Luca F. Pavarino

Neumann-Neumann algorithms for spectral elements in three dimensions

NEUMANN-NEUMANN ALGORITHMS FOR SPECTRAL ELEMENTS IN THREE DIMENSIONS (*)

by Luca F. PAVARINO (¹)

Abstract — *In recent years, domain decomposition algorithms of Neumann-Neumann type have been proposed and studied for h-version finite element discretizations The goal of this paper is to extend this family of algorithms to spectral element discretizations of elliptic problems in three dimensions Neumann Neumann methods provide parallel and scalable preconditioned iterative methods for the linear systems resulting from the spectral discretization In the same Schwarz framework successfully employed for h-version finite elements, quasi-optimal bounds are proved for the conditioning of the iteration operator These bounds depend polylogarithmically on the spectral degree p and are independent of the number and size of subdomains and the jumps in the coefficients of the elliptic operator on the element interfaces*

Key words Domain decomposition, iterative substructuring, spectral elements, p-version finite elements
AMS(MOS) subject classifications. 65N30, 65N55

Résumé — *Les méthodes de décomposition de domaine du type Neumann-Neumann ont été récemment proposées et étudiées pour la discrétisation h-version des éléments finis Dans cet article, nous étendons cette famille de méthodes aux discrétisations par éléments spectraux pour problèmes elliptiques tridimensionnels Les méthodes de Neumann Neumann fournissent des préconditionneurs parallèles et évolutifs (« scalable ») pour la résolution itérative des systèmes linéaires obtenus par discrétisation spectrale Dans le même cadre de Schwarz employé avec succès pour la h-version des éléments finis, nous obtenons des estimations quasi-optimales du conditionnement de l'opérateur d'itération Ces estimations dépendent du logarithme du degré spectral p et elles sont indépendantes du nombre et du diamètre des éléments ainsi que des sauts des coefficients de l'opérateur elliptique à travers les interfaces des éléments*

1. INTRODUCTION

Neumann-Neumann algorithms are nonoverlapping domain decomposition methods (also known as iterative substructuring methods) that have been extensively studied in recent years ; see Dryja and Widlund [7], [8], Le Tallec, De Roeck, and Vidrascu [11], Mandel [12], Mandel and Brezina [13], and the

references therein. For a general overview of domain decomposition methods, see Chan and Mathew [5], Dryja, Smith, and Widlund [6], Le Tallec [10] and the book by Smith, Bjørstad and Gropp [20]. A general introduction to spectral methods can be found in the books by Canuto, Hussaini, Quarteroni and Zang [3] and by Bernardi and Maday [1].

Iterative substructuring algorithms provide preconditioned iterative methods for the solution of the large linear systems arising from the discretization of elliptic partial differential equations with finite and spectral elements. The Neumann-Neumann preconditioner, in its additive form, is the sum of terms corresponding to the solution of local Neumann problems on each subdomain and an additional term corresponding to the solution of a coarse problem. This last term is necessary to obtain a method with convergence independent of the number of subdomains. The resulting method is therefore highly parallel, scalable and well-suited for the emerging parallel computing architectures.

In the current literature, Neumann-Neumann methods have been considered for the standard h-version of the finite element method. In his Master's thesis, Pahl [14] proposed and studied numerically Neumann-Neumann methods for spectral elements, but did not give mathematical proofs of his conjectures. In this paper, we extend our previous work on substructuring methods to Neumann-Neumann algorithms for spectral element discretizations and give complete proofs of the results. The main result of the paper is the proof of quasi-optimal bounds on the convergence rate of the proposed Neumann-Neumann algorithms. Our analysis is based on the Schwarz framework for iterative substructuring methods and is directly inspired by the h-version results of Dryja and Widlund [8]. In the Schwarz framework, a domain decomposition method is determined by a decomposition of the discrete space into local and coarse subspaces and by bilinear forms defined over these subspaces ; see Section 4. We rely heavily on the technical tools developed in our previous work on wire basket based algorithms for spectral elements (Pavarino and Widlund [16], [15]). Some of the results presented here were announced without proofs in our review paper Pavarino and Widlund [17]. Numerical experiments in two dimensions suggesting these results have been conducted by Pahl [14]. Large scale three-dimensional parallel experiments with other iterative methods using spectral element discretizations can be found in Fischer and Rønquist [9] and Rønquist [18], [19]. For recent work on spectral element preconditioners, based on different proofs, see Casarin [4].

This paper is organized as follows. In the next section, we introduce two model problems and their spectral element discretizations. In Section 3, the matrix form of the Neumann-Neumann preconditioner is introduced in order to clarify and motivate the more abstract convergence analysis that will follow. In Section 4, the classical Schwarz framework used to describe and analyze our algorithms is reviewed briefly. The local spaces of the Neumann-Neumann algorithm are defined in Section 5. Some technical results needed in the

analysis are given in Section 6. A standard piecewise linear coarse space for the first model problem is introduced in Section 7 and a first quasi-optimal bound for the resulting algorithm is proved. In Section 8, this simple coarse space is replaced by a more sophisticated one of minimal dimension yielding the same convergence bound. Finally, in Section 9, a coarse space is introduced for the second model problem, which has highly varying coefficients, and an analogous quasi-optimal bound is proved.

Numerical experiments in two dimensions with Neumann-Neumann preconditioners for spectral elements can be found in Pahl [14], Chapter 4, pp. 54-58.

2. THE CONTINUOUS AND DISCRETE MODEL PROBLEMS

The bounded domain $\Omega \subset R^3$ is decomposed into nonoverlapping subdomains Ω_j, $j = 1, ..., N$. We focus on the case where the subdomains Ω_j form a finite element decomposition of Ω of mesh size H. The subdomains are all cubes, or images of the reference cube $\Omega_{ref} = (-1, 1)^3$ under smooth mappings. This assumption allows the introduction of efficient spectral element discretizations, but is not necessary in the construction of Neumann-Neumann methods, which can have subdomains of quite arbitrary shapes. The boundary conditions are assumed to change type only at the boundary between two subregions. We consider linear, selfadjoint, elliptic problems on Ω, with zero Dirichlet boundary conditions on a part $\partial\Omega_D$ of the boundary $\partial\Omega$.

Model Problem I : Find $u \in V = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega_D\}$ such that

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx = f(v) \quad \forall v \in V.$$

Model Problem II : Find $u \in V$ such that

$$a^{(\rho)}(u, v) = \int_\Omega \rho(x) \nabla u \cdot \nabla v \, dx = f(v) \quad \forall v \in V.$$

Here $\rho(x) > 0$ can be discontinuous, with very different values for different subregions, but we assume this coefficient to vary only moderately within each subregion Ω_j. In fact, without decreasing the generality of our results, we will only consider the piecewise constant case of $\rho(x) = \rho_j$, for $x \in \Omega_j$.

We discretize these problems with conforming spectral elements, where we associate each subdomain to an element Ω_j. Let $Q_p(\Omega_{ref})$ be the tensor product of three copies of the space of degree p polynomials on $[-1, 1]$, i.e.

$$Q_p(\Omega_{ref}) = \{\text{polynomials of degree at most p in each variable}\}.$$

The discrete space $V^p \subset V$ is the space of continuous, piecewise Q_p elements

$$V^p = \left\{ v \in V : v|_{\Omega_j} \in Q_p(\Omega_j), j = 1, ..., N \right\}.$$

By restricting u and the test function v to the space V^p, we obtain a conforming Galerkin method for each model problem :

Find $u \in V^p$ such that

(1) $$a^{(\rho)}(u, v) = f(v) \quad \forall v \in V^p,$$

where $\rho \equiv 1$ for Model Problem I. Among the well-known bases for V^p used in the p-version and spectral element literature, we will consider the special basis introduced in Pavarino and Widlund [16], which is very convenient for the theoretical analysis. On the reference cube, this basis consists of vertex, edge, face and interior basis functions ; see Section 5.

A second conforming discretization is obtained by using numerical quadrature at the Gauss-Lobatto-Legendre points ; see Bernardi and Maday [1]. On the reference cube Ω_{ref}, let $\Xi = \left\{ \xi_i, \xi_j, \xi_k \right\}_{i,j,k=0}^p$ be the set of Gauss-Lobatto-Legendre points and let σ_i be the weight associated with ξ_i. The L^2-inner product is replaced with the quadrature-based inner product

(2) $$(u, v)_{Q, \Omega_{ref}} = \sum_{i=0}^p \sum_{j=0}^p \sum_{k=0}^p u(\xi_i, \xi_j, \xi_k) v(\xi_i, \xi_j, \xi_k) \sigma_i \sigma_j \sigma_k.$$

This inner product is uniformly equivalent to the standard L^2-inner product on $Q_p(\Omega_{ref})$. By using the smooth mappings between Ω_{ref} and Ω_j, we can define a quadrature-based inner product analog to (2) on each Ω_j. The original bilinear form $a^{(\rho)}(\cdot , \cdot)$ is replaced, element by element, by

$$a_Q^{(\rho)}(u, v) = \sum_{j=1}^N \rho_j (\nabla u, \nabla v)_{Q, \Omega_j} \quad \forall u, v \in V^p.$$

The same quadrature rule is used for the integrals in the right-hand side of (1), obtaining an equivalent functional denoted by $f_Q(\cdot)$ and the discrete problem :

Find $u \in V^p$ such that

(3) $$a_Q^{(\rho)}(u, v) = f_Q(v) \quad \forall v \in V^p,$$

where $\rho \equiv 1$ for Model Problem I. A detailed analysis of this discrete problem, including a discussion of existence, uniqueness, and error estimates for an individual element, is given in Bernardi and Maday [1]. The discrete

problem with numerical quadrature is of practical importance because the Gauss-Lobatto-Legendre mesh provides a very convenient tensorial basis for V^p. Such a basis is constructed, locally on the reference element, by introducing tensor products

$$l_i(x)\,l_j(y)\,l_k(z)\,, \quad 0 \leqslant i, j, k \leqslant p\,,$$

of Lagrange interpolating polynomials $l_i(x)$ defined by $l_i(\xi_j) = \delta_{ij}$, $0 \leqslant i,\ j \leqslant p$. This provides a nodal basis associated with the Gauss-Lobatto-Legendre nodes Ξ in the sense that, on the reference element,

$$u(x, y, z) = \sum_{\Xi} u(\xi_i, \xi_j, \xi_k)\,l_i(x)\,l_j(y)\,l_k(z)\,.$$

Having chosen a basis for V^p, the discrete problems (1) and (3) are then turned into linear systems of algebraic equations

(4)
$$K\underline{u} = \underline{f},$$

where K is the stiffness matrix, $\underline{u}$ and $\underline{f}$ are the vectors representing u and f in the given basis. A Neumann-Neumann method provides a preconditioner for the iterative solution of the Schur complement S of K with respect to the interface Γ, defined in the next section.

3. MATRIX FORM OF THE NEUMANN-NEUMANN PRECONDITIONER

We define the interface Γ of the decomposition of Ω into the subdomains Ω_j by $\Gamma = \bigcup_{j=1}^{N} \partial\Omega_j \backslash \partial\Omega_D$. Let I denote the set of indices of the basis functions with support interior to each subdomain and let B denote the set of indices of the remaining basis functions (associated with the interface Γ). Each vector can be partitioned as $\underline{u} = (\underline{u}_I, \underline{u}_B)$ and the system (4) can be rewritten as

$$\begin{pmatrix} K_{II} & K_{IB} \\ K_{IB}^T & K_{BB} \end{pmatrix} \begin{pmatrix} \underline{u}_I \\ \underline{u}_B \end{pmatrix} = \begin{pmatrix} \underline{f}_I \\ \underline{f}_B \end{pmatrix}.$$

By eliminating the variables associated with the interior of the subdomains, we obtain the equivalent system

(5)
$$\begin{pmatrix} K_{II} & K_{IB} \\ 0 & S \end{pmatrix} \begin{pmatrix} \underline{u}_I \\ \underline{u}_B \end{pmatrix} = \begin{pmatrix} \underline{f}_I \\ \underline{f}_B - K_{IB}^T K_{II}^{-1} \underline{f}_I \end{pmatrix},$$

where $S = K_{BB} - K_{IB}^T K_{II}^{-1} K_{IB}$ is the Schur complement of K with respect to the interface Γ. $\underline{u}_B$ is found by solving with an iterative method the reduced system

(6) $$S\underline{u}_B = \tilde{f}_B = f_B - K_{IB}^T K_{II}^{-1} f_I,$$

and $\underline{u}_I$ can be found from the first block of (5) as $\underline{u}_I = K_{II}^{-1}(f_I - K_{IB} \underline{u}_B)$. K_{II} is block diagonal with one block per subdomain, since the interior basis functions with support in different subdomains have zero $a(\,.\,,\,.\,)$-inner product. Therefore the application of K_{II}^{-1} to a vector can be computed by solving independent local problems on each subdomain. Moreover, S does not need to be formed explicitly, since in the iterative solution process only its action on a vector is needed. This again requires the application of K_{II}^{-1}. Approximate local solvers for the interior problems could be used (see Dryja, Smith and Widlund [6]), but in this paper we will restrict our attention to exact interior solvers. In the same way as the original stiffness matrix K could be built by subassembly from the local contributions K^j of each subdomain Ω_j, also the reduced problem can be built by subassembly as

$$\begin{pmatrix} K_{II} & K_{IB} \\ 0 & S \end{pmatrix} \begin{pmatrix} u_I \\ \underline{u}_B \end{pmatrix} = \sum_{j=1}^N \begin{pmatrix} K_{II}^j & K_{IB}^j \\ 0 & S^j \end{pmatrix} \begin{pmatrix} u_I^j \\ u_B^j \end{pmatrix} = \sum_{j=1}^N \begin{pmatrix} f_I^j \\ f_B^j - K_{IB}^{jT} K_{II}^{j-1} f_I^j \end{pmatrix}.$$

Here we used the convention of padding local vectors by zeros when they are needed as global vectors and $S^j = K_{BB}^j - K_{IB}^{jT} K_{II}^{j-1} K_{IB}^j$ is the local Schur complement associated with Ω_j.

The Neumann-Neumann preconditioner for S (without a coarse problem) is defined as

(7) $$B^{-1} = \sum_{j=1}^N D_j^{-1} (S^j)^{-1} D_j^{-1},$$

where D_j^{-1} are diagonal matrices with nonzero elements only for the components associated with $\partial\Omega_j$. An additional term representing the solution of an appropriate coarse problem will be defined in Section 7 (and alternative coarse problems will be introduced in Sections 8 and 9). In computing the action of $(S^j)^{-1}$ there is no need to compute explicitly the Schur complement S^j, since

$$(S^j)^{-1} = (0 \quad I) \begin{pmatrix} K_{II}^j & K_{IB}^j \\ K_{IB}^{jT} & K_{BB}^j \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \underline{v} \end{pmatrix},$$

which corresponds to the extraction of the values on $\partial \Omega_j$ of the solution of a local problem on Ω_j with v as Neumann boundary condition on $\partial \Omega_j$. S^j can be singular if $\partial \Omega_j \cap \partial \Omega_D = \emptyset$. In that case, we define a pseudoinverse $(\hat{S}^j)^{-1}$ of S^j, where $\hat{S}^j$ is the Schur complement of the nonsingular matrix $\hat{K}^j = K^j + \dfrac{1}{H^2} M_j$ and M_j is the local mass matrix.

We will see in Section 5 that $B^{-1} S$ corresponds to an additive Schwarz operator T defined by a decomposition of V^p into subspaces. Our convergence analysis of the Neumann-Neumann method will be based on the variational formulation of the preconditioner and on the application of the Schwarz framework, briefly reviewed in the next section.

4. CLASSICAL SCHWARZ THEORY

In this section, we recall the abstract Schwarz theory, very useful for the analysis of both overlapping and iterative substructuring domain decomposition methods. We refer to Dryja and Widlund [8] for a more complete treatment. Let V be a finite dimensional Hilbert space with inner product $b(\cdot , \cdot)$ and let f be a linear functional over V. We want to solve the discrete problem :

Find $u \in V$ such that

$$(8) \qquad b(u, v) = f(v) \quad \forall v \in V .$$

Let V be decomposed into $N + 1$ subspaces :

$$V = V_0 + V_1 + \cdots + V_N.$$

For each subspace, we assume that there is a symmetric, positive definite bilinear form

$$b_i(\cdot , \cdot) : V_i \times V_i \to R$$

and define the operator $T_i : V \to V_i$ by

$$(9) \qquad b_i(T_i v, w) = b(v, w) \quad \forall w \in V_i.$$

If $b_i(v, w) = b(v, w)$, then $T_i = P_i$, the orthogonal projection on V_i in the inner product $b(\cdot , \cdot)$. Define the additive Schwarz operator

$$T = T_0 + T_1 + \cdots + T_N$$

and replace the original problem (8) with

$$(10) \qquad Tu = g , \quad g = \sum_{i=0}^{N} g_i = \sum_{i=0}^{N} T_i u .$$

This problem has the same solution as (8), since the right-hand side is constructed by

$$b_i(g_i, v) = b(u, v) = f(v) \quad \forall v \in V_i.$$

In the variational formulation of our Neumann-Neumann preconditioner, we will see that equation (10) will correspond exactly to the preconditioned system $B^{-1} S$ defined in the previous section.

The following basic convergence result is proved in Dryja and Widlund [8].

THEOREM 4.1 : *Let there exist*

i) *a constant* C_0 *such that* $\forall v \in V$ *there exists a decomposition* $v = \sum_{i=0}^{N} v_i$, $v_i \in V_i$, *such that*

$$\sum_{i=0}^{N} b_i(v_i, v_i) \leq C_0^2 b(v, v) ;$$

ii) *a constant* ω *such that for* $i = 0, 1, ..., N$,

$$b(v, v) \leq \omega b_i(v, v) \quad \forall v \in V_i ;$$

iii) *constants* ϵ_{ij}, *for* $i, j = 1, ..., N$ *such that*

$$b(v_i, v_j) \leq \epsilon_{ij} b(v_i, v_i)^{1/2} b(v_j, v_j)^{1/2} \quad \forall v_i \in V_i, \forall v_j \in V_j.$$

Then

$$C_0^{-2} b(v, v) \leq b(Tv, v) \leq (\rho(\mathcal{E}) + 1) \omega b(v, v) \quad \forall v \in V.$$

$\rho(\mathcal{E})$ *is the spectral radius of the matrix* $\mathcal{E} = \{\epsilon_{ij}\}_{i,j=1}^{N}$.

Our convergence proofs will be based on rewriting the Neumann-Neumann preconditioner in a variational form in the Schwarz framework, estimating these theer parameters C_0, $\rho(\mathcal{E})$, and ω and applying Theorem 4.1. The

estimates of this theorem provide an upper bound for the condition number of T: $cond(T) \leq C_0^2 \, \omega(\rho(\mathscr{E}) + 1)$. The square root of this bound is therefore an upper bound for the number of iterations of a conjugate gradient-like method for (10).

5. VARIATIONAL FORM OF THE PRECONDITIONER AND LOCAL SUBSPACES

In this section, we introduce the subspaces associated with the local solvers of our Neumann-Neumann preconditioner. A coarse global space will be introduced in the next section, thus completing the definition of the algorithm. In our analysis, we will concentrate on the discrete problem (1) and will use the special base for V^p described in Pavarino and Widlund [16]. Analogous analysis and results can be given for the second discrete problem (3), as we have done in Pavarino and Widlund [15] for wire basket based algorithms.

Let $s(\, . \, , \, . \,)$ and $s_j(\, . \, , \, . \,)$ be the bilinear forms defined by the Schur complements

(11) $$s(u, v) = \underline{u}_B^T S \underline{v}_B \quad \text{and} \quad s_j(u, v) = \underline{u}_B^{jT} S^j \underline{v}_B^j.$$

Let $V^p(\Gamma)$ be the subspace of piecewise discrete harmonic functions of V^p, i.e. the subspace consisting of functions $u \in V^p$ satisfying

$$a(u, v) = 0 \quad \forall v \in V^p \cap H_0^1(\Omega_j), \quad j = 1, ..., N.$$

In matrix form, this orthogonality condition between $V^p(\Gamma)$ and $V^p \cap H_0^1(\Omega_j)$ becomes $K_{II} \underline{u}_I + K_{IB} \underline{u}_B = 0$. Discrete harmonic functions $v \in V^p(\Gamma)$ are completely defined by their values on the interface Γ and $s(v, v) = a(v, v)$. The reduced problem (6) for the Schur complement can be rewritten as

(12) $$s(u, v) = \tilde{f}(v) \quad \forall v \in V^p(\Gamma).$$

We can then work with the abstract Schwarz framework of the previous section by taking $V = V^p(\Gamma)$ and $b(\, . \, , \, . \,) = s(\, . \, , \, . \,)$. We will now define a decomposition of $V^p(\Gamma)$ into local subspaces $V_i(\Gamma) \subset V^p(\Gamma)$ associated with each Ω_i and a bilinear form $b_i(\, . \, , \, . \,)$ for each $V_i(\Gamma)$:

$$V_i(\Gamma) = \text{span} \left\{ \text{basis functions of } V^p \text{ associated with } \partial\Omega_i \right\}.$$

If numerical quadrature at the Gauss-Lobatto-Legendre points is used, then $V_i(\Gamma)$ is spanned by the basis functions of V^p vanishing at the Gauss-Lobatto-Legendre points outside $\partial\Omega_i$. We remark that functions in $V_i(\Gamma)$ have support in the union of the 26 subdomains that are neighbors of Ω_i. In order to define

the bilinear forms $b_t(\ \cdot\ ,\ \cdot\ )$, we need to recall the decomposition of a discrete harmonic polynomial function $u \in V_i(\Gamma)$ into vertex, edge, and face components introduced in Pavarino and Widlund [16]

$$u = u_{V_i} + u_{E_i} + u_{F_i} .$$

Suppose for simplicity that $\Omega_i = \Omega_{ref}$.

• The vertex component u_{V_i} is given as the sum of eight terms, one for each vertex. The one associated with vertex $V_i^{(1)} = (1, 1, 1)$ is given by

$$(13) \qquad u_{V_i}^{(1)}(x, y, z) = u(1, 1, 1)\ \varphi_0(x)\ \varphi_0(y)\ \varphi_0(z) ,$$

where φ_0 is the degree p polynomial on $[-1, 1]$ which satisfies $\varphi(1) = 1$, $\varphi(-1) = 0$ and has minimal L^2-norm (see Lemmas 4.1 and 4.2 in [16]).

• The edge component u_{E_i} is the sum of twelve terms, one for each edge. The one associated with edge $E_i^{(1)} = \{(1, 1, z) : z \in (-1, 1)\}$ is given by

$$(14) \qquad u_{E_i}^{(1)}(x, y, z) = \varphi_0(x)\ \varphi_0(y)\ (u(1, 1, z) - u_{V_i}(1, 1, z)) .$$

• The face component u_{F_i} is the sum of six terms, one for each face. The one associated with face $F_i^{(1)} = \{(1, y, z) : (y, z) \in (-1, 1)^2\}$ is given by the discrete harmonic extension inside Ω_{ref} of $u(1, y, z) - u_{V_i}(1, y, z) - u_{E_i}(1, y, z)$.

More details and motivations for this construction can be found in [16], Sections 4.2-4.4. In [15], we have also detailed this construction in the numerical quadrature case. Using the same notation found in these papers, we define the wire basket W_i of the subdomain Ω_i as the union of the edges and vertices of Ω_i. The sum of the vertex and edge components, known as the wire basket component, will be denoted by $\bar{I}^{W_i} u = u_{V_i} + u_{E_i}$, from which $u = \bar{I}^{W_i} u + u_{F_i}$.

Let $v_i : V^p(\Gamma) \to V_i(\Gamma)$ be operators defined by

$$v_i(u) = 8\ u_{V_i} + 4\ u_{E_i} + 2\ u_{F_i}.$$

This definition is motivated by the fact that, under our assumptions on the subdomains, each vertex belongs to eight subdomains, each edge to four and each face to two. In a more general geometry, these operators could still be defined by replacing these numbers by the number of subdomains on which

each interface basis function has support. We note that $v_t(u) = \bar{I}^{W_t} v_t(u) + 2 u_{F_t}$. The « inverses » $v_t^+ : V^p(\Gamma) \to V_t(\Gamma)$ of these operators will be needed to scale properly the different contributions to the preconditioner :

$$v_t^+(u) = \frac{1}{8} u_{V_t} + \frac{1}{4} u_{E_t} + \frac{1}{2} u_{F_t}.$$

Clearly, $u = \sum_{t=1}^{N} v_t^+(u)$, $\forall u \in V^p(\Gamma)$ and $v_t(v_t^+(u)) = u$, $\forall u \in V_t(\Gamma)$. In order to deal with singular Neumann problems associated with interior subdomains, we need to define the new inner product

$$\hat{a}_t(u, v) = \int_{\Omega_t} \nabla u \cdot \nabla v \, dx + \frac{1}{H^2} \int_{\Omega_t} uv \, dx,$$

where H is the mesh size of the subdomains, and the associated weighted norm

$$\|u\|_{H^1(\Omega_t)}^2 = |u|_{H^1(\Omega_t)}^2 + \frac{1}{H^2} \|u\|_{L^2(\Omega_t)}^2.$$

The local stiffness matrix associated with $\hat{a}_t(\,.\,,\,.\,)$ is the matrix $\hat{K}^t = K^t + \frac{1}{H^2} M_t$ defined in Section 3. Let $\hat{s}_t(\,.\,,\,.\,)$ be the bilinear form given by the local Schur complement $\hat{S}^t$ with respect to $\hat{a}_t(\,.\,,\,.\,)$, i.e. $\hat{s}_t(u, v) = \underline{u}_B^{tT} \hat{S}^t \underline{v}_B^t$. Then, we are finally able to define the bilinear forms on $V_t(\Gamma) \times V_t(\Gamma)$ by

$$(15) \qquad b_t(u, v) = \hat{s}_t(v_t(u), v_t(v)) \equiv \hat{a}_t(\mathcal{H}_t(v_t(u)), \mathcal{H}_t(v_t(u))),$$

where $\mathcal{H}_t w$ is the discrete harmonic extension, with respect to $\hat{a}_t(\,.\,,\,.\,)$, of a polynomial w defined on $\partial \Omega_t$. In other words, $\mathcal{H}_t w$ is the solution of the Dirichlet problem

$$(16) \qquad \hat{a}_t(\mathcal{H}_t w, v) = 0 \quad \forall v \in V^p \cap H_0^1(\Omega_t)$$

with the Dirichlet boundary condition $\mathcal{H}_t w = w$ on $\partial \Omega_t$.

   This completes the definition of the local part of our algorithm in the abstract Schwarz framework (a coarse space V_0 will be introduced in Section 7). In fact, an additive Schwarz operator (without a coarse space) is now defined by

$$T = T_1 + \cdots + T_N,$$

where each T_i is defined by

(17) $b_i(T_i u, v) = s(u, v) \quad \forall v \in V_i(\Gamma)$.

The reduced problem (12) is then replaced by the preconditioned problem

$$Tu = g = \sum_{i=0}^{N} g_i$$

where g_i, $i = 1, ..., N$ are computed as in (10). This problem has the matrix form $B^{-1} S$, where B^{-1} is the Neumann-Neumann preconditioner (without a coarse space) introduced in eq. (7), Section 3. In fact, the variational system (17) has the matrix form

$$\underline{v}_{B_i}^T D_i \hat{S}^i D_i \underline{T_i u} = \underline{v}_{B_i}^T S \underline{u}_B,$$

and therefore

$$\underline{T_i u} = D_i^{-1} (\hat{S}^i)^{-1} D_i^{-1} S \underline{u}_B,$$

where $D_i \underline{v}_B$ is the matrix representation of $v_i(v)$, $v \in V^p(\Gamma)$ and D_i is the diagonal matrix with nonzero elements only for the components on $\partial \Omega_i$ and equal to the corresponding coefficients of $v_i(v)$ (i.e. 8, 4, 2 for the vertex, edge, and face components, respectively). Analogously, the « inverse » D_i^{-1} is diagonal with nonzero elements only for the components on $\partial \Omega_i$ and equal to the corresponding coefficients of $v_i^+(v)$ (i.e. 1/8, 1/4, 1/2 for the vertex, edge, and face components, respectively).

In order to apply Theorem 4.1, we will need some technical results obtained in the next section. In the following, we will sometimes use two indices (instead of one) to denote a geometric object shared by two elements ; for example, a face shared by the elements Ω_i and Ω_j will be denoted by F_{ij}.

6. TECHNICAL RESULTS

Using the technical tools developed in our papers [16] and [15], we are able to translate the h-version technical results of Dryja and Widlund [8], Section 4, into spectral ones. In the following, C will denote a generic positive constant independent of p, H and N.

LEMMA 6.1 : *Let W be the wire basket of the reference cube Ω_{ref}, $u \in Q_p(\Omega_{ref})$ and denote by $\bar{u}_W = \int_W uds / \int_W ds$ the average of u over W. Then*

$$\|u\|^2_{L^2(W)} \leq C(1 + \log p) \|u\|^2_{H^1(\Omega_{ref})}$$

and

$$\|u - \bar{u}_W\|^2_{L^2(W)} \leq C(1 + \log p) |u|^2_{H^1(\Omega_{ref})}.$$

Proof : This is Lemma 5.3 in [16].

An essential result is the following analog of Lemma 4 in [8] :

LEMMA 6.2 : *For all $u \in V_i(\Gamma)$*

$$s(u, u) \leq C(1 + \log p)^2 b_i(u, u).$$

Proof : $u \in V_i(\Gamma)$ has support in Ω_i and in the 26 neighboring elements Ω_j sharing a face or an edge or a vertex with Ω_i. We divide the integral accordingly,

$$(18) \qquad s(u, u) = s_i(u, u) + \sum_{j \neq i} s_j(u, u),$$

and we bound each term separately.

a) Since $u = \dfrac{1}{2} v_i(u) + \dfrac{1}{2}(2u - v_i(u))$, then

$$(19) \quad s_i(u, u) \leq \frac{1}{2} s_i(v_i(u), v_i(u)) + \frac{1}{2} s_i(2u - v_i(u), 2u - v_i(u)).$$

The first term in (19) is bounded by $\dfrac{1}{2} \hat{s}_i(v_i(u), v_i(v))$. To bound the second term, we note that

$$2u - v_i(u) = 2u_{V_i} + 2u_{E_i} + 2u_{F_i} - 8u_{V_i} - 4u_{E_i} - 2u_{F_i} = -6u_{V_i} - 2u_{E_i}.$$

We recall that from Lemma 5.4 and 5.6 in [16] we have for all $u \in V_i(\Gamma)$

$$|u_{V_i}|^2_{H^1(\Omega_i)} \leq C\|u\|^2_{L^2(W_i)} \quad \text{and} \quad |u_{E_i}|^2_{H^1(\Omega_i)} \leq C\|u\|^2_{L^2(W_i)},$$

where W_i is the wire basket of Ω_i. Therefore,

$$s_i(2u - v_i(u), 2u - v_i(u)) = |2u - v_i(u)|^2_{H^1(\Omega_i)} = |6u_{V_i} + 2u_{E_i}|^2_{H^1(\Omega_i)}$$

$$\leq 72|u_{V_i}|^2_{H^1(\Omega_i)} + 8|u_{E_i}|^2_{H^1(\Omega_i)} \leq C\|u\|^2_{L^2(W_i)} \leq C\|v_i(u)\|^2_{L^2(W_i)},$$

because the vertex and edge components are $L^2(W_t)$-orthogonal (see Section 4.3 in [16]). The values of $v_i(u)$ on $\partial \Omega_i$ can be extended inside Ω_t by using the discrete harmonic extension $\mathscr{H}_i$; see eq. (16). By Lemma 6.1 applied to $\mathscr{H}_i(v_i(u))$, we then have

$$\|v_i(u)\|^2_{L^2(W_t)} \leq C(1 + \log p) \|\mathscr{H}_i(v_i(u))\|^2_{H^1(\Omega_t)}$$

$$= C(1 + \log p)\, \hat{a}_i(\mathscr{H}_i(v_i(u)), \mathscr{H}_i(v_i(u)))$$

and we conclude that

$$s_t(u, u) \leq C(1 + \log p)\, b_t(u, u)\,.$$

b) Consider an element Ω_j with only a vertex V_{ij} in common with Ω_t and suppose, for simplicity, that both elements are translations of the reference cube. By Lemma 5.4 of [16]

$$s_j(u, u) = |u_{V_{ij}}|^2_{H^1(\Omega_j)} \leq C\|u_{V_{ij}}\|^2_{L^2(W_j)}$$

$$= 3\, Cu^2(V_{ij})\, \|\varphi_0\|^2_{L^2(-1, 1)} = C\|u_{V_{ij}}\|^2_{L^2(W_t)},$$

where φ_0 is the polynomial defined in (13). We then conclude as in a) that this is bounded by $C(1 + \log p)\, b_i(u, u)$.

c) Consider an element Ω_j with only an edge E_{ij} in common with Ω_t. Let V_1 and V_2 be the endpoints of $\bar{E}_{ij}$. Again by Lemma 5.4 of [16] and the estimates just obtained in b)

$$s_j(u, u) = |u_{E_{ij}} + u_{V_1} + u_{V_2}|^2_{H^1(\Omega_j)} \leq 3(|u_{E_{ij}}|^2_{H^1(\Omega_j)} + |u_{V_1}|^2_{H^1(\Omega_j)} + |u_{V_2}|^2_{H^1(\Omega_j)})$$

$$\leq C(\|u_{E_{ij}}\|^2_{L^2(W_j)} + \|u_{V_1}\|^2_{L^2(W_j)} + \|u_{V_2}\|^2_{L^2(W_j)})$$

$$= C(\|u_{E_{ij}}\|^2_{L^2(W_t)} + \|u_{V_1}\|^2_{L^2(W_t)} + \|u_{V_2}\|^2_{L^2(W_t)})$$

and we conclude, as in a), that this is bounded by $C(1 + \log p)\, b_i(u, u)$.

d) Consider an element Ω_j with a face F_{ij} in common with Ω_i. Let $u_{E_{ij}}$ be the sum of the four edge components of u associated with the four edges of $\bar{F}_{ij}$ and let $u_{V_{ij}}$ be the sum of the four vertex components of u associated with the four vertices of $\bar{F}_{ij}$. Then,

$$s_j(u, u) = |u_{F_{ij}} + u_{E_{ij}} + u_{V_{ij}}|^2_{H^1(\Omega_j)} \leq 2|u_{F_{ij}}|^2_{H^1(\Omega_j)} + 2|u_{E_{ij}} + u_{V_{ij}}|^2_{H^1(\Omega_j)}.$$

The last term is bounded as in c). The first term is bounded by using Corollary 5.5 and Lemma 5.7 in [16], the definitions of $\tilde{I}^{W_t}$ and $\mathscr{H}_t$:

$$|u_{F_y}|^2_{H^1(\Omega_j)} \leqslant C\|u_{F_y}\|^2_{H^{1/2}_{00}(F_y)} \leqslant C\|2\,u_{F_y}\|^2_{H^{1/2}_{00}(F_y)}$$

$$= C\|v_t(u) - \tilde{I}^{W_t}\,v_t(u)\|^2_{H^{1/2}_{00}(F_y)}$$

$$\leqslant C(1 + \log p)^2\|\mathscr{H}_t(v_t(u))\|^2_{H^1(\Omega_j)}$$

$$= C(1 + \log p)^2\hat{a}_t(\mathscr{H}_t(v_t(u)), \mathscr{H}_t(v_t(u))).$$

Therefore, this term $s_j(u, u)$ is bounded by $C(1 + \log p)^2 b_t(u, u)$ and the lemma is proved.

7. METHOD NN_1 WITH A STANDARD COARSE SPACE FOR MODEL PROBLEM I

We now introduce a coarse space in order to make the Neumann-Neumann algorithm independent of the number of subdomains N. For Model Problem I, we can use a standard coarse space V_0 consisting of piecewise linear functions over the coarse mesh defined by the subdomains Ω_t, i.e. $V_0 = V^1(\Omega)$. Let $V_t(\Gamma)$ be the local spaces defined above. On V_0, define the bilinear form $b_0(u, v) = s(u, v)$ (the same $s(\,.\,,.\,)$ defined in (11)), while on $V_t(\Gamma)$ we will use the bilinear form $b_t(\,.\,,.\,)$ defined in (15). Let

$$T_{NN_1} = T_0 + T_1 + \cdots + T_N,$$

where the T_t's are associated with the bilinear forms $b_t(\,.\,,.\,)$ by eq. (17). The abstract Theorem 4.1 becomes in this context

THEOREM 7.1 : *For all* $u \in V^p(\Gamma)$

$$cs(u, u) \leqslant s(T_{NN_1} u, u) \leqslant C(1 + \log p)^2 s(u, u),$$

where c and C are constants independent of p, H, and N.

Proof : We estimate the three parameters involved in the three assumptions of Theorem 4.1.

Assumption i). Define $u_0 = Q_0 u$, where Q_0 is the L^2-projection from $V^p(\Gamma)$ into V_0, with the properties

$$|Q_0 u|_{H^1(\Omega)} \leqslant C|u|_{H^1(\Omega)} \quad \text{and} \quad \|u - Q_0 u\|_{L^2(\Omega)} \leqslant CH|u|_{H^1(\Omega)};$$

see Bramble and Xu [2]. Define $w = u - u_0$ and $u_i = v_i^+(w)$. Clearly, $u_i \in V_i(\Gamma)$ and $\sum_1^N u_i = w$. Moreover,

$$b_0(u_0, u_0) = s(u_0, u_0) = |Q_0 u|_{H^1(\Omega)}^2 \leqslant C|u|_{H^1(\Omega)}^2$$

and

$$b_i(u_i, u_i) = \hat{s}_i(v_i(v_i^+(w)), v_i(v_i^+(w)))$$

$$= \hat{s}_i(w, w) \leqslant |w|_{H^1(\Omega_i)}^2 + \frac{1}{H^2} \|w\|_{L^2(\Omega_i)}^2.$$

Therefore,

$$\sum_{i=1}^N b_i(u_i, u_i) \leqslant |w|_{H^1(\Omega)}^2 + \frac{1}{H^2} \|w\|_{L^2(\Omega)}^2 \leqslant C|u|_{H^1(\Omega)}^2 = Cs(u, u) ,$$

and assumption i) is satisfied with $C_0^2 = C$, a constant independent of p, H and N.

Assumption ii). $b_0(u_0, u_0) = s(u_0, u_0)$ implies $\|T_0\|_s = 1$. By Lemma 6.2,

$$b(u_i, u_i) \leqslant C(1 + \log p)^2 b_i(u_i, u_i) .$$

Therefore, assumption ii) is satisfied with $\omega \leqslant C(1 + \log p)^2$.

Assumption iii). Since any point $x \in \Omega$ belongs to the closure of at most eight elements Ω_i, then $\rho(\mathscr{E})$ is bounded by a constant independent of p, H and N (this is a standard argument in domain decomposition theory ; see Chan and Mathew [5], Theorem 12).

8. METHOD NN_2 WITH A COARSE SPACE OF MINIMAL DIMENSION FOR MODEL PROBLEM I

We now consider an alternative coarse space with only one degree of freedom for each subdomain. The set of substructures indices is partitioned according to the intersection between each boundary $\partial\Omega_i$ and $\partial\Omega_D$:

$i \in N_I$ if $\partial\Omega_i \cap \partial\Omega_D = \phi$;

$i \in N_{B,F}$ if $\partial\Omega_i \cap \partial\Omega_D$ contains a face ;

$i \in N_{B,E}$ if $\partial\Omega_i \cap \partial\Omega_D$ contains only edges ;

$i \in N_{B,V}$ if $\partial\Omega_i \cap \partial\Omega_D$ contains only vertices ;

We then define $N_B = N_{B,F} \cup N_{B,E} \cup N_{B,V}$. The new coarse basis functions are the images $v_i^+(1)$ of the constant function equal to 1 on $\partial\Omega_i$:

$$V_0 = span\{v_i^+(1)\}_{i \in \mathcal{N}_I}.$$

The bilinear form for V_0 is now

$$b_0(u, v) = (1 + \log p)^{-2} s(u, v),$$

while the local spaces $V_\iota(\Gamma)$, the local bilinear forms $b_\iota(. , .)$ and operators T_ι are as before. For the operator $T_{NN2} = T_0 + T_1 + \cdots + T_N$, we can then prove the same bound as in Theorem 7.1.

THEOREM 8.1 : *For all $u \in V^p(\Gamma)$*

$$cs(u, u) \leq s(T_{NN2} u, u) \leq C(1 + \log p)^2 s(u, u),$$

where c and C are constants independent of p, H, and N.

Proof : Assumption i). Define $u_0 = \sum_\iota \bar{u}_\iota v_\iota^+(1)$, where

(20)
$$\bar{u}_\iota = \begin{cases} \left(\int_{\partial\Omega_\iota} u \, ds \right) / \left(\int_{\partial\Omega_\iota} ds \right) & \text{if } i \in N_I \\ 0 & \text{if } i \in N_B \end{cases}$$

and $u_\iota = v_\iota^+(u - \bar{u}_\iota) = v_\iota^+(u) - \bar{u}_\iota v_\iota^+(1)$, so that $u = u_0 + \sum_{\iota=1}^N u_\iota$. Then

(21)
$$b_\iota(u_\iota, u_\iota) = \hat{s}_\iota(v_\iota(u_\iota), v_\iota(u_\iota)) = \hat{s}_\iota(u - \bar{u}_\iota, u - \bar{u}_\iota)$$

$$\leq \hat{a}_\iota(u - \bar{u}_\iota, u - \bar{u}_\iota) = \|\nabla u\|^2_{L^2(\Omega_\iota)} + \frac{1}{H^2} \|u - \bar{u}_\iota\|^2_{L^2(\Omega_\iota)}.$$

Let $i \in N_I$. Then

$$\|\bar{u}_\iota\|^2_{L^2(\Omega_\iota)} = H^3 \bar{u}_\iota^2 \leq C \frac{1}{H} \left(\int_{\partial\Omega_\iota} u \, ds \right)^2 \leq CH \|u\|^2_{L^2(\partial\Omega_\iota)}.$$

By a trace theorem $\|u\|^2_{L^2(\partial\Omega_\iota)} \leq CH\hat{a}_\iota(u, u)$ and therefore

$$\|\bar{u}_\iota\|^2_{L^2(\Omega_\iota)} \leq CH^2 \hat{a}_\iota(u, u) \quad \text{and} \quad \frac{1}{H^2} \|u - \bar{u}_\iota\|^2_{L^2(\Omega_\iota)} \leq C\hat{a}_\iota(u, u).$$

Since the left-hand side of the last inequality does not change if u is shifted by a constant, we can apply Poincaré's inequality and obtain

$$b_\iota(u_\iota, u_\iota) \leq Cs_\iota(u, u).$$

If $i \in N_B$ (since in this case $\bar{u}_i = 0$), we can extend the region of integration in the last term of (21) to include the substructures that are neighbors of Ω_i. Since $\partial\Omega_D$ has positive measure, from our assumptions it follows that one of these substructures has a face on $\partial\Omega_D$. By using Friedrichs' inequality we can then remove the L^2-term again and obtain

$$(22) \qquad \sum_{i=1}^{N} b_i(u_i, u_i) \leq C \sum_{i=1}^{N} s_i(u, u) = Cs(u, u) .$$

It remains to bound the coarse term $b_0(u_0, u_0) = (1 + \log p)^{-2} s(u_0, u_0)$ by $Cs(u, u)$. Let $w = \sum_{i=1}^{N} u_i$, so that $u_0 = u - w$. Since $s(u_0, u_0) \leq 2 s(u, u) + 2 s(w, w)$, we have only to bound $s(w, w)$. But the support of each u_i is contained in the union of Ω_i and the 26 elements which are neighbors of Ω_i. Since most pairs of subspaces are mutually orthogonal

$$(23) \qquad s(w, w) \leq C \sum_{i=1}^{N} s(u_i, u_i) .$$

By Lemma 6.2 the right hand side of (23) is bounded by $C(1 + \log p)^2 \sum_{i=1}^{N} b_i(u_i, u_i)$ and this last sum has already been estimated in the first part of this proof; see (22). Therefore,

$$s(u_0, u_0) \leq C(1 + \log p)^2 s(u, u)$$

and assumption i) is then proved with $C_0^2 = C$.

Assumption ii): the estimate $\omega \leq C(1 + \log p)^2$ follows directly from Lemma 6.2 for $1 \leq i \leq N$ and from the definition of b_0 for $i = 0$.

Assumption iii): as in Theorem 7.1.

9. METHOD NN_3 WITH A COARSE SPACE OF MINIMAL DIMENSION FOR MODEL PROBLEM II

For Model Problem II, where the coefficients ρ_i can be discontinuous across element interfaces, we replace the operators ν_i by the operators $\mu_i : V^p(\Gamma) \to V_i(\Gamma)$ defined by

$$(24) \qquad \mu_i(u) = \sum_{k=1}^{6} c_{ik}^F u_{F_{ik}} + \sum_{k=1}^{12} c_{ik}^E u_{E_{ik}} + \sum_{k=1}^{8} c_{ik}^V u_{V_{ik}},$$

where we recall that $u = \sum_{k=1}^{6} u_{F_{ik}} + \sum_{k=1}^{12} u_{E_{ik}} + \sum_{k=1}^{8} u_{V_{ik}}$ is the decomposition of u into face, edge and vertex components defined in Section 5. The coefficients are

$$c_{i_k}^F = \sqrt{\rho_{i_{k_1}}} + \sqrt{\rho_{i_{k_2}}}$$

(25)
$$c_{i_k}^E = \sqrt{\rho_{i_{k_1}}} + \sqrt{\rho_{i_{k_2}}} + \sqrt{\rho_{i_{k_3}}} + \sqrt{\rho_{i_{k_4}}}$$

$$c_{i_k}^V = \sqrt{\rho_{i_{k_1}}} + \cdots + \sqrt{\rho_{i_{k_8}}},$$

where the sums are respectively taken over the two elements sharing the face F_{i_k}, the four elements sharing the edge E_{i_k} and the eight elements sharing the vertex V_{i_k}. For Model Problem I where $\rho_{i_k} = 1$, $i_k = 1, ..., N$, we would obtain the old coefficients $c_{i_k}^F = 2$, $c_{i_k}^E = 4$, $c_{i_k}^V = 8$, i.e. $\mu_i = \nu_i$.

The « inverses » of the operators (24) are then defined by

$$\mu_i^+(u) = \sum_{k=1}^{6} \frac{1}{c_{i_k}^F} u_{F_{ik}} + \sum_{k=1}^{12} \frac{1}{c_{i_k}^E} u_{E_{ik}} + \sum_{k=1}^{8} \frac{1}{c_{i_k}^V} u_{V_{ik}}$$

and the coarse space by

$$V_0 = span\left\{ \sqrt{\rho_i}\, \mu_i^+(1) \right\}_{i \in \mathcal{N}_i}.$$

We still have the partition of unity property $u = \sum_{i=1}^{N} \sqrt{\rho_i}\, \mu_i^+(u)$ and $\mu_i(\mu_i^+(u)) = u$, $\forall u \in V_i(\Gamma)$. For the coarse space, we use the bilinear form

$$b_0(u, v) = (1 + \log p)^{-2} \sum_{i=1}^{N} \rho_i s_i(u, v),$$

while on the local spaces $V_i(\Gamma)$, we use

$$b_i(u, v) = \hat{s}_i(\mu_i(u), \mu_i(v)) \equiv \hat{a}_i(\mathcal{H}_i(\mu_i(u)), \mathcal{H}_i(\mu_i(u))).$$

As before, these bilinear forms define operators T_i's, $i = 0, 1, ..., N$ and let

$$T_{NN3} = T_0 + T_1 + \cdots + T_N$$

be the additive Schwarz operator. In order to prove the analog of Theorem 8.1 for T_{NN3}, we need first to prove an analog of Lemma 6.2 :

LEMMA 9.1 : *For Model Problem II and for all $u \in V_i(\Gamma)$*

$$s^{(p)}(u, u) \leq C(1 + \log p)^2 b_i(u, u).$$

Proof : We consider the different contributions to the integral,

$$s^{(\rho)}(u, u) = \rho_i s_i(u, u) + \sum_{j \neq i} \rho_j s_j(u, u) ,$$

and estimate each term separately.

a) We first estimate $\rho_i s_i(u, u)$. $u \in V_i(\Gamma)$ is decomposed in the standard way as

$$u = \sum_{k=1}^{6} u_{F_{ik}} + \sum_{k=1}^{12} u_{E_{ik}} + \sum_{k=1}^{8} u_{V_{ik}}.$$

It follows from the definition (25) that ρ_i is less than any of the coefficients $(c_{i_k}^F)^2$, $(c_{i_k}^E)^2$, $(c_{i_k}^V)^2$. By applying Corollary 5.5 and Lemma 5.7 in [16], we have

$$\rho_i s_i(u_{F_{ik}}, u_{F_{ik}}) = \rho_i |u_{F_{ik}}|_{H^1(\Omega_i)}^2 \leqslant C\rho_i \|u_{F_{ik}}\|_{H_{00}^{1/2}(F_{ik})}^2 \leqslant C \|c_{i_k}^F u_{F_{ik}}\|_{H_{00}^{1/2}(F_{ik})}^2$$

$$= C \|\mu_i(u) - \tilde{I}^{W_i} \mu_i(u)\|_{H_{00}^{1/2}(F_{ik})}^2$$

$$\leqslant C(1 + \log p)^2 \|\mathscr{H}_i(\mu_i(u))\|_{H^1(\Omega_i)}^2$$

$$= C(1 + \log p)^2 b_i(u, u) .$$

The wire basket component $\tilde{I}^{W_i} u = \sum_{k=1}^{12} u_{E_{ik}} + \sum_{k=1}^{8} u_{V_{ik}}$ can be estimated as in Lemma 5.6 of [16].

$$\rho_i s_i(\tilde{I}^{W_i} u, \tilde{I}^{W_i} u) \leqslant 20 \left(\sum_{k=1}^{12} \rho_i s_i(u_{E_{ik}}, u_{E_{ik}}) + \sum_{k=1}^{8} \rho_i s_i(u_{V_{ik}}, u_{V_{ik}}) \right)$$

$$\leqslant 20 \left(\sum_{k=1}^{12} |c_{i_k}^E u_{E_{ik}}|_{H^1(\Omega_i)}^2 + \sum_{k=1}^{8} |c_{i_k}^V u_{V_{ik}}|_{H^1(\Omega_i)}^2 \right)$$

$$\leqslant C \left(\sum_{k=1}^{12} \|c_{i_k}^E u_{E_{ik}}\|_{L^2(W_i)}^2 + \sum_{k=1}^{8} \|c_{i_k}^V u_{V_{ik}}\|_{L^2(W_i)}^2 \right)$$

$$\leqslant C \|\mu_i(u)\|_{L^2(W_i)}^2 \leqslant C(1 + \log p) \|\mathscr{H}_i(\mu_i(u))\|_{H^1(\Omega_i)}^2$$

$$= C(1 + \log p) b_i(u, u) .$$

b) We next estimate $\rho_j\, s_j(u, u)$ when Ω_j has only a vertex V_{ij} in common with Ω_i. As in Theorem 7.1, we can prove that $s_j(u, u) \leq C\|u_{V_{ij}}\|^2_{L^2(W_i)}$. Therefore,

$$\rho_j\, s_j(u, u) \leq C\rho_j\|u_{V_{ij}}\|^2_{L^2(W_i)} \leq C\|c^V_{ij}\, u_{V_{ij}}\|^2_{L^2(W_i)}$$

and as in part a) this is bounded by

$$C\|\mu_i(u)\|^2_{L^2(W_i)} \leq C(1 + \log p)\, b_i(u, u) .$$

c) and d). Estimates of $\rho_j\, s_j(u, u)$ when Ω_j has only an edge or a face in common with Ω_i can be obtained from Theorem 7.1 as shown in b).

We can then prove a bound for the condition number of T_{NN3}.

THEOREM 9.2 : *If $N_B = N_{B, F}$, then for all $u \in V^p(\Gamma)$*

$$cs^{(p)}(u, u) \leq s^{(p)}(T_{NN3}\, u, u) \leq C(1 + \log p)^2\, s^{(p)}(u, u) .$$

Here c and C are constants independent of p, N, H and the discontinuities of $\rho(x)$ across element interfaces.

Proof: Assumption i). Define $u_0 = \sum_i \bar{u}_i\, \rho_i^{1/2}\, \mu_i^+(1)$, where $\bar{u}_i$ is defined in (20) and $\bar{u}_i = 0$ for $i \in N_B$. The other terms of the decomposition of u are $u_i = \sqrt{\rho_i}\, \mu_i^+(u - \bar{u}_i)$, so that $u = u_0 + \sum_{i=1}^{N} u_i$. For $i \in N_I$, we have

$$b_i(u_i, u_i) \leq \rho_i \hat{a}_i(u - \bar{u}_i, u - \bar{u}_i) ,$$

which has been estimated by $C\rho_i\, s_i(u, u)$ in the proof of Theorem 8.1 of the previous section. Estimates for $i \in N_B$ and the bound $b_0(u, u) \leq Cs^p(u, u)$ can be established as in Theorem 8.1.

Assumption ii). We can repeat the proof of Theorem 8.1 by replacing $s(\,.\,,.\,)$ with $s^p(\,.\,,.\,)$ and Lemma 6.2 with Lemma 9.1.

Assumption iii). As in the previous proof of Theorem 7.1.

Alternative coarse spaces (without the restriction $N_B = N_{B, F}$ or with inexact coarse solver yielding a reduced stencil), for h-version finite elements, can be found in Dryja and Widlund [8].

10. CONCLUSION

We have introduced and analized Neumann-Neumann preconditioners for spectral element discretizations of elliptic problems in three dimensions. Using the abstract Schwarz framework, we have proved that the conditioning of the resulting iteration operator is bounded by the square of the logarithm of the spectral degree p, while it is independent of the number of spectral elements

N and their size H. This is analogous to the results for Neumann-Neumann preconditioners for standard h-version finite elements. Three different coarse spaces have been studied : a standard piecewise linear space, a piecewise constant space and a special piecewise constant space for problems with discontinuous coefficients across element interfaces.

Numerical experiments with Neumann-Neumann preconditioners for spectral elements in two dimensions can be found in Chapter 4 of Pahl's Master thesis [14], pp. 54-58. The iteration counts presented seem to confirm the bounds obtained in this paper and show that a piecewise linear coarse space yields better iteration counts than a piecewise constant coarse space. However, the latter is still of great interest because it is the only one that can be used for unstructured (nonconforming) triangulations.

REFERENCES

[1] C. BERNARDI and Y. MADAY, 1992, *Approximations Spectrales de Problèmes aux Limites Elliptiques,* vol. 10 of Mathématiques & Applications, Springer-Verlag France, Paris.

[2] J. H. BRAMBLE and J. XU, 1991, *Some estimates for a weighted L^2 projection,* Math. Comp., **56,** pp. 463-476.

[3] C. CANUTO, M. Y. HUSSAINI, A. QUARTERONI and T. A. ZANG, 1988, *Spectral Methods in Fluid Dynamics,* Springer-Verlag.

[4] M. CASARIN, 1995, *Quasi-optimal Schwarz methods for the conforming spectral element discretization,* in 1995 Copper Mountain Conference on Multigrid Methods, N. D. Melson, T. A. Manteuffel and S. F. McCormick, eds., NASA, 1995.

[5] T. F. CHAN and T. P. MATHEW, 1994, *Domain decomposition algorithms,* Acta Numerica, Cambridge University Press, pp. 61-143.

[6] M. DRYJA, B. F. SMITH and O. B. WIDLUND, 1994, *Schwarz analysis of iterative substructuring algorithms for elliptic problems in three dimensions,* SIAM J. Numer. Anal., **31,** pp. 1662-1694.

[7] M. DRYJA and O. B. WIDLUND, 1990, *Towards a unified theory of domain decomposition algorithms for elliptic problems,* in Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, T. Chan, R. Glowinski, J. Périaux and O. Widlund, eds., SIAM, Philadelphia, PA, pp. 3-21.

[8] M. DRYJA and O. B. WIDLUND, 1995, *Schwarz methods of Neumann-Neumann type for three-dimensional elliptic finite element problems*, Comm. Pure Appl. Math., **48**, pp. 121-155.

[9] P. F. FISCHER and E. M. RØNQUIST, 1994, *Spectral element methods for large scale parallel Navier-Stokes calculations*, Comp. Meth. Appl. Mech. Engr., **116**, pp. 69-76.

[10] P. LE TALLEC, 1994, *Domain decomposition methods in computational mechanics*, in Computational Mechanics Advances, J. T. Oden, ed., vol. 1 (2), North-Holland, pp. 121-220.

[11] P. LE TALLEC, Y.-H. DE ROECK and M. VIDRASCU, 1991, *Domain-decomposition methods for large linearly elliptic three dimensional problems*, J. of Computational and Applied Mathematics, **35**.

[12] J. MANDEL, *Balancing domain decomposition*, 1993, Comm. Numer. Meth. Engrg., **9**, pp. 233-241.

[13] J. MANDEL and M. BREZINA, 1993, *Balancing domain decomposition : Theory and computations in two and three dimensions*, tech. rep., Computational Mathematics Group, University of Colorado at Denver, UCD/CCM TR 2.

[14] S. S. PAHL, *Schwarz type domain decomposition methods for spectral element discretizations*, Master's thesis, Department of Computational and Applied Mathematics, University of Wittwatersrand, Johannesburg, South Africa, December 1993.

[15] L. F. PAVARINO and O. B. WIDLUND, 1997, *Iterative substructuring methods for spectral elements : Problems in three dimensions based on numerical quadrature*. Computers & Mathematics with Applications, **33**, pp. 193-209.

[16] L. F. PAVARINO and O. B. WIDLUND, 1996, *A polylogarithmic bound for an iterative substructuring method for spectral elements in three dimensions*, SIAM J. Numer. Anal., **33**, pp. 1303-1335.

[17] L. F. PAVARINO and O. B. WIDLUND, 1995, *Preconditioned conjugate gradient solvers for spectral elements in 3D*, in Solution Techniques for Large-Scale CFD Problems, W. Habashi, ed., John Wiley & Sons, pp. 249-270.

[18] E. M. RØNQUIST, 1992, *A domain decomposition method for elliptic boundary value problems : Application to unsteady incompressible fluid flow*, in Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations, T. F. Chan, D. E. Keyes, G. A. Meurant, J. S. Scroggs and R. G. Voigt, eds., Philadelphia, PA, SIAM.

[19] E. M. RØNQUIST, 1995, *A domain decomposition solver for the steady Navier-Stokes equations*, Comput. Methods Appl. Mech. Engrg. To appear.

[20] B. F. SMITH, P. E. BJØRSTAD and W. D. GROPP, 1996, *Domain Decomposition : Parallel Multilevel Methods for Elliptic Partial Differential Equations*, Cambridge University Press.