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NUMERICAL APPROXIMATION OF AXISYMMETRIC POSITIVE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS IN AXISYMMETRIC DOMAINS OF $\mathbb{R}^3$ (*)

by N. ACHTAICH [1]

Abstract. — *We propose an algorithm which approaches positive solutions of elliptic semilinear equations with subcritical nonlinearity in axisymmetric domains of* $\mathbb{R}^3$ *with a degenerate boundary.*

Key words : Nonlinear elliptic equations, limiting Sobolev exponent, axisymmetric domains, degenerate boundary.

Résumé. — *Nous proposons un algorithme approchant les solutions positives d'équations elliptiques semi-linéaires présentant une non linéarité sous critique, dans des domaines axisymétriques de* $\mathbb{R}^3$ *à frontière dégénérée.*

Mots-Clés :

1. INTRODUCTION

We are concerned with the numerical approximation of the nonlinear elliptic equation

$$\begin{cases} -\ \Delta u = u^p & \text{in } \Omega \\ \quad u > 0 & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where p is a constant depending on the geometry of $\Omega = D \times \mathbb{R}/2\pi\mathbb{Z}$ and D is defined by

$$D = \{(r, z) : 0 < r < R \text{ and } - g(r) < z < g(r)\}$$

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R is a positive constant and g a function defined on $]0, R[$ satisfying

$$\lim_{r \to 0^+} \frac{g(r)}{r} = 0 . \tag{1.2}$$

In [1], we proved some Sobolev embeddings and obtain with classical methods (see P. L. Lions [12] and the abondunt list of references there in), the existence of a solution of problem (1.1). The solutions of problem (1.1) correspond to the nontrivial critical points (see [3], [7]) of the functional

$$J(v) = \frac{1}{2} \int_D |\nabla v|^2 \, rdr \, dz - \frac{1}{p+1} \int_D |v|^{p+1} \, rdr \, dz . \tag{1.3}$$

Our motivation for investigating problem (1.1) comes from the fact that it resembles some variational problems in physics with spherical symmetry. The main purpose of this paper is to propose an algorithm based on the constatation that the function

$$\lambda \in \,]0, \infty[\, \mapsto J(\lambda \bar{u})$$

achives its maximum at $\lambda = 1$, where $\bar{u}$ is a solution of problem (1.1). Therefore we maximise J in the direction of $\bar{u}$ and determine the zero of J' in the other directions, this corresponds to a Newton iteration. The presence of more or less stable nontrivial solutions of (1.1) is not excluded. The numerical computations makes probable this possibility and suggests that there are at least three nontrivial solutions of (1.1).

Our paper is organized as follows. In Section 2, we describe the above mentionned algorithm. We first recall the principal result in [1] and give a sketch of the classical proof of the existence result of solutions of (1.1) (Theorem 2.2). Then we describe the algorithm. In Section 3, we establish the convergence of algorithm (Theorem 3.1). The proof involves a combination of rather various technical ingredients. In Section 4, we give some numerical results and interpret them from the viewpoint of C. Bandle and A. Brillard [4].

2. DESCRIPTION OF THE ALGORITHM

As indicated above, D has a degenerate boundary. We denote by $\mathscr{D}(D)$ the space of infinitely differentiable functions on D with compact support in D. Then consider,

$$H_0^1(D, r) = \Big\{ u \in \mathscr{D}'(D) : \exists \{u_n\} \subset \mathscr{D}(D)$$

$$\text{such that } \lim_{n \to \infty} \int_D |\nabla(u_n - u)|^2 \, r \, dr \, dz = 0 \Big\} \tag{2.1}$$

and for $q > 1$,

$$L_r^q(D) = \left\{ u : \exists \{u_n\} \subset \mathscr{D}(D) \text{ such that } \lim_{n \to \infty} \int_D |(u_n - u)|^q \, r \, dr \, dz = 0 \right\}. \tag{2.2}$$

THEOREM 2.1 : *[1] We have the following compact embedding*

$$H_0^1(D, r) \hookrightarrow L_r^q(D) \quad \forall q < q_c \tag{2.3}$$

where q_c depends on D. □

For example, if $g(r)$ is equivalent to r^α, in a neighbourhood of 0, then

$$q_c = 2(\alpha + 2) \; (\textit{see } [1]) . \tag{2.4}$$

THEOREM 2.2 : *If $p + 1 < q_c$ and $p \neq 1$ then (1.1) has at least a nontrivial solution $\bar{u} \in H_0^1(D, r)$.* □

Sketch of the proof

Consider the functional

$$M_1(u) = \int_D |\nabla u|^2 \, r \, dr \, dz$$

where $u \in K = \left\{ v \in H_0^1(D, r) : \int_D |v|^{p+1} \, r \, dr \, dz = 1 \right\}$.

It's know (see [10]) that there exists $\tilde{u} \in K$ such that

$$M_1(\tilde{u}) = \inf_{u \in K} M_1(u) .$$

We deduce the existence of $u \in H_0^1(D, r) - \{0\}$ such that

$$M(u) = \frac{\left(\int_D |\nabla u|^2 \, r \, dr \, dz \right)^{p+1}}{\left(\int_D |u|^{p+1} \, r \, dr \, dz \right)^2} = \inf_{v \neq 0} M(v)$$

and consequently u satisfies

$$\begin{cases} -\Delta u = \dfrac{\left(\int_D |\nabla u|^2 \, r \, dr \, dz\right)}{\left(\int_D |u|^{p+1} \, r \, dr \, dz\right)} |u|^{p-1} u & \text{on } D \\ u = 0 & \text{on } \partial D . \end{cases}$$

Let $\bar{u} = \left(\dfrac{\int_D |\nabla u|^2 \, r \, dr \, dz}{\int_D |u|^{p+1} \, r \, dr \, dz} \right)^{\frac{1}{p-1}} u.$

Then we have

$$\begin{cases} -\Delta \bar{u} = |\bar{u}|^{p-1} \bar{u} & \text{on } D \\ \bar{u} = 0 & \text{on } \partial D . \quad \square \end{cases}$$

Now we propose an algorithm which approximates a solution $\bar{u}$ of problem (1.1).

First, we remark that the functional J defined for every $u \in H_0^1(D, r)$ by

$$J(u) = \frac{1}{2} \int_D |\nabla u|^2 \, r \, dr \, dz - \frac{1}{p+1} \int_D |u|^{p+1} \, r \, dr \, dz$$

satisfies

$$J(0) = 0 \quad \text{and} \quad J(-u) = J(u), \quad \forall u \in H_0^1(D, r).$$

The Sobolev embedding implies that $J > 0$ in a neighbourhood of 0. On the other hand $u = 0$ is the unique local minimizer of J.

Indeed, if we denote by

$$J'_u(u) = \lim_{\theta \to 0} \frac{J(u + \theta u) - J(u)}{\theta}$$

and

$$J''_u(u, u) = \lim_{\theta \to 0} \frac{J'_u(u + \theta u) - J'_u(u)}{\theta}$$

then we have

$$\begin{aligned} J''_{\bar{u}}(\bar{u}, \bar{u}) &= \int_D |\nabla \bar{u}|^2 \, r \, dr \, dz - p \int_D |\bar{u}|^{p+1} \, r \, dr \, dz \\ &= \|\bar{u}\|^2 - p |\bar{u}|_{p+1}^{p+1} \\ &= (1-p) \|\bar{u}\|^2 \\ &< 0 \end{aligned}$$

where

$$\|u\| = \left(\int_D |\nabla u|^2 \, r \, dr \, dz \right)^{\frac{1}{2}},$$

$$|u|_{p+1} = \left(\int_D |u|^{p+1} \, r \, dr \, dz \right)^{\frac{1}{p+1}}$$

and

$$((u, v)) = \int_D \nabla u \, \nabla v r \, dr \, dz \, .$$

Moreover it follows from [3] that the functional J possesses nontrivial critical points.

Then we propose the following algorithm

o Let u_0 be a given element of $H_0^1(D, r)$.

o We derive u_{n+1} from u_n in two steps

- Step 1 : find a maximum $u_{n+\frac{1}{2}}$ of J in the direction u_n.
- Step 2 : find w_n in the other directions such that

$$J'_{2\, u_{n+\frac{1}{2}}}(w_n) = 0$$

where

$$J_2(u_{n+\frac{1}{2}} + x_n) = J(u_{n+\frac{1}{2}}) + J'_{u_{n+\frac{1}{2}}}(x_n) + J''_{u_{n+\frac{1}{2}}}(x_n, x_n)$$

and write

$$u_{n+1} = u_{n+\frac{1}{2}} + w_n .$$

In step 1, the maximum of $J(\lambda_n u_n)$ is given by

$$\lambda_n = \left[\frac{\|u_n\|^2}{|u_n|_{p+1}^{p+1}}\right]^{\frac{1}{p-1}} \tag{2.5}$$

and then we define

$$u_{n+\frac{1}{2}} = \lambda_n u_n. \tag{2.6}$$

In step 2, we determine

$$w_n \in u_n^{\perp} = \{z_n \in H_0^1(D, r) : ((u_n, z_n)) = 0\}$$

such that

$$\int_D \nabla w_n \nabla z_n r\, dr\, dz - p \int_D u_{n+\frac{1}{2}}^{p-1} w_n z_n r\, dr\, dz$$
$$= \int_D u_{n+\frac{1}{2}}^{p} z_n r\, dr\, dz \quad \forall z_n \in u_n^{\perp} \tag{2.7}$$

which becomes

$$\int_D \nabla w_n \nabla z_n r\, dr\, dz - p \int_D u_{n+\frac{1}{2}}^{p-1} w_n z_n r\, dr\, dz$$

$$= \mu_n \int_D \nabla u_{n+\frac{1}{2}} \nabla z_n r\, dr\, dz + \int_D u_{n+\frac{1}{2}}^{p} z_n r\, dr\, dz \quad \forall z_n \in H_0^1(D, r) \tag{2.8}$$

where $\mu_n \in \mathbb{R}$ is a Lagrange multiplier.

By (2.8) we have

$$\begin{cases} -\Delta w_n - p u_{n+\frac{1}{2}}^{p-1} w_n = -\mu_n \Delta u_{n+\frac{1}{2}} + u_{n+\frac{1}{2}}^{p} \text{ on } D & (2.9) \\ ((u_{n+\frac{1}{2}}, w_n)) = 0 & (2.10) \\ w_n = 0 \quad \text{on } \partial D & (2.11) \end{cases}$$

Finally, we define

$$u_{n+1} = u_{n+\frac{1}{2}} + w_n . \tag{2.12}$$

Remark : Combining (2.9) and (2.12) we get

$$(-\Delta - p u_{n+\frac{1}{2}}^{p-1}) u_{n+1} = (-\Delta - p u_{n+\frac{1}{2}}^{p-1}) u_{n+\frac{1}{2}} - (\mu_n \Delta u_{n+\frac{1}{2}} - u_{n+\frac{1}{2}}^p)$$

and this step is a generalized Newton method in $u_n^{\frac{1}{2}}$ (see [8]).

This explain the quadratic convergence that we will prove in Section 3.

3. CONVERGENCE OF THE ALGORITHM

THEOREM 3.1 : *Suppose $\| \bar{u} - u_0 \|$ is small enough, where $\bar{u}$ is a solution of problem (1.1), then the above described algorithm is convergent.* □

We denote by $z_n = \bar{u} - u_n$ the error at the n^{th} iteration, where u_n is given by the algorithm.

LEMMA 3.2 : *We have*

$$\lambda_n = 1 + \frac{((\bar{u}, z_n))}{\| \bar{u} \|^2} + \frac{1}{p-1} \frac{\| z_n \|^2}{\| \bar{u} \|^2} - \frac{1}{2} \frac{p(p+1)}{p-1} \frac{\int_D \bar{u}^{p-1} z_n^2 \, r \, dr \, dz}{\| \bar{u} \|^2}$$

$$+ \frac{3p+2}{2} \frac{((\bar{u}, z_n))^2}{\| \bar{u} \|^4} + o(\| z_n \|^2) . \quad \square \tag{3.1}$$

Proof : We have

$$\lambda_n = \left[\frac{\| u_n \|^2}{| u_n |_{p+1}^{p+1}} \right]^{\frac{1}{p-1}} = \left[\frac{\| \bar{u} - z_n \|^2}{\| \bar{u} - z_n \|_{p+1}^{p+1}} \right]^{\frac{1}{p-1}} .$$

Since $\|\bar{u}\|^2 = |\bar{u}|_{p+1}^{p+1}$, $\int_D \bar{u}^p z_n\, r\, dr\, dz = \int_D \nabla\bar{u}\, \nabla z_n\, r\, dr\, dz$ and z_n small enough, we obtain

$$\lambda_n = \left(\frac{\|\bar{u}\|^2 - 2((\bar{u}, z_n)) + \|z_n\|^2}{|\bar{u}|_{p+1}^{p+1} - (p+1)\int_D \bar{u}^p z_n\, r\, dr\, dz + \frac{p(p+1)}{2}\int_D \bar{u}^{p-1} z_n^2\, r\, dr\, dz + o(\|z_n\|^2)} \right)^{\frac{1}{p-1}}$$

$$= \left(\frac{1 - 2\frac{((\bar{u}, z_n))}{\|\bar{u}\|^2} + \frac{\|z_n\|^2}{\|\bar{u}\|^2}}{1 - (p+1)\frac{((\bar{u}, z_n))}{\|\bar{u}\|^2} + \frac{1}{2}p(p+1)\frac{\int_D \bar{u}^{p-1} z_n^2\, r\, dr\, dz}{\|\bar{u}\|^2} + o(\|z_n\|^2)} \right)^{\frac{1}{p-1}}$$

$$= \left(1 + (p-1)\frac{((\bar{u}, z_n))}{\|\bar{u}\|^2} + \frac{\|z_n\|^2}{\|\bar{u}\|^2} - \frac{p(p+1)}{2}\frac{\int_D \bar{u}^{p-1} z_n^2\, r\, dr\, dz}{\|\bar{u}\|^2} + h(z_n) \right)^{\frac{1}{p-1}}$$

where

$$h(z_n) = (p^2 - 1)\frac{((\bar{u}, z_n))^2}{\|\bar{u}\|^4} + o(\|z_n\|^2).$$

Expanding $(1 + x)^\alpha$ for x in some neighbourhood of 0 and α constant, we obtain

$$\lambda_n = 1 + \frac{((\bar{u}, z_n))}{\|\bar{u}\|^2} + \frac{1}{p-1}\frac{\|z_n\|^2}{\|\bar{u}\|^2} - \frac{1}{2}\frac{p(p+1)}{p-1}\frac{\int_D \bar{u}^{p-1} z_n^2\, r\, dr\, dz}{\|\bar{u}\|^2}$$

$$+ \frac{3p+2}{2}\frac{((\bar{u}, z_n))^2}{\|\bar{u}\|^4} + o(\|z_n\|^2). \quad \square$$

COROLLARY 3.3 : *We have*

$$u_{n+\frac{1}{2}} = \bar{u} + \frac{((\bar{u}, z_n))}{\|\bar{u}\|^2}\bar{u} - z_n + G_2(z_n) + o(\|z_n\|^2) \tag{3.2}$$

where

$$G_2(z_n) = \frac{1}{p-1}\frac{\|z_n\|^2}{\|\bar{u}\|^2}\bar{u} - \frac{1}{2}\frac{p(p+1)}{p-1}\frac{\int_D \bar{u}^{p-1} z_n^2\, r\, dr\, dz}{\|\bar{u}\|^2}\bar{u}$$
$$+ \frac{3p+2}{2}\frac{((\bar{u}, z_n))^2}{\|\bar{u}\|^4}\bar{u} - \frac{((\bar{u}, z_n))}{\|\bar{u}\|} z_n. \quad \square \tag{3.3}$$

Proof: Here we use (2.6) and (3.1). The conclusion of corollary 3.3 follows. □

The next lemma is a crucial step in the proof of Theorem 3.1.

LEMMA 3.4 : *We have*

$$w_n = z_n - \frac{((\bar{u}, z_n))}{\|\bar{u}\|^2}\bar{u} + O(\|z_n\|^2) \tag{3.4}$$

and

$$\mu_n = -1 - p\frac{\|z_n\|^2}{\|\bar{u}\|^2} + p^2\frac{\int_D \bar{u}^{p-1} z_n^2\, r\, dr\, dz}{\|\bar{u}\|^2}$$
$$- p(p-1)\frac{((\bar{u}, z_n))^2}{\|\bar{u}\|^4} + o(\|z_n\|^2). \quad \square \tag{3.5}$$

Proof : We start by the problem (2.9), (2.10) and (2.11).

We shall find L_0, $L_1(z_n)$, $L_2(z_n)$, a_0, $a_1(z_n)$ and $a_2(z_n)$ such that

$$w_n = L_0 + L_1(z_n) + L_2(z_n) + o(\|z_n\|^2) \tag{3.6}$$

and

$$\mu_n = a_0 + a_1(z_n) + a_2(z_n) + o(\|z_n\|^2) \tag{3.7}$$

where

$$L_1(z_n) = O(\|z_n\|),\ a_1(z_n) = O(\|z_n\|) \quad \text{and} \quad a_2(z_n) = O(\|z_n\|^2).$$

It follows from (2.9) that

$$\begin{cases} - \Delta(L_0 + L_1 + L_2) - p\left(\bar{u} + \dfrac{((\bar{u}, z_n))}{\|\bar{u}\|^2}\bar{u} - z_n + G_2\right)^{p-1}(L_0 + L_1 + L_2) = \\ - (a_0 + a_1 + a_2)\left(\Delta\bar{u} + \dfrac{((\bar{u}, z_n))}{\|\bar{u}\|^2}\Delta\bar{u} - \Delta z_n + \Delta G_2\right) \\ + \left(\bar{u} + \dfrac{((\bar{u}, z_n))}{\|\bar{u}\|^2}\bar{u} - z_n + G_2\right)^p + o(\|z_n\|^2) \end{cases}$$

and therefore we have

$$\begin{cases} -\Delta L_0 - \Delta L_1 - \Delta L_2 - p\bar{u}^{p-1}\left[1 + (p-1)\dfrac{((\bar{u}, z_n))}{\|\bar{u}\|^2} - (p-1)\dfrac{z_n}{\bar{u}} + (p-1)\dfrac{G_2}{\bar{u}}\right. \\ \left. + \dfrac{(p-1)(p-2)}{2}\left(\dfrac{((\bar{u}, z_n))^2}{\|\bar{u}\|^4} + \dfrac{z_n^2}{\bar{u}^2} - 2\dfrac{((\bar{u}, z_n))^2}{\bar{u}\|\bar{u}\|^2} z_n\right)\right](L_0 + L_1 + L_2) = -a_0\Delta\bar{u} \\ - a_0\dfrac{((\bar{u}, z_n))}{\|\bar{u}\|^2}\Delta\bar{u} + a_0\Delta z_n - a_0\Delta G_2 - a_1\Delta\bar{u} - a_1\dfrac{((\bar{u}, z_n))}{\|\bar{u}\|^2}\Delta\bar{u} + a_1\Delta z_n - a_2\Delta\bar{u} \\ + \bar{u}^p\left[1 + p\dfrac{((\bar{u}, z_n))}{\|\bar{u}\|^2} - p\dfrac{z_n}{\bar{u}} + p\dfrac{G_2}{\bar{u}} + \dfrac{p(p-1)}{2}\left(\dfrac{((\bar{u}, z_n))^2}{\|\bar{u}\|^4} + \dfrac{z_n^2}{\bar{u}^2} - 2\dfrac{((\bar{u}, z_n))^2}{\bar{u}\|\bar{u}\|^2} z_n\right)\right] \\ + o(\|z_n\|^2). \end{cases} \tag{3.8}$$

We deduce from (3.8) that

$$- \Delta L_0 - p\bar{u}^{p-1} L_0 = - a_0 \Delta\bar{u} + \bar{u}^p \tag{3.9}$$

$$\begin{cases} - \Delta L_1 - p(p-1)\bar{u}^{p-1}\dfrac{((\bar{u}, z_n))}{\|\bar{u}\|^2} L_0 + p(p-1)\bar{u}^{p-1}\dfrac{z_n}{\bar{u}} L_0 - p\bar{u}^{p-1} L_1 = \\ - a_0\dfrac{((\bar{u}, z_n))}{\|\bar{u}\|^2}\Delta\bar{u} + a_0\Delta z_n - a_1\Delta\bar{u} + p\bar{u}^p\dfrac{((\bar{u}, z_n))}{\|\bar{u}\|^2} - p\bar{u}^{p-1} z_n \end{cases} \tag{3.10}$$

and

$$\begin{cases} - \Delta L_2 - p(p-1)\,\bar{u}^{p-2} L_0 G_2 - \dfrac{p(p-1)(p-2)}{2}\dfrac{((\bar{u}, z_n))^2}{\|\bar{u}\|^4}\bar{u}^{p-1} L_0 \\ + \dfrac{p(p-1)(p-2)}{2}\bar{u}^{p-3} L_0 z_n^2 + p(p-1)(p-2)\dfrac{((\bar{u}, z_n))}{\|\bar{u}\|^2}\bar{u}^{p-2} L_0 z_n \\ - p(p-1)\dfrac{((\bar{u}, z_n))}{\|\bar{u}\|^2}\bar{u}^{p-1} L_1 + p(p-1)\,\bar{u}^{p-2} L_1 z_n \\ - p\bar{u}^{p-1} L_2 = - a_0 \Delta G_2 - a_1 \dfrac{((\bar{u}, z_n))}{\|\bar{u}\|^2}\Delta\bar{u} + a_1 \Delta z_n - a_2 \Delta\bar{u} \\ + p\bar{u}^{p-1} G_2 + \dfrac{p(p-1)}{2}\dfrac{((\bar{u}, z_n))^2}{\|\bar{u}\|^4}\bar{u}^p + \dfrac{p(p-1)}{2}\bar{u}^{p-2} z_n^2 - p(p-1)\dfrac{((\bar{u}, z_n))}{\|\bar{u}\|^2}\bar{u}^{p-1} z_n . \end{cases} \tag{3.11}$$

From (2.10) we deduce that

$$((\bar{u}, L_0)) = 0 \tag{3.12}$$

$$((\bar{u}, L_1)) = ((z_n, L_0)) \tag{3.13}$$

and

$$((\bar{u}, L_2)) = ((z_n, L_1)) + \frac{((\bar{u}, z_n))}{\|\bar{u}\|^2}((\bar{u}, L_1)) + ((G_2, L_0)) . \tag{3.14}$$

Combining (3.9), (3.12) and the fact that $w_n = 0$ on ∂D, we obtain

$$\begin{cases} - \Delta L_0 - p\bar{u}^{p-1} L_0 = - a_0 \Delta\bar{u} + \bar{u}^p \quad \text{on } D & (3.15) \\ ((\bar{u}, L_0)) = 0 & (3.16) \\ L_0 = 0 \quad \text{on } \partial D & (3.17) \end{cases}$$

Multiply (3.15) by $\bar{u}$, integrate it by parts and use the fact that $\bar{u}$ is a solution of problem (1.1), we obtain

$$(1-p)((\bar{u}, L_0)) = (1 + a_0)\|\bar{u}\|^2 \tag{3.18}$$

so that by (3.16)

$$a_0 = -1 . \tag{3.19}$$

Then the problem (3.15), (3.16) and (3.17) becomes

$$\begin{cases} -\Delta L_0 - p\bar{u}^{p-1} L_0 = 0 & \text{on } D \\ ((\bar{u}, L_0)) = 0 & \\ L_0 = 0 & \text{on } \partial D \end{cases} \tag{3.20}$$

and therefore

$$L_0 = 0 \text{ on } D . \tag{3.21}$$

By the same arguments we establish that,

$$a_1 = 0 \tag{3.22}$$

$$L_1 = z_n - \frac{((\bar{u}, z_n))}{\|\bar{u}\|^2} \bar{u} \tag{3.23}$$

$$a_2 = -p \frac{\|z_n\|^2}{\|\bar{u}\|^2} + p^2 \frac{\int_D \bar{u}^{p-1} z_n^2 \, r \, dr \, dz}{\|\bar{u}\|^2} - p(p-1) \frac{((\bar{u}, z_n))^2}{\|\bar{u}\|^4} \tag{3.24}$$

and

$$L_2 = O(\|z_n\|^2) . \quad \square \tag{3.25}$$

Proof of theorem (3.1) : Since $u_{n+1} = u_{n+\frac{1}{2}} + w_n$ then by (3.2) and (3.4) we have

$$u_{n+1} = \bar{u} + L_2 + \frac{1}{p-1} \frac{\|z_n\|^2}{\|\bar{u}\|^2} \bar{u} - \frac{1}{2} \frac{p(p+1)}{p-1} \frac{\int_D \bar{u}^{p-1} z_n^2 \, r \, dr \, dz}{\|\bar{u}\|^2} \bar{u}$$

$$+ \frac{3p+2}{2} \frac{((\bar{u}, z_n))^2}{\|\bar{u}\|^4} \bar{u} - \frac{((\bar{u}, z_n))}{\|\bar{u}\|^2} z_n + o(\|z_n\|^2)$$

and therefore

$$z_{n+1} = -L_2 - \frac{1}{p-1}\frac{\|z_n\|^2}{\|\bar{u}\|^2}\bar{u} + \frac{1}{2}\frac{p(p+1)}{p-1}\frac{\int_D \bar{u}^{p-1} z_n^2 \, r \, dr \, dz}{\|\bar{u}\|^2}\bar{u}$$

$$-\frac{3p+2}{2}\frac{((\bar{u}, z_n))^2}{\|\bar{u}\|^4}\bar{u} + \frac{((\bar{u}, z_n))}{\|\bar{u}\|^2} z_n + o(\|z_n\|^2)$$

$$= A(z_n)$$

where A is a quadratic operator. We remark in consequence that if $z_n = O(\varepsilon)$ then $z_{n+1} = O(\varepsilon^2)$.

Conclusion : the algorithm converges and its convergence is quadratic.

4. NUMERICAL RESULTS AND CONCLUSION

If $g(r)$ is equivalent to r^2, in a neighbourhood of 0, then by (2.4) $q_c = 8$ and the corresponding problem (1.1) possesses a solution if $p < 7$. The algorithm proposed in Section 2 is applied. It's efficient because the convergence occurs after two or three iterations. The results obtained for several p such that $p < 7$ are gathered in the table (A). In table (B), we propose for the same p, three different solutions. In [4], C. Bandle and A. Brillard are studied in particular the asymptotic behaviour of the sequence $(u_\varepsilon)_\varepsilon$ where u_ε is a solution of the problem

$$\begin{cases} -\Delta u = u^{5-\varepsilon} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

and Ω is a bounded, smooth and open subset of $\mathbb{R}^3$. They use epi-convergence arguments and the classical methods to prove that the sequence $(u_\varepsilon)_\varepsilon$ converges to 0 in the weak topology of $H_0^1(\Omega)$.

Our numerical result suggests that if $p_1 < p_2 < 7$ then $\|u_{p_2}\|_2 < \|u_{p_1}\|_2$ where $\|\cdot\|_2$ is the Euclidian norm and u_{p_i} is a numerical solution of the problem

$$\begin{cases} -\Delta u = u^{p_i} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.2}$$

Table (A). — The values above are the functional values of $\bar{u}$ in the knots which are neighbour on the peak of D.

knot	$p_1 = 6.5$	$p_2 = 6.65$	$p_3 = 6.8$	$p_4 = 6.9$	$p_5 = 6.96$	$p_6 = 6.99$	$p_7 = 6.9975$
1	0.0004	0.0004	0.0004	0.0003	0.0003	0.0003	0.0003
2	0.0013	0.0012	0.0012	0.0012	0.0012	0.0011	0.0011
3	0.0032	0.0031	0.0031	0.0029	0.0029	0.0029	0.0029
4	0.0065	0.0062	0.0063	0.0061	0.0060	0.0060	0.0060
5	0.0091	0.0089	0.0087	0.0087	0.0086	0.0086	0.0086
6	0.0098	0.0095	0.0094	0.0092	0.0091	0.0091	0.0091
7	0.0188	0.0184	0.0181	0.0179	0.0177	0.0177	0.0177
8	0.0199	0.0194	0.0190	0.0188	0.0187	0.0187	0.0186
9	0.0259	0.0253	0.0248	0.0246	0.0244	0.0243	0.0243
10	0.0398	0.0389	0.0382	0.0377	0.0374	0.0373	0.0373
11	0.0405	0.0395	0.0388	0.0383	0.0381	0.0379	0.0379
12	0.0478	0.0468	0.0459	0.0454	0.0450	0.0449	0.0449
13	0.0701	0.0686	0.0672	0.0664	0.0660	0.0658	0.0657
14	0.0705	0.0690	0.0677	0.0669	0.0664	0.0662	0.0662
15	0.0778	0.0761	0.0746	0.0737	0.0732	0.0730	0.0730
16	0.0857	0.0838	0.0822	0.0813	0.0807	0.0804	0.0804
17	0.1157	0.1132	0.1110	0.1096	0.1089	0.1085	0.1085
18	0.1335	0.1305	0.1280	0.1265	0.1256	0.1252	0.1251
19	0.1351	0.1321	0.1296	0.1281	0.1272	0.1268	0.1267
20	0.1587	0.1553	0.1523	0.1505	0.1494	0.1489	0.1488
21	0.1707	0.1670	0.1638	0.1618	0.1607	0.1602	0.1601
22	0.2032	0.1988	0.1950	0.1926	0.1913	0.1907	0.1906

where Ω is defined in Section 1. We have a result, which means that we obtain the same claim concerning the sequence $(u_\varepsilon)_\varepsilon$, where u_ε is a solution of problem (1.1), as in [4] if $H_0^1(D, r) \not\subset L_r^{q_c}(D)$ (see [1]). We investigate the same conclusion as in [4] if $H_0^1(D, r) \hookrightarrow L_r^{q_c}(D)$. This last condition is in itself an interesting result because it extend a principal theorem proved in [1]. But, the open problem posed in [1] is not completely solved. Other ideas, for solving the above problem, using the eigenvalues of $-\Delta$ with conditions of Dirichlet are under investigation.

Table (B). — Here $p = 6.8$.

knot	u_1	u_2	u_3
1	0.0004	0.0005	0.0004
2	0.0014	0.0018	0.0012
3	0.0037	0.0044	0.0031
4	0.0075	0.0089	0.0063
5	0.0108	0.0127	0.0087
6	0.0115	0.0135	0.0094
7	0.0222	0.0262	0.0181
8	0.0235	0.0276	0.0190
9	0.0305	0.0361	0.0248
10	0.0467	0.0553	0.0382
11	0.0477	0.0564	0.0388
12	0.0563	0.0665	0.0459
13	0.0827	0.0975	0.0672
14	0.0817	0.0979	0.0677
15	0.0908	0.1082	0.0746
16	0.1007	0.1193	0.0822
17	0.1365	0.1610	0.1110
18	0.1565	0.1857	0.1280
19	0.1549	0.1870	0.1296
20	0.1843	0.2207	0.1523
21	0.2016	0.2377	0.1638
22	0.2378	0.2830	0.1950

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