq \rho , \quad \alpha = 1, 2 ,$$

for all $y \in \bar{\omega}$.

We recall here that it is possible to solve a *reduced problem,* posed in terms of the unknowns ζ_1 and ζ_2. Let $\bar{\boldsymbol{\zeta}} = (\zeta_\alpha)$; then we have the following result, proved in Ciarlet & Sanchez-Palencia [1996] :

THEOREM 1 : *Assume that the surface S is uniformly elliptic in the sense of (1.16), and let*

$$d \overset{\text{def}}{=} a^{\alpha\beta\rho\sigma} b_{\rho\sigma} b_{\alpha\beta} . \tag{1.17}$$

Then there exists a constant d_0 *such that*

$$d(y) \geq d_0 > 0 \quad \textit{for all } y \in \bar{\omega} . \tag{1.18}$$

Let $\zeta = (\zeta_i)$ *a solution of the variational problem (1.9). Then* $\tilde{\zeta} = (\zeta_\alpha)$ *solves the following reduced variational problem :*

$$\tilde{\zeta} \in \tilde{\mathbf{V}} \quad and \quad \tilde{B}(\tilde{\zeta}, \tilde{\eta}) = L(\tilde{\eta}) \quad for\ all\ \tilde{\eta} \in \tilde{\mathbf{V}}\,, \tag{1.19}$$

where the space $\tilde{\mathbf{V}}$ *is defined as :*

$$\tilde{\mathbf{V}} = \{\tilde{\eta} = (\eta_\alpha)\ ;\ \eta_\alpha \in H_0^1(\omega)\} = H_0^1(\omega) \times H_0^1(\omega)\,, \tag{1.20}$$

the symmetric bilinear form $\tilde{B}$ *is defined by :*

$$\tilde{B}(\tilde{\zeta}, \tilde{\eta}) = \int_\omega \varepsilon \tilde{a}^{\alpha\beta\rho\sigma} \tilde{\gamma}_{\rho\sigma}(\tilde{\zeta})\, \tilde{\gamma}_{\alpha\beta}(\tilde{\eta}) \sqrt{a}\, dy\,, \tag{1.21}$$

with

$$\tilde{a}^{\alpha\beta\rho\sigma} = a^{\alpha\beta\rho\sigma} - \frac{1}{d}(b_{\varphi\psi}\, a^{\varphi\psi\alpha\beta}\, a^{\xi\eta\rho\sigma}\, b_{\xi\eta})\,, \tag{1.22}$$

$$\tilde{\gamma}_{\alpha\beta}(\tilde{\eta}) = \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - \Gamma^p_{\alpha\beta}\, \eta_p\,, \tag{1.23}$$

and

$$\tilde{L}(\tilde{\eta}) = \int_\omega \left\{ p^\alpha\, \eta_\alpha + \frac{p^3}{d}(a^{\alpha\beta\rho\sigma}\, b_{\rho\sigma} \tilde{\gamma}_{\alpha\beta}(\tilde{\eta})) \right\} \sqrt{a}\, dy\,. \tag{1.24}$$

Conversely, if $\tilde{\zeta} = (\zeta_\alpha)$ *solves the reduced problem (1.19), then* $\zeta = (\zeta_i)$, *where*

$$\zeta_3 \overset{\text{def}}{=} \frac{1}{d}\left(a^{\alpha\beta\rho\sigma}\, b_{\rho\sigma} \tilde{\gamma}_{\alpha\beta}(\tilde{\zeta}) + \frac{p^3}{\varepsilon} \right),$$

solves the problem (1.9).

If we assume that γ is of class $\mathscr{C}^3$ and that φ is analytic in an open set containing $\bar{\omega}$, *the reduced problem* (1.19) *has a unique solution,* and consequently the variational problem (1.9) also *has one and only one solution* (see Ciarlet & Sanchez-Palencia [1996, Theorem 6.2] and Ciarlet & Lods [1996, Theorem 5]).

2. STRONG ELLIPTICITY OF THE REDUCED PROBLEM

The reduced problem (1.19) can be written as follows :

$$\int_\omega \varepsilon \tilde{a}^{\alpha\beta\rho\sigma} \tilde{\gamma}_{\rho\sigma}(\tilde{\zeta})\, \tilde{\gamma}_{\alpha\beta}(\tilde{\eta}) \sqrt{a}\, dy$$

$$= \int_\omega \left\{ p^\alpha\, \eta_\alpha + \frac{p^3}{d}(a^{\alpha\beta\rho\sigma}\, b_{\rho\sigma} \tilde{\gamma}_{\alpha\beta}(\tilde{\eta})) \right\} \sqrt{a}\, dy \quad \text{for all } \tilde{\eta} \in \tilde{\mathbf{V}}\,, \tag{2.1}$$

or, if we substitute $\tilde{\gamma}_{\alpha\beta}(\tilde{\boldsymbol{\eta}})$ by its expression given in (1 23) :

$$\int_\omega \varepsilon \tilde{a}^{\alpha\beta\rho\sigma} \tilde{\gamma}_{\rho\sigma}(\tilde{\zeta})\,(\partial_\beta \eta_\alpha - \Gamma^\xi_{\alpha\beta} \eta_\xi)\sqrt{a}\,dy$$

$$= \int_\omega \left\{ p^\alpha \eta_\alpha + \frac{p^3}{d}\,(a^{\alpha\beta\rho\sigma}\, b_{\rho\sigma}(\partial_\beta \eta_\alpha - \Gamma^\xi_{\alpha\beta}\eta_\xi))\right\}\sqrt{a}\,dy\,.$$

We deduce that the variational problem (2 1) is *formally* equivalent to the following boundary-value problem :

$$\begin{cases} (2.2) & -\varepsilon\left[\frac{1}{\sqrt{a}}\,\partial_\beta\left(\tilde{a}^{\alpha\beta\rho\sigma}\tilde{\gamma}_{\rho\sigma}(\tilde{\zeta})\sqrt{a}\right) + \tilde{a}^{\tau\beta\rho\sigma}\tilde{\gamma}_{\rho\sigma}(\tilde{\zeta})\,\Gamma^\alpha_{\tau\beta}\right] \\ & = p^\alpha - \frac{1}{\sqrt{a}}\,\partial_\beta\left[\frac{p^3}{d} - a^{\alpha\beta\rho\sigma}\, b_{\rho\sigma}\sqrt{a}\right] - \frac{p^3}{d}\, a^{\tau\beta\rho\sigma}\, b_{\rho\sigma}\,\Gamma^\alpha_{\tau\beta} \\ & \text{in } \omega \text{ for } \alpha = 1, 2\,, \\ (2.3) & \zeta_\alpha = 0 \quad \text{on } \gamma, \quad \alpha = 1, 2\,. \end{cases} \qquad (2.4)$$

The purpose of Section 2 is to prove the following result :

THEOREM 2 : *The second-order system (2.4) of partial differential equations and boundary condition with respect to the unknowns ζ_1 and ζ_2, is a « uniformly », and « strongly elliptic » system that satisfies the « supplementary condition on L » and the « complementing boundary condition », in the sense of Agmon, Douglis & Nirenberg [1964]*

Proof · The proof is divided in five steps. As we will often use notations introduced by Agmon, Douglis & Nirenberg [1964], any reference to a page, or equation, number of this paper will be simply identified by the sign #.

(i) Let

$$M \stackrel{\text{def}}{=} -\varepsilon\left[\frac{1}{\sqrt{a}}\,\partial_\beta\left(\tilde{a}^{\alpha\beta\rho\sigma}\tilde{\gamma}_{\rho\sigma}(\tilde{\zeta})\sqrt{a}\right) + \tilde{a}^{\tau\beta\rho\sigma}\tilde{\gamma}_{\rho\sigma}(\tilde{\zeta})\,\Gamma^\alpha_{\tau\beta}\right]$$

denote the left member of equation (2 2). We can write :

$$\partial_\beta\left(\tilde{a}^{\alpha\beta\rho\sigma}\tilde{\gamma}_{\rho\sigma}(\tilde{\zeta})\sqrt{a}\right) = \sqrt{a}\,\partial_\beta(\tilde{a}^{\alpha\beta\rho\sigma}\,\partial_\rho\zeta_\sigma - \tilde{a}^{\alpha\beta\rho\sigma}\,\Gamma^\xi_{\rho\sigma}\zeta_\xi)$$

$$+ (\tilde{a}^{\alpha\beta\rho\sigma}\,\partial_\rho\zeta_\sigma - \tilde{a}^{\alpha\beta\rho\sigma}\,\Gamma^\xi_{\rho\sigma}\zeta_\xi)\,\frac{\partial_\beta a}{2\sqrt{a}}\,,$$

so that :

$$\frac{1}{\sqrt{a}}\,\partial_\beta\Big(\tilde{a}^{\alpha\beta\rho\sigma}\tilde{\gamma}_{\rho\sigma}(\bar{\zeta})\sqrt{a}\Big) = \left(\partial_\beta(\tilde{a}^{\alpha\beta\rho\sigma}\,\Gamma^\xi_{\rho\sigma}) + \tilde{a}^{\alpha\beta\rho\sigma}\,\Gamma^\alpha_{\rho\sigma}\,\frac{\partial_\beta a}{2\,a}\right)\zeta_\xi$$

$$+\left(\partial_\beta\tilde{a}^{\alpha\beta\rho\sigma} + \frac{\partial_\beta a}{2\,a}\,\tilde{a}^{\alpha\beta\rho\sigma}\right)\partial_\rho\,\zeta_\rho$$

$$-\,(\tilde{a}^{\alpha\beta\rho\sigma}\,\Gamma^\xi_{\rho\sigma})\,\partial_\beta\,\zeta_\xi$$

$$+\,(\tilde{a}^{\alpha\beta\rho\sigma})\,\partial_{\beta\rho}\,\zeta_\sigma\,,$$

and :

$$\tilde{a}^{\tau\beta\rho\sigma}\tilde{\gamma}_{\rho\sigma}(\bar{\zeta})\,\Gamma^\alpha_{\tau\beta} = (\tilde{a}^{\tau\beta\rho\sigma}\,\Gamma^\alpha_{\tau\beta})\,\partial_\rho\,\zeta_\sigma - (\tilde{a}^{\tau\beta\rho\sigma}\,\Gamma^\alpha_{\tau\beta})\,\zeta_\xi\,.$$

Thus

$$M = -\,\varepsilon\Bigg\{-\left[\partial_\beta(\tilde{a}^{\alpha\beta\rho\sigma}\,\Gamma^\xi_{\rho\sigma}) + \tilde{a}^{\alpha\beta\rho\sigma}\,\Gamma^\xi_{\rho\sigma}\,\frac{\partial_\beta a}{2\,a} + \tilde{a}^{\tau\beta\rho\sigma}\,\Gamma^\alpha_{\tau\beta}\,\Gamma^\xi_{\rho\sigma}\right]\zeta_\xi$$

$$+\left[\partial_\beta\tilde{a}^{\alpha\beta\rho\sigma} + \frac{\partial_\beta a}{2\,a}\,\tilde{a}^{\alpha\beta\rho\sigma} + \tilde{a}^{\tau\beta\rho\sigma}\,\Gamma^\alpha_{\tau\beta}\right]\partial_\rho\,\zeta_\sigma$$

$$-\,[\tilde{a}^{\alpha\beta\rho\sigma}\,\Gamma^\xi_{\rho\sigma}]\,\partial_\beta\,\zeta_\xi$$

$$+\,[\tilde{a}^{\alpha\beta\rho\sigma}]\,\partial_{\beta\rho}\,\zeta_\sigma\Bigg\}\,,$$

which can be written as :

$$M = -\,\varepsilon[\tilde{a}^{\alpha\beta\rho\sigma}\,\partial_{\beta\rho}\,\zeta_\sigma + H^{\alpha\rho\sigma}\,\partial_\rho\,\zeta_\sigma + V^{\alpha\sigma}\,\zeta_\sigma]\,,$$

with

$$H^{\alpha\rho\sigma} = \left[\partial_\beta\tilde{a}^{\alpha\beta\rho\sigma} + \frac{\partial_\beta a}{2\,a}\,\tilde{a}^{\alpha\beta\rho\sigma} + \tilde{a}^{\tau\beta\rho\sigma}\,\Gamma^\alpha_{\tau\sigma} - \tilde{a}^{\alpha\rho\beta\xi}\,\Gamma^\sigma_{\beta\xi}\right],$$

$$V^{\alpha\sigma} = -\left[\partial_\beta(\tilde{a}^{\alpha\beta\rho\xi}\,\Gamma^\sigma_{\rho\xi}) + \tilde{a}^{\alpha\beta\rho\xi}\,\Gamma^\sigma_{\rho\xi}\,\frac{\partial_\beta a}{2\,a} + \tilde{a}^{\tau\beta\rho\xi}\,\Gamma^\alpha_{\tau\beta}\,\Gamma^\sigma_{\rho\xi}\right].$$

Let

$$P^\alpha \overset{\text{def}}{=} p^\alpha - \frac{1}{\sqrt{a}}\,\partial_\beta\left[\frac{p^3}{d}\,a^{\alpha\beta\rho\sigma}\,b_{\rho\sigma}\sqrt{a}\right] - \frac{p^3}{d}\,a^{\tau\beta\rho\sigma}\,b_{\rho\sigma}\,\Gamma^\alpha_{\tau\beta}\,;$$

the system (2.4) reads :

$$\begin{cases} -\varepsilon[\tilde{a}^{\alpha\beta\rho\sigma}\,\partial_{\beta\rho}\zeta_\sigma + H^{\alpha\rho\sigma}\,\partial_\rho\zeta_\sigma + V^{\alpha\sigma}\zeta_\sigma] = P^\alpha & \text{in } \omega \\ \zeta_\alpha = 0 & \text{on } \gamma . \end{cases} \tag{2.5}$$

We deduce that for every $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$, and for every $y \in \bar{\omega}$, the matrix $(l_{\alpha\beta}(y, \xi))$ of page 38# is given by :

$$l_{\alpha\beta}(\,.\,,\xi) = -\varepsilon[V^{\alpha\beta} + H^{\alpha1\beta}\,\xi_1 + H^{\alpha2\beta}\,\xi_2 + \tilde{a}^{\alpha11\beta}\,\xi_1^2 + (\tilde{a}^{\alpha21\beta} + \tilde{a}^{\alpha12\beta})\,\xi_1\,\xi_2 + \tilde{a}^{\alpha22\beta}\,\xi_2^2] . \tag{2.6}$$

(ii) *Uniform ellipticity of the system* (2.4).

The integers t'_α of page 43# are chosen as follows :

$$t'_1 = 1 \quad \text{and} \quad t'_2 = 1 ,$$

so that

$$\deg l_{\alpha\beta}(\,.\,,\xi) = t'_\alpha + t'_\beta \quad \text{and} \quad t'_\alpha \geqslant 0 .$$

We denote by $l'_{\alpha\beta}$ the terms in $(l_{\alpha\beta})$ which are just of the order $t'_\alpha + t'_\beta$.

Let

$$L(\,.\,,\xi) \overset{\text{def}}{=} \det l'_{\alpha\beta}(\,.\,,\xi) .$$

In order to prove the uniform ellipticity of the system (2.4), we must verify that there exists a constant A such that :

$$A > 0 \quad \text{and} \quad A^{-1}|\xi|^4 \leqslant |L(y,\xi)| \leqslant A|\xi|^4 \tag{2.7}$$

for all $y \in \bar{\omega}$ and for all $\xi = (\xi_1, \xi_2)$, the integer m of (1.6)# being here equal to $\frac{1}{2}\deg L(y,\xi) = 2$. We first verify that the system (2.4) is *elliptic* in the sense of (1.5)#, i.e. that :

$$L(y,\xi) \neq 0 \quad \text{for all } \xi \in R^2, \xi \neq \mathbf{0} . \tag{2.8}$$

We have :

$$L(\,.\,,\xi\,) = \varepsilon^2[\,(\,\tilde{a}^{1111}\tilde{a}^{1212} - (\,\tilde{a}^{1112}\,)^2\,)\,\xi_1^4 + (\,\tilde{a}^{2222}\tilde{a}^{1212} - (\,\tilde{a}^{2212}\,)^2\,)\,\xi_2^4$$
$$+ 2(\,\tilde{a}^{1111}\tilde{a}^{2212} - \tilde{a}^{1122}\tilde{a}^{1112}\,)\,\xi_1^3\,\xi_2$$
$$+ 2(\,\tilde{a}^{2222}\tilde{a}^{1112} - \tilde{a}^{2212}\tilde{a}^{1122}\,)\,\xi_1\,\xi_2^3$$
$$+ (\,\tilde{a}^{1111}\tilde{a}^{2222} + 2\,\tilde{a}^{1112}\tilde{a}^{2212} - 2\,\tilde{a}^{1122}\tilde{a}^{1212} - (\,\tilde{a}^{1122}\,)^2\,)\xi_1^2\,\xi_2^2\,]\,.$$

For conciseness, let :

$$K = \frac{4\,\lambda\mu}{\lambda + 2\,\mu} + 4\,\mu\,, \quad K_1 = 2\,\mu\,,$$

After some calculations, we get :

$$L(\,.\,,\xi\,) = \varepsilon^2\,\frac{4\,K_1^2}{a^3\,d}\,(\,K - K_1\,)\,[\,(\,b_{22}\,)^2\,\xi_1^4 + (\,b_{11}\,)^2\,\xi_2^4 - 4\,b_{12}\,b_{22}\,\xi_1^3\,\xi_2$$
$$- 4\,b_{12}\,b_{11}\,\xi_1\,\xi_2^3 + 2(\,b_{11}\,b_{22} + 2(\,b_{12}\,)^2\,)\,\xi_1^2\,\xi_2^2]\,.$$

Let

$$\mathscr{K}(\,y\,) \stackrel{\text{def}}{=} 4\,\varepsilon^2\,\frac{K_1^2(K - K_1\,)}{[a(\,y\,)]^3[d(\,y\,)]}\,.$$

We have :

$$K_1 = 2\,\mu > 0\,,$$
$$K - K_1 = \frac{4\,\lambda\mu}{\lambda + 2\,\mu} + 2\,\mu > 0\,,$$
$$a(\,y\,) \geqslant a_0 > 0 \quad \text{for all } y \in \bar{\omega}\ (\textit{cf.}\ (\,1.5\,)\,)\,,$$
$$a(\,y\,) \geqslant d_0 > 0 \quad \text{for all } y \in \bar{\omega}\ (\textit{cf.}\ (\,1.18\,)\,)\,.$$

Thus there exists a positive constant $\mathscr{K}_0$ such that :

$$\mathscr{K}(\,y\,) \geqslant \mathscr{K}_0 > 0 \quad \text{for all } y \in \bar{\omega}\,.$$

Then we can write

$$L(\,.\,,\xi\,) = \mathscr{K}\,[\,b_{22}\,\xi_1^2 - 2\,b_{12}\,\xi_1\,\xi_2 + b_{11}\,\xi_2^2]^2\,.$$

Property (2.8) is then a direct consequence of assumption (1.16).

In order to prove the stronger property of uniform ellipticity, we only have to verify that property (2.7) is true for all $\xi \in \mathbf{R}^2$ such that $|\xi| = 1$, since for all $y \in \bar{\omega}$, the polynomial $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2 \mapsto L(y, \xi)$ is homogeneous of degree 4. It is therefore sufficient to prove that there exists a constant A such that

$$A > 0 \quad \text{and} \quad A^{-1} \leqslant |L(y, \xi)| \leqslant A$$

for all $y \in \bar{\omega}$, for all $\xi \in \mathbf{R}^2$ such that $|\xi| = 1$. This follows from the continuity of the functions $y \in \bar{\omega} \mapsto a_{\alpha\beta}(y)$ and $y \in \bar{\omega} \mapsto b_{\alpha\beta}(y)$, from the compactness of the set $\bar{\omega} \times \{\xi \in \mathbf{R}^2 ; |\xi| = 1\}$ and from property (2.8). Hence, property (2.7) is established, and the system (2.4) is uniformly elliptic in the sense of Agmon, Douglis & Nirenberg [1964], as stated.

(iii) *Strong ellipticity of the system* (2.4).

We then show that the system (2.4) is strongly elliptic in the sense of (2.5)# (see also Lions & Magenes [1968]), i.e. the following property is satisfied : There exists $k > 0$ such that, for any $\xi \in \mathbf{R}^2$, $\xi \neq \mathbf{0}$, and for any $\boldsymbol{\eta} \in \mathbf{C}^2$, $\boldsymbol{\eta} \neq \mathbf{0}$,

$$\mathcal{R}e\,[-l'_{11}(y,\xi)\,|\eta_1|^2 - l'_{12}(y,\xi)\,(\eta_1\bar{\eta}_2 + \eta_2\bar{\eta}_1) - l'_{22}(y,\xi)\,|\eta_2|^2] \geqslant k|\xi|^2(|\eta_1|^2 + |\eta_2|^2)\,.$$

Since the functions $l'_{\alpha\beta}(y, \xi)$ are real in our case, it suffices to show that, for all $\xi \in \mathbf{R}^2$, $\xi \neq \mathbf{0}$, and for all $\boldsymbol{\eta} \in \mathbf{R}^2$, $\boldsymbol{\eta} \neq \mathbf{0}$,

$$R(y, \xi, \boldsymbol{\eta}) \stackrel{\text{def}}{=} -l'_{11}(y,\xi)\,\eta_1^2 - 2\,l'_{12}(y,\xi)\,\eta_1\,\eta_2 - l'_{22}(y,\xi)\,\eta_2^2 \geqslant k|\xi|^2|\boldsymbol{\eta}|^2\,. \tag{2.9}$$

We have

$$\begin{aligned} R(y, \xi, \boldsymbol{\eta}) = \varepsilon[&(\tilde{a}^{1111}\,\xi_1^2 + 2\,\tilde{a}^{1112}\,\xi_1\,\xi_2 + \tilde{a}^{1212}\,\xi_2^2)\,\eta_1^2 \\ &+ 2(\tilde{a}^{1112}\,\xi_1^2 + (\tilde{a}^{1212} + \tilde{a}^{1122})\,\xi_1\,\xi_2 + \tilde{a}^{2212}\,\xi_2^2)\,\eta_1\,\eta_2 \\ &+ (\tilde{a}^{1212}\,\xi_1^2 + 2\,\tilde{a}^{2212}\,\xi_1\,\xi_2 + \tilde{a}^{2222}\,\xi_2^2)\,\eta_2^2]\,. \end{aligned}$$

For conciseness, let :

$$B^{\alpha\beta} = b_{\varphi\psi}\,a^{\varphi\psi\alpha\beta}\,.$$

Then we have :

$$R(y, \xi, \boldsymbol{\eta}) = \frac{\varepsilon}{d}\Big[(Kd(a^{11})^2 - (B^{11})^2)\, \xi_1^2 + 2(Kda^{11} a^{12} - B^{11} B^{12})\, \xi_1 \xi_2$$

$$+ \left(Kd(a^{12})^2 - (B^{12})^2 + d\frac{K_1}{a} \right) \xi_2^2 \Big] \eta_1^2$$

$$+ 2\frac{\varepsilon}{d}\, [(Kda^{11} a^{12} - B^{11} B^{12})\, \xi_1^2$$

$$+ \left(Kd(a^{12})^2 - (B^{12})^2 + Kda^{11} a^{22} - B^{11} B^{22} - d\frac{K_1}{a} \right) \xi_1 \xi_2$$

$$+ (Kda^{12} a^{22} - B^{12} B^{22})\, \xi_2^2\, \eta_1 \eta_2$$

$$+ \frac{\varepsilon}{d}\Big[\left(Kd(a^{12})^2 - (B^{12})^2 + d\frac{K_1}{a} \right) \xi_1^2$$

$$+ 2(Kda^{22} a^{12} - B^{22} B^{12})\, \xi_1 \xi_2$$

$$+ (Kd(a^{22})^2 - (B^{22})^2)\, \xi_2^2 \Big] \eta_2^2 . \tag{2.10}$$

We then compute the coefficients of ξ_1^2, ξ_2^2 and $\xi_1 \xi_2$:

$$dK(a^{11})^2 - (B^{11})^2 = 4\frac{K_1}{a}\left[K(a^{11} b_{12} + a^{12} b_{22})^2 + \frac{1}{a}(K - K_1)(b_{22})^2 \right], \tag{2.11}$$

$$dK(a^{22})^2 - (B^{22})^2 = 4\frac{K_1}{a}\left[K(a^{12} b_{11} + a^{22} b_{12})^2 + \frac{1}{a}(K - K_1)(b_{11})^2 \right], \tag{2.12}$$

$$dK(a^{12})^2 - (B^{12})^2 + d\frac{K_1}{a} = \frac{K_1}{a}\Big[K(a^{11} b_{11} - a^{22} b_{22})^2 + \frac{4}{a}(K - K_1)\, b_{11} b_{22} \Big], \tag{2.13}$$

$$dKa^{11} a^{12} - B^{11} B^{12} = $$

$$= 2\frac{K_1}{a}\left[K(a^{22} b_{22} - a^{11} b_{11})(a^{12} b_{22} + a^{11} b_{12}) + \frac{2}{a}(K_1 - K)\, b_{12} b_{22} \right], \tag{2.14}$$

$$dKa^{12}a^{22} - B^{12}B^{22}$$

$$= 2\frac{K_1}{a}\left[K(a^{12}b_{11} - a^{22}b_{12})(a^{11}b_{11} - a^{22}b_{22}) + \frac{2}{a}(K_1 - K)\,b_{11}\,b_{12}\right], \tag{2.15}$$

$$dKa^{11}a^{22} - B^{11}B^{22} + dK(a^{12})^2 - (B^{12})^2 + d\frac{K_1}{a} = \frac{K_1}{a}\,[K(a^{11}b_{11} - a^{22}b_{22})^2$$

$$\left[-4\,K(a^{11}b_{12} + a^{12}b_{22})(a^{12}b_{11} + a^{22}b_{12}) + \frac{8}{a}(K - K_1)(b_{12})^2\right]. \tag{2.16}$$

Substituting (2.11)-(2.16) in (2.10), and ordering certain terms, we rewrite (2.10) in the following way :

$$R(y, \xi, \eta) = \frac{\varepsilon K K_1}{ad}\,[\eta_1[a^{11}(2\,b_{12}\,\xi_1 - b_{11}\,\xi_2) + b_{22}(2\,a^{12}\,\xi_1 + a^{22}\,\xi_2)]$$

$$+ \eta_2[a^{22}(-2\,b_{12}\,\xi_2 + b_{22}\,\xi_1) + b_{11}(-2\,a^{12}\,\xi_2 - a^{11}\,\xi_1)]]^2$$

$$+ 4\,\frac{\varepsilon K(K - K_1)}{a^2\,d}\,(b_{22}\,\xi_1^2 + 2\,b_{12}\,\xi_1\,\xi_2 + b_{11}\,\xi_2^2)$$

$$\times (b_{22}\,\eta_1^2 + 2\,b_{12}\,\eta_1\,\eta_2 + b_{11}\,\eta_2^2)\,.$$

Since the middle surface is uniformly elliptic by assumption, we see by applying (1.16) that

$$R(y, \xi, \eta) \geqslant \frac{4\,\varepsilon K(K - K_1)}{a^2\,d}\,|\xi|^2|\eta|^2\,,$$

and thus inequality (2.9) is established. Therefore, the reduced problem is strongly elliptic in the sense of Agmon, Douglis & Nirenberg [1964]. We note that the assumption of uniform ellipticity of the shell is definitely needed in this proof.

(iv) *« Supplementary condition on L » :* We must verify that, for each $y \in \bar{\omega}$ and for any pair of linearly independent vectors $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$ and $\eta = (\eta_1, \eta_2) \in \mathbf{R}^2$, the polynomial

$$\tau \in \mathbf{C} \mapsto L(y, \xi + \tau\eta)$$

which is of degree 4, has no real root and thus has exactly two roots $\tau_1^+(y, \xi, \boldsymbol{\eta})$ and $\tau_2^+(y, \xi, \boldsymbol{\eta})$ verifying

$$\mathcal{I}m\, \tau_\alpha^+(y, \xi, \boldsymbol{\eta}) > 0 \quad \alpha = 1, 2 .$$

We recall that :

$$L(\,.\,, \xi) = \mathcal{K}[b_{22}\, \xi_1^2 - 2\, b_{12}\, \xi_1\, \xi_2 + b_{11}\, \xi_2^2]^2 .$$

Let

$$E(y) = \sqrt{\mathcal{K}(y)} \begin{pmatrix} b_{22}(y) & -b_{12}(y) \\ -b_{12}(y) & b_{11}(y) \end{pmatrix} . \tag{2.17}$$

Then :

$$L(y, \xi) = \{E(y)\, \xi \,.\, \xi\}^2 . \tag{2.18}$$

Consequently,

$$L(y, \xi + \tau\boldsymbol{\eta}) = [E(y)\,(\xi + \tau\boldsymbol{\eta})\,.\,(\xi + \tau\boldsymbol{\eta})]^2 ,$$

$$L(y, \xi + \tau\boldsymbol{\eta}) = (\tau^2\{E(y)\,\boldsymbol{\eta}\,.\,\boldsymbol{\eta}\} + 2\,\tau\{E(y)\,\xi\,.\,\boldsymbol{\eta}\} + \{E(y)\,\xi\,.\,\xi\})^2 .$$

The matrix $E(y)$ is positive definite for, according to (2.9),

$$E(y)\,\xi\,.\,\xi = \sqrt{L(y, \xi)} \geqslant \sqrt{A^{-1}}\,|\xi|^2 \quad \text{for any vector } \xi \in \mathbf{R}^2$$

Then $\{E(y)\,\xi\,.\,\boldsymbol{\eta}\}$ defines an inner product and in particular, ξ and $\boldsymbol{\eta}$ being two linearly independent vectors :

$$\{E(y)\,\xi\,.\,\boldsymbol{\eta}\}^2 < \{E(y)\,\xi\,.\,\xi\}\,\{E(y)\,\xi\,.\,\xi\} .$$

This shows that the polynomial $\tau \mapsto L(y, \xi + \tau\boldsymbol{\eta})$ has no real root, and thus the supplementary condition on L is established.

(v) *« Complementing boundary condition » :* It remains to verify that the « complementing boundary condition » of page 42[#] is satisfied (see also Lions & Magenes [1968, p. 240]) for the problem (2.4). This property can be proved in detail, but in fact it follows from the strong ellipticity of the system, because in this case Dirichlet conditions are always complementing (*cf.* Agmon, Douglis & Nirenberg [1964, p. 44]). This finishes the proof of Theorem 2.□

3. A REGULARITY RESULT IN THE LINEAR CASE

We first recall that existence and uniqueness of a solution for the reduced problem (1.19) are established in two cases :

$$\gamma \text{ is of class } \mathscr{C}^3 \text{ and } \varphi \text{ is analytic in an open set containing } \bar{\omega}, \tag{3.1}$$

$$\gamma \text{ is of class } \mathscr{C}^4 \text{ and } \varphi \in \mathscr{C}^5(\bar{\omega})\ ; \tag{3.2}$$

see Ciarlet & Sanchez-Palencia [1996] and Ciarlet & Lods [1996].

THEOREM 3 : *Assume that assumption (3.1) is satisfied, that* $p^\alpha \in L^q(\omega)$ *and* $p^3 \in W^{1,q}(\omega)$, *where* $q \geq 2$. *Then the solution* $\zeta \in H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega)$ *of the membrane problem (1.9) is in the space* $W^{2,q}(\omega) \times W^{2,q}(\omega) \times W^{1,q}(\omega)$. *Let* m *be an integer* ≥ 1. *If the boundary* γ *is of class* $\mathscr{C}^{m+3}$ *and if* $p^\alpha \in W^{m,q}(\omega)$, $p^3 \in W^{m+1,q}(\omega)$, *then the solution* $\zeta \in H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega)$ *of the membrane problem is in the space* $W^{m+2,q}(\omega) \times W^{m+2,q}(\omega) \times W^{m+1,q}(\omega)$.

Proof: The proof is similar to the one in Ciarlet [1988, Sect. 6.3] for the pure displacement problem in linearized three-dimensional elasticity. The proof is divided in four steps :

(i) We recall that the reduced problem can be written as a system of partial differential equations :

$$\begin{cases} -\ \varepsilon\left[\dfrac{1}{\sqrt{a}}\,\partial_\beta\left(\tilde{a}^{\alpha\beta\rho\sigma}\tilde{\gamma}_{\rho\sigma}(\tilde{\zeta})\sqrt{a}\right) + \tilde{a}^{\tau\beta\rho\sigma}\,\tilde{\gamma}_{\rho\sigma}(\tilde{\zeta})\,\Gamma^\alpha_{\tau\beta}\right] = P^\alpha & \text{in } \omega\,, \\ \zeta_1 = \zeta_2 = 0 & \text{on } \gamma\,, \end{cases}$$

The first equations can be written :

$$-\frac{\varepsilon}{\sqrt{a}}\,\partial_\beta\left(\tilde{a}^{\alpha\beta\rho\sigma}\,\partial_\rho\,\zeta_\sigma\sqrt{a}\right) = P^\alpha - \frac{\varepsilon}{\sqrt{a}}\,\partial_\beta(\tilde{a}^{\alpha\beta\rho\sigma}\,\Gamma^\xi_{\rho\sigma}\,\zeta_\xi) + \varepsilon\tilde{a}^{\tau\beta\rho\sigma}\tilde{\gamma}_{\rho\sigma}(\zeta)\,\Gamma^\alpha_{\tau\beta}\,.$$

We note that, if $\varphi \in \mathscr{C}^3(\bar\omega)$, then $[\partial_\beta(\tilde a^{\alpha\beta\rho\sigma}\,\Gamma^\xi_{\rho\sigma}\,\zeta_\xi)] \in L^2(\omega)$, and $[\tilde a^{\tau\beta\rho\sigma}\tilde\gamma_{\rho\sigma}(\tilde\zeta)\,\Gamma^\alpha_{\tau\beta}] \in L^2(\omega)$ Likewise, if $p^\alpha \in L^2(\omega)$, $p^3 \in H^1(\omega)$, and $\varphi \in \mathscr{C}^3(\bar\omega)$, we have

$$P^\alpha = p^\alpha - \frac{1}{\sqrt a}\,\partial_\beta\left[\frac{p^3}{d}\,a^{\alpha\beta\rho\sigma}\,b_{\rho\sigma}\sqrt a\right] - \frac{p^3}{d}\,a^{\tau\beta\rho\sigma}\,b_{\rho\sigma}\,\Gamma^\alpha_{\tau\beta} \in L^2(\omega)$$

Finally, if $\varphi \in \mathscr{C}^3(\bar\omega)$, $p^\alpha \in L^2(\omega)$, $p^3 \in H^1(\omega)$, then

$$f^\alpha \overset{\text{def}}{=} P^\alpha - \frac{\varepsilon}{\sqrt a}\,\partial_\beta(\tilde a^{\alpha\beta\rho\sigma}\,\Gamma^\xi_{\rho\sigma}\,\zeta_\xi) + \varepsilon\tilde a^{\tau\beta\rho\sigma}\tilde\gamma_{\rho\sigma}(\tilde\zeta)\,\Gamma^\alpha_{\tau\beta} \in L^2(\omega)$$

In particular, if (3 1) or (3 2) is satisfied, the solution $\tilde\zeta = (\zeta_\alpha)$ of the reduced problem also verifies

$$\begin{cases} -\dfrac{\varepsilon}{\sqrt a}\,\partial_\beta\left(\tilde a^{\alpha\beta\rho\sigma}\,\partial_\rho\,\zeta_\sigma\sqrt a\right) = f^\alpha \text{ in } \omega, \\ \zeta_\alpha = 0 \text{ on } \gamma \end{cases} \qquad (3\ 3)$$

The previous sections show that this system is uniformly and strongly elliptic, satisfies the supplementary condition on L and the complementing boundary condition, in the sense of Agmon, Douglis & Nirenberg [1964]

(ii) The system (3 3) is strongly elliptic and, as in Ciarlet & Sanchez-Palencia [1996], we also have the $\tilde{\mathbf V}$-*ellipticity* under the assumption (3 1) Hence we can apply Lemma 3 2 of Nečas [1967, p 260] , If the boundary γ is of class $\mathscr{C}^2$, and if $f^\alpha \in L^2(\omega)$, then

$$\tilde\zeta \in \mathbf{H}^1_0(\omega) \cap \mathbf{H}^2(\omega)$$

Thus the result is established for $m = 0$ and $q = 2$, the required regularity of ζ_3 being a consequence of the relation

$$\zeta_3 = \frac{1}{d}\left(a^{\alpha\beta\rho\sigma}\,b_{\rho\sigma}\tilde\gamma_{\alpha\beta}(\tilde\zeta) + \frac{p^3}{\varepsilon}\right)$$

(iii) Define the space

$$\tilde{\mathbf V}^q = \{\tilde{\boldsymbol\eta} = (\eta_\alpha) \in W^{2\ q}(\omega)\ , \eta_\alpha = 0 \text{ sur } \gamma\}\ ,$$

and consider the mapping

$$\tilde{\mathscr{A}}\ '(\mathbf 0)\quad \tilde{\boldsymbol\eta} \in \tilde{\mathbf V}^q \to \left\{-\frac{\varepsilon}{\sqrt a}\,\partial_\beta\left(\tilde a^{\alpha\beta\rho\sigma}\,\partial_\rho\,\eta_\sigma\sqrt a\right)\right\} \in L^q(\omega)$$

We show that we can apply theorem 3.5 of Geymonat [1965]. Since this theorem refers to certain notations introduced by Agmon, Douglis & Nirenberg [1964], we first recall that the integers s_α and t_β of page 39# are chosen as

$$s_1 = s_2 = 0 \quad \text{and} \quad t_1 = t_2 = 2 ,$$

so that they verify :

$$2 = \deg l_{\alpha\beta}(\,\cdot\, , \xi) \leqslant s_\alpha + t_\beta = 2 , \quad s_\alpha \leqslant 0 \leqslant t_\beta .$$

Besides, the boundary conditions being

$$\zeta_1 = 0 \quad \text{on } \gamma , \quad \zeta_2 = 0 \quad \text{on } \gamma ,$$

the matrix $(B_{\alpha\beta}(y, \xi))$ of page 42# is given by

$$B_{\alpha\beta}(y, \xi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

the associated integers being $r_1 = r_2 = -2$, so that :

$$\deg B_{\alpha\beta}(y, \xi) \leqslant r_\alpha + t_\beta .$$

Let us now verify the assumptions of theorem 3.5 of Geymonat [1965] :

Assumption 1 : Regularity of the bounded open set ω : It should have a boundary γ of class $\mathscr{C}^{l_1 + t + 1}$, with $l_1 = \max(0, r_1 + 1, r_2 + 1)$, $t = \max(t_1, t_2)$.

Assumption 1 thus requires that γ be of class $\mathscr{C}^3$, which is the case here.

Assumption 2 : Regularity of the functions $\tilde{a}^{\alpha\beta\rho\sigma}$: If we let

$$l_{\alpha\beta}(y, \xi) = \sum_{|J| \leqslant s_\alpha + t_\beta} A_{\alpha\beta J}\, \xi^J ,$$

with $J = (\rho, \sigma)$, and $\xi^J = \xi_1^\rho \xi_2^\sigma$, we should also have :

$$A_{\alpha\beta J} \in \mathscr{C}^{l_1 - s_\alpha + 1}(\bar{\omega}) \quad \text{for} \quad |J| = s_\alpha + t_\beta ,$$

$$A_{\alpha\beta J} \in W_\infty^{l_1 - s_\alpha + 1}(\bar{\omega}) \quad \text{for} \quad |J| < s_\alpha + t_\beta .$$

In our case,

$$l_{\alpha\beta}(y, \xi) = -\varepsilon\{[\tilde{a}^{\alpha 11\beta}\xi_1^2 + (\tilde{a}^{\alpha 21\beta} + \tilde{a}^{\alpha 12\beta})\xi_1\xi_2 + \tilde{a}^{\alpha 22\beta}\xi_1^2]$$
$$+ \left[\partial_1(\tilde{a}^{\alpha 11\beta}\sqrt{a}) + \partial_2(\tilde{a}^{\alpha 21\beta}\sqrt{a})\right]\xi_1$$
$$+ \left[\partial_1(\tilde{a}^{\alpha 12\beta}\sqrt{a}) + \partial_2(\tilde{a}^{\alpha 22\beta}\sqrt{a})\right]\xi_2\Big\},$$

which we can write as follows :

$$l_{\alpha\beta}(y, \xi) = \sum_{|J| \leq s_\alpha + t_\beta} A_{\alpha\beta J}\,\xi^J .$$

Moreover, we already saw that $s_1 = s_2 = 0$ and $l_1 = 0$. Since we assumed that $\varphi \in \mathscr{C}^3(\bar{\omega})$, then the functions $\tilde{a}^{\alpha\beta\rho\sigma}$ are in $\mathscr{C}^1(\bar{\omega})$ and assumption 2 is satisfied.

Assumption 3 : Regularity of the coefficients of the boundary operator : If we let

$$B_{\alpha\beta} = \sum_{|J| \leq r_\alpha + t_\beta} b_{\alpha\beta J}(y)\,\xi^J ,$$

then we should have

$$b_{\alpha\beta J} \in \mathscr{C}^{l_1 - r_\alpha + 1}(\gamma) \quad \text{for} \quad |J| = s_\alpha + t_\beta ,$$
$$b_{\alpha\beta J} \in W_\infty^{l_1 - r_\alpha + 1}(\gamma) \quad \text{for} \quad |J| < s_\alpha + t_\beta .$$

In our case,

$$B_{\alpha\beta}(y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

hence assumption 3 is satisfied. Besides, we already pointed out that the system (3.3) is uniformly elliptic, and that it satisfies the supplementary condition on L and the complementing boundary condition. It then follows from Geymonat [1965, Theorem 3.5] that the mapping

$$\tilde{\mathscr{A}}\,'(\mathbf{0}) : \tilde{\boldsymbol{\eta}} \in \tilde{\mathbf{V}}^q \mapsto \left\{-\frac{\varepsilon}{\sqrt{a}}\partial_\beta\left(\tilde{a}^{\alpha\beta\rho\sigma}\,\partial_\rho\eta_\sigma\sqrt{a}\right)\right\} \in L^q(\omega)$$

has an index ind $\tilde{\mathscr{A}}\,'(\mathbf{0})$ independent of $q \in]1, \infty[$.

We already know from step (ii) that ind $\tilde{\mathscr{A}}'(\mathbf{0}) = \mathbf{0}$ when $q = 2$, for $\tilde{\mathscr{A}}'(\mathbf{0})$ is a bijection in this case. Besides $\tilde{\mathbf{V}}^q \hookrightarrow \mathbf{H}_0^1(\omega)$ when $q \geqslant 2$, and consequently, the mapping $\tilde{\mathscr{A}}'(\mathbf{0}) : \tilde{\mathbf{V}}^q \to L^q(\omega)$ is injective for these values of q, but since ind $\tilde{\mathscr{A}}'(\mathbf{0}) = \mathbf{0}$, then $\tilde{\mathscr{A}}'(\mathbf{0})$ is also surjective when $q \geqslant 2$ Concerning the unknown ζ_3, we conclude as in step (ii) Hence we have proved the regularity result for $m = 0$ and $q \geqslant 2$.

(iv) In order to establish the regularity result for $m \geqslant 1$, we apply theorem 10.5 of Agmon, Douglis & Nirenberg [1964] : Once we know that each $\|\zeta_\alpha\|_{W^{2,q}}$ is finite, we have .

$$\mathbf{f} \in \mathbf{W}^{m,q}(\omega) \Rightarrow \tilde{\zeta} \in \mathbf{W}^{2+m,q}(\omega),$$

and thus there exists a constant C such that

$$\|\zeta_\alpha\|_{W^{2+m,q}} \leqslant C \sum_\beta \left(\|f_\beta\|_{W^{m,q}(\omega)} + \|\zeta_\beta\|_{L^2(\omega)} \right)$$

But if $p^\alpha \in W^{m,q}(\omega)$ and $p^3 \in W^{m+1,q}(\omega)$, it follows that $\mathbf{f} \in \mathbf{W}^{m,q}(\omega)$. This completes the proof of the theorem. □

Remark · There exists a *nonlinear membrane shell model,* which is obtained from an asymptotic analysis of the nonlinear elastic shell problem for Saint Venant-Kirchhoff materials with suitable scalings and assumptions , this asymptotic analysis is due to Miara [1994] The two-dimensional nonlinear variational problem found in this fashion reads : Find

$$\zeta \in \mathscr{V}(\omega) = \{\boldsymbol{\eta} = (\eta_i) \in W^{1,4}(\omega), \eta_i = 0 \text{ on } \gamma\}$$

such that

$$\varepsilon \int_\omega a^{\alpha\beta\rho\sigma} E_{\rho\|\sigma}(\zeta)\, F_{\alpha\|\beta}(\boldsymbol{\eta}, \zeta)\sqrt{a}\, dy = \int_\omega p^i \eta_i \sqrt{a}\, dy \quad \text{for all } \boldsymbol{\eta} \in \mathscr{V}(\omega),$$

where

$$E_{\rho\|\sigma}(\zeta) = \frac{1}{2}(\zeta_{\rho\|\sigma} + \zeta_{\sigma\|\rho}) + \frac{1}{2} a^{ms} \zeta_{m\|\rho} \zeta_{s\|\sigma},$$

$$F_{\alpha\|\beta}(\boldsymbol{\eta}, \zeta) = \eta_{\alpha\|\beta} + a^{pq} \eta_{p\|\alpha} \zeta_{q\|\beta},$$

$$\zeta_{\alpha\|\beta} = \partial_\beta \zeta_\alpha - \Gamma^\rho_{\alpha\beta} \zeta_\rho - b_{\alpha\beta} \zeta_3,$$

$$\zeta_{3\|\beta} = \partial_\beta \zeta_3 - \Gamma^\rho_{\beta 3}(0) \zeta_\rho$$

This variational problem can be written as a boundary-value problem :

$$\begin{cases} \mathscr{A}(\zeta) = \mathbf{p} & \text{in } \omega\,, \\ \zeta^\alpha = 0 & \text{on } \gamma\,, \end{cases}$$

where the nonlinear operator $\mathscr{A} = (\mathscr{A}_i)$ is given by :

$$\mathscr{A}_\alpha(\zeta) = -\varepsilon\Big[\frac{1}{\sqrt{a}}\,\partial_\beta\Big(a^{\alpha\beta\rho\sigma}\,E_{\rho\|\sigma}(\zeta)\sqrt{a}\Big) + (a^{\tau\beta\rho\sigma}\,\Gamma^\alpha_{\tau\beta}\,E_{\rho\|\sigma}(\zeta))$$

$$+\frac{1}{\sqrt{a}}\,\partial_\beta\Big(a^{\tau\beta\rho\sigma}\,a^{p\alpha}\,E_{\rho\|\sigma}(\zeta)\,\zeta_{p\|\tau}\sqrt{a}\Big)$$

$$+(a^{\tau\beta\rho\sigma}\,a^{pk}\,E_{\rho\|\sigma}(\zeta)\,\zeta_{p\|\tau})\Big]$$

$$\mathscr{A}_3(\zeta) = -\varepsilon\Big[\frac{1}{\sqrt{a}}(a^{\tau\beta\rho\sigma}\,\Gamma^3_{\tau\beta}\,E_{\rho\|\sigma}(\zeta)) + \frac{1}{\sqrt{a}}\,\partial_\beta\Big(a^{\tau\beta\rho\sigma}\,a^{p3}\,E_{\rho\|\sigma}(\zeta)\,\zeta_{p\|\tau}\sqrt{a}\Big)$$

$$+(a^{\tau\beta\rho\sigma}\,a^{p\gamma}\,\Gamma^3_{\gamma\beta}\,E_{\rho\|\sigma}(\zeta)\,\zeta_{p\|\tau})\Big]\,.$$

One can easily verify that the *linear part* of $\mathscr{A}(\zeta)$ coincides with the operator associated with the *linear membrane shell problem.*

However, contrary to the three-dimensional case (see Ciarlet [1988, Sect. 6.4]), it is *not* possible to obtain an existence theorem for this nonlinear membrane model by using the regularity of Theorem 3 and applying the implicit function theorem. Indeed, $\mathscr{A}$ is infinitely differential between

$$X_1 = \{\zeta = (\zeta_i) \in W^{3,q}(\omega)\,;\, \zeta_\alpha = 0 \text{ on } \gamma\} \text{ and } X_2 = W^{1,q}(\omega)\times$$

$$\times\, W^{1,q}(\omega) \times W^{1,q}(\omega)\,,$$

and thus $\mathscr{A} : X_1 \mapsto X_2$ is differentiable at $\mathbf{0}$. Besides, $\mathscr{A}$ is still differentiable between X_1 and $L^q(\omega)\times L^q(\omega)\times W^{1,q}(\omega)$ with the same derivative $\mathscr{A}'(\mathbf{0})$, but $\mathscr{A}$ is no longer differentiable in the space

$$\{\zeta = (\zeta_i) \in W^{2,q}(\omega)\times W^{2,q}(\omega)\times W^{1,q}(\omega)\,;\, \zeta_\alpha = 0 \text{ on } \gamma\}\,. \qquad \square$$

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REFERENCES

S. AGMON, A. DOUGLIS, L. NIRENBERG, 1964, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II, *Comm. Pure Appl. Math.*, **17**, pp. 35-92.

M. BERNADOU, P. G. CIARLET, 1976, Sur l'ellipticité du modèle de W. T. Koiter, in *Computing Methods in Applied Sciences and Engineering* (R. Glowinski & J. L. Lions, editors), pp. 89-136, Lecture Notes in Economics and Mathematical Systems, Vol. 134, Springer-Verlag, Heidelberg.

M. BERNADOU, P. G. CIARLET, B. MIARA, 1994, Existence theorems for two dimensional linear shell theories, *J. Elasticity*, **34**, pp. 111-138.

P. G. CIARLET, 1988, *Mathematical Elasticity, Vol. I : Three-Dimensional Elasticity*, North Holland, Amsterdam.

P. G. CIARLET, 1996, *Mathematical Elasticity, Vol. II : Plates and Shells*, North Holland, Amsterdam.

P. G. CIARLET, V. LODS, 1996, On the ellipticity of linear membrane shell equations, *J. Math. Pures Appl.*, **75**, pp. 107-124.

P. G. CIARLET, V. LODS, 1994a, Analyse asymptotique des coques linéairement élastiques. I. Coques « membranaires », *C.R. Acad. Sci. Paris*, **318**, Série I, pp. 863-868.

P. G. CIARLET, V. LODS, 1994b, Analyse asymptotique des coques linéairement élastiques. III. Une justification du modèle de W. T. Koiter, *C.R. Acad. Sci. Paris*, **319**, Série I, pp. 299-304.

P. G. CIARLET, V. LODS, B. MIARA, 1994, Analyse asymptotique des coques linéairement élastiques. II. Coques « en flexion », *C.R. Acad. Sci. Paris*, **319**, Série I, pp. 95-100.

P. G. CIARLET, B. MIARA, 1992, On the ellipticity of linear shell models, *Z. Angew. Math. Phys.*, **43**, pp. 243-253.

P. G. CIARLET, E. SANCHEZ-PALENCIA, 1996, An existence and uniqueness theorem for the two-dimensional linear membrane shell equations, *J. Math. Pures Appl.*, **75**, pp. 51-67.

P. DESTUYNDER, 1980, Sur une Justification des Modèles de Plaques et de Coques par les Méthodes Asymptotiques, Doctoral Dissertation, Université Pierre et Marie Curie.

P. DESTUYNDER, 1985, A classification of thin shell theories, *Acta Applicandae Mathematicae*, **4**, pp. 15-63.

G. GEYMONAT, 1965, Sui problemi ai limiti per i sistemi lineari ellitici, *Ann. Mat. Pura Appl.*, **69**, pp. 207-284.

W. T. KOITER, 1970, On the foundations of the linear theory of thin elastic shells, *Proc. Kon. Ned. Akad. Wetensch.*, **B73**, pp. 169-195.

J. L. LIONS, E. MAGENES, 1968, *Problèmes aux Limites Non Homogènes et Applications*, Vol. I, Dunod, Paris.

B. MIARA, 1994, Analyse asymptotique des coques membranaires non linéairement élastiques, *C.R. Acad. Sci. Paris,* **318,** Série I, pp. 689-694.

NEČAS, J., 1967, *Les Méthodes Directes en Théorie des Équations Elliptiques,* Masson, Paris.