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SOLUTION OF A TWO-DIMENSIONAL STATIONARY INDUCTION HEATING PROBLEM WITHOUT BOUNDEDNESS OF THE COEFFICIENTS (*) (**)

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Abstract — *We consider in this paper a system of equations modelling a quasi-stationary induction heating process Existence of a solution is obtained in Sobolev spaces using estimations in L^∞-norm Using a truncation technique, we build a sequence of truncated problems the solutions of which converge to a solution of the initial unbounded coefficient problem*

Résumé — *On considère, dans cet article, un système d'équations modélisant un procédé quasi-stationnaire de chauffage par induction Les coefficients physiques intervenant dans la modélisation ne sont pas bornés On construit une suite de problèmes en utilisant une méthode de troncature Nous montrons l'existence d'une solution dans les espaces de Sobolev pour le problème tronqué et donnons des estimations en norme L^∞ des solutions On montre alors que les solutions des problèmes tronqués convergent vers une solution du problème à coefficients non bornés*

1. INTRODUCTION

We are concerned in the present paper with the existence of a solution of the following nonlinear system of elliptic equations:

$$\begin{cases} i\mu\omega H - \nabla \cdot (\rho(\theta)\, \nabla H) = 0 & \text{in } \Omega, \\ - \nabla \cdot (\lambda(\theta)\, \nabla\theta) = \frac{1}{2}\rho(\theta)\, |\nabla H|^2 & \text{in } \Omega \end{cases} \tag{1.1}$$

where $|\nabla H|$ is the modulus of the vector valued function ∇H, i.e.,

$$|\nabla H|^2 = \partial_{x_1} H\, \partial_{x_1} H^* + \partial_{x_2} H\, \partial_{x_2} H^*,$$

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and v^* is the complex conjugate function of u. The domain Ω is an open and bounded subset of $\mathbb{R}^2$.

Here above, the first equation describes the evolution of a time harmonic magnetic field H in a conductor and the second is the steady state heat equation. In fact, the stationarity of the temperature field results here from an averaging process of the power source term on a period of length $2\pi/\omega$ whence the factor $\frac{1}{2}$. Typically, this system governs an induction heating process (*cf.* [Cl], [ClTo], [CRST]). It belongs to a large class of problems that were recently studied in Lewandowski [Le], Murat [Mu] and Gallouët-Herbin [GaHe].

The main goal of the present paper is to prove that Problem (1.1), provided with Dirichlet boundary conditions on H and θ, has at least one solution (H, θ). The main difficulty we treat here is that we do not assume that the physical coefficients of the equations are bounded. This corresponds to realistic situations where, for instance, polynomial functions of the temperature are used to take account for the physical properties of the material. To handle this difficulty, we first transform in Section 2 the initial problem using the Kirchhoff transformation thus writing the heat equation using the Laplace operator. This enables us to prove regularity of the solutions of the heat equation. Subsequently, in Section 3, we define a truncated problem whose conductivity coefficient ρ is truncated. We then prove, using the Schauder's fixed point theorem, the existence of at least one solution of the resulting truncated problem.

Section 4 is devoted to the proof of some lemmas concerning the estimate in L^∞-norm of the solution of an elliptic problem in function of the regularity of the right-hand side.

Section 5 gives the proof of the main result of the paper. We prove that the temperature is bounded in $L^\infty(\Omega)$ independently of the truncation. For this, the results obtained in Section 4 are used.

2. THE MODEL

In References [Cl], [ClTo], [CRST], we have studied the well-posedness of the following system of equations:

$$\begin{cases} \mu \dfrac{\partial H}{\partial t} - \nabla \cdot (\rho(\theta)\, \nabla H) = 0 & \text{in } (0, T) \times \Omega\,, \\ C_p(\theta)\, \dfrac{\partial \theta}{\partial t} - \nabla \cdot (\lambda(\theta)\, \nabla \theta) = \rho(\theta)\, |\nabla H|^2 & \text{in } (0, T) \times \Omega\,, \end{cases}$$

where μ is a positive constant and λ, ρ, C_p are bounded and smooth real valued functions that stand respectively for magnetic permeability of the free space, thermal conductivity, density and specific heat. The expressions $\rho(\theta)$, $\lambda(\theta)$ and $C_p(\theta)$ denote the composition of the functions ρ, λ and C_p with the function θ.

Here we are concerned with the so-called quasi-stationary process. In other words, since the magnetic field evolves at a high rate, we can assume that the temperature field, as a first order approximation, depends only on the time averaged power density. It is therefore possible to seek a time-independent temperature field and a magnetic field of the form

$$H(t,x) = \mathrm{Re}\,(\exp(i\omega t)\,H(x))\,,$$

where ω is the angular frequency of the current and i is the unit imaginary number.

These considerations lead to the following model:

$$\text{Find } H : \Omega \to \mathbb{C},\ \theta : \Omega \to \mathbb{R} \text{ such that :}$$

$$\begin{cases} i\mu\omega H - \nabla\,.\,(\rho(\theta)\,\nabla H) = 0 & \text{in } \Omega\,, \\ H = H_0, & \text{on } \partial\Omega\,, \end{cases} \tag{2.1}$$

$$\begin{cases} -\nabla\,.\,(\lambda(\theta)\,\nabla\theta) = \frac{1}{2}\rho(\theta)\,|\nabla H|^2 & \text{in } \Omega\,, \\ \theta = \theta_0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.2}$$

Here above, the temperature on the boundary of the domain Ω is assumed to be equal to a constant θ_0. The term $\frac{1}{2}\rho(\theta)\,|\nabla H|^2$ in the right-hand side of Equation (2.2) corresponds to the mean power density supplied to the induced conductor. In addition, the development of the induction heating model (*cf.* [Cl], [CRST]) shows that the magnetic field in the free space and on the boundary of the conductors is independent of the space variables. That is, H_0 is a constant.

We now suppose that the electric resistivity and the thermal conductivity satisfy the following conditions:

$$\lambda \in C^0(\mathbb{R}),\ \lambda > 0\,, \tag{2.3}$$

$$\rho \in C^0(\mathbb{R})\,, \tag{2.4}$$

$$\text{There exists } \alpha > 0 \text{ such that } \rho(t) \geqslant \alpha \quad \forall t > 0\,, \tag{2.5}$$

$$\rho \text{ is nondecreasing}\,. \tag{2.6}$$

We furthermore assume that the domain Ω is bounded and that its boundary is of class C^2.

We now make the change of variable:

$$\beta(\theta) \overset{\text{def}}{=} \int_{\theta_0}^{\theta} \lambda(s)\,ds\,.$$

Since the function λ is positive the function β is one-to-one. Let us define:

$$u \overset{\text{def}}{=} \beta(\theta),$$

$$r(u) = r \circ u \overset{\text{def}}{=} \rho \circ \theta = \rho(\theta),$$

Equation (2.2) becomes:

$$\begin{cases} -\ \Delta u = \frac{1}{2} r(u)\ |\nabla H|^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Setting $h = H - H_0$, Equation (2.1) yields:

$$\begin{cases} i\omega\mu h - \nabla \cdot (r(u)\ \nabla h) = -\ i\omega\mu H_0 & \text{in } \Omega, \\ h = 0 & \text{on } \partial\Omega. \end{cases}$$

In what follows f stands for the complex number $-\ i\omega\mu H_0$. Let us notice at this point that the whole analysis developed in the present paper can be extended to the case where f is a function of $L^2(\Omega)$.

The final problem consists then in seeking a pair $(h, u) : \Omega \to \mathbb{C} \times \mathbb{R}$ such that:

$$\begin{cases} i\omega\mu h - \nabla \cdot (r(u)\ \nabla h) = f & \text{in } \Omega, \\ h = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.7}$$

$$\begin{cases} -\ \Delta u = \frac{1}{2} r(u)\ |\nabla h|^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

Hypotheses (2.3)-(2.6) imply that the function r satisfies the following properties:

$$r \text{ is nondecreasing, unbounded}, \tag{2.9}$$

$$r \text{ is continuous}, \tag{2.10}$$

$$r(t) \geqslant \alpha > 0 \text{ for all } t. \tag{2.11}$$

In the following section, Problem (2.7)-(2.8) is approximated using a truncation technique where the truncated problem has bounded coefficients. We first prove that the truncated problem has at least one solution. The L^∞-estimate on the solution of the truncated problem enables us then to prove the existence of at least one solution of (2.7)-(2.8).

3. THE TRUNCATED PROBLEM

Let k denote a real positive number. We define the truncated function r_k by:

$$r_k(s) \stackrel{\text{def}}{=} \begin{cases} r(s) & \text{if } -\infty < s \leqslant k\,, \\ r(k) & \text{if } k < s < \infty\,. \end{cases}$$

Notice that r_k is a continuous and bounded function.

We now consider the truncated problem $(\mathscr{P}_k)$ consisting in seeking two functions $h_k : \Omega \to \mathbb{C}$ and $u_k : \Omega \to \mathbb{R}$ such that:

$$\begin{cases} i\omega\mu h_k - \nabla \cdot (r_k(u_k)\, \nabla h_k) = f & \text{in } \Omega\,, \\ h_k = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{3.1}$$

$$\begin{cases} -\ \Delta u_k = \frac{1}{2}\, r_k(u_k)\, |\nabla h_k|^2 & \text{in } \Omega\,, \\ u_k = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{3.2}$$

Using the Schauder's fixed point theorem, we shall prove that Problem (3.1)-(3.2) has at least one solution. For this end, let us define the following operators:

To each measurable function v we associate the function $\phi \in W^{1,2}(\Omega)$ solution of the problem:

$$\begin{cases} i\omega\mu\phi - \nabla \cdot (r_k(v)\, \nabla\phi) = f & \text{in } \Omega\,, \\ \phi = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{3.3}$$

Let T_1 denote the operator defined by:

$$v \mapsto \phi \stackrel{\text{def}}{=} T_1(v)\,.$$

Assume now that v and ϕ are given; we associate to them the function $g(v, \phi)$ defined by

$$g : (v, \phi) \mapsto \frac{1}{2}\, r_k(v)\, |\nabla\phi|^2\,.$$

Finally, let $p \in (1, \infty)$; we denote by T_2 the inverse of the Laplace operator defined from $W^{-1,p}(\Omega)$ to $W_0^{1,p}(\Omega)$ (This is well defined thanks to the smoothness of the boundary). We can formulate Problem (3.1)-(3.2) as the following fixed point problem.

Find $u_k : \Omega \to \mathbb{R}$ such that :

$$u_k = T(u_k) \stackrel{\text{def}}{=} T_2(g(u_k, T_1(u_k)))\,.$$

We notice here that the operators T and T_1 are formally defined. A more precise definition using appropriate function spaces will be done later;

We now prove some preliminary results. First, we shall present a version of a regularity result of N. G. Meyers for an elliptic operator. Its proof can be found in the appendix of the present paper. The main feature of this version is that we give sharp estimates of the regularity constants appearing in the Meyer's result. These estimates are essential to prove the existence of a solution (h, u) such that u is bounded.

LEMMA 3.1: *Let a denote a function of $L^\infty(\Omega)$ such that*

$$0 < \alpha \leqslant a \leqslant \beta < \infty \quad a.e.\ in\ \Omega .$$

Then, there exists $p = p(\alpha, \beta, \Omega) > 2$ such that if $f \in W^{-1,p}(\Omega)$ then the following problem:

$$\begin{cases} -\nabla . (a \nabla u) = f & \text{in } \Omega , \\ u = 0 & \text{on } \partial\Omega , \end{cases} \tag{3.4}$$

has a unique solution $u \in W_0^{1,p}(\Omega)$. Moreover, we have the estimate:

$$\|u\|_{W_0^{1,p}(\Omega)} \leqslant C_{\alpha,\beta} \|f\|_{W^{-1,p}(\Omega)} . \tag{3.5}$$

In addition, there exists two constants $\chi = \chi(\Omega) > 1$ and $C_0 = C_0(\Omega) > 0$, such that for

$$\beta > \frac{\alpha(2\chi - 1)}{2(\chi - 1)} \tag{3.6}$$

the values of p and $C_{\alpha,\beta}$ can be chosen in the following way:

$$p = \frac{4 \ln \chi}{2 \ln \chi - \ln \left(\frac{2\beta - \alpha}{2\beta - 2\alpha} \right)} > 2 , \tag{3.7}$$

$$C_{\alpha,\beta} = \frac{C_0}{\alpha} . \tag{3.8}$$

Notice that the definition of p given by the expression (3.7) implies that $p < 4$. (See the proof of Lemma 3.1 in the appendix.)

It is clear that applying Lemma 3.1 to Problem (3.2) fixes the value of p. This value depends on α and on $r(k)$. We choose then a real number q such that:

$$q \in \left(2, \frac{2p}{4 - p} \right) . \tag{3.9}$$

For this choice the imbedding of $W^{2,p/2}(\Omega)$ into $W^{1,q}(\Omega)$ is compact. We shall now show that the operator T fulfills the conditions of the Schauder's fixed point theorem (*cf.* [GiTr]) in the Banach space $W_0^{1,q}(\Omega)$.

LEMMA 3.2: *Let Ω denote an open subset of $\mathbb{R}^2$ and r a continuous function on $C^0(\mathbb{R})$. Let $\tilde{r}: L^\infty(\Omega) \to L^\infty(\Omega)$ the mapping defined by*

$$\tilde{r}: v \mapsto \tilde{r}(v) = r \circ v .$$

Then, $\tilde{r}$ is continuous.

Proof: Let v denote an arbitrary element of $L^\infty(\Omega)$ and let us shown that $\tilde{r}$ is continuous at v.

We denote by $K = \|v\|_{L^\infty(\Omega)}$. The function r is uniformly continuous on the compact $[-2K, 2K]$. Therefore, for each $\varepsilon > 0$ there is a real number $\delta > 0$ such that for all t, $s \in [-2K, 2K]$ with $|t-s| \leqslant \delta$ we have $|r(t) - r(s)| \leqslant \varepsilon$.

We choose an $\varepsilon > 0$ and impose $\delta \leqslant K$. Let $w \in L^\infty(\Omega)$ such that $\|w - v\|_{L^\infty(\Omega)} \leqslant \delta$. We have $|w(x) - v(x)| \leqslant \delta$ for almost all $x \in \Omega$. Using the uniform continuity of r we have $|r(w(x)) - r(v(x))| \leqslant \varepsilon$ for a.e. x. Thus

$$\|\tilde{r}(w) - \tilde{r}(v)\|_{L^\infty(\Omega)} \leqslant \varepsilon ,$$

whence the continuity. □

To simplify the notations we set r instead of $\tilde{r}$.

LEMMA 3.3: *The mapping $T_1 : W_0^{1,q}(\Omega) \to W_0^{1,p}(\Omega)$ is continuous.*

Proof: Let v be a function of $W_0^{1,q}(\Omega)$. The function $r_k(v) = r_k \circ v$ satisfies the condition $\alpha \leqslant r_k(v) \leqslant r(k)$. Using the Lax-Milgram theorem we deduce the existence and uniqueness of a solution ϕ to Equation (3.3) in $W_0^{1,2}(\Omega)$ with the following estimate:

$$\|\phi\|_{W_0^{1,2}(\Omega)} \leqslant C_1 |f| . \tag{3.10}$$

Equation (3.3) can be rewritten by placing the term $i\omega\mu\phi$ in the right-hand side. Applying Lemma 3.1 we obtain thanks to inequality (3.5) the estimate:

$$\|\phi\|_{W_0^{1,p}(\Omega)} \leqslant C_2(|f| + \|\phi\|_{W^{-1,p}(\Omega)}) .$$

Using estimate (3.10) and the continuous imbedding of $W_0^{1,2}(\Omega)$ into $W^{-1,p}(\Omega)$ we have:

$$\|\phi\|_{W_0^{1,p}(\Omega)} \leqslant C_3 |f| , \tag{3.11}$$

the constants C_1, C_2, C_3 being independent of v.

On the other hand, let v and w denote two functions of $W_0^{1,q}(\Omega)$ and set $\phi = T_1(v)$, $\psi = T_1(w)$, $\varphi = \phi - \psi$. We have:

$$- \nabla . (r_k(v) \nabla \varphi) + i\omega\mu\varphi = \nabla . ((r_k(v)) \nabla \psi) . \tag{3.12}$$

Using the energy inequrality and (3.11) we have:

$$\begin{aligned}\|\varphi\|_{W_0^{1,2}(\Omega)} &\leqslant C_4 \|(r_k(v) - r_k(w)) \nabla\psi\|_{L^2(\Omega)} \\ &\leqslant C_4 \|r_k(v) - r_k(w)\|_{L^\infty(\Omega)} \|\nabla\psi\|_{L^2(\Omega)} \\ &\leqslant C_3\, C_4 |f| \ \|r_k(v) - r_k(w)\|_{L^\infty(\Omega)} .\end{aligned}$$

We apply again Lemma 3.1 to Equation (3.12). We obtain using the previous inequality:

$$\begin{aligned}\|\varphi\|_{W_0^{1,p}(\Omega)} &\leqslant C_5(\|(r_k(v) - r_k(w)) \nabla\psi\|_{L^p(\Omega)} + \omega\mu \|\varphi\|_{W^{-1,p}(\Omega)}) \\ &\leqslant C_6(\|r_k(v) - r_k(w)\|_{L^\infty(\Omega)} \|\nabla\psi\|_{L^p(\Omega)} + \|\varphi\|_{W^{1,2}(\Omega)}) \\ &\leqslant C_7 \|r_k(v) - r_k(w)\|_{L^\infty(\Omega)} .\end{aligned}$$

From Lemma 3.2, we deduce that the mapping $v \mapsto \tilde{r}_k(v) = r_k \circ v$ is continuous from $L^\infty(\Omega)$ to $L^\infty(\Omega)$. The continuity of the imbedding of $W^{1,q}(\Omega)$ into $L^\infty(\Omega)$ implies the continuity of the operator T_1. □

LEMMA 3.4: *The mapping* $T_2\, g : W_0^{1,q}(\Omega) \times W_0^{1,p}(\Omega) \mapsto W_0^{1,q}(\Omega)$ *is continuous and compact.*

Proof: The operator T_2 is linear and continuous from $L^{p/2}(\Omega)$ to $W^{2,p/2}(\Omega) \cap W_0^{1,p/2}(\Omega)$ for $p > 2$ [Gr]. It follows that T_2 is linear continuous and compact from $L^{p/2}(\Omega)$ to $W_0^{1,q}(\Omega)$. It remains then to prove that the mapping $g : W_0^{1,q}(\Omega) \times W_0^{1,p}(\Omega) \to L^{p/2}(\Omega)$ is continuous. Let (v, ϕ) and (w, ψ) denote two elements of $W_0^{1,q}(\Omega) \times W_0^{1,p}(\Omega)$. We have:

$$\begin{aligned}\|g(v, \phi) - g(w, \psi)\|_{L^{p/2}(\Omega)} &= \frac{1}{2} \|r_k(v)\ |\nabla\phi|^2 - r_k(\omega)\ |\nabla\psi|^2\|_{L^{p/2}(\Omega)} \\ &\leqslant \frac{1}{2}(\|(r_k(v) - r_k(w))\ |\nabla\phi|^2\|_{L^{p/2}(\Omega)} \\ &\quad + \|r_k(w)\ (|\nabla\phi|^2 - |\nabla\psi|^2)\|_{L^{p/2}(\Omega)} \\ &\leqslant \frac{1}{2}(\|r_k(v) - r_k(w)\|_{L^\infty(\Omega)} \|\ |\nabla\phi|^2\|_{L^{p/2}(\Omega)} \\ &\quad + \|r_k(w)\|_{L^\infty(\Omega)} \|\ |\nabla\phi|^2 - |\nabla\psi|^2\|_{L^{p/2}(\Omega)}) \\ &\leqslant C(\|r_k(v) - r_k(w)\|_{L^\infty(\Omega)} \\ &\quad + \|\nabla\phi - \nabla\psi\|_{L^p(\Omega)}(\|\nabla\phi\|_{L^p(\Omega)} + \|\nabla\psi\|_{L^p(\Omega)}) .\end{aligned}$$

Using the continuity of the mapping $w \in L^\infty(\Omega) \mapsto r_k(w) \in L^\infty(\Omega)$ and the continuity of the imbedding of $W^{1,q}(\Omega)$ into $L^\infty(\Omega)$ we deduce the continuity of g. □

We are now able to state the main theorem of this section.

THEOREM 3.1: *There exists $p > 2$ such that Problem (3.1)-(3.2) has at least one solution $(h_k, u_k) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ for all q satisfying (3.9). Furthermore, we have the estimate:*

$$\| h_k \|_{W_0^{1,p}(\Omega)} + \| u_k \|_{W_0^{1,q}(\Omega)} \leq C|f|^2 , \tag{3.13}$$

where C is independent of u_k and h_k.

Proof: From Lemmas 3.3 and 3.4, we deduce that the operator $T : W_0^{1,q}(\Omega) \to W_0^{1,q}(\Omega)$ is continuous and compact. It remains to prove that $T(v)$ is uniformly bounded in $W_0^{1,q}(\Omega)$. Inequality (3.11) shows that $T_1(v)$ is bounded independently of v. We can write:

$$\begin{aligned} \| T(v) \|_{W_0^{1,q}(\Omega)} &\leq \| T_2\, g(v, T_1(v)) \|_{W_0^{2,p/2}(\Omega)} \\ &\leq C_1 \| r_k(v)\ |\nabla T_1(v)|^2 \|_{L^{p/2}(\Omega)} \\ &\leq C_2 \| T_1(v) \|^2_{W_0^{1,p}(\Omega)} \\ &\leq C_3 |f|^2 \end{aligned}$$

where C_3 is independent of v.

Choosing for D the ball with radius $C_3|f|^2$ and center 0 we clearly satisfy the conditions of the Schauder's fixed point theorem. We then obtain a solution $u_k \in W_0^{1,q}(\Omega)$ of the equation $T(u_k) = u_k$ and the pair $(T_1(u_k), u_k)$ is a solution of Problem (3.1)-(3.2). □

Remark 3.1: Since the right-hand side of Equation (3.2) is nonnegative, the weak maximum principle implies that the function u_k given by Theorem 3.1 is nonnegative for all k.

Remark 3.2: From Theorem 3.1, the function u_k is an element of $W^{2,p/2}(\Omega)$. We deduce that $u_k \in W^{1,s}(\Omega)$ with $s = \dfrac{2p}{4-p}$. Since $s > 2$, the Morrey's Theorem (*cf.* [Bre], p. 166) implies that the function u_k is continuous on $\bar{\Omega}$.

4. SOME TECHNICAL LEMMAS

We have proved in the previous section that Problem (3.1)-(3.2) has at least one solution (h_k, u_k). We shall now study the sequence of functions $(h_k, u_k)_k$ and show the existence of a limit pair (h, u) satisfying (2.7)-(2.8).

If the function r is bounded, the pair (h_k, u_k) is solution of Problem (2.7)-(2.8) with $k = \|r\|_{L^\infty(\mathbb{R})}$. If r is not bounded, we can no more conclude that the sequence $(r_k(u_k))_k$ is bounded in $L^\infty(\Omega)$. In this case, we shall seek a solution of Problem (2.7)-(2.8) as a limit of the sequence of solutions of the truncated problems $\mathscr{P}_k$.

Our main goal is to prove the existence of a solution pair (h, u) such that $u \in L^\infty(\Omega)$. Notice that if this is the case, Problem (2.7)-(2.8) is well defined since the function $r(u)$ is bounded. An alternative approach was considered by Murat [Mu] in which the notion of a renormalized solution is introduced.

We shall, in the present section, prove a series of lemmas that will enable us to control the L^∞-norm of u_k in terms of k. We shall prove that, under some conditions, there exists a real number K such that $\|u_K\|_{L^\infty(\Omega)} \leq K$.

Remark 4.1: Let $K > 0$ and let (h_K, u_K) denote a solution of Problem (3.1)-(3.2) such that

$$\|u_K\|_{L^\infty(\Omega)} \leq K . \tag{4.1}$$

Then $r_k \circ u_K = r_K \circ u_K$ a.e. for all $k \geq K$. Therefore, (h_K, u_K) is a solution of Problem (2.7)-(2.8).

It is now sufficient to prove the existence of $K > 0$ such that (4.1) holds. From Remark 3.1, the function u_k is nonnegative.

The next lemma gives a first *a priori* estimate on u_k.

LEMMA 4.1: *Let (h_k, u_k) denote a solution of Problem (3.1)-(3.2) in the space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. We have*

$$\int_\Omega r_k(u_k)\, |\nabla h_k|^2\, dx \leq \frac{1}{\mu\omega} |f|^2\, dx . \tag{4.2}$$

Proof: Multiplying the first equation of (3.1) by the complex conjugate function of h_k and integrating on the domain Ω, we obtain:

$$i\omega\mu \int_\Omega |h_k|^2\, dx + \int_\Omega r_k(u_k)\, |\nabla h_k|^2\, dx = \int_\Omega f h_k^*\, dx .$$

Thus

$$\omega\mu \int_\Omega |h_k|^2\, dx \leq \int_\Omega \operatorname{Im}(f h_k^*)\, dx \leq \int_\Omega |f|\, |h_k|\, dx .$$

Using the Cauchy-Schwarz inequality, we get:

$$\left(\int_{\Omega} |h_k|^2\, dx\right)^{1/2} \leqslant \frac{1}{\mu\omega}\left(\int_{\Omega} |f|^2\, dx\right)^{1/2}.$$

Using again the Cauchy-Schwarz inequality we have:

$$\int_{\Omega} r_k(u_k)\, |\nabla h_k|^2\, dx = \int_{\Omega} \operatorname{Re}\,(f h_k^*)\, dx \leqslant \frac{1}{\mu\omega}\int_{\Omega} |f|^2\, dx\,,$$

which achieves the proof. □

Remark 4.2: From (4.2) and Hypothesis (2.5) we get:

$$\int_{\Omega} |\nabla h_k|^2\, dx \leqslant \frac{1}{\alpha}\,\mu\omega \int_{\Omega} |f|^2\, dx\,. \tag{4.3}$$

Therefore, the sequence $(h_k)_k$ is bounded in $W_0^{1,2}(\Omega)$.

In order to prove that the existence of a solution (h, u) such that u is bounded, we shall use a method based of results of Stampacchia [St]. The main idea consists in measuring the subsets of Ω on which the function u is "very large". We then show, under suitable conditions, that these subsets are of measure zero.

In the remaining part of this section we suppose that k is a fixed positive real number such that:

$$r(k) > \frac{\alpha(2\chi - 1)}{2(\chi - 1)} \tag{4.4}$$

where χ is given by Lemma 3.1. Notice that this is possible since the function r is positive and unbounded.

Let now l denote a positive real number. We define the following subsets of Ω:

$$A_l = \{x \in \Omega\,;\, u_k(x) > l\}\,,$$

$$\Omega_l = \{x \in \Omega\,;\, l < u_k(x) \leqslant l + 1\}\,.$$

Let, in addition, v_l and w_l denote the functions:

$$v_l = \max\,(u_k - l, 0)\,, \quad w_l = \min\,(v_l, 1)\,.$$

Here above, we omitted to index the sets A_l and Ω_l and the functions v_l and w_l with k for the sake of conciseness.

Notice that the functions $v_{l|A_l}$ and $w_{l|A_l}$ are elements of $W_0^{1,2}(A_l)$ and that, thanks to the continuity of u_k (*cf.* Remark 3.2), A_l is an open subset of $\mathbb{R}^2$.

Remark 4.3: If for $l > 0$ we have $|A_l| = 0$, then $\|u_k\|_{L^\infty(\Omega)} \leqslant l$.

The previous remark justifies the choice of the used method. Studying the sets A_l will enable us to evaluate the norm of u_k in $L^\infty(\Omega)$. In what follows, we first estimate the measure of the domains A_l in function of l independently of k. The next lemma allows to control the size of the domains A_l in function of l.

LEMMA 4.2: *Let l_0 denote a real positive number and $\phi : [l_0, \infty) \to [0, \infty)$ a nonincreasing function. We assume the existence of $\gamma \in (0, 1)$ such that:*

$$\forall l \in [l_0, \infty), \quad \phi(l+1) \leqslant \gamma \phi(l).$$

Then

$$\phi(l) \leqslant C_3 \gamma^l, \quad \text{with } C_3 = \phi(l_0)\left(\frac{1}{\gamma}\right)^{l_0+1}.$$

Proof: Let $l \geqslant l_0$ and $n \in \mathbb{N}$ such that $l = l_0 + n + \delta$, $0 \leqslant \delta < 1$. We have:

$$\phi(l) = \phi(l_0 + n + \delta) \leqslant \gamma^n \phi(l_0 + \delta) \leqslant \gamma^n \phi(l_0).$$

On the other hand we can write:

$$\gamma^n = \gamma^{l - l_0 - \delta} = \gamma^l \left(\frac{1}{\gamma}\right)^{l_0 + \delta} \leqslant \gamma^l \left(\frac{1}{\gamma}\right)^{l_0 + 1}.$$

We then deduce (4.5). □

Lemma 4.2 will lead us to a first estimate of the measure of A_l. The following result gives an estimate of the behaviour of the function $l \mapsto |A_l|$.

LEMMA 4.3: *Let (h_k, u_k) denote a solution of Problem (3.1)-(3.2) given by Theorem 3.1. Then there exist $\eta \in (0, 1)$ and $C_4 > 0$ such that:*

$$|A_l| \leqslant C_4 \eta^l \quad \forall l > 0.$$

The constants η and C_4 are independent of k and l. They depend on $|\Omega|$ and $|f|$.

Proof: Multiplying Equation (3.2) by w_l and integrating on the domain Ω, we obtain:

$$\int_{\Omega_l} \nabla u_k \cdot \nabla u_k \, dx = \frac{1}{2} \int_{A_l} r_k(u_k) \, |\nabla h_k|^2 \, w_l \, dx$$

$$\leqslant \frac{1}{2} \int_{A_l} r_k(u_k) \, |\nabla h_k|^2 \, dx .$$

Then

$$\int_{\Omega_l} |\nabla u_k|^2 \, dx \leqslant \frac{1}{2} \int_{A_l} r_k(u_k) \, |\nabla h_k|^2 \, dx . \tag{4.6}$$

Let $q \in (1, 2)$, we can write thanks to Hölder's inequality:

$$\left(\int_{\Omega_l} |\nabla u_k|^q \, dx \right)^{\frac{1}{q}} \leqslant |\Omega_l|^{\frac{1}{q} - \frac{1}{2}} \left(\int_{\Omega_l} |\nabla u_k|^2 \, dx \right)^{\frac{1}{2}} .$$

Let q^* denote the real number satisfying $1/q^* = 1/q - 1/2$. The previous inequality and the Sobolev's imbedding of $W^{1,q}(A_l)$ into $L^{q^*}(A_l)$ give:

$$\left(\int_{A_l} |w_l|^{q^*} \, dx \right)^{\frac{1}{q^*}} \leqslant C_{q^*} \left(\int_{\Omega_l} |\nabla u_k|^q \, dx \right)^{\frac{1}{q}}$$

$$\leqslant C_{q^*} |\Omega_l|^{\frac{1}{q} - \frac{1}{2}} \left(\int_{\Omega_l} |\nabla u_k|^2 \, dx \right)^{\frac{1}{2}} .$$

Noticing that $w_l = 1$ on A_{l+1} and using (4.6) and Lemma 4.1 we get

$$|A_{l+1}|^{\frac{1}{q^*}} \leqslant C_{q^*} |\Omega_l|^{\frac{1}{q} - \frac{1}{2}} \left(\int_{\Omega_l} |\nabla u_k|^2 \, dx \right)^{\frac{1}{2}}$$

$$\leqslant C_{q^*} |\Omega_l|^{\frac{1}{q} - \frac{1}{2}} \left(\int_{A_l} \frac{1}{2} r_k(u_k) \, |\nabla h_k|^2 \, dx \right)^{\frac{1}{2}}$$

$$\leqslant C_{q^*} |\Omega_l|^{\frac{1}{q} - \frac{1}{2}} \left(\frac{1}{2 \, \mu \omega} \int_{\Omega} |f|^2 \, dx \right)^{\frac{1}{2}} .$$

Therefore

$$|A_{l+1}| \leqslant |\Omega_l| \, C_q^{q^*}\left(\frac{1}{2\,\mu\omega}\int_\Omega |f|^2\,dx\right)^{\frac{q^*}{2}}.$$

The value of q can be arbitrarily fixed (e.g. $q=\frac{3}{2}$). We then deduce the existence of a constant $C>0$ independent of k and of l such that

$$|A_{l+1}| \leqslant C|\Omega_l| \leqslant C(|A_l| - |A_{l+1}|)\,.$$

Hence

$$|A_{l+1}| \leqslant \frac{C}{1+C}|A_l|\,.$$

Applying Lemma 4.2 to the function $\phi(l) = |A_l|$ we obtain the desired result with

$$\eta = \frac{C}{1+C}, \quad C_4 = |\Omega|\,\frac{C}{1+C},$$

the constants η and C_4 being independent of k and l. □

The previous estimates are not sufficient to determine a solution (h, u) of Problem (2.7)-(2.8) with u bounded. We shall now prove a further result using the sharp estimates obtained for the Meyers' theorem. For this we need the next lemma the proof of which is given in ([St], p. 93).

LEMMA 4.4: *Let l_0 denote a real positive number and $\phi : [l_0, \infty) \to [0, \infty)$ a nonincreasing function. We assume the existence of positive real numbers γ, C_5 and of $\beta > 1$ such that:*

$$\phi(t) \leqslant \frac{C_5}{(t-l)^\gamma}\phi(l)^\beta \quad \forall l, t, \quad l_0 \leqslant l < t\,. \tag{4.7}$$

Then there exists $d>0$ such that $\phi(l_0+d)=0$ with

$$d^\gamma = C_5\, 2^{\frac{\gamma\beta}{\beta-1}}\,\phi(l_0)^{\beta-1}\,. \tag{4.8}$$

We shall apply Lemma 4.4 to the function $\phi(l) = |A_l|$. We show that there exists $l_1 > 0$ such that $\phi(l_1) = 0$. From Remark 4.3 we have $\|u\|_{L^\infty(\Omega)} \leqslant l_1$.

LEMMA 4.5: *Let $\mathcal{O}$ a bounded open set of $\mathbb{R}^2$ and let $s > 2$. The imbedding of $W^{1,2}(\mathcal{O})$ into $L^s(\mathcal{O})$ is continuous with*

$$\|u\|_{L^s(\mathcal{O})} \leqslant \frac{s}{2}\,|\mathcal{O}|^{1/s}\,\|u\|_{W^{1,2}(\mathcal{O})} \qquad \forall u \in W^{1,2}(\mathcal{O})\,. \tag{4.9}$$

Proof: Let s^* such that $1/s^* = 1/s + 1/2$. We have $s^* < 2$. From (Brezis [Bre], p. 162) we have the following Sobolev inequality:

$$\|u\|_{L^s(\mathcal{O})} \leqslant C_{s^*}\|u\|_{W^{1,s^*}(\mathcal{O})}\,, \tag{4.10}$$

where $C_{s^*} \leqslant s^*/(2 - s^*)$. In [Ta], the optimal value of the constant involved in (4.10) is given. The value of s is given by $s = 2\,s^*/2 - s^*$. Hence

$$\begin{aligned}\|u\|_{L^s(\mathcal{O})} &\leqslant \frac{1}{2}\,s\,\|u\|_{W^{1,s^*}(\mathcal{O})}\\ &\leqslant \frac{1}{2}\,s\,|\mathcal{O}|^{\frac{1}{s^*}-\frac{1}{2}}\|u\|_{W^{1,2}(\mathcal{O})}\\ &\leqslant \frac{1}{2}\,s\,|\mathcal{O}|^{\frac{1}{s}}\|u\|_{W^{1,2}(\mathcal{O})}\,,\end{aligned}$$

whence (4.9). □

We now state the principal result of this section. We obtain an estimate of the measure of A_l, for given $l > 0$. This estimate takes into account the regularity result obtained from the Meyer's lemma.

LEMMA 4.6: *Let $l > 0$ and assume that function r satisfies hypotheses (2.9)-(2.11). Then, for $t > l$ and for all $q > 2$, we have:*

$$|A_t| \leqslant \left(\frac{C_6}{t-l}\,r(k)\,\frac{2\,pq}{p-2}\right)^q |A_l|^{1+\frac{q(p-2)}{p}}\,, \tag{4.11}$$

where p is chosen according to (3.7). In particular, p depends on k. The constants C_6 is independent of k, l and t.

Proof: Let g_k denote the function $\frac{1}{2}\,r_k(u_k)\,|\nabla h_k|^2$ and let $l \geqslant 0$. Multiplying Equation (3.2) by v_l and integrating over Ω we get

$$\int_{A_l}|\nabla u_k|^2\,dx \leqslant \int_{A_l} g_k\,v_l\,dx\,.$$

Lemma 3.1 implies that g_k is an element of $L^{p/2}(\Omega)$. Setting $s = \frac{2p}{2+p}$ we check that $1 < s < p/2$ since $p > 2$. If s' is the conjugate number of $s(1/s + 1/s' = 1)$, we can write using Lemma 4.5 and the Poincaré's inequality:

$$\int_{A_l} |\nabla u_k|^2 \, dx \leqslant \left(\int_{A_l} |g_k|^s \, dx\right)^{1/s} \left(\int_{A_l} |v_l|^{s'} \, dx\right)^{1/s'}$$

$$\leqslant \frac{1}{2} s' |A_l|^{\frac{1}{s} - \frac{2}{p}} \| v_l \|_{W_0^{1,2}(A_l)} |A_l|^{1/s'} \left(\int_{A_l} |g_k|^{p/2} \, dx\right)^{2/p}$$

$$\leqslant \frac{1}{2} s' |A_l|^{\frac{1}{s} - \frac{2}{p} + \frac{1}{s'}} S \left(\int_{A_l} |\nabla u_k|^2 \, dx\right)^{1/2} \left(\int_{A_l} |g_k|^{p/2} \, dx\right)^{2/p},$$

where S is the constant of the Poincaré's inequality. We obtain:

$$\left(\int_{A_l} |\nabla u_k|^2 \, dx\right)^{1/2} \leqslant \frac{1}{2} S s' |A_l|^{1 - \frac{2}{p}} \left(\int_{A_l} |g_k|^{p/2} \, dx\right)^{2/p}. \qquad (4.12)$$

Let now $q > 2$. Using again Lemma 4.5 we obtain:

$$\left(\int_{A_l} |v_l|^q \, dx\right)^{1/q} \leqslant \frac{1}{2} q |A_l|^{\frac{1}{q}} \| v_l \|_{W^{1,2}(A_l)}$$

$$\leqslant \frac{1}{2} q S |A_l|^{\frac{1}{q}} \left(\int_{A_l} |\nabla u_k|^2 \, dx\right)^{1/2}. \qquad (4.13)$$

Let $t > l$. We have $A_t \subset A_l$. In the domain A_t we have $v_l \geqslant t - l$. We then write:

$$(t-l)\,|A_t|^{\frac{1}{q}} \leqslant \left(\int_{A_t} |v_l|^q\,dx\right)^{1/q} \leqslant \left(\int_{A_l} |v_l|^q\,dx\right)^{1/q}.$$

From the previous estimate and from relationships (4.12), (4.13), we get:

$$|A_l|^{\frac{1}{q}}(t-l) \leqslant \frac{1}{4}\,q s' S^2 |A_l|^{1-\frac{2}{p}+\frac{1}{q}}\left(\int_{A_l} |g_k|^{p/2}\,dx\right)^{2/p}. \tag{4.14}$$

Since the function $r(t)$ is nondecreasing, we have $r_k(u_k) \leqslant r(k)$.

Consequently, we obtain from Estimate (3.8) the inequality:

$$\left(\int_{A_l} |g_k|^{p/2}\,dx\right)^{2/p} = \frac{1}{2}\left(\int_{A_l} r_k(u_k)^{p/2}|\nabla h_k|^p\,dx\right)^{2/p}$$

$$\leqslant \frac{1}{2}\,r(k)\left(\int_{\Omega} |\nabla h_k|^p\,dx\right)^{\frac{2}{p}}$$

$$\leqslant \frac{1}{2}\,r(k)\,\frac{C_0^2}{\alpha^2}\,|f|^2.$$

Since $s' = 2p/(p-2)$, we obtain using inequality (4.14):

$$|A_t| \leqslant |A_l|^{1+\frac{(p-2)q}{p}}\left(\frac{S^2\,C_0^2|f|^2}{8\,\alpha^2(t-l)}\right)^q r^q(k)\left(\frac{2pq}{p-2}\right)^q.$$

We then deduce (4.11) with

$$C_6 = \frac{S^2\,C_0^2|f|^2}{8\,\alpha^2},$$

the constant C_6 being independent of k, l and t. □

5. EXISTENCE OF A BOUNDED SOLUTION

This section is devoted to the proof of the existence of a solution (h, u) of Problem (2.7)-(2.8) with u bounded. As we have mentioned in the previous section (Remark 4.1), it is sufficient to show that there exists $K > 0$ such that $\|u_K\|_{L^\infty(\Omega)} \leqslant K$.

THEOREM 5.1: *Let k be a given real number satisfying (4.4). There exist constants $C_7 > 0$ and $\eta \in (0, 1)$ independent of k such that the following inequality holds:*

$$\|u_k\|_{L^\infty(\Omega)} \leqslant \frac{k}{2} + C_7\left(\frac{p}{p-2}\right)^2 r(k)\, \eta^{\frac{k}{2}\frac{p-2}{p}}, \tag{5.1}$$

where p is given by (3.7).

Proof: Set $l_0 = k/2$. Relationship (4.11) satisfies Hypothesis (4.7) with $\phi(l) = |A_l|$. We deduce that $|A_l| = 0$ for $l = l_0 + d$ where d is given by (4.8). Wet get:

$$\|u_k\|_{L^\infty(\Omega)} \leqslant \frac{k}{2} + d\,.$$

Using Lemma 4.4, Estimate (4.8) gives with $\gamma = q$, $\beta = 1 + \dfrac{q(p-2)}{p}$, $C_5 = C_6^q\, r^q(k)\, (2pq/p-2)^q$:

$$d^q = 2^{\frac{q\beta}{\beta-1}}\, C_6^q\left(\frac{2pq}{p-2}\right)^q r^q(k)\, |A_{\frac{k}{2}}|^{\frac{q(p-2)}{p}}\,.$$

We have:

$$d = 2^{\frac{\beta}{\beta-1}}\, C_6\, \frac{2pq}{p-2}\, r(k)\, |A_{\frac{k}{2}}|^{\frac{p-2}{p}}\,.$$

If we take, in Lemma 4.6, $q = \dfrac{p}{p-2}$, we find $\beta = 2$. We then deduce the following:

$$d = 8\, C_6\left(\frac{p}{p-2}\right)^2 r(k)\, |A_{\frac{k}{2}}|^{\frac{p-2}{p}}\,.$$

Therefore

$$\|u_k\|_{L^\infty(\Omega)} \leqslant \frac{k}{2} + 8\, C_6\left(\frac{p}{p-2}\right)^2 r(k)\, |A_{\frac{k}{2}}|^{\frac{p-2}{p}}\,.$$

From Theorem 4.1, there exist $\eta \in (0,1)$ and $C_4 > 0$ such that:

$$|A_{\frac{k}{2}}| \leqslant C_4 \eta^{\frac{k}{2}},$$

where the constants η and C_4 are independent of k. We get:

$$\|u_k\|_{L^\infty(\Omega)} \leqslant \frac{k}{2} + C_7\left(\frac{p}{p-2}\right)^2 r(k)\, \eta^{\frac{k}{2}\frac{p-2}{p}}.$$

We thus have the desired estimate. □

The next theorem gives the main result. Its corollary will be more interesting from a practical point of view.

THEOREM 5.2: *Let r denote a function satisfying hypotheses (2.9)-(2.11). We assume furthermore that r satisfies the condition:*

$$\lim_{k\to\infty}\left(\frac{\ln\chi}{\ln\left(1+\frac{\alpha}{2\,r(k)}\right)}\right)^2 r(k)\exp\left(\ln(\eta)\frac{k\ln\left(1+\frac{\alpha}{2\,r(k)}\right)}{4\ln\chi}\right) = 0\,. \quad (5.2)$$

Then, there exists $p>2$ for which Problem (2.7)-(2.8) has a solution (h,u) in $W_0^{1,p}(\Omega)\times(W_0^{1,p/2}(\Omega)\cap W^{2,p/2}(\Omega))$, thus $u\in L^\infty(\Omega)$. The constants χ and η are given by Lemmas 3.1 and 4.1. In particular, we have $\eta\in(0,1)$ and $\chi>1$.

Proof: Relationship (3.7) gives:

$$p = \frac{4\ln\chi}{2\ln\chi - \ln\left(\frac{2\,r(k)-\alpha}{2\,r(k)-2\,\alpha}\right)}.$$

whence

$$\frac{p}{p-2} = \frac{2\ln\chi}{\ln\left(\frac{2\,r(k)-\alpha}{2\,r(k)-2\,\alpha}\right)} = \frac{2\ln\chi}{\ln\left(1+\frac{\alpha}{2\,r(k)-2\,\alpha}\right)} \leqslant \frac{2\ln\chi}{\ln\left(1+\frac{\alpha}{2\,r(k)}\right)}.$$

Using (5.1), we obtain:

$$\|u_k\|_{L^\infty(\Omega)} \leqslant \frac{k}{2} + C_7\left(\frac{2\ln\chi}{\ln\left(1+\frac{\alpha}{2\,r(k)}\right)}\right)^2 r(k)\exp\left(\ln(\eta)\frac{k\ln\left(1+\frac{\alpha}{2\,r(k)}\right)}{4\ln\chi}\right).$$

If r satisfies the condition:

$$\lim_{k \to \infty} \left(\frac{2 \ln \chi}{\ln \left(1 + \frac{\alpha}{2 r(k)} \right)} \right)^2 r(k) \exp\left(\ln (\eta) \frac{k \ln \left(1 + \frac{\alpha}{2 r(k)} \right)}{4 \ln \chi} \right) = 0 ,$$

then there exists a constant K that fulfills relationship (4.4) such that $\| u_K \|_{L^\infty(\Omega)} \leqslant K$. Applying Remark 4.1, we conclude that (h_K, u_K) is solution of Problem (2.7)-(2.8) with u_K bounded. Applying Lemma 3.1, we obtain a solution $(h, u) = (h_K, u_K)$ in $W_0^{1,p}(\Omega) \times (W_0^{1,p/2}(\Omega) \cap W^{2,p/2}(\Omega))$ for a $p > 2$. The value of p depends on α and on $r(K)$. □

The following consequence of the above result gives an information on the behaviour at the infinity of functions satisfying condition (5.2).

COROLLARY 5.1: *Let r denote a function satisfying Hypotheses (2.9)-(2.11). If there exists $\zeta \in (0, 1)$ such that:*

$$\lim_{k \to \infty} r(k)\, k^{-\zeta} < \infty , \tag{5.3}$$

then r satisfies condition (5.2).

Proof: Using a development of $\ln(1 + \alpha / 2\, r(k))$ at first order we obtain the following equivalence for $k \to \infty$:

$$\left(\frac{1}{\ln \left(1 + \frac{\alpha}{2 r(k)} \right)} \right)^2 \exp\left(\ln (\eta) \frac{k \ln \left(1 + \frac{\alpha}{r(k)} \right)}{4 \ln \chi} \right) \sim 4 \left(\frac{r(k)}{\alpha} \right)^2 \exp\left(\ln (\eta) \frac{k\alpha}{4\, r(k) \ln \chi} \right) .$$

From (5.3) we deduce the existence of a constant C_8 such that for k large enough, we have:

$$\left(\frac{\ln \chi}{\ln \left(1 + \frac{\alpha}{2 r(k)} \right)} \right)^2 r(k) \exp\left(\ln (\eta) \frac{k \ln \left(1 + \frac{\alpha}{r(k)} \right)}{4 \ln \chi} \right) \leqslant C_8\, k^{3\zeta} \exp(\ln (\eta)\, C_8\, k^{1-\zeta}) .$$

The value of $\ln(\eta)$ is negative since $\eta \in (0, 1)$. Consequently condition (5.2) is satisfied. □

Clearly this result does not give the existence for $\zeta = 1$. This is due to the roughness of the estimate of p given by (3.7). It remains an open problem to show the existence for $\zeta \geq 1$.

Interpreting Relationship (5.3) with respect to the temperature, the condition (5.2) becomes: for all $\zeta \in [0, 1)$ there exists θ_0 and $C_\zeta > 0$ such that

$$\rho(\theta) \leq C_\zeta \left(\int_0^\theta \lambda(s)\, ds \right)^\zeta$$

for $\theta \geq \theta_0$. In this case, Hypothesis (5.2) is satisfied.

The monotonicity condition imposed to the function r is not necessary. It simplifies the proof. The following lemma will enable us to avoid this constraint.

LEMMA 5.1: *Let r denote a continuous function that satisfies the following conditions:*

$$\lim_{t \to \infty} r(t) = \infty\,, \tag{5.4}$$

Then, there exists a sequence of real nonnegative numbers $(k_i)_i$ such that:

$$\lim_{i \to \infty} k_i = +\infty\,, \tag{5.5}$$

$$\sup_{t \leq k_i} r(t) = r(k_i)\,, \quad \forall i \in \mathbb{N}\,. \tag{5.6}$$

Proof: We define the function:

$$\hat{r}(t) = \sup_{s \in [0, t]} r(s)\,,$$

and the set $S = \{t \geq 0\,; \hat{r}(t) = r(t)\}$. Notice that S is closed since the functions $\hat{r}(t)$ and $r(t)$ are continuous. Furthermore, the function $\hat{r}$ is non-decreasing. We shall first prove that S is not bounded.

Assume that S is bounded and let M denote its upper bound. We have $\hat{r}(M) = r(M)$ since S is closed and

$$\hat{r}(t) > r(t)\,,\quad \forall t > M\,.$$

We shall prove that $\hat{r}(t) = \hat{r}(M)$ for $t \geq M$. We have, by definition of M, for $t \geq M$:

$$\hat{r}(t) = \sup_{s \in [0, t]} r(s) = \sup_{s \in [M, t]} r(s)\,.$$

The continuity of r implies that, for given $t > M$, there exists $t_1 \in [M, t]$ such that $\hat{r}(t) = r(t_1)$. If $t_1 > M$ then $\hat{r}(t) = r(t_1) < \hat{r}(t_1)$ since M is the upper bound of S. This is contradictory with the monotonicity of $\hat{r}$. Consequently, we have $t_1 = M$ and $\hat{r}(t) = \hat{r}(M)$ for all $t \geqslant M$. This implies that the function r is bounded by $r(M)$ which is contradictory with Hypothesis (5.4).

The set S is therefore not bounded. We can take a nondecreasing sequence of points $(k_i)_i$ of S satisfying the property (5.5). The monotonicity of $\hat{r}$ implies that the sequence $r(k_i)_i$ is nondecreasing. From (5.4) we deduce that its limit is $+\infty$. By construction, this sequence is therefore convenient. □

It is then sufficient to take in the proofs of the preceding sections the values of the sequence $(k_i)_i$. The properties (5.5) and (5.6) replace the hypothesis of monotonicity of r.

APPENDIX

We give here a proof of Lemma 3.1.

Let X_p stand for the Banach space $W_0^{1,p}(\Omega)$ equipped with the norm:

$$\| u \|_{X_p} = \left(\int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{p}},$$

the set Ω begin here a domain of $\mathbb{R}^n$. Let Y_p denote the Banach space $W^{-1,p}(\Omega)$ equipped with the norm:

$$\| u \|_{Y_p} = \min \left\{ \| g \|_{(L^p(\Omega))^n} \text{ such that } g \in (L^p(\Omega))^n \text{ with } \nabla . g = u \right\}.$$

Let $a \in L^\infty(\Omega)$ with

$$0 < \alpha \leqslant a(x) \leqslant \beta < \infty \quad \text{a.e. } x \in \Omega .$$

We consider the following elliptic operator:

$$Au = \frac{1}{\beta} Lu = -\nabla \cdot \left(\frac{a}{\beta} \nabla u \right).$$

Equation (3.4) can be written in the form:

$$Au = (A + \Delta)\, u - \Delta u = g \overset{\text{def}}{=} \frac{1}{\beta} f, \tag{6.1}$$

Clearly, the operator $-\Delta$ is an isomorphism from X_p onto Y_p for all $p \in (1, \infty)$ (*cf.* [Si]). Let G_p denote its inverse defined from Y_p onto X_p. We have:

$$(G_p(A + \Delta) + I)\, u = G_p\, g ,$$

where I is the identity operator on X_p.

It is clear that if for a given p:

$$\| G_p(A + \Delta) \|_{\mathscr{L}(X_p, X_p)} < 1 , \tag{6.2}$$

then the operator $A + \Delta$ is a one-to-one mapping from X_p onto Y_p. Since Δ is an isomorphism from X_p onto Y_p we can deduce that A is also an isomorphism from X_p onto Y_p. Our proof will then consist in determining the values of p for which inequality (6.2) holds.

Given $u \in X_p$, we have:

$$\begin{aligned} \| (A + \Delta) u \|_{Y_p} &= \left\| \nabla \cdot \left(1 - \frac{a}{\beta} \right) \nabla u \right\|_{Y_p} \\ &\leqslant \left\| \left(1 - \frac{a}{\beta} \right) \nabla u \right\|_{L^p(\Omega)} \\ &\leqslant \sup_{x \in \Omega} \left| \frac{\beta - a(x)}{\beta} \right| \| \nabla u \|_{L^p(\Omega)} \\ &\leqslant \frac{\beta - \alpha}{\beta} \| u \|_{X_p} . \end{aligned}$$

Therefore:

$$\| A + \Delta \|_{\mathscr{L}(X_p, Y_p)} \leqslant \frac{\beta - \alpha}{\beta} . \tag{6.3}$$

Let now h denote an element of $(L^p(\Omega))^n$. We can associate to h the function $v \in X_p$ satisfying $-\Delta v = \nabla . h$. This defines a linear and continuous mapping:

$$\pi : h \in (L^p(\Omega))^n \mapsto \pi(h) = \nabla v \in (L^p(\Omega))^n .$$

Consequently, there exists a constant $C_p > 0$ such that

$$\| \nabla v \|_{(L^p(\Omega))^n} \leqslant C_p \| h \|_{(L^p(\Omega))^n} .$$

From the definition of the norms of the spaces X_p and Y_p, the previous inequality implies that

$$\| G_p \|_{\mathscr{L}(Y_p, X_p)} \leqslant C_p$$

with $C_2 = 1$. Choosing the value $p = 4$ we obtain:

$$\| \nabla v \|_{(L^4(\Omega))^n} \leqslant \chi \| h \|_{(L^4(\Omega))^n} ,$$

where $\chi > 1$. Notice that the choice $p = 4$ is arbitrary but avoids technical difficulties. More generally on can choose any $p_0 > 2$.

The mapping $\pi : L^4(\Omega) \to L^4(\Omega)$ is linear and continuous. It is also continuous from $L^2(\Omega)$ to itself. From an interpolation theorem (*cf.* [Bre], p. 128) we deduce that L is a linear and continuous mapping from $L^p(\Omega)$ to itself with

$$\frac{1}{p} = \frac{\theta}{4} + \frac{1-\theta}{2}, \quad \theta \in [0, 1].$$

Moreover, the norm C_p of the operator π in $L^p(\Omega)$ satisfies the following inequality:

$$C_p \leqslant \chi^{\theta}.$$

We find $\theta = 2(p-2)/p$. Therefore, using (6.3) we obtain an estimate of the norm of the operator $G_p(A + \Delta)$, that is:

$$\| G_p(A + \Delta) \|_{\mathscr{L}(X_p, X_p)} \leqslant \chi^{\frac{2(p-2)}{p}} \frac{\beta - \alpha}{\beta}.$$

We then choose the value of p such that the norm of the operator $G_p(A + \Delta)$ is bounded by $\frac{2\beta - \alpha}{2\beta}$, *i.e.*,

$$\chi^{\frac{2(p-2)}{p}} \frac{\beta - \alpha}{\beta} = \frac{2\beta - \alpha}{2\beta}.$$

For $p \in [2, 4]$, we have $\dfrac{2(p-2)}{p} \in [0, 1]$. Whence:

$$1 \leqslant \chi^{\frac{2(p-2)}{p}} \leqslant \chi.$$

Consequently a choice of p is possible if β satisfies the condition

$$\chi \geqslant \frac{2\beta - \alpha}{2\beta - 2\alpha},$$

i.e.

$$\beta \geqslant \frac{\alpha(2\chi - 1)}{2(\chi - 1)}.$$

We can then evaluate p in function of α and β. We find:

$$p = \frac{4 \ln \chi}{2 \ln \chi - \ln\left(\dfrac{2\beta - \alpha}{2\beta - 2\alpha}\right)}.$$

Hence:

$$\| G_p(A + \Delta) \|_{\mathscr{L}(X_p, X_p)} \leqslant \frac{2\beta - \alpha}{2\beta} < 1 .$$

The mapping $G_p(A + \Delta) + I$ is then invertible and the norm of $(I + G_p(A + \Delta))^{-1}$ is given by

$$\| (I + G_p(A + \Delta))^{-1} \|_{\mathscr{L}(X_p, X_p)} \leqslant \frac{1}{1 - \frac{2\beta - \alpha}{2\beta}} \leqslant \frac{2\beta}{\alpha} .$$

We deduce then that $G_p(A + \Delta)$ is invertible and consequently A is an isomorphism from X_p onto Y_p and

$$\| (G_p A)^{-1} \|_{\mathscr{L}(X_p, X_p)} = \| A^{-1} \Delta \|_{\mathscr{L}(X_p, X_p)} \leqslant \frac{2\beta}{\alpha} .$$

Using the relation:

$$\| A^{-1} \|_{\mathscr{L}(Y_p, X_p)} = \| A^{-1} \Delta G_p \|_{\mathscr{L}(Y_p, X_p)} \leqslant \| A^{-1} \Delta \|_{\mathscr{L}(X_p, X_p)} \| G_p \|_{\mathscr{L}(Y_p, X_p)} ,$$

we obtain the bound:

$$\| A^{-1} \|_{\mathscr{L}(Y_p, X_p)} \leqslant C_p \frac{2\beta}{\alpha} \leqslant \chi \frac{2\beta}{\alpha} .$$

Then, for each $g \in Y_p$, Problem (6.1) has a unique solution u in X_p and we have the estimate:

$$\begin{aligned} \| \nabla u \|_{(L^p(\Omega))^n} &\leqslant \chi \frac{2\beta}{\alpha} \| g \|_{W^{-1,p}(\Omega)} , \\ &\leqslant \frac{2\chi}{\alpha} \| f \|_{W^{-1,p}(\Omega)} . \end{aligned}$$

Using the Poincaré's inequality we have:

$$\| u \|_{W_0^{1,p}(\Omega)} \leqslant S \| \nabla u \|_{(L^p(\Omega))^n} \leqslant \frac{2 S\chi}{\alpha} \| f \|_{W^{-1,p}(\Omega)} ,$$

which yields the desired result. □

REFERENCES

[BLP] A. BENSOUSSAN, J. L. LIONS, G. PAPANICOLAOU, 1978, *Asymptotic Analysis for Periodic Structures,* North Holland.

[BoGa] L. BOCCARDO, T. GALLOUËT, 1989, *Nonlinear Elliptic and Parabolic Equations Involving Measure Data,* Journal of Functional Analysis, **87, No 1**.

[Bre] H. BREZIS, 1987, *Analyse fonctionnelle,* Masson, Paris.

[Cl] S. CLAIN, 1994, *Analyse mathématique et numérique d'un modèle de chauffage par induction,* PhD Thesis, EPFL, Lausanne.

[ClTo] S. CLAIN, R. TOUZANI, *A two-Dimensional Stationary Induction Heating Problem,* Math. Meth. Appl. Sci., 1997, **20,** 759-766.

[CRST] S. CLAIN, J. RAPPAZ, M. SWIERKOSZ, R. TOUZANI, 1993, *Numerical Modelling of Induction Heating for Two-Dimensional Geometries,* Math. Mod. Meth. Appl. Sci. **3, no 6,** 905-822.

[GaHe] T. GALLOUËT, R. HERBIN, 1994, *Existence of a solution to a Coupled Elliptic System* **7, No 2,** Appl. Math. Lett., 49-55.

[GiTr] D. GILBARG, N. TRUDINGER, 1977, *Ellipic Partial Differential Equations of Second Order,* Springer Verlag.

[Lew] R. LEWANDOWSKI, *The Mathematical Analysis of a Coupling of a Turbulent Kinetic Energy Equation to the Bavier-Stokes Equation with an Addy Viscosity,* Paper in preparation.

[Mu] F. MURAT, 1994, Private Communication.

[Si] C. G. SIMADER, 1972. *On Dirichlet's Boundary Value Problem,* vol. 268, Lecture Notes in Mathematics, Springer Verlag.

[St] G. STAMPACCHIA, 1966, *Equations elliptiques du second ordre à coefficients discontinus,* Presses Universitaires de Montréal.

[Ta] G. TALENTI, *Best Constants in Sobolev inequality,* vol. 110, Ann. Mat. Pura Appl., 1976.

[Ve] V. VESPRI, *Semigroup Theory and Application,* Lecture notes in pure and applied mathematics, vol. 116, 1989.