RAIRO
MODÉLISATION MATHÉMATIQUE
ET ANALYSE NUMÉRIQUE

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RAIRO – Modélisation mathématique et analyse numérique, tome 31, nº 5 (1997), p. 615-641.

<http://www.numdam.org/item?id=M2AN_1997__31_5_615_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol 31, n° 5, 1997, p 615 à 641)

A KINETIC EQUATION FOR GRANULAR MEDIA (*)

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Abstract — *We introduce and analyze a kinetic equation for a system of particles in one dimension performing inelastic collisions Such a system has been proposed as a microscopic model for granular media*

Résumé — *On analyse une équation cinétique pour un système de particules à une dimension sujettes à des collisions inélastiques Ce système est présenté comme modèle microscopique pour les milieux granulaires*

1. THE KINETIC EQUATION

One-dimensional particle systems performing inelastic collisions have been recently studied as a model for the time evolution of granular media [1-4]. The main features of these systems are the possibility of the occurrence of inelastic collapses (namely infinitely many collisions in a finite time) and the tendency of the system to clusterize, that is to create states of concentration of the density, as sand grains over a shaken sheet of paper.

When the number of particles under consideration is large, it seems natural to apply the methods of the kinetic theory to understand the general behavior of the system under suitable scaling limits. In dimension greater than 1 it has been proposed a Boltzmann-like Equation [5].

We consider a one-dimensional system constituted by N particles on the line, colliding inelastically. Then we rescale suitably the degree of inelasticity, as well as the total number of particles (which is assumed to diverge), to obtain a kinetic equation for the one-particle probability density. Such a derivation is purely formal and will be presented in this section. Then, in the next section, we start a rigorous analysis of this kinetic equation in an homogeneous regime. We establish existence and uniqueness of the solution and determine, in a very precise way, the asymptotic behavior as the time goes to infinity. In Section 3 we approach the non-homogeneous problem and establish an existence and uniqueness theorem for small times as well as a global theorem under suitable smallness assumptions. The last section of the paper is devoted to general considerations. Most of the straightforward technicalities are confined in Appendix.

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We now establish more precisely the model. Consider N identical particles on the real line and denote by $x_1, ..., x_N$ and by $v_1, ..., v_N$ their positions and velocities respectively. The dynamics of the system is defined in the following way. The particles goes freely up to the first instant in which two of them are in the same point. Then they collide according to the rule :

$$v' = v_1 + \varepsilon(v - v_1), \quad v_1' = v - \varepsilon(v - v_1), \tag{1.1}$$

where v', v_1' and v, v_1 are the outgoing and ingoing velocities respectively and ε is a real parameter measuring the degree of inelasticity of the collision. Notice that the total momentum is conserved in the collision, while the modulus of the relative velocity decreases by a fixed rate for any collision. Then the particles go on up to the instant of the next collision which is performed by the same rule and so on.

Since the particles are assumed to be identical, the physics does not change if we replace the law (1.1) by the following one :

$$v' = v - \varepsilon(v - v_1), \quad v_1' = v_1 + \varepsilon(v - v_1), \tag{1.2}$$

which is the same as Eq. (1.1) with the names of the particles exchanged after the collision. It is often easier to do computations using (1.2), so that we shall assume as collision rule Eq. (1.2) in place of Eq. (1.1).

The ordinary differential cquation governing the time evolution of the system is :

$$\dot{x}_i = v_i, \quad \dot{v}_i = \varepsilon \sum_{j=1}^{N} \delta(x_i - x_j)\,(v_j - v_i)|v_j - v_i|\,. \tag{1.3}$$

Notice that $\varepsilon(v_j - v_i)$ is the jump performed by the particle i after a collision with the particle j, while $\delta(x_i - x_j)|v_j - v_i| = \delta(t - t_{i,j})$, being $t_{i,j}$ the instant of the impact between the particle i and j.

Let $\mu^N = \mu^N(x_1, v_1, ..., x_N, v_N)$ be a probability density for the system. The Liouville equation describing its time evolution reads as :

$$\left(\partial_t + \sum_{i=1}^{N} v_i\, \partial_{x_i}\right) \mu^N(x_1, v_1, ..., x_N, v_N) =$$

$$-\varepsilon \sum_{i \neq j} \delta(x_i - x_j)\, \partial_{v_i}[\phi(v_j - v_i)\, \mu^N(x_1, v_1, ..., x_N, v_N)] \tag{1.4}$$

where $\phi(\bar{v} - v) = (\bar{v} - v)|\bar{v} - v|$.

Proceeding as in the derivation of the BBKGY hierarchy for Hamiltonian systems, we introduce the j-particle distribution functions :

$$f_j^N(x_1, v_1, ..., x_j, v_j) = \int dx_{j+1}\, dv_{j+1} ... dx_N\, dv_N\, \mu^N(x_1, v_1, ..., x_N, v_N) \quad (1.5)$$

and integrating Eq. (1.4) over the last variables, we obtain the following hierarchy of equations :

$$\left(\partial_t + \sum_{i=1}^{J} v_i\, \partial_{x_i}\right) f_j^N(x_1, v_1, ..., x_j, v_j) =$$

$$- \varepsilon \sum_{\substack{i=1 \\ i \neq k}}^{J} \delta(x_i - x_k)\, \partial_{v_i}[\phi(v_k - v_i) f_j^N(x_1, v_1, ..., x_j, v_j)]$$

$$- \varepsilon(N-j) \sum_{i=1}^{J} \partial_{v_i} \int dv_{j+1}\, \phi(v_{j+1} - v_i) f_{j+1}^N(x_1, v_1, ..., x_{j+1}, v_{j+1}) . \quad (1.6)$$

An inspection of Eq. (1.6) suggest the scaling limit $\varepsilon \to 0$, $N \to \infty$ in such a way that $N\varepsilon \to \lambda$, where λ is a positive parameter. If f_j^N have a limit (say f_j) they are expected to satisfy the following (infinite) hierarchy of equations :

$$\left(\partial_t + \sum_{i=1}^{J} v_i\, \partial_{x_i}\right) f_j(x_1, v_1, ..., x_j, v_j) =$$

$$- \lambda \sum_{i=1}^{J} \partial_{v_i} \int dv_{j+1}\, \phi(v_{j+1} - v_i) f_{j+1}(x_1, v_1, ..., x_{j+1}, v_{j+1}) . \quad (1.7)$$

Finally, if the initial state is chaotic, namely if initially :

$$f_j(x_1, v_1, ..., x_j, v_j) = \prod_{i=1}^{J} f_0(x_i, v_i) ,$$

then we expect that the dynamics does not creates correlations (propagation of chaos) so that :

$$f_j(x_1, v_1, ..., x_j, v_j ; t) = \prod_{i=1}^{J} f(x_i, v_i, t) ,$$

by which we obtain, for the one particle distribution function, the kinetic equation :

$$(\partial_t + v\, \partial_x) f(x, v) = - \lambda\, \partial_v (Ff) , \quad (1.8)$$

where :

$$F(v,t)=\int d\bar{v}\ \phi(\bar{v}-v)f(\bar{v},t)\,. \tag{1.9}$$

In facts products of solutions of Eq. (1.8) are solutions of the hierarchy (1.7) as follows by a simple algebraic computation. Scope of the present paper is a preliminary study of Eq. (1.8).

2. THE HOMOGENEOUS EQUATION

The mathematical analysis of Eq. (1.8) is considerably simplified whenever the medium is spatially homogeneous. In this case we have :

$$\partial_t f(v,t)+\partial_v(Ff)\,(v,t)=0 \tag{2.1}$$

where :

$$F(v,t)=\int d\bar{v}\ \phi(\bar{v}-v)f(\bar{v},t)=-\phi*f(v,t)\,. \tag{2.2}$$

Here we set $\lambda=1$, being λ only a time scale.

The most remarkable feature of Eq. (2.1) is the decreasing in time of the momenta of the solution as expected by the dissipativity of the collision rule. Namely, suppose $f=f(v,t)$ be a smooth probability density, solution to Eq. (2.1). Then, for all $p\geqslant 1$, we have :

$$\frac{d}{dt}\int v^p f(v,t)\,dv=p\int v^{p-1}(Ff)\,(v)\,dv$$

$$=p\int dv\,d\bar{v}\ v^{p-1}\,\phi(\bar{v}-v)f(v,t)f(\bar{v},t)$$

$$=-\frac{p}{2}\int dv\,d\bar{v}\ (\bar{v}^{\,p-1}-v^{p-1})\,\phi(\bar{v}-v)f(v,t)f(\bar{v},t)\leqslant 0 \tag{2.3}$$

where, in the last step, we used the antisymmetry of ϕ.

Notice that $\int vf(v,t)\,dv=p_0=const.$ Without loss of generality we shall put in this section $p_0=0$. As a consequence :

$$\int v^2 f(v)\,dv=\frac{1}{2}\iint dv\,d\bar{v}\,f(v)f(\bar{v})\,(v-\bar{v})^2\,. \tag{2.4}$$

Moreover, by (2.3), (2.4) and the Hölder inequality :

$$\frac{d}{dt}\int v^2 f(v)\,dv = -\iint dv\,d\bar{v}\,f(v)f(\bar{v})|v-\bar{v}|^3 \leqslant$$

$$-\left(\iint dv\,d\bar{v}\,f(v)f(\bar{v})|v-\bar{v}|^2\right)^{3/2} = -(2)^{3/2}\left(\int v^2 f(v)\right)^{3/2}. \quad (2.5)$$

Denoting by $T(t) = \int dv\, v^2 f(v,t)$ twice the kinetic energy of the system, by (2.5) we have that :

$$T_2(t) \leqslant \frac{T_2(0)}{\left(1+t\sqrt{2T_2(0)}\right)^2}. \quad (2.6)$$

The initial value problem associated to Eq. (2.1) is easy to solve in the natural setting of the space of the probability measures with a suitable finite momentum.

Denote by $\mathcal{M}_0$ the space of all Borel probability measures in $\mathbb{R}$ and by :

$$\mathcal{M}_{2p} = \left\{\mu \in \mathcal{M}_0 \,\middle|\, \int v^{2p}\mu(dv) < +\infty, p \geqslant 0\right\}, \quad (2.7)$$

equipped with the topology of the weak convergence of the measures, the space of Borel probability measures of finite momentum of order $2p$.

We can prove :

THEOREM 2.1 : *Let $\mu_0 \in \mathcal{M}_2$ with*

$$T_{2\log} \equiv \int \mu_0(dv)\, v^2(1+\log(1+|v|)) < \infty.$$

Then for any $T > 0$, there exists a unique measure valued function $\mu \in C([0,T];\mathcal{M}_2) \cap C([(0,T];\mathcal{M}_{2p})$ satisfying the following properties.

(i) *Denoting by $V(v,t)$ the solution of the initial value problem :*

$$\dot{V}(t,v) = -\phi * \mu(t)(V(t,v)), \quad V(0,v) = v, \quad (2.8)$$

we have that :

$$\int u(v)\,\mu(dv,t) = \int u(V(t,v))\,\mu_0(dv). \quad (2.9)$$

for all bounded continuous function $u = u(v)$.

Moreover supp $(\mu(t)) \subset [-1/t, 1/t]$. *In particular* $\mu(t) \in \mathcal{M}_{2p}$ *for all* p *(assuming only that* $T_{2\log} < +\infty$*)* and

$$T_{2p}(t) \equiv \int \mu(dv, t)\, v^{2p} \leqslant \frac{1}{t^{2p}}. \tag{2.10}$$

(ii) *The solution* $\mu(t)$ *is continuous with respect to the initial data, that is, if* $\mu_n \to \mu$ *then* $\mu_n(t) \to \mu(t)$ *(both convergences in the weak sense).*
(iii) $T_{2p}(t)$ *is decreasing in time and the kinetic energy* $T_2(t)$ *satisfies the bound (2.6). As consequence*

$$\lim_{t \to \infty} \mu(t, dv) = \delta(dv) \tag{2.11}$$

in the sense of the weak convergence of the measures.
(iv) *If* μ_0 *has a density, also* $\mu(t)$ *has a density (with respect to* dv*). Moreover if* $\mu_0(dv) = f_0(v)\, dv$ *with* $f_0 \in C^k(\mathbb{R})$ *for* $k \geqslant 1$, *then* $\mu(dv, t) = f(v, t)\, dv$ *with* $f \in C^1([0, T]\,; C^{k-1}(\mathbb{R})) \cap C([0, T]\,; C^k(\mathbb{R})$ *and* $f(t)$ *solves Eq. 2.1 classically.*

Remark : From (i) we argue that $\mu(t)$ satisfies Eq. (2.1) in the following weak sense :

$$\frac{d}{dt}\mu(u, t) = \mu(F\, \partial_v u, t) \tag{2.12}$$

where $\mu(u, t) = \int \mu(dv, t)\, u(v)$ and $u = u(v)$ is any smooth test function. In particular if $\mu = \sum_{i=1}^{N} \alpha_i \delta_{v_i}$, then $\mu(t) = \sum_{i=1}^{N} \alpha_i \delta_{v_i(t)}$ where :

$$\dot{v}_i = \sum_{i=1}^{N} \alpha_j\, \phi(v_j - v_i)\,. \tag{2.13}$$

The dynamics (2.13) corresponds to the time evolution of a system of N particles (not spatially localized) performing collisions at random times.

The above theorem will be proved in Appendix.

The previous analysis achieves the problem of the asymptotic behavior of the solution for $t \to \infty$. More detailed informations however, can be obtained by a suitable rescaling. Indeed, since we know that the support of the velocity concentrates around zero (see *i*) in Theorem 2.1), it is natural to scale the probability distribution $f = f(v, t)$ and define a new unknown g in the following way :

$$g(\xi, t) = \frac{1}{t} f\left(\frac{\xi}{t}, t\right), \tag{2.14}$$

obtaining for g the following equation :

$$t\,\partial_t g + g + \partial_\xi(gG) + \xi\,\partial_\xi g = 0\,, \tag{2.15}$$

where :

$$G = -\,\phi * g\,.$$

Changing now the time scale by putting $\tau = \log t$, we finally obtain :

$$\dot{g} + [g(\xi + G)]' = 0 \tag{2.16}$$

where the dot and the prime denote the derivatives with respect to τ and v respectively.

For sake of simplicity we have assumed that f (and hence g) are absolutely continuous. The general case can be handled easily by considering Eq. (2.16) in a weak sense (see the above remark).

We are now interested to the asymptotic behavior of $g(t)$ as $t \to \infty$. (From now on we shall use again the symbol t for the rescaled time instead of τ). This will explain us the way how the true solution approaches the distribution δ and we shall find an universal behavior.

We first notice that, according to Proposition 2.1 all momenta of $g(t)$ are uniformly bounded. This allows us to introduce the following functional :

$$A(g(t)) = \frac{1}{6}\int d\xi\, d\bar{\xi} g(\xi, t)\, g(\bar{\xi}, t)|\xi - \bar{\xi}|^3 - \frac{1}{2}\int d\xi g(\xi, t)\, \xi^2\,. \tag{2.17}$$

An easy computation shows that :

$$\dot{A}(t) = -\int d\xi g(\xi, t)\,(\xi + G)^2 \leqslant 0\,. \tag{2.18}$$

Hence we are led to consider the set $\mathscr{V} = \{g \in \mathscr{M}_2 | \dot{A}(g) = 0\}$ as good candidate to describe the asymptotic set of $g(t)$. First we characterize the set $\mathscr{V}$.

PROPOSITION 2.2 : *If $g \in \mathscr{V}$ then :*

$$g(d\xi) = \sum_{i=1}^{3} \alpha_i\, \delta_{\xi_i}(d\xi) \tag{2.19}$$

with $\alpha_i \geqslant 0$ *and* $\sum_i \alpha_i = 1$.

Proof : Consider the function :

$$H(\xi) = \xi - \phi * g(\xi)\,. \tag{2.20}$$

If $g \in \mathscr{V}$ then $H = 0$ on the support of g. Notice that H is two times differentiable a.e. and :

$$H'(\xi) = 1 - 2 \int |\bar{\xi} - \xi| \, g(d\bar{\xi}) \tag{2.21}$$

$$H''(\xi) = 2(m^+(\xi) - m^-(\xi)) \text{ a.e.} \tag{2.22}$$

where $m^-(\xi) = \int_{-\infty}^{\xi} g(d\bar{\xi})$ and $m^+(\xi) = \int_{\xi}^{\infty} g(d\bar{\xi})$. Hence H'' is decreasing. As matter of facts there are two possibilities : either

(i) H has at most three zeros, or

(ii) the set $I = \{\xi : H(\xi) = 0\}$ is a closed interval.

In case (i) the proof is achieved since the support of g consists of a set of points whose cardinality does not exceed 3.

Consider now the case (ii). We have necessarily $g(\mathbb{R}/I) = 0$ so that the support of g is contained in I. On the other hand, since $H'' = 0$ in the interior of I, by (2.18) the support of g must be confined on the boundary of I so that g is a convex combination of two δ. □

We now show that $\mathscr{V}$ is asymptotically approached by the solution. We assume, now and in the rest of the section, that the initial condition $g_0(\xi) = g(\xi, 0) = f(\xi, 1)$ is fixed.

PROPOSITION 2.3 : *The cluster points of $g(t)$ (for the topology of the weak convergence of the measures) are contained in $\mathscr{V}$.*

Proof : Let $A_\infty = \lim_{t \to \infty} A(g(t))$. Since the functional $\dot{A}$ is weakly continuous (the momenta of $g(t)$ are uniformly bounded) we have that any cluster point $g^* = \lim_{t_k \to \infty} g(t_k)$ satisfies :

$$\dot{A}(g^*) = \lim_{t_k \to \infty} \dot{A}(g(t_k)), \tag{2.23}$$

for some diverging sequence t_k.

Suppose now that $\dot{A}(g^*) < -\varepsilon < 0$. By the control of the momenta we can easily show that :

$$|A(g(t+s)) - A(g(t))| = o(s). \tag{2.24}$$

Hence :

$$\sup_{s \in [0, \delta]} \dot{A}(g(t_k + s)) \leq \dot{A}(g(t_k)) - o(\delta) \leq -\frac{\varepsilon}{2} \tag{2.25}$$

for k large enough and δ sufficiently small. This makes impossible the convergence of the integral :

$$-\int_0^\infty dt\, \dot{A}(g(t)) = A(g(0)) - A_\infty . \tag{2.26}$$

Therefore $\dot{A}(g^*) = 0$. □

We now can outline more precisely the asymptotic behaviour by establishing a more detailed (and universal) description of the limit (2.11). To do this (see Theorem 2.2 below), we first need a preparatory Lemma which will be proven in the Appendix.

We set :

$$\mathscr{V} = \{\delta_0\} \cup \mathscr{V}_2 \cup \mathscr{V}_3 \tag{2.27}$$

where

$$\mathscr{V}_2 = \sum_{i=1}^{2} \alpha_i \delta_{\xi i}, \quad \alpha_i > 0 , \tag{2.28}$$

and

$$\mathscr{V}_3 = \sum_{i=1}^{3} \alpha_i \delta_{\xi i}, \quad \alpha_i > 0 . \tag{2.29}$$

LEMMA 2.1 : *If* $g \in \mathscr{V}_3$ *and* $g = \sum_{i=1}^{3} \alpha_i \delta_{\xi i}$ *with* $\xi_1 < \xi_2 < \xi_3$ *then :*

$$\max(\alpha_1, \alpha_2) < \frac{1}{2} . \tag{2.30}$$

If $g \in \mathscr{V}_2$ *then :*

$$g \equiv g_\mu = \mu \delta_{\mu - 1} + (1 - \mu)\, \delta_\mu \tag{2.31}$$

for $\mu \in (0, 1)$.

We now prove :

THEOREM 2.2 : *Suppose* $g_0 \in L_1(\mathbb{R})$. *Then :*

$$\lim_{t \to \infty} g(t) = \frac{1}{2} \delta_{-\frac{1}{2}} + \frac{1}{2} \delta_{\frac{1}{2}} \tag{2.32}$$

in the sense of weak convergence of measures.

Proof : For a given initial condition g_0 denote by $\mathscr{C} \subset \mathscr{V}$ the set of the cluster points of $g(t)$.

We first prove that

$$\mathscr{C} \cap \mathscr{V}_3 = \Phi \ . \tag{2.33}$$

The idea of the proof is that if $g^* \in \mathscr{C}$ and $g^* = \sum \alpha_i \delta_{\xi_i}$ with $\xi_1 < \xi_2 < \xi_3$, then $H(\xi_2) = 0$, $H'(\xi_2) > 0$ so that H is a repulsive field in ξ_2 and hence cannot attract mass.

Consider the characteristics problem associated to Eq. (2.16), namely :

$$\dot{V}(\xi, t) = H(V(\xi, t), t) \ , \quad V(\xi, 0) = 0 \ , \tag{2.34}$$

and the special (time dependent) point $\xi_0(t)$ which separate exactly the masses, that is :

$$m^+(\xi_0(t), t) = m^-(\xi_0(t), t) \ . \tag{2.35}$$

Then (see 2.22 and 2.23) $\xi_0(t)$ is a maximum point for $H'(\ .\ , t)$. Moreover ξ_0 is carried by the characteristics :

$$V(\xi_0, t) = \xi_0(t) \ , \tag{2.36}$$

because, if not, it would be a mass flow through $\xi_0(t)$. Moreover the characteristics are fully ordered in the sense that $V(\xi_1, t) < V(\xi_2, t)$ if $\xi_1 < \xi_2$. Finally we note that the characteristics are differentiable with respect to the initial conditions and that :

$$\partial_\xi V(\xi, t) = \exp \int_0^t ds H'(V(\xi, s)) \ . \tag{2.37}$$

Therefore, since $H'(V(\xi, t))$ is increasing in ξ, for $\xi < \xi(0)$, we have that :

$$\partial_\xi V(\xi_1, t) < \partial_\xi V(\xi_2, t) \tag{2.38}$$

if $\quad \xi_1 < \xi_2 < \xi_0$.

Assume now that, for some diverging sequence $\{t_k\}$, we have $\lim_{k \to \infty} g(t_k) = g_\infty \in \mathscr{V}_3$. Let $\xi_1 < \xi_2 < \xi_3$ end α_1, α_2, α_3 be the support and the masses of g_∞. Due to the full ordering of the trajectories, we can outline two points η_1 and η_2 and three regions $I_1 = (-\infty, \eta_1)$, $I_2 = (\eta_1 \eta_2)$, $I_3 = (\eta_2, \infty)$ for which :

$$\lim_{k \to \infty} V(\xi, t_k) = \xi_i, \quad \xi \in I_i. \tag{2.39}$$

In particular, by (2.30) $\xi_0(t) \to \xi_2$.

Taking now two points $\eta_1' \in I_1$, $\eta_2' \in I_2$ with $\eta_2' < \xi_0$, we have :

$$\frac{V(\xi_0, t) - V(\eta_2', t)}{\xi_0 - \eta_2'} > \frac{V(\eta_2', t) - V(\eta_1', t)}{\eta_2' - \eta_1'} \tag{2.40}$$

by (2.38). On the other hand the left hand side of (2.45) converges to zero, while the right hand side converges to $\frac{\xi_2 - \xi_1}{\eta_2' - \eta_1'}$ which is strictly positive. This contradicts the assumption and (2.38) is proven.

The next step is to prove that

$$\mathscr{C} \cap \mathscr{V}_2 = \frac{1}{2}\delta_{-\frac{1}{2}} + \frac{1}{2}\delta_{\frac{1}{2}}. \tag{2.41}$$

Indeed this can be proven by using the same argument as before. Suppose that :

$$\lim_{k \to \infty} g(t_k) = \mu\delta_{\mu-1} + (1-\mu)\,\delta_\mu \tag{2.42}$$

for some sequence $\{t_k\}$ and some $\mu \neq 1/2$. Then there exists η such that $V(\xi, t_k) \to \mu - 1$ for $\xi < \eta$ and $V(\xi, t_k) \to \mu$ for $\xi > \eta$. Suppose also that $\xi_0 < \eta$ (otherwise use the forthcoming argument on the right part of the line). Choosing now two points η_1 and η_2 such that $\eta_1 < \eta < \eta_2 < \xi_0$, we have :

$$\frac{V(\xi_0, t_k) - V(\eta_2, t_k)}{\xi_0 - \eta_2} > \frac{V(\eta_2, t_k) - V(\eta_1, t_k)}{\eta_2 - \eta_1}, \tag{2.43}$$

but also that the left hand side of (2.43) converges to zero while the right hand side converges to $(\eta_2 - \eta_1)^{-1} > 0$. Hence (2.41) is proven.

The last step is to show that $\delta_0 \notin \mathscr{C}$. Indeed suppose that $\lim_{t \to \infty} g(t) = \delta_0$. Then, since $\xi_0(t) \to 0$, $H'(\xi(t)) \to 1$, we find a contradiction, because $H'(\xi(t), t)$ must be definitively negative to attract all the trajectories. So the only possibility we have in order that $\delta_0 \in \mathscr{C}$ is that $g(t)$ has at least another cluster point in $\mathscr{V}_2$ which cannot fail to be $1/2\,\delta_{-1/2} + 1/2\,\delta_{1/2}$ for the previous step. By the explicit analysis of $\mathscr{V}_2$ given by (2.31), we compute the values of the functional A finding $A(\delta_0) = 0$ and $A(1/2\,\delta_{-1/2} + 1/2\,\delta_{1/2}) = -\frac{1}{12}$. Therefore by the monotonicity of $A(g(t))$ we conclude that there is a unique cluster point given by the two symmetric δ. □

Remark : The proof of Theorem 2.2 works as well for initial measures g_0 enjoying the property that there exists a point ξ_0 such that $m^-(\xi_0) = 1/2$ and ξ_0 is a continuity point of m^-. For instance if $g(d\xi) = \sum_{i=1}^{N} \alpha_i \delta_{v_i}$ then it must exists an index i_0 for which :

$$\sum_{i=1}^{i_0} \alpha_i = \frac{1}{2}. \tag{2.44}$$

It can also be proven that, whenever condition (2.44) fails, the asymptotic behavior can be anomalous. Indeed the cluster set for such initial datum may not include the two symmetric delta, but only the stationary solutions of Eq. (2.16) consisting in three delta and the asymmetric two delta described in appendix.

3. THE INHOMOGENEOUS EQUATION

We start by establishing a local theorem which will be proven in Appendix.

THEOREM 3.1 : *Consider the initial value problem associated to Eq. (1.8) with a non-negative initial datum* $f_0 \in C^1(\mathbb{R} \times \mathbb{R})$, $f_0 \in L_\infty(x, v)$ *and* $\nabla f_0 \in L_\infty(x, v)$. *Suppose also that supp* $f_0 \subset \mathbb{R} \times [-\eta, \eta]$. *Then there exists* $T > 0$ *depending on* $\|f_0\|_{L_\infty}$ *and* $\|\nabla f_0\|_{L_\infty}$, *and a unique (classical) non-negative solution* $f \in C^1([0, T) \times \mathbb{R} \times [-\eta, \eta])$ *to Eq. (1.8).*

Natural questions arise :

(i) Do weak solutions exist globally in time ?

(ii) Can occur singularities in a finite time and if it is so, can the solution concentrate (that is the production of a component of δ type in the x, v-space) ?

(iii) If there are many weak solutions, do a criterion (like the entropy conditions for the conservation laws) which select the physically relevant solutions exist ?

We do not know how to answer to these questions even at a heuristic level. The only extra a-priori information we have on the solution, is a reverse H-theorem, which describes the obvious tendency of the system to concentrate. Namely, defining $H(f) = \int dx\, dv (f \log f)(x, v, t)$, then :

$$\dot{H}(f) = \int dx\, dv\, d\bar{v}\, |v - \bar{v}|\, f(x, v, t) f(x, \bar{v}, t) \geq 0. \tag{3.1}$$

Unfortunately Eq. (3.1) does not seem to exclude or guarantee the concentration feature in a finite time. However if λ is sufficiently small compared with the size of the initial condition f_0, the dispersivity of the free flow allows us to prevent singularities. Indeed we can prove :

THEOREM 3.2 : *Suppose f_0 with compact support and $\|f_0\| + \|\nabla f_0\| = a < +\infty$. Suppose also that $\lambda < \lambda_0$ where λ_0 depends on a and the diameter of the support of f_0. Then there exists a unique classical global solution to Eq. (1.8). Moreover there exists a constant C, depending only of f_0 and λ, such that :*

$$\sup_{t \in [0,\infty)} \|f(t)\|_\infty + \|\partial_x f(t)\|_\infty \leq C, \tag{3.2}$$

$$\|\partial_v f(t)\|_\infty \leq C(1+t), \tag{3.3}$$

$$\|\rho(t)\|_\infty \leq \frac{C}{1+t}. \tag{3.4}$$

(here $\rho(x,t) = \int dv\, f(x,v,t)$ is the spatial density).

Notice that inequalities (3.2), (3.3) and (3.4) are those satisfied by the free transport solutions.

Proof: We assume that there exists a classical solution $f = f(x, v, t)$ to Eq. (1.8) with the same regularity in Theorem 3.1 and we shall prove estimates (3.2-5). After this it is straightforward to construct such a solution by means of the usual fixed point argument.

Let us introduce the characteristic system :

$$\dot{X}(x,v,t) = V(x;v,t), \quad \dot{V}(x,v,t) = \lambda F(X(x,v,t), V(x,v,t), t) \tag{3.5}$$

$$X(x,v,0) = x \quad V(x,v,0) = v. \tag{3.6}$$

Since f_0 has compact support, there exists a square $Q_0 = [-L, L]^2$ in which this support is contained. Consider its time evolution under the action of the free flow :

$$Q(t) = \{(x + vt, v) \mid (x,v) \in Q_0\}. \tag{3.7}$$

We shall show that the support of $f(t)$ is contained in $Q(t)$. This is equivalent to show that, denoted $\tilde{X}(x,v,t) = X(x,v,t) - V(x,v,t)\,t$, we have

$$(\tilde{X}(x,v,t), V(x,v,t)) \in Q_0 \text{ if } (x,v) \in Q_0.$$

By definition :

$$\dot{\tilde{X}}(x, v, t) = -tF(\tilde{X}(x, v, t) + V(x, v, t)\, t, V(x, v, t), t)$$

$$\dot{V}(x, v, t) = F(\tilde{X}(x, v, t) + V(x, v, t)\, t, V(x, v, t), t)\,. \tag{3.8}$$

By a direct inspection, if $\operatorname{supp} f(\,.\,, t) \subset Q(t)$, for $(x, v) \in Q_0$:

$$F(L + vt, v, t) \leqslant 0 \quad F(x + vt, L, t) \leqslant 0$$

$$F(-L + vt, v, t) \geqslant 0 \quad F(x + vt, -L, t) \geqslant 0\,,$$

then $(\tilde{X}, V)$ cannot exit from Q_0. Then $\operatorname{supp} f(\,.\,, t) \subset Q(t)$ for all time.

Introducing now the function :

$$h(t) = L \text{ if } t < 2\,, \quad h(t) = \frac{2L}{t} \text{ if } t \geqslant 2 \tag{3.9}$$

it follows that the sections $S(x, t) = \{v \mid (x, v) \in Q(t)\}$ satisfy meas $S(x, t) \leqslant h(t)$. This allows us to establish the following elementary estimates, for $(x, v) \in Q(t)$.

$$|F(x, v, t)| \leqslant \|f(t)\|_\infty \int_{S(x,t)} \phi(\bar{v} - v)\, d\bar{v} \leqslant \|f(t)\|_\infty \int_0^{2h(t)} \xi^2\, d\xi$$

$$= \frac{8}{3} \|f(t)\|_\infty h(t)^3\,. \tag{3.10}$$

similarly :

$$|\partial_x F(x, v, t)| \leqslant \frac{8}{3} \|\partial_x f(t)\|_\infty h(t)^3, \quad |\partial_x F(x, v, t)| \leqslant \frac{8}{3} \|\partial_v f(t)\|_\infty h(t)^3. \tag{3.11}$$

But also :

$$|\partial_v F(x, v, t)| = 2 \int_{S(x,t)} |\bar{v} - v|\, f(x, \bar{v}, t)\, d\bar{v} \leqslant 4\, h(t)^2 \|f(t)\|_\infty\,, \tag{3.12}$$

$$|\partial_v^2 F(x, v, t)| \leqslant 4\, h(t)\, \|f(t)\|_\infty, \quad |\partial_v^2 F(x, v, t)| \leqslant 4\, h(t)^2 \|\partial_v f(t)\|_\infty \tag{3.13}$$

and

$$|\partial_{x,v}^2 F(x, v, t)| \leqslant 4\, h(t)^2 \|\partial_x f(t)\|_\infty\,. \tag{3.14}$$

We now estimate $f(t)$ and its first derivative and, to this end, we write Eq. 1.8 in the following form:

$$\partial_t f + v\,\partial_x f + \lambda F\,\partial_v f = \lambda A$$

$$\partial_t\,\partial_x f + v\,\partial_x^2 f + \lambda F\,\partial_v\,\partial_x f = \lambda B$$

$$\partial_t\,\partial_v f + v\,\partial_x\,\partial_v f + \lambda F\,\partial_v^2 f = \lambda D \tag{3.15}$$

where

$$A = -f\,\partial_v F$$

$$B = -\left(\partial_v f\,\partial_x F + \partial_x f\,\partial_v F + f\,\partial_{x,v}^2 F\right)$$

$$D = -\left(\partial_x f + 2\,\partial_v f\,\partial_v F + f\,\partial_v^2 F\right). \tag{3.16}$$

Equations (3.15) have to be understood as time derivatives along the characteristic curves. For instance for the first equation we have:

$$\frac{df(X(t), V(t))}{dt} = \lambda A(X(t), V(t), t) \tag{3.17}$$

where in $X(t)$, $V(t)$ we skipped the dependence of the trajectories by the initial data for notational simplicity. Note that in this way we avoid the dependence on the second derivatives of $f(t)$, so that we can work in a C^1 setup only.

We now estimate A, B, D. By (3.12), for $(x, v) \in Q(t)$:

$$|A(x, v, t)| \leq 4\,h(t)^2 \|f(t)\|_\infty^2. \tag{3.18}$$

By (3.11) and (3.14):

$$|B(x, v, t)| \leq \frac{8}{3} h(t)^3 \|\partial_v f(t)\|_\infty \|\partial_x f(t)\|_\infty + 8\,h(t)^2 \|f(t)\|_\infty \|\partial_x f(t)\|_\infty. \tag{3.19}$$

By (3.12) and (3.13)

$$|D(x, v, t)| \leq \|\partial_x f(t)\|_\infty + 12\,h(t)^2 \|f(t)\|_\infty \|\partial_v f(t)\|_\infty. \tag{3.20}$$

By Eqs. (3.15) and (3.18) we can write:

$$f(X(x, v, t), V(x, v, t)) = f_0(x, v) + \lambda \int_0^t 4\,h(s)^2 \|f(s)\|_\infty^2 \tag{3.21}$$

from which, defining $\delta = \sup_{t \in \mathbb{R}^+} \|f(t)\|_\infty$ and $\gamma_2 = \int_0^{+\infty} h(t)^2\, dt$, we get :

$$\delta \leqslant \|f_0\|_\infty + 4\,\lambda\gamma_2\,\delta^2 \tag{3.22}$$

and hence :

$$\delta \leqslant 2\,\|f_0\|_\infty \tag{3.23}$$

provided that :

$$\lambda \leqslant \frac{1}{16\,\gamma_2\,\|f_0\|_\infty}\,. \tag{3.24}$$

Moreover by Eqs. (3.15) and (3.20) :

$$h(t)\,|(\partial_v f)\,(X(x, v, t), V(x, v, t))| \leqslant L|\partial_v f_0| + \lambda h(t) \int_0^t ds(\,\|\partial_x f(s)\|_\infty)$$
$$+\, 12\, h(s)^2\,\|f(s)\|_\infty\,\|\partial_v f(s)\|_\infty\,. \tag{3.25}$$

We now set :

$$\alpha = \sup_{t \in \mathbb{R}^+} h(t)\,\|\partial_v f(t)\|_\infty\,, \quad \beta = \sup_{t \in \mathbb{R}^+} \|\partial_x f\|_\infty\,, \tag{3.26}$$

and from (3.25) and (3.23) we obtain :

$$\alpha \leqslant L\,\|\partial_v f_0\|_\infty + 2\,\lambda L\beta + 24\,\lambda\,\|f_0\|_\infty\,\alpha\gamma_1\,, \tag{3.27}$$

after setting :

$$\sup_{t \in \mathbb{R}^+} h(t) \int_0^t ds h(s) = \gamma_1\,. \tag{3.28}$$

For :

$$\lambda < \frac{1}{24\,\gamma_1\,\|f_0\|_\infty\,L}\,, \tag{3.29}$$

we have :

$$\alpha \leqslant \frac{L\,\|\partial_v f_0\|_\infty + 2\,\lambda\beta}{1 - 24\,\lambda\gamma_1\,\|f_0\|_\infty\,L}\,. \tag{3.30}$$

After having estimated α in terms of β we conclude by setting (again from (3.15) and (3.19)) :

$$\beta \leq \| \partial_x f_0 \|_\infty + \frac{8}{3} \lambda \gamma_2 \alpha \beta + 16 \lambda \gamma_2 \| f_0 \|_\infty \beta . \tag{3.31}$$

Inserting in (3.31) estimate (3.30), we can solve with respect to β, provided that λ is sufficiently small. This proves (3.2) and (3.3) of Theorem 3.2.

We conclude with the proof of (3.4) :

$$\rho(x, t) = \int_{S(x,t)} f(x, v, t)\, dv \leq 2\, h(t) \| f_0 \|_\infty \leq \frac{C}{1+t} \qquad \square$$

4. NUMERICAL SIMULATIONS AND ADDITIONAL REMARKS

In this section we consider some simulations performed to see if and when an homogeneous behaviour is realized for the particle system. We consider $N = 1\,000$ particles in the box $[-1, 1]$ with an homogeneous initial condition (i.e. the velocity and the position of the particles are randomly extracted with a uniform distribution in $[-1, 1]^2$) for different values of λ. The collisions with the wall are inelastic : if a particle hits the wall with velocity v then it is reflected from the wall with velocity $-(1 - 2\varepsilon) v$. For λ sufficiently large (for example if $\lambda = 4$) the system exhibits an inelastic collapse after a certain time.

For λ sufficiently small, for instance $\lambda = 0.5$, we can distinguish two different regimes.

In the first regime, that is up to about $N_c = 5 \cdot 10^6$ collisions, the homogeneous behaviour is qualitatively fulfilled. In fact, as time goes on, the gas remains spatially homogeneous, while the velocity distributions support concentrates as t^{-1}, and the rescaled distribution goes toward a two peaks distribution. In *figure 1a* it is plotted the phase space density after $5.4 \cdot 10^6$ collisions and in *figure 1b* is given an histogram of the velocity density.

After this, for a larger number of collisions, the homogeneous situation seems to break down and correlation between the particle becomes important. In particular the gas looses also its spatial homogeneity. In *figure 2* it is plotted the Kantarovich-Rubinstein distance between the computed rescaled distribution and the two peaks distribution $\frac{1}{2}\left[\delta\left(v - \frac{1}{2}\right) + \delta\left(v + \frac{1}{2}\right)\right]$.

This results are in agreement with what found by Esipov and Poshel [5] who proposed (and discuss the validity of) a Boltzmann Equation in order to describe an inelastic gas of particles in dimension greater than 1. In particular they find that, in the homogeneous case, it is expected a universal behaviour of the velocity distribution (not gaussian) but that the homogeneous description is numerically unstable at very long times.

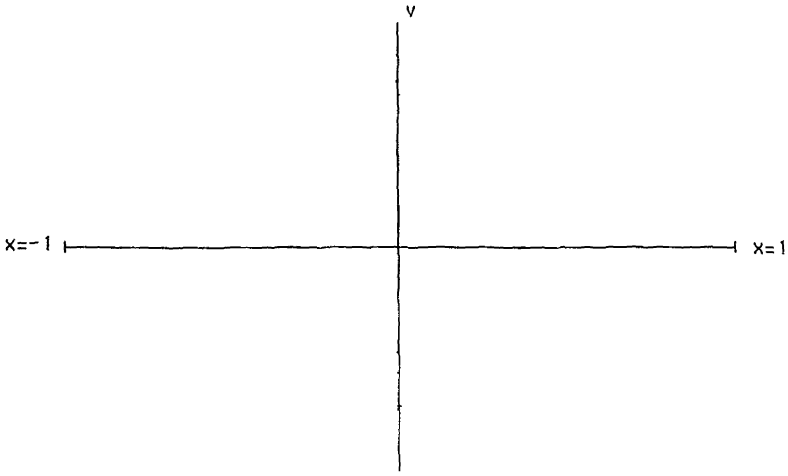

Figure 1a.

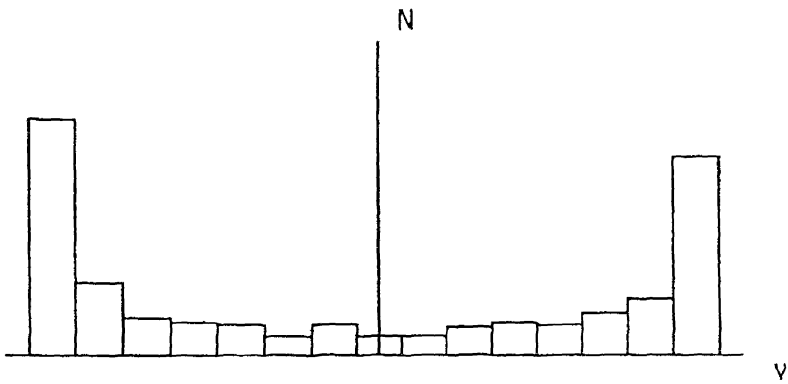

Figure 1b.

Coming back to the one dimensional case and to the kinetic equation (1.8) what we can conclude ? In our opinion, at least for λ small (we shall come back on this point later on) the kinetic equation is a good description of a gas of inelastic particle in the kinetic limit under consideration. In fact also if, after a certain time, the homogeneous behaviour breaks down because of the correlations becomes important, it seems reasonable that such a critical time tends to ∞ as $N \to \infty$. This obviously does not means that the kinetic equation (1.8) is able to characterize the asymptotic behaviour of a gas of N

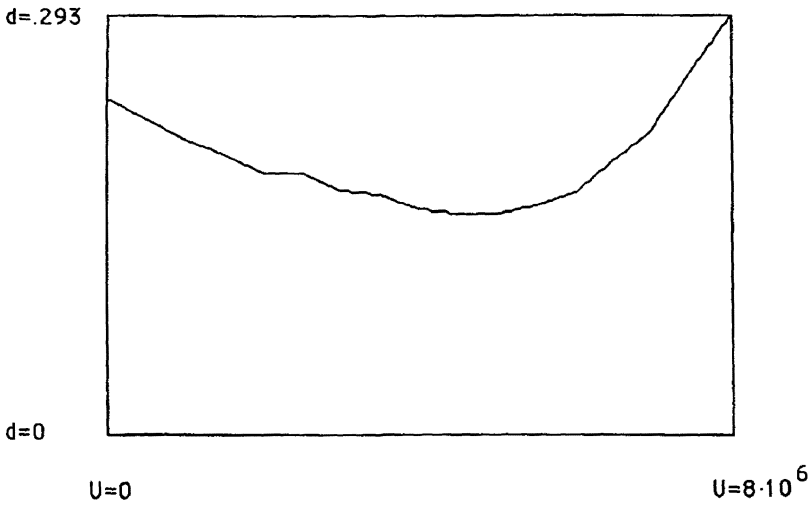

Figure 2.

particles. Here we want only to stress that the universal two-peaks distribution (instead of a Maxwellian) as asymptotic behavior for the homogeneous case could be important when trying to model a gas of inelastic particles by means of hydrodynamical equations.

In all cases it seems difficult to provide a rigorous proof of the validity of the kinetic equation (1.8) under the scaling limit we have formally considered in Section 1, for a given arbitrary interval of time.

Finally, once more, we want to outline the problem of the global existence of the solutions for Eq. (1.8). This problem seems to be strictly related to the problem of the existence of the dynamics for N particles in the kinetic limit. In particular fixed N let us define $\varepsilon_c(N)$ as the minimum number of particles required to have an inelastic collapse, and $\lambda_c(N) = N\varepsilon_c(N)$. In the paper by Bernu and Mazighi [2] it is suggested that, as $N \to \infty$, ε_c behaves as $\frac{1}{N}$. If this is true, in the kinetic limit it would be possible to define a critical values of λ, λ_c, that separate the two different behaviour. Nevertheless Mc Namara and Young [1], on the basis of the analysis of a particular model, suggest that the minimum number of particles N_c in order to have a collapse for a fixed value of ε, behaves asymptotically as $N_c = \frac{1}{\varepsilon}\log\left(\frac{1}{\varepsilon}\right)$. In this case $\lambda_c = \varepsilon N_c = \log\left(\frac{1}{\varepsilon}\right)$ therefore it would diverge as $N \to \infty$. Then we would expect that the kinetic equation admits a global solution for any values of λ since the concentration is unexpected at a finite time. The problem of characterizing rigorously $\lambda_c(N)$ is, as far as we know, open.

ACKNOWLEDGMENTS

It is a pleasure for us to thank Peter Constantin for having introduced us to the field of granular media.

NOTE ADDED IN PROOF

After this work was completed we knew that Eq. (2.1-2) has previously considered from an heuristic point of view by S. Mc Namara and W. R. Young in the paper « Kinetics of a one-dimensional granular medium in the quasi-elastic limit », Phys. Fluids A **5** (1) p. 34-45, 1993 and by Y. Du, H. Li and L. P. Kadanoff in the paper « Breakdown of Hydrodynamics in a One-Dimensional System of Inelastic Particles », Phys. Rev. Lett. **74** (8) p. 1268-1271, 1995.

APPENDIX

Proof of Theorem 2.1

The proof of the existence and uniqueness of the pair $(\mu(t), V(t, v))$ (see (2.8) and (2.9)) for an initial datum μ_0 with compact support, is straightforward.

Indeed suppose that μ_0 has support contained in $[-v_0, v_0]$. Then there exists $v^+(0) = \inf\{v \,|\, \mu_0[(v, +\infty)] = 0\}$. It is easy to realize that, denoting by $V^+(t)$ the solution of (2.8) with initial datum $v^+(0)$, we have :

$$\begin{aligned}\dot{V}^+(t) &= \int \mu(d\bar{v}, t)\, \phi(\bar{v} - V^+(t)) = \\ &- \int \mu(d\bar{v}, t) |\bar{v} - V^+(t)|^2 = \\ &- T_2(t) - V^+(t)^2 \leq - V^+(t)^2 . \end{aligned} \tag{A.1}$$

Therefore, noticing that $V^+(t) > 0$ by the conservation of the momentum,

$$V^+(t) \leq \frac{1}{\dfrac{1}{v^+(0)} + t} \leq \frac{1}{t} . \tag{A.2}$$

Introducing $V^{-}(t)$ the left extreme velocity, we find an analogous bound and conclude that :

$$\max\left(|V^{-}(t)|^{p}, V^{+}(t)^{p}\right) \leq \frac{1}{t}, \tag{A.3}$$

With these a-priori estimates we see that $F(v, t)$ is bounded and Lipschitz continuous in v, uniformly in t and v on the support of $\mu(dv, t)$. This allows us to construct a global, unique solution, by the contraction mapping principle (for instance in the space of the probability measures equipped with a suitable metric, equivalent to the metric of the weak convergence of the measures).

For μ_0 with $T_{2\log}(0) < +\infty$, we can obtain a solution as limit for $n \to +\infty$ of the solutions $\mu^N(t)$ with initial data $\mu_0^n(dv) = \dfrac{\mu_0(dv)\,\chi\{|v| \leq n\}}{\int \mu_0(dv)\,\chi\{|v| \leq n\}}$.

Indeed the associated trajectories $V^n(v, t)$ are bounded and equicontinuous on the compact sets of v and t. In facts the force F is bounded and continuous in μ with respect to the week topology for the measures, if $T_{2\log}$ is bounded :

$$|F(v)| \leq \int |\bar{v} - v|^2 \mu(d\bar{v}) \leq T_2 + v^2 \tag{A.4}$$

and

$$|\phi * \mu_1 - \phi * \mu_2| \leq \frac{2\,T_{2\log}}{\log(1+K)} + \left|\int_{|\bar{v}|<K} [\mu_1(d\bar{v}) - \mu_2(d\bar{v})]\,\phi(\bar{v} - v)\right|, \tag{A.5}$$

because $T_{2\log}(t) = \int \mu(dv, t)\, v^2(1 + \log(1 + |v|))$ is decreasing in time.

Moreover $F(\,.\,, t)$ is uniformly Lipschitz continuous in v, for $t \in (\delta, T]$ for any $\delta > 0$, as a consequence of (A.2). This allows us to show that, passing to subsequence if necessary, the limit of (V^n, μ^n) satisfies (2.8) and (2.9).

Now we prove the uniqueness to achieve statement (i) of Theorem 2.1.

Notice that

$$F(v, t) = -v^2 - \int \mu(d\bar{v})\,\bar{v}^2 + 2\int_v^{+\infty} \mu(d\bar{v})\,(v - \bar{v})^2 \leq v^2 + T_2. \tag{A.6}$$

The last inequality is consequence of the bound :

$$\int_v^{+\infty} \mu(d\bar{v}\,, t)\,(v - \bar{v}\,)^2 \leqslant \int_v^{+\infty} \mu(d\bar{v}\,, t)\,\bar{v}^{\,2}\,. \tag{A.7}$$

Using now the same argument leading to (A.2), we argue that the support of $\mu(t)$ is contained in $\left[-\left(\sqrt{2\,T_2} + \frac{2}{t}\right), \sqrt{2\,T_2} + \frac{2}{t}\right]$.

We now improve (A.7) by showing that there exists $k > 0$ such that, if $v \geqslant k$ then

$$F(v, t) \leqslant -\,v^2\,. \tag{A.8}$$

In other words, we have to show that :

$$-\,T_2(t) + 2\int_v^{+\infty} \mu(d\bar{v}\,, t)\,\bar{v}^{\,2} \leqslant 0\,. \tag{A.9}$$

This is in fact obvious by the previous property on the support of $\mu(t)$ if $t > 0$ and v is sufficiently large. On the other hand, by the inequalities :

$$\dot{T}_1(t) \geqslant -\,2\,T_2(t)\,, \tag{A.10}$$

we have :

$$T_1(t) - T_1(0) \geqslant -\,2\,T_2(0)\,t\,;$$

$$T_2(t) \geqslant (T_1(t))^2 \geqslant (T_1(0) - 2\,T_2(0)\,t)^2 \tag{A.11}$$

for $t \leqslant \dfrac{T_1(0)}{2\,T_2(0)}$. Since we have also that :

$$2\int_v^{+\infty} \mu(d\bar{v}\,, t) \leqslant \frac{2\,T_{2\,\log}}{1 + \log\,(1 + |v|\,)}$$

we conclude that :

$$-\,T_2(t) + 2\int_v^{+\infty} \mu(d\bar{v}\,, t)\,\bar{v}^{\,2}$$

$$\leqslant \frac{2\,T_{2\,\log}}{1 + \log\,(1 + |v|\,)} - (T_1(0) - 2\,T_2(0)\,t)^2 \leqslant 0 \tag{A.13}$$

for $t < t_1$ and $v > k$ with a suitable t_1 and k chosen accordingly.

For $t \geqslant t_1$ one can choose an even larger k (namely $k > \sqrt{2\,T_2(0)} + 2/t_1$) for which v is outside the support of μ and the identity :

$$\int_v^{+\infty} \mu(d\bar{v}\,, t)\,\bar{v}^{\,2} = 0 \tag{A.14}$$

holds.

Thus (A.8) is proven. From this fact follows that :

$$|V(v,t)| \leqslant k + \frac{|v|}{1+|v|\,t} \tag{A.15}$$

Now, let us assume that $\mu(dv, t)$ and $\tilde{\mu}(dv, t)$, are two continuous solutions with the same initial datum μ_0 ; let us denote by $V_t(v)$ and $\tilde{V}_t(v)$ the corresponding flows, and $F_t = -\phi * \mu(\,.\,, t)$ and $\tilde{F}_t = -\phi * \tilde{\mu}(\,.\,, t)$.

By the definitions :

$$|V_t(v) - \tilde{V}_t(v)| \leqslant \int_0^t ds(\,|F_s(V_s(v)) - F_s(\tilde{V}_s(v))| + |F_s(\tilde{V}_s(v)) - \tilde{F}_s(\tilde{V}_s(v))|\,)\,. \tag{A.16}$$

Denoting by $\delta(v,t) = |V_t(v) - \tilde{V}_t(v)|$, and by c any constant depending only on μ_0, from the definition of F :

$$|F_s(V_s(v)) - F_s(\tilde{V}_s(v))| \leqslant 2\,\delta(v,s) \int \mu(d\bar{v}\,, t)\,(\,|\bar{v}\,| + \max\,(V_s(v), \tilde{V}_s(v)))\,, \tag{A.17}$$

then, using (A.15) :

$$|F_s(V_s(v)) - F_s(\tilde{V}_s(v))| \leqslant \left(c + 2\,\frac{|v|}{1+|v|\,s}\right)\delta(v,s)\,. \tag{A.18}$$

In a similar way, we obtain :

$$|F_s(\tilde{V}_s(v)) - \tilde{F}_s(\tilde{V}_s(v))| \leqslant c\left(1 + \frac{|v|}{1+|v|\,s}\right)\int \mu(d\bar{v}\,, s)\,\delta(\bar{v}\,, s) + c\int \mu(d\bar{v}\,, s)\,\frac{|\bar{v}\,|}{1+|\bar{v}\,|\,s}\,\delta(\bar{v}\,, s)\,. \tag{A.19}$$

Applying the Gronwall Lemma to the collection (A.17)-(A.19), we have :

$$\delta(v,t) \leqslant ce^{ct}(1+|v|\,t)^2 \int_0^t ds\left(\left(1+\frac{|v|}{1+|v|\,s}\right)\int \mu(d\bar{v}\,,s)\,\delta(\bar{v}\,,s)\right.$$

$$\left. + \int \mu(d\bar{v}\,,s)\,\frac{|\bar{v}\,|}{1+|\bar{v}\,|\,s}\,\delta(\bar{v}\,,s)\right). \tag{A.20}$$

Setting $A(v,t) = \int \mu_0(dv)\,\frac{1+|v|}{1+|v|\,t}\,\delta(v,t)$, fixed $T>0$ and denoting by c any constant depending only on T, we have :

$$A(v,t) \leqslant c\int_0^t ds\,A(v,s)\int \mu_0(d\bar{v}\,)\,\frac{(1+|\bar{v}\,|)^3}{1+|\bar{v}\,|\,s}. \tag{A.21}$$

Even if $T_3(0) = +\infty$, we have that

$$\int_0^t ds\int \mu_0(d\bar{v}\,)\,\frac{(1+|\bar{v}\,|)^3}{1+|\bar{v}\,|\,s} = \int \mu_0(d\bar{v}\,)\,(1+|\bar{v}\,|)^3\,\frac{\log(1+|\bar{v}\,|\,t)}{|\bar{v}\,|} \leqslant cT_{2\log}.$$

Then we can use the Gronwall Lemma in (A.21) so that $A(v,t)=0$, i.e. $V(v,t) = \tilde{V}(v,t)\,\mu_0$ – a.e. Then the solution is unique.

The continuity of the solution with respect to the initial datum can be treated in the same way. Therefore we proved (i) and (ii).

Let us now discuss the properties (iii) and (iv) for the solutions.

Let $\mu_0(dv)$ be a measure with $T_{2\log} < +\infty$ and $V(v,t)$ be the unique corresponding flow. Notice that, thanks to (A.3), $|V(v,t)| < 1/t$. Therefore $T_{2p} = \int \mu_0(dv)\,V^{2p}(v,t)$ is finite for $t>0$. Proceeding as in (2.3)-(2.6), (in the weak form), we obtain that T_{2p} decreases if $p \geqslant \frac{1}{2}$, and that $T = T_2$ verifies the bound (2.6). As consequence :

$$\left|\int \mu(dv,t)\,\psi(v) - \psi(0)\right| \leqslant \int \mu(dv,t)\,|\psi(v)-\psi(0)|$$

$$\leqslant \frac{\|\psi'\|_\infty\,T(0)^{\frac{1}{2}}}{1+t\sqrt{2\,T(0)}}. \tag{A.22}$$

Taking the limit for $t \to +\infty$, we prove (2.11).

If $\mu_0(dv) = f_0(v)\,dv$, with $f_0 \in \mathbf{C}^k(\mathbb{R})$, by the continuity of $V(v,t)$ with respect to $V(v,t)$ with respect to v, it follows that $\mu(dv,t)$ is a absolutely continuous with respect the Lebesgue measure with density denoted by $f(v,t)$.

The regularity of $f(v,t)$ follows by the fact that if $f \in \mathbf{C}([0,\infty);\mathbf{C}^k(\mathbb{R}))$ then $F \in \mathbf{C}([0,\infty);\mathbf{C}^{k+2}(\mathbb{R}))$, $V \in \mathbf{C}([0,\infty);\mathbf{C}^{k+1}(\mathbb{R}))$. Moreover $\partial_v(fF) \in \mathbf{C}([0,\infty);\mathbf{C}^k(\mathbb{R}))$ then $f \in \mathbf{C}^1([0,\infty);\mathbf{C}^{k-1}(\mathbb{R}))$ and solves classically Eq. (2.1).

Proof of Lemma 2.1

By direct computation $g(d\xi) \in \mathscr{V}_2$ iff

$$g(d\xi) = (\mu\delta_{\mu-1} + (1-\mu)\,\delta_\mu)\,d\xi\,, \tag{A.23}$$

where $\mu \in (0,1)$. Moreover $g(d\xi) \in \mathscr{V}_3$ iff

$$g(d\xi) = \left(\mu_1\,\delta_{\frac{-\mu_2}{1-4\mu_1\mu_3}} + \mu_2\,\delta_{\frac{\mu_1-\mu_3}{1-4\mu_1\mu_3}} + \mu_3\,\delta_{\frac{\mu_2}{1-4\mu_1\mu_3}}\right)d\xi\,, \tag{A.24}$$

where $\mu_1,\ \mu_3 \in \left(0,\frac{1}{2}\right)$, $\mu_2 = 1-\mu_1-\mu_3$.

Proof of Theorem 3.1

Let us start by showing the following a-priori estimates for regular solutions of Eq. (1.8).

We suppose that, initially $\sup_x \int_{|v|>\eta} f(x,v) = 0$ for some $\eta > 0$. Suppose also that $\|f_0\|_\infty$, $\|\nabla f_0\|_\infty < +\infty$. We claim that there exists $T > 0$ and $C > 0$, depending on $\|f_0\|_\infty$, $\|\nabla f_0\|_\infty$ and η, such that $\|f(\,.\,,.\,,t)\|_\infty$, $\|\nabla f(\,.\,,.\,,t)\|_\infty < C$ for $t \leqslant T$.

Indeed let

$$\begin{aligned} \dot{X}(x,v,t) &= V(x,v,t) \\ \dot{V}(x,v,t) &= \lambda F(X(x,v,t),V(x,v,t),t)\,, \end{aligned} \tag{A.25}$$

where

$$F(x,v,t) = \int d\bar{v}\ \phi(\bar{v}-v)\,f(x,\bar{v},t)\,, \tag{A.26}$$

and let

$$J(x, v, t) = \begin{pmatrix} \partial_x X & \partial_v X \\ \partial_x V & \partial_v V \end{pmatrix} \quad A(x, v, t) = \begin{pmatrix} 0 & 1 \\ \lambda \partial_x F(x, v, t) & \lambda \partial_v F(x, v, t) \end{pmatrix}. \tag{A.27}$$

Sometimes in the following, we shall not specify the dependence of X, and V on x, v, t for sake of notational simplicity.

By (A.25) and (A.26) it follows that

$$\dot{J}(x, v, t) = A(X, V, t)\, J(x, v, t)$$

$$\dot{J}^{-1}(x, v, t) = -J^{-1}(x, v, t)\, A(X, V, t) \tag{A.28}$$

$$\det J(x, v, t) = e^{\int_0^t \operatorname{tr} A(X(x, v, s), V(x, v, s), s)}.$$

From Eq. (1.8) and (A.24) it follows that, for any test function $\psi \in \mathbf{C}_0^\infty(\mathbb{R}^2)$:

$$\int \psi(x, v) f(x, v, t)\, dx\, dv = \int \psi(X(x, v, t), V(x, v, t)) f_0(x, v)\, dx\, dv, \tag{A.29}$$

and then:

$$f(X(x, v, t), V(x, v, t), t) = f_0(x, v)\, e^{-\int_0^t \operatorname{tr} A(X(x, v, s), V(x, v, s), s)}. \tag{A.30}$$

Taking the gradient with respect (x, v), we have:

$$(\nabla f)(X(t), V(t)) = (J^t(t))^{-1} \nabla f_0(x, v)\, e^{-\int_0^t \operatorname{tr} A(X(s), V(s), s)} \tag{A.31}$$

$$+ f_0(x, v)\, e^{-\int_0^t \operatorname{tr} A(X(s), V(s), s)} (J^t(t))^{-1} \int_0^t ds (J^t(s))\, (\nabla \operatorname{tr} A)(X(s), V(s), s).$$

First we observe that, by virtue of our assumption on the compactness of the support of f in the v variable, such a support can only decrease in time. Namely if $v > \eta$, $F(x, v, t) < 0$ and if $v < -\eta$, $F(x, v, t) > 0$. Then :

$$-\operatorname{tr} A(x, v, t) = 2\lambda \int d\bar{v}\ |v - \bar{v}\ |\ f(x, \bar{v}\ , t) \leqslant 4\eta^2 \|f(t)\|_\infty = c\|f(t)\|_\infty$$

$$\|A(x, v, t)\| \leqslant c(1 + \|\nabla f(t)\|_\infty)$$

$$\|\nabla \operatorname{tr} A(x, v, t)\| \leqslant c\|\nabla f(t)\|_\infty$$

$$\|J^{-1}(x, v, t)\| \leqslant e^{ct + c\int_0^t ds \|\nabla f(s)\|_\infty}. \tag{A.32}$$

Using (A.32) in (A.30), (A.31) :

$$\|f(t)\|_\infty + \|\nabla f(t)\|_\infty \leqslant (\|f_0\|_\infty + \|\nabla f_0\|_\infty)\, e^{c\int_0^t ds(\|f(s)\|_\infty + \|\nabla f(s)\|_\infty)}, \tag{A.33}$$

which implies

$$\|f(t)\|_\infty + \|\nabla f(t)\|_\infty \leqslant \frac{\|f_0\|_\infty + \|\nabla f_0\|_\infty}{1 - ct(\|f_0\|_\infty + \|\nabla f_0\|_\infty)}. \tag{A.34}$$

With this a-priori estimate is not difficult to construct, via the standard iterative procedure, a unique solution in $C^1([0, T] \times \mathbb{R} \times [-\eta, \eta])$ for the initial value problem.

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