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FADING MEMORY EFFECTS IN ELASTIC-VISCOELASTIC COMPOSITES (*)

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Abstract — *We study the macroscopic behavior of an elastic-viscoelastic mixture, in the non periodic case The oscillating problem presents an elastic term and a short memory term At the macroscopic level a long memory term, of the convolution type, also appears The main result consists in establishing the limit equation and characterizing its coefficients in terms of the oscillations of the domains occupied by each component of the mixture*

Key words Homogenization, elasticity, viscoelasticity, memory effect

Résumé — *On étudie le comportement macroscopique d'un mélange élastique-viscoélastique Le problème oscillant présente un terme élastique et un terme à mémoire courte Au niveau macroscopique un terme à mémoire longue apparaît aussi Le résultat principal consiste à établir l'équation limite et à caractériser ses coefficients en fonction des oscillations des domaines occupés par les composantes du mélange*

1. INTRODUCTION

This work is concerned with the global behavior of a composite formed by a fine mixture of an elastic material with a viscoelastic one.

The study of a global behavior is relevant in structural Engineering, when dealing with polymer based composites, in Biomechanics or even in food industries, where elastic and viscoelastic materials, or tissues, are often to be found together.

The macroscopic behavior of heterogeneous materials is the main goal of the mathematical method known as homogenization. The problem to solve is to find the constitutive laws satisfied by a composite material, when the microscopic structure and constitutive laws of each constituent are known.

When dealing with linearized elasticity the homogenization theory characterizes the macroscopic behavior of a mixture as a new elastic material, where the elastic coefficients depend, sometimes in a very implicit way, on the

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coefficients of the initial components and on their oscillations. Although we obtain a new elastic material, we remain in the same class of elastic materials. These results are classic and homogenization theory provides the asymptotic expansion method for periodic oscillations (see [1], [6], [9], [14]), and the H-convergence for the non periodic case (see [13], [16]).

Something quite different happens when we mix different kinds of viscoelastic materials. In a linear setting, a differential equation with an integral term (fading memory effect) may arise from the homogenization of an equation with a purely differential structure. The macroscopic behavior of a Kelvin-Voigt material is such an example. The integral term is, very often, of the convolution type and its corresponding kernel is entirely characterized by the mixture components and its oscillations. These characterizations were performed in [14], [7], in the periodic case, and in [10], in the non periodic case.

In this paper we study the macroscopic behavior of an elastic-viscoelastic mixture, in the non periodic case; this problem is still untreated in the literature, at our knowledge. We also recover the periodic case characterization but in a more detailed version than the one presented in [14].

From a mechanical point of view, when we deal with an elastic-viscoelastic mixture as, for instance, a fiber-reinforced polymer, the viscoelastic character of the matrix plays also an important part. The macroscopic behavior of such a composite also depends, apart from the microscopic coefficients and the geometric structure, on the history, or memory, of the strain. The mathematical difficulty in studying such an elastic-viscoelastic mixture arises from the fact that the two materials have different behaviors at the microscale, which prevent us from applying the standard techniques. In fact the operator corresponding to the viscoelastic material degenerates.

Specifically, we consider a bounded viscoelastic medium Ω with elastic inclusions T^ε. The balance and constitutive equations are the following:

$$\rho \ddot{u}_i^\varepsilon = \frac{\partial \sigma_{ij}^\varepsilon}{\partial x_j} + \rho f_i,$$

$$\sigma_{ij}^\varepsilon = [a_{ijkl}^1\, e_{kl}(u^\varepsilon) + b_{ijkl}^1\, e_{kl}(\dot{u}^\varepsilon)]\chi^\varepsilon + a_{ijkl}^0\, e_{kl}(u^\varepsilon)\,(1-\chi^\varepsilon)\,,$$

where χ^ε is the characteristic function of the viscoelastic medium $\Omega \backslash T^\varepsilon$. The parameter ε represents, as ε tends to zero, the refinement of the mixture. Quantities a_{ijkl}^1 and a_{ijkl}^0 represent the elastic coefficients of the viscoelastic and elastic parts, respectively, and b_{ijkl}^1 the short memory coefficients of the viscoelastic material. The scalar term ρ stands for the density and f_i for the components of the given body forces. We note that it is, in fact, the characteristic function χ^ε which represents the oscillations of the problem.

The main result (Theorem 3.2) consists in establishing that the limit problem, in the whole of Ω, as ε goes to zero, is the following:

$$\rho \ddot{u}_i = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i,$$

$$\sigma_{ij} = a_{ijkl}\, e(u) + b_{ijkl}\, e(\dot{u}) + \int_0^t k_{ijkl}(t-s)\, e_{kl}(\dot{u}(s))\, ds\,,$$

and to characterize the tensors k, a and b in terms of the initial tensors a^1, a^0 and b^1, and of the oscillating sequence χ^{ε}. We are especially interested in non periodic oscillations of the function χ^{ε}.

The method used is, essentially, the H-convergence. We solve the problem by introducing a perturbation coefficient δ in the elastic part, in order to reduce the problem to the known setting of a viscoelastic mixture (see [10]). Passing to the limit in the perturbation δ, we prove that the homogenized medium is a viscoelastic one, with a long memory term where the effect of the micro-structure is visible. Some relevant examples are presented.

For the sake of simplicity we consider here a scalar version of the elastic-viscoelastic problem; we emphasize that it can be extended to the elastic vectorial case with no major problems than the price of a heavy notation.

We now present a brief summary of the paper. In Section 2, we give the general setting of the problem for a composite formed by a viscoelastic matrix with elastic inclusions. After that we introduce a perturbed problem by adding a small viscosity term in the inclusions. Thus the problem reduces to the known case of a viscoelastic composite (see [10]).

In Section 3 we obtain the homogenized problem for the elastic-viscoelastic composite, by passing to the limit when the additional viscosity term tends to zero in the homogenized system satisfied by the viscoelastic composite. The long memory term is still present in the limit problem.

In Section 4 we study some relevant examples, where explicit formulas for the homogenized materials may be derived: the periodic media, the quasi-periodic media and the layered media.

In the Appendix we recall some classical results concerning H-convergence, also adapted here to the case of complex matrices.

2. THE ELASTIC-VISCOELASTIC PERTURBED PROBLEM

2.1. The general setting of the problem

As we referred in the Introduction, we study the problem of a mixture occupying a bounded domain Ω in $\mathbb{R}^N$, formed by a viscoelastic matrix with elastic inclusions occupying a measurable subset T^ε of Ω. We recall the balance and constitutive equations:

$$\rho \ddot{u}_i^\varepsilon = \frac{\partial \sigma_{ij}^\varepsilon}{\partial x_j} + \rho f_i, \tag{2.1}$$

$$\sigma_{ij}^\varepsilon = [a_{ijkl}^1 e_{kl}(u^\varepsilon) + b_{ijkl}^1 e_{kl}(\dot{u}^\varepsilon)] \chi^\varepsilon + a_{ijkl}^0 e_{kl}(u^\varepsilon)(1 - \chi^\varepsilon), \tag{2.2}$$

where χ^ε is the characteristic function of the viscoelastic medium $\Omega^\varepsilon = \Omega \backslash T^\varepsilon$. As usual $(\sigma_{ij}^\varepsilon)$ represents the stress tensor, u^ε the displacement tensor and $e_{kl}(u^\varepsilon)$ the strain tensor; a_{ijkl}^1 and a_{ijkl}^0 represent the elastic coefficients of the viscoelastic and elastic parts, respectively, and b_{ijkl} the short memory coefficients of the viscoelastic material. The scalar term ρ stands for the density and f_i for the components of the given body forces.

We are interested in the asymptotic behavior of equations (2.1)-(2.2), as ε tends to zero.

The tensors $A^1 = (a_{ijkl}^1)$, $A^0 = (a_{ijkl}^0)$ and $B^1 = (b_{ijkl}^1)$ satisfy the following coercivity and symmetry conditions, for α and β in $\mathbb{R}^+$,

$$\alpha \xi_{ij} \leqslant a_{ijkl}^1 \xi_{ij} \xi_{kl} \leqslant \beta \xi_{ij} ; \alpha \xi_{ij} \leqslant a_{ijkl}^0 \xi_{ij} \xi_{kl} \leqslant \beta \xi_{ij} ;$$

$$\alpha \xi_{ij} \leqslant b_{ijkl}^1 \xi_{ij} \xi_{kl} \leqslant \beta \xi_{ij} ;$$

$$a_{ijkl}^1 = a_{klij}^1 = a_{jikl}^1 ; a_{ijkl}^0 = a_{klij}^0 = a_{jikl}^0 ; b_{ijkl}^1 = b_{klij}^1 = b_{jikl}^1 .$$

When the inclusions are also viscoelastic, i.e. the tensor B is defined and coercive in the inclusions T^ε, the problem of a periodic mixture was studied in the general framework of homogenization by Sanchez-Palencia [14] and by Francfort and Suquet [7]. These results were extended to the general non periodic case by Mascarenhas [10]. The main feature in the studies is the presence of a fading memory term in the limit, or macroscopic, constitutive equation. In the present problem, the difficulty comes from the fact that the viscosity tensor degenerates in the inclusions.

For the reader's convenience we treat the corresponding scalar version of problems (2.1) and (2.2) minding that the extension to an elastic-viscoelastic mixture follows essentially the same steps. We consider the problem in the general case of a non periodic mixture.

We consider, instead of problems (2.1) and (2.2), the following scalar case:

$$\begin{cases} \ddot{u}_\varepsilon - \operatorname{div}(A^\varepsilon \nabla u^\varepsilon + B^\varepsilon \nabla \dot{u}^\varepsilon) = f & \text{in } \Omega \times [0, +\infty] \\ u^\varepsilon(0) = 0, \quad \dot{u}^\varepsilon(0) = 0 & \text{in } \Omega , \\ u^\varepsilon = 0 \quad \text{on } \partial\Omega \times [0, +\infty[, \end{cases} \tag{2.3}$$

with

$$\text{(2.4)} \qquad \begin{cases} A^{\varepsilon} = A^1 \chi^{\varepsilon} + A^0(1 - \chi^{\varepsilon}) \\ B^{\varepsilon} = B^1 \chi^{\varepsilon}, \end{cases}$$

where the symmetric matrices $A^1 = (a^1_{ij})$, $A^0 = (a^0_{ij})$ and $B^1 = (b^1_{ij})$ satisfy the following coercivity conditions, for α and β in $\mathbb{R}^+$,

$$\text{(2.5)} \qquad \alpha|\xi|^2 \leqslant a^1_{ij}\xi_i\xi_j \leqslant \beta|\xi|^2 ; \alpha|\xi|^2 \leqslant a^0_{ij}\xi_i\xi_j \leqslant \beta|\xi|^2;$$

$$\alpha|\xi|^2 \leqslant b^1_{ij}\xi_i\xi_j \leqslant \beta\,|\xi|^2.$$

This means that, in the equilibrium equation

$$\ddot{u}^{\varepsilon} - \operatorname{div} \sigma^{\varepsilon} = f,$$

the stress σ^{ε} is given by $\sigma^{\varepsilon} = A^1\,\nabla u^{\varepsilon} + B^1\,\nabla \dot{u}^{\varepsilon}$ in Ω^{ε}, and by $\sigma^{\varepsilon} = A^0\,\nabla u^{\varepsilon}$ in T^{ε}, i.e., we deal with a scalar version of a mixture of an elastic material, occupying T^{ε}, with a Kelvin-Voigt viscoelastic material, occupying Ω^{ε}.

The existence and uniqueness of a solution of problem (2.3)-(2.4) is classical: the following proposition is an immediate consequence of [15], Ch. VI, Thm. 2D and of standard *a priori* estimates.

PROPOSITION 2.1: *If the coercivity conditions (2.5) are satisfied, if f belongs to $C^1([0, +\infty[\,; L^{\infty}(\Omega))$ and is bounded in $[0, +\infty[$, then there exists a unique solution of problem (2.3)-(2.4), satisfying*

$$u^{\varepsilon} \in C([0, +\infty[\,; H^1_0(\Omega)) \cap C^1(\,]0, +\infty[; H^1_0(\Omega))$$

$$\cap\, C^1([0, +\infty[\,; L^2(\Omega)) \cap C^2(\,]0, +\infty[\,; L^2(\Omega))$$

and, for a positive constant C.

$$\text{(2.6)} \qquad \|\nabla u^{\varepsilon}(t)\|_{L^2(\Omega)} \leqslant C, \quad \|\dot{u}^{\varepsilon}(t)\|_{L^2(\Omega)} \leqslant C, \quad \forall t > 0.$$

Define, for each $\lambda \in \mathbb{C}$, $\Re\lambda > 0$, the matrix

$$\text{(2.7)} \qquad C^{\varepsilon}(\lambda) := A^{\varepsilon} + \lambda B^{\varepsilon}.$$

In view of the symmetry of A^{ε} and B^{ε} and of the ellipticity conditions (2.5) we see that $C^{\varepsilon}(\lambda)$ is a sequence in the space $M(\alpha, (1 + |\lambda|)\beta\,; \Omega\,; \mathbb{C})$, introduced in the Appendix. By the definition of H-convergence and by the compactness Theorem A.2, also stated in the Appendix, we may guarantee, for each $\lambda \in \mathbb{C}$, $\Re\lambda > 0$, the existence of a subsequence ε' of ε, depending on λ, and of a matrix $C(\lambda) \in M(\alpha, (1 + |\lambda|)\beta ; \Omega ; \mathbb{C})$, such that

$$C^{\varepsilon'}(\lambda) \xrightarrow{H} C(\lambda) . \tag{2.8}$$

By a diagonalization process, we can prove the existence of a subsequence ε' such that (2.8) holds for a countable dense subset of $\{\lambda \in \mathbb{C} : \Re\lambda > 0\}$ and, using Theorem A.4, we obtain the same convergence for all $\lambda \in \mathbb{C}$, $\Re\lambda > 0$ (for details see the proof of Proposition 2.2). Then, with no loss of generality, we can suppose that (2.8) holds for the whole sequence ε and for all $\lambda \in \mathbb{C}$, $\Re\lambda > 0$; so, the matrix $C(\lambda)$ is well defined as the H-limit of $C^{\varepsilon}(\lambda)$, that we suppose to exist.

Since the solutions u^{ε} of problem (2.3)-(2.4) satisfy estimates (2.6), using the Laplace transform we obtain:

$$\lambda^2 \widehat{u^{\varepsilon}}(\lambda) - \operatorname{div}\left[(A^{\varepsilon} + \lambda B^{\varepsilon})\, \nabla \widehat{u^{\varepsilon}}\right] = \hat{f}(\lambda) ,$$

as well as, up to a subsequence, the following weak convergences, as ε goes to zero and for all $t > 0$,

$$u^{\varepsilon}(t) \rightharpoonup u(t) \quad \text{in } H_0^1(\Omega), \quad \text{and} \quad \dot{u}^{\varepsilon}(t) \rightharpoonup \dot{u}(t) \quad \text{in } L^2(\Omega) . \tag{2.9}$$

Consequently, for all $\lambda \in \mathbb{C}$, $\Re\lambda > 0$,

$$\widehat{u^{\varepsilon}}(\lambda) \rightharpoonup \hat{u}(\lambda) \text{ in } H_0^1(\Omega) ,$$

where $\hat{u}(\lambda) \in H_0^1(\Omega)$ is the solution of

$$\lambda^2 \hat{u}(\lambda) - \operatorname{div}\left[C(\lambda)\, \nabla \hat{u}\right] = \hat{f}(\lambda) . \tag{2.10}$$

The uniqueness of the solution of problem (2.10) as well as the uniqueness of the Laplace transform, imply that convergences (2.9) hold for the whole sequence ε. We would like to consider (2.10) as the Laplace transform of the limit equation corresponding to the asymptotic problem (2.3)-(2.4); therefore we will have to identify $C(\lambda)$ as the Laplace transform of some time dependent matrix and relate it with the limit behavior of the sequences (A^{ε}) and (B^{ε}).

2.2. The perturbed problem

As we recalled in Section 2.1, homogenization results are already known when there is a viscosity coercive matrix in the inclusions. This is the reason why we add another small perturbation parameter, δ, in the inclusions. Our main point is to study the dependence of the homogenized operators on the small perturbation and to establish convergence results, when this parameter tends to zero.

Consider the matrix defined by

$$B_\delta^\varepsilon = B^1 \chi^\varepsilon + \delta I(1 - \chi^\varepsilon) , \tag{2.11}$$

where I is the identity matrix. Replacing B^ε by B_δ^ε in (2.3) we obtain a medium formed by two viscoelastic materials and so with the same microscopic behavior.

Consider the corresponding matrix

$$C_\delta^\varepsilon(\lambda) := A^\varepsilon + \lambda B_\delta^\varepsilon . \tag{2.12}$$

PROPOSITION 2.2.: *Let $C_\delta^\varepsilon(\lambda)$ be given by (2.12). There exists a subsequence ε' of ε and a family $C_\delta(\lambda)$ of matrices in $M(\alpha, (1 + |\lambda|)\beta ; \Omega ; \mathbb{C})$, such that, for each δ in $\mathbb{R}^+$ and $\lambda \in \mathbb{C}$, $\Re\lambda > 0$, $C_\delta^{\varepsilon'}(\lambda)$ H-converges to $C_\delta(\lambda)$. Moreover, if $C^\varepsilon(\lambda)$, defined in (2.7), H-converges to $C(\lambda)$, for all $\lambda \in \mathbb{C}$, $\Re\lambda > 0$, then*

$$|C(\lambda) - C_\delta(\lambda)| \leqslant |\lambda| \, \delta \frac{\beta}{\alpha} (1 + |\lambda|), \quad \text{a.e. in } \Omega , \tag{2.13}$$

which implies, as δ goes to zero,

$$C_\delta(\lambda) \to C(\lambda) \quad \text{in } [L^\infty(\Omega)]^{N \times N} . \tag{2.14}$$

Proof: Using Theorem A.2, together with a diagonalization process, we obtain the existence of a subsequence ε' of ε and of a family $C_p(q)$, $p \in \mathbb{Q}^+$, $q = r + is$, $r \in \mathbb{Q}^+$, $s \in \mathbb{Q}$, of matrices in $M(\alpha, (1 + |q|)\beta ; \Omega ; \mathbb{C})$, such that, for all p, q,

$$C_p^{\varepsilon'}(q) \xrightarrow{H} C_p(q) . \tag{2.15}$$

Since, from the definition of $C_\delta^\varepsilon(\lambda)$, one has

$$|C_p^{\varepsilon'}(q) - C_{\bar p}^{\varepsilon'}(\bar q)| \leqslant |q - \bar q| \, \beta + |qp - \bar q \bar p|, \quad \text{a.e. in } \Omega , \tag{2.16}$$

for any $\varepsilon' > 0$, p, $\bar p \in \mathbb{Q}^+$, $q = r + is$, $\bar q = \bar r + i\bar s$, r, $\bar r \in \mathbb{Q}^+$, s, $\bar s \in \mathbb{Q}$, and almost everywhere in Ω, Theorem A.4 yields

$$|C_p(q) - C_{\bar p}(\bar q)| \leqslant (|q - \bar q| \, \beta + |qp - \bar q \bar p|) \cdot \frac{\beta}{\alpha} \sqrt{(1 + |q|)(1 + |\bar q|)}, \quad \text{a.e. in } \Omega , \tag{2.17}$$

for any $p, q, \bar p, \bar q$. In view of (2.17) we can define $C_\delta(\lambda)$ by a density argument, for all $\delta \in \mathbb{R}^+$, $\lambda \in \mathbb{C}$, $\Re\lambda > 0$. One has $C_\delta^{\varepsilon'}(\lambda) \xrightarrow{H} C_\delta(\lambda)$. Indeed, by Theorem A.2, There exists a subsequence ε'' of ε' and $\bar C_\delta(\lambda)$ such that

$C_\delta^{\varepsilon''}(\lambda) \xrightarrow{H} \bar{C}_\delta(\lambda)$. Let (2.17) hold for δ and λ instead of $\bar{p}$ and $\bar{q}$ and for p and q such that $p \to \delta$ and $q \to \lambda$. Then $|\bar{C}_\delta(\lambda) - C_p(q)| \to 0$ and, consequently, $\bar{C}_\delta(\lambda) = C_\delta(\lambda)$. We also conclude that the hole sequence ε' satisfies $C_\delta^{\varepsilon'}(\lambda) \xrightarrow{H} C_\delta(\lambda)$.

Moreover, since $C^\varepsilon(\lambda) \xrightarrow{H} C(\lambda)$ and $|C^\varepsilon(\lambda) - C_\delta^\varepsilon(\lambda)| \leqslant |\lambda|\,\delta$, using again Theorem A.4, we obtain (2.13) and, consequently, (2.14). □

Since δ is a small parameter we may suppose that $\delta < \beta$. For all $\varepsilon > 0$ we will have

$$A^\varepsilon \in M(\alpha, \beta\,; \Omega\,; \mathbb{R}), \quad B_\delta^\varepsilon \in M(\delta, \beta\,; \Omega\,; \mathbb{R}), \tag{2.18}$$

$$C_\delta^\varepsilon(\lambda) \in M(\alpha, (1 + |\lambda|)\,\beta\,; \Omega\,; \mathbb{C}).$$

Remark 2.3: In view of Theorem A.2 and Proposition 2.2, we define, up to a subsequence of ε, A in $M(\alpha, \beta\,; \Omega\,; \mathbb{R})$, B_δ in $M(\delta, \beta\,; \Omega\,; \mathbb{R})$, $C_\delta(\lambda)$ and $C(\lambda)$ in $M(\alpha, (1 + |\lambda|)\,\beta\,; \Omega\,; \mathbb{C})$, such that, for all δ in $\mathbb{R}^+$ and λ in $\mathbb{C}$, $\Re\lambda > 0$,

$$A^\varepsilon \xrightarrow{H} A, \quad B_\delta^\varepsilon \xrightarrow{H} B_\delta, \quad C_\delta^\varepsilon(\lambda) \xrightarrow{H} C_\delta(\lambda), \quad C^\varepsilon(\lambda) \xrightarrow{H} C(\lambda). \tag{2.19}$$

□

We are now in the framework of H-convergence for a mixture of two viscoelastic materials, treated in [10]. If convergences (2.19) hold, following [10], Thm. 4.2, there exists a symmetric matrix K_δ in $[C^\infty(\mathbb{R}^+\,; L^\infty(\Omega))]^{N \times N}$, analytic in t, such that its Laplace transform satisfies, for all δ in $\mathbb{R}^+$ and λ in $\mathbb{C}$, $\Re\lambda > 0$,

$$\hat{K}_\delta(\lambda) = \frac{C_\delta(\lambda)}{\lambda} - \frac{A}{\lambda} - B_\delta\,. \tag{2.20}$$

Moreover there exists a positive constant c_δ such that

$$|K_\delta(t)| \leqslant c_\delta, \quad \forall t > 0\,. \tag{2.21}$$

3. THE HOMOGENIZED PROBLEM FOR THE ELASTIC-VISCOELASTIC MIXTURE

We are now in position to identify the matrix $C(\lambda)$, the H-limit of the sequence $C^\varepsilon(\lambda)$, and to establish the limit problem.

The following proposition gives us a sufficient condition for a function to be a Laplace transform (see [14], Ch. 4, and [8], Ch. 3):

PROPOSITION 3.1: *Let $F(\lambda)$ be a holomorphic function of the complex variable λ, with values in a reflexive separable Banach space $\mathcal{B}$. Suppose that for certain ξ_0, $c \in \mathbb{R}^+$ and N, positive integer,*

$$\|F(\lambda)\|_{\mathcal{B}} \leqslant c|\lambda|^N, \quad \forall \lambda \in \mathbb{C}, \quad \Re\lambda > \xi_0 .$$

Then $\dfrac{F(\lambda)}{\lambda^{N+2}}$ is the Laplace transform of f defined by

$$f(t) = \frac{1}{2\pi i}\int_{\Re\lambda=\gamma} \frac{F(\lambda)}{\lambda^{N+2}} e^{\lambda t}\, d\lambda ,$$

where $\gamma > \xi_0$, for all $\lambda \in \mathbb{C}$, $\Re\lambda > \xi_0$.

Consider $\mathcal{B}$ as the complex space $[L^2(\Omega)]^{N\times N}$. Since $C_\delta(\lambda) = A + \lambda B_\delta + \lambda \hat{K}_\delta(\lambda)$ is holomorphic in the half plane $\Re\lambda > 0$ and, in view of estimate (2.13), the same happens with $C(\lambda)$. In fact, in each compact of the complex half plane $\Re\lambda > 0$, $C(\lambda)$ is the uniform limit of the holomorphic functions $C_\delta(\lambda)$, as δ goes to zero, and thus, holomorphic.

Suppose that B_δ converges in $[L^2(\Omega)]^{N\times N}$, as δ goes to zero, to an element B. Since B_δ is bounded in $[L^\infty(\Omega)]^{N\times N}$, independently of δ, B will be an element of $[L^\infty(\Omega)]^{N\times N}$. Since $C_\delta(\lambda) \to C(\lambda)$ in $[L^\infty(\Omega)]^{N\times N}$ and $C_\delta(\lambda) - A - \lambda B_\delta = \lambda \hat{K}_\delta(\lambda)$, we will have $\hat{K}_\delta(\lambda) \to \bar{K}(\lambda)$, in $[L^2(\Omega)]^{N\times N}$, with $\bar{K}$ satisfying

$$C(\lambda) - A - \lambda B = \lambda \bar{K}(\lambda) . \tag{3.1}$$

From equality (3.1) we conclude that $\bar{K}(\lambda)$ is holomorphic in the half plane $\Re\lambda > 0$. Moreover the following estimates hold, for some positive constant c,

$$\begin{aligned} \|\hat{K}_\delta(\lambda) - \bar{K}(\lambda)\|_{\mathcal{B}} &\leqslant \frac{1}{|\lambda|}\|C_\delta(\lambda) - C(\lambda)\|_{\mathcal{B}} + \|B - B_\delta\|_{\mathcal{B}} \\ &\leqslant c\,\delta(1 + |\lambda|) + \|B - B_\delta\|_{\mathcal{B}} , \end{aligned} \tag{3.2}$$

for all $\lambda \in \mathbb{C}$, $\Re\lambda > 0$.

From (3.2), fixing $\delta = \delta_0 > 0$, we will have

$$\|\bar{K}(\lambda)\|_{\mathcal{B}} \leqslant \|\hat{K}_\delta(\lambda) - \bar{K}(\lambda)\|_{\mathcal{B}} + \|\hat{K}_{\delta_0}(\lambda)\|_{\mathcal{B}}$$

and, from estimate (2.21),

$$\|\hat{K}_{\delta_0}(\lambda)\|_{\mathscr{B}} \leqslant \frac{c_{\delta_0}}{\xi_0},$$

which implies

$$\|\bar{K}(\lambda)\|_{\mathscr{B}} \leqslant c\,\delta_0(1+|\lambda|) + \|B - B_\delta\|_{\mathscr{B}} + \frac{c_{\delta_0}}{\xi_0},$$

for all $\lambda \in \mathbb{C}$, $\Re\lambda > \xi_0$, where ξ_0 is an arbitrary fixed positive real number.

Using Proposition 3.1 we conclude to the existence of a function G, defined by

$$G(t) = \frac{1}{2\pi i}\int_{\Re\lambda=\gamma} \frac{\bar{K}(\lambda)}{\lambda^3} e^{\lambda t}\, d\lambda, \tag{3.3}$$

where $\gamma > 0$, for all $\lambda \in \mathbb{C}$, $\Re\lambda > 0$, and such that

$$\bar{K}(\lambda) = \lambda^3 \hat{G}(\lambda). \tag{3.4}$$

Defining

$$K := D^3 G, \tag{3.5}$$

where D stands for the distributional derivative in t, we will have, for all $\lambda \in \mathbb{C}$, $\Re\lambda > 0$,

$$C(\lambda) = A + \lambda B + \lambda\hat{K}(\lambda). \tag{3.6}$$

The limit equation corresponding to problem (2.3)-(2.4) is, then, given by the following theorem:

THEOREM 3.2: *Consider problem (2.3)-(2.4), where $f \in C^1([0,+\infty[\,; L^\infty(\Omega))$ is bounded in $[0,+\infty[$ and the sequences A^ε defined by (2.4)-(2.5), $B^\varepsilon_\delta = B^1\chi^\varepsilon + \delta I(1-\chi^\varepsilon)$ defined by (2.11)-(2.5) and $C^\varepsilon(\lambda)$ defined by (2.7), satisfy the following H-convergences, for all $\delta > 0$ and $\lambda \in \mathbb{C}$, $\Re\lambda > 0$ (c.f. (2.19)):*

$$A^\varepsilon \xrightarrow{H} A, \quad B^\varepsilon_\delta \xrightarrow{H} B_\delta, \quad C^\varepsilon(\lambda) \xrightarrow{H} C(\lambda). \tag{3.7}$$

Furthermore, suppose that, as δ goes to zero,

$$B_\delta \to B, \quad in\ [L^2(\Omega)]^{N\times N}. \tag{3.8}$$

Then the solutions $u^{\varepsilon}(t)$ of problem (2.3)-(2.4) converge, weakly in $H_0^1(\Omega)$, for all $t > 0$, as ε goes to zero, to $u(t)$ satisfying

$$(3.9) \qquad \begin{cases} \ddot{u} - \operatorname{div}(A\,\nabla u + B\,\nabla \dot{u} + K_*\nabla \dot{u}) = f & \text{in } \Omega \times [0, +\infty[\,, \\ u(0) = 0, \quad \dot{u}(0) = 0 & \text{in } \Omega\,, \\ u = 0 \quad \text{on } \partial\Omega \times [0, +\infty[\,, & \end{cases}$$

where $*$ denotes the convolution in time t and K is defined by (3.5) and satisfies (3.6).

Proof: At the end of Section 2.1 we established the weak convergence of the sequence $u^{\varepsilon}(t)$ to $u(t)$, whose Laplace transform satisfies (2.10).

Since (3.6) holds for all $\lambda \in \mathbb{C}$, $\Re\lambda > 0$, where $K(\lambda)$ is defined by (3.5) and (3.3), using the uniqueness of the Laplace transform, we complete the proof. □

4. EXAMPLES

4.1. Periodic and quasi-periodic media

First we describe what we mean by periodic and quasi-periodic media.

Let Ω be an open, bounded, connected and lipschitzian subset of $\mathbb{R}^N$, and $Y =]0, 1[^N$.

Let $T \subset Y$ be the closure of a regular, open, connected subset of Y, and $Y^* = Y \backslash T$.

Consider χ the characteristic set function of Y^* and, keeping the same notation, extend it to the whole of $\mathbb{R}^N$.

For each small positive parameter ε, $\mathbb{R}^N$ is covered by squares $Y^{\varepsilon k} = \varepsilon Y + \varepsilon k$, where $k \in \mathbb{Z}^N$.

Let $\mathscr{Z}^{\varepsilon}$ represent the set of all $k \in \mathbb{Z}^N$ such that $Y^{\varepsilon k}$ is included in $\bar{\Omega}$. Define:

$$\chi^{\varepsilon}(x) = \begin{cases} \chi\left(\frac{x}{\varepsilon}\right), & x \in Y^{\varepsilon k}, \quad k \in \mathscr{Z}^{\varepsilon}, \\ 1, & x \in \Omega \cap Y^{\varepsilon k}, \quad k \notin \mathscr{Z}^{\varepsilon}. \end{cases}$$

The subset $\Omega^{\varepsilon} \subset \Omega$, defined by the characteristic function χ^{ε}, corresponds to a εY-periodic perforation of Ω, all the holes having the same size and shape: we say that Ω is periodically perforated.

Classical homogenization results allow us to treat asymptotically, i.e., as ε goes to zero, an extense class of P.D.E. problems (see [5] and [9]).

The case where the size and shape of the holes varies from cell to cell, is called quasi-periodic and has been treated in [11]. We briefly summarize here the homogenization results obtained in [11], our present setting being slightly different, but more adapted to the classical methods of control in domains and including the N dimensional case. The same proofs hold, with minor modifications (c.f. [4]).

In this last case we consider, instead of a unique reference perforated cell Y^*, a family of perforated cells $\{Y^*(x)\}_{x \in \Omega}$, i.e., the reference holes vary with the zone of the perforation.

We would like to say that the function

$$x \in \bar{\Omega} \mapsto Y^*(x) \subset Y \tag{4.1}$$

is the *microstructure* of the perforation. More precisely, we will define a *microstructure* as an element

$$B \in \mathbf{C} = \mathscr{C}^1(\bar{\Omega}\,; \mathbf{\Phi}_0)\,, \tag{4.2}$$

where $\mathbf{\Phi}_0 \subset W^{1,\infty}(Y\,; \mathbb{R}^N)$ is the set of all the bilipschitzian homeomorphims of Y into Y, that coincide with the identity on the boundary ∂Y of Y, and such that the image of a fixed lipschitzian subdomain Y_0^* of Y is still lipschizian. We suppose that Y_0^* is open, and that $T_0 = Y \backslash Y_0^*$ is connected and contained in the interior of Y. The set $\mathbf{\Phi}_0$ is equipped with the usual norm of $W^{1,\infty}(Y\,; \mathbb{R}^N)$.

We denote, then,

$$Y^*(x) = B(x)\,(Y_0^*) = \{B(x)\,(z) : z \in Y_0^*\}\,, \tag{4.3}$$

and by χ the characteristic function of $\bigcup_{x \in \Omega} [\{x\} \times B(x)\,(Y_0^*)]$, in $\Omega \times Y$, extended by periodicity in the second variable, to the whole of $\Omega \times \mathbb{R}^N$. Let $T(x) = Y \backslash Y^*(x)$.

As in the periodic case, we define $\Omega^\varepsilon \subset \Omega$ by the following characteristic function:

$$\chi^\varepsilon(x) = \begin{cases} \chi\left(x, \frac{x}{\varepsilon}\right), & x \in Y^{\varepsilon k}, \quad k \in \mathscr{Z}^\varepsilon, \\ 1, & x \in \Omega \cap Y^{\varepsilon k}, \quad k \notin \mathscr{Z}^\varepsilon. \end{cases} \tag{4.4}$$

Once defined the characteristic function χ^ε, we consider problem (2.3)-(2.4), corresponding to the viscoelastic matrix occupying Ω^ε, with elastic inclusions occupying $\Omega \backslash \Omega^\varepsilon$, and where the matrices A^ε and B^ε are symmetric and satisfy the ellipticity conditions (2.5).

In this case it is well known that the sequences (A^ε), (B_δ^ε), $(C_\delta^\varepsilon(\lambda))$ and $(C^\varepsilon(\lambda))$ are H-convergent, for fixed δ and λ, as ε goes to zero, to the matrices defined respectively by $A = (a_{ij})$, $B_\delta = (b_{\delta ij})$, $C_\delta(\lambda) = (c_{\delta ij}^\lambda)$ and $C(\lambda) = (c_{ij}^\lambda)$, where, for each $x \in \Omega$,

$$(4.5) \qquad a_{ij}(x) = \int_{Y^*(x)} (A^1\, \nabla_y(\theta_i^a(x,y) - y_i), \nabla_y(\theta_j^a(x,y) - y_j))\, dy$$

$$+ \int_{T(x)} (A^0\, \nabla_y(\theta_i^a(x,y) - y_i), \nabla_y(\theta_j^a(x,y) - y_j))\, dy\,,$$

$$(4.6) \qquad b_{\delta ij}(x) = \int_{Y^*(x)} (B^1\, \nabla_y(\theta_i^{b_\delta}(x,y) - y_i), \nabla_y(\theta_j^{b_\delta}(x,y) - y_j))\, dy$$

$$+ \delta \int_{T(x)} (\nabla_y(\theta_i^{b_\delta}(x,y) - y_i), \nabla_y(\theta_j^{b_\delta}(x,y) - y_j))\, dy\,,$$

$$(4.7) \qquad c_{\delta ij}^\lambda(x) = \int_{Y^*(x)} ((A^1 + \lambda B^1)\, \nabla_y(\theta_i^{c_\delta^\lambda}(x,y) - y_i), \nabla_y(\theta_j^{c_\delta^\lambda}(x,y) - y_j))\, dy$$

$$+ \int_{T(x)} ((A^0 + \lambda\delta I)\, \nabla_y(\theta_i^{c_\delta^\lambda}(x,y) - y_i), \nabla_y(\theta_j^{c_\delta^\lambda}(x,y) - y_j))\, dy\,,$$

$$(4.8) \qquad c_{ij}^\lambda(x) = \int_{Y^*(x)} ((A^1 + \lambda B^1)\, \nabla_y(\theta_i^{c^\lambda}(x,y) - y_i), \nabla_y(\theta_j^{c^\lambda}(x,y) - y_j))\, dy$$

$$+ \int_{T(x)} (A^0\, \nabla_y(\theta_i^{c^\lambda}(x,y) - y_i), \nabla_y(\theta_j^{c^\lambda}(x,y) - y_j))\, dy\,,$$

where the functions θ_i^a, $\theta_i^{b_\delta}$, $\theta_i^{c_\delta^\lambda}$ and $\theta_i^{c^\lambda}$ $(i = 1, \dots, N)$, are defined as follows. Consider the space $H^1_\#(Y)$ of the functions in $H^1(Y)$ whose traces coincide in the opposite faces of the unit cube Y. For each x in Ω, $\theta_i^a(x, .)$, $\theta_i^{b_\delta}(x, .)$, $\theta_i^{c_\delta^\lambda}(x, .)$ and $\theta_i^{c^\lambda}(x, .)$ are the unique solutions of the following cell problems, where $(e_i)_{i=1, \dots, N}$ stands for the canonic basis in $\mathbb{R}^N$:

$$
(4.9) \quad \begin{cases} \theta_i^a(x, .) \in H^1_\#(Y), \displaystyle\int_{Y^*(x)} \theta_i^a(x, y)\, dy = 0, \quad \forall \varphi \in H^1_\#(Y), \\ \displaystyle\int_{Y^*(x)} (A^1 \nabla_y \theta_i^a(x, y), \nabla_y \varphi(y))\, dy + \int_{T(x)} (A^0 \nabla_y \theta_i^a(x, y), \nabla_y \varphi(y))\, dy = \\ -\displaystyle\int_{Y^*(x)} (A^1 \nabla_y \varphi, e_i(y))\, dy - \int_{T(x)} (A^0 \nabla_y \varphi, e_i(y))\, dy, \end{cases}
$$

$$
(4.10) \quad \begin{cases} \theta_i^{b_\delta}(x, .) \in H^1_\#(Y), \displaystyle\int_{Y^*(x)} \theta_i^{b_\delta}(x, y)\, dy = 0, \quad \forall \varphi \in H^1_\#(Y), \\ \displaystyle\int_{Y^*(x)} (B^1 \nabla_y \theta_i^{b_\delta}(x, y), \nabla_y \varphi(y))\, dy + \delta \int_{T(x)} (\nabla_y \theta_i^{b_\delta}(x, y), \nabla_y \varphi(y))\, dy = \\ -\displaystyle\int_{Y^*(x)} (B^1 \nabla_y \varphi, e_i)\, dy - \delta \int_{T(x)} (\nabla_y \varphi, e_i)\, dy, \end{cases}
$$

$$
(4.11) \quad \begin{cases} \theta_i^{c_\delta^\lambda}(x, .) \in H^1_\#(Y), \displaystyle\int_{Y^*(x)} \theta_i^{c_\delta^\lambda}(x, y)\, dy = 0, \quad \forall \varphi \in H^1_\#(Y), \\ \displaystyle\int_{Y^*(x)} ((A^1 + \lambda B^1) \nabla_y \theta_i^{c_\delta^\lambda}(x, .), \nabla \varphi)\, dy + \\ \displaystyle\int_{T(x)} ((A^0 + \lambda\delta I) \nabla_y \theta_i^{c_\delta^\lambda}(x, .), \nabla_y \varphi)\, dy = \\ -\displaystyle\int_{Y^*(x)} ((A^1 + \lambda B^1) \nabla_y \varphi, e_i)\, dy - \int_{T(x)} ((A^0 + \lambda\delta I) \nabla_y \varphi, e_i)\, dy, \end{cases}
$$

$$
(4.12) \quad \begin{cases} \theta_i^{c^\lambda}(x,.) \in H^1_\#(Y), \displaystyle\int_{Y^*(x)} \theta_i^{c^\lambda}(x,y)\,dy = 0, \quad \forall \varphi \in H^1_\#(Y), \\ \displaystyle\int_{Y^*(x)} ((A^1 + \lambda B^1)\,\nabla_y\,\theta_i^{c^\lambda}(x,.), \nabla_y\,\varphi)\,dy \\ + \displaystyle\int_{T(x)} (A^0\,\nabla_y\,\theta_i^{c^\lambda}(x,.), \nabla_y\,\varphi)\,dy = \\ - \displaystyle\int_{Y^*(x)} ((A^1 + \lambda B^1)\,\nabla_y\,\varphi, e_i)\,dy - \int_{T(x)} (A^0\,\nabla_y\,\varphi, e_i)\,dy . \end{cases}
$$

Also for the sequence (B^ε) of degenerate matrices, there exists a homogenized limit matrix $B = (b_{ij})$, in the sense first introduced in [5], for the periodic case, in [11], for the quasi-periodic case, and recently generalized in [3], under the name of H^0-convergence:

$$
(4.13) \quad b_{ij}(x) = \int_{Y^*(x)} (B^1\,\nabla_y(\theta_i^b(x,y) - y_i), \nabla_y(\theta_j^b(x,y) - y_j))\,dy ,
$$

for each x in Ω, where $\theta_i^b(x,.)$, $i = 1, ..., N$, are the unique solutions of:

$$
(4.14) \quad \begin{cases} \theta_i^b(x,.) \in H^1_\#(Y^*(x)), \displaystyle\int_{Y^*(x)} \theta_i^b(x,y)\,dy = 0, \quad \forall \varphi \in H^1_\#(Y^*(x)) \\ \displaystyle\int_{Y^*(x)} (B^1\,\nabla_y\,\theta_i^b(x,y), \nabla_y\,\varphi(y))\,dy = - \int_{Y^*(x)} (B^1\,\nabla_y\,\varphi, e_i)\,dy . \end{cases}
$$

The following proposition holds:

PROPOSITION 4.1: *As δ goes to zero,*

$$
(4.15) \qquad B_\delta \to B, \quad \text{and} \quad C_\delta(\lambda) \to C(\lambda), \quad in\ [L^2(\Omega)]^{N \times N},
$$

where B_δ, B, $C_\delta(\lambda)$ and $C(\lambda)$, are the matrices whose elements were introduced in (4.6)-(4.10), (4.13)-(4.14), (4.7)-(4.11) and (4.8)-(4.12), respectively.

Proof: For each $x \in \Omega$ and $i = 1, \ldots, N$, let $\varphi = \theta_i^{b_\delta}(x, .)$, in (4.10). Using conditions (2.5), Hölder's inequality and the fact that δ is small, we obtain, for some positive constant c, independent of δ:

$$\begin{aligned} \| \nabla_y \theta_i^{b_\delta}(x, .) \|^2_{L^2(Y^*(x))} + \frac{\delta}{\alpha} \| \nabla_y \theta_i^{b_\delta}(x, .) \|^2_{L^2(T(x))} & \\ \leqslant c \frac{\beta}{\alpha} \| \nabla \theta_i^{b_\delta}(x, .) \|_{L^2(Y^*(x))} & \\ + c \frac{\delta}{\alpha} \| \nabla_y \theta_i^{b_\delta}(x, .) \|_{L^2(T(x))} & \\ \leqslant c \frac{\beta}{\alpha} \| \nabla \theta_i^{b_\delta}(x, .) \|_{L^2(Y^*(x))} & \\ + c \sqrt{\frac{\delta}{\alpha}} \| \nabla_y \theta_i^{b_\delta}(x, .) \|_{L^2(T(x))} . & \end{aligned}$$

Consequently both $\| \nabla_y \theta_i^{b_\delta}(x, .) \|_{L^2(Y^*(x))}$ and $\sqrt{\delta} \| \nabla_y \theta_i^{b_\delta}(x, .) \|_{L^2(T(x))}$ are bounded, independently of δ. Since, by the inequality of Poincaré, $\| \theta_i^{b_\delta}(x, .) \|_{H^1_\#(Y^*(x))}$ is a bounded sequence, we can extract a subsequence satisfying, for some $\eta_i^b(x, .) \in H^1_\#(Y^*(x))$, as δ' goes to 0,

$$\theta_i^{b_{\delta'}}(x, .) \rightharpoonup \eta_i^b(x, .), \quad \text{weakly in } H^1_\#(Y^*(x)) . \tag{4.16}$$

We also have, for some constant $\bar{c}$:

$$\left| \delta \int_{T(x)} (\nabla_y \theta_i^{b_\delta}(x, y), \nabla_y \varphi(y))\, dy \right| \leqslant \sqrt{\delta}\, \bar{c} \| \nabla_y \varphi \|_{L^2(T(x))} \to 0 , \tag{4.17}$$

as δ goes to 0.

Passing to the limit in (4.10), with the help of (4.16) and (4.17), we see that $\eta_i^b(x, .)$ is solution of problem (4.14) and, consequently, equal to $\theta_i^b(x, .)$. Since the limit of any subsequence of $\theta_i^{b_{\delta'}}(x, .)$ is the same, the whole sequence converges to $\theta_i^b(x, .)$.

Letting $\varphi = \theta_i^{b_\delta}(x, .)$ in the variational formulation (4.10), we see that the elements $b_{\delta ij}(x)$ may also be given by:

$$b_{\delta ij}(x) = \int_{Y^*(x)} b^1_{ik} \frac{\partial(\theta_j^{b_\delta}(x, y) - y_j)}{\partial y_k}\, dy + \delta \int_{T(x)} \frac{\partial(\theta_j^{b_\delta}(x, y) - y_j)}{\partial y_i}\, dy , \tag{4.18}$$

and, analogously,

$$b_{ij}(x) = \int_{Y^*(x)} b^1_{ik} \frac{\partial(\theta_j^{b}(x, y) - y_j)}{\partial y_k}\, dy . \tag{4.19}$$

Since

$$\theta_i^{b_\delta}(x, .) \rightharpoonup \theta_i^b(x, .), \text{ weakly in } H^1_{\#}(Y^*(x)), \tag{4.20}$$

we have $b_{\delta ij}(x) \to b_{ij}(x)$, for each x, and, by the Theorem of Dominated Convergence, the result follows.

The proof of $C_\delta(\lambda) \to C(\lambda)$ is analogous. □

In view of Proposition 4.1 we can apply Theorem 3.2, obtaining the desired homogenized equation. However, in this case, we can define the homogenized kernel in a more precise way, directly from the cell problem, as it was done in [7], for the periodic and non degenerate case. This will be carried out by Proposition 4.2.

Define, for each $x \in \Omega$,

$$V_x := \left\{ \varphi \in H^1_{\#}(Y) \,\middle|\, \int_{Y^*(x)} \varphi \, dy = 0 \right\},$$

$$D_x := \left\{ v \in V_x \,\middle|\, \exists c > 0 \int_{Y^*(x)} A^1 \nabla_y v \nabla_y \varphi \, dy + \int_{T(x)} A^0 \nabla_y v \nabla_y \varphi \, dy \right.$$

$$\left. \leqslant c \left[\int_{Y^*(x)} B^1 \nabla_y \varphi \nabla_y \varphi \, dy \right]^{\frac{1}{2}}, \forall \varphi \in V_x \right\}.$$

The theory of implicit degenerate evolution equations (see [15]) guarantees the existence of a unique solution $w_i(x, ., .)$ such that

$$\begin{cases} w_i(x, ., .) \in C([0, +\infty[\,; V_x) \cap C^1(]0, +\infty[\,; V_x)\,; \quad w_i(x, t, .) \in D_x, \forall t > 0\,; \\ \displaystyle\int_{Y^*(x)} (B^1 \nabla_y \dot{w}_i(x, t, y), \nabla_y \varphi(y))\, dy + \int_{Y^*(x)} (A^1 \nabla_y w_i(x, t, y), \nabla_y \varphi(y))\, dy \\ \displaystyle + \int_{T(x)} (A^0 \nabla_y w_i(x, t, y), \nabla_y \varphi(y))\, dy = 0, \quad \forall \varphi \in V_x\,; \\ w_i(x, 0, .) = \theta_i^b - \theta_i^a. \end{cases} \tag{4.21}$$

Define, for each $x \in \Omega$ and $t > 0$, the matrix $K(x, t)$:

$$K(x, t)\, e_i := \int_{Y^*(x)} [B^1 \nabla_y \dot{w}_i(x, t, y) + A^1 \nabla_y w_i(x, t, y)]\, dy + \int_{T(x)} A^0 \nabla_y w_i(x, t, y)\, dy. \tag{4.22}$$

PROPOSITION 4.2: *The matrix K introduced in (4.22) satisfies (c.f. (3.6)), for all $\lambda > 0$ (it is enough to consider real values of λ),*

$$C(\lambda) = A + \lambda B + \lambda \hat{K}(\lambda),$$

where the matrices A, B and $C(\lambda)$ have their coefficients given by (4.5), (4.13) and (4.8), respectively. Consequently, K is the homogenized kernel that appears in the limit equation (3.9).

Proof: Letting $v = \dot{w}_\iota$, in the variational formulation, integrating in time and using conditions (2.5), we obtain that $\| \nabla_y w_\iota(x, t, .) \|_{L^2(Y)}$ is bounded independently of t. Applying the Laplace transform to (4.21) we get

$$\int_{Y^*(x)} ((A^1 + \lambda B^1) \nabla_y \hat{w}_\iota(\lambda), \nabla_y \varphi) \, dy + \int_{T(x)} (A^0 \nabla_y \hat{w}_\iota(\lambda), \nabla_y \varphi) \, dy = \int_{Y^*(x)} (B^1 \nabla_y (\theta^b_\iota - \theta^a_\iota), \nabla_y \varphi) \, dy, \quad \forall \varphi \in V_x . \tag{4.23}$$

Applying also the Laplace transform to (4.22) we have

$$\hat{K}(\lambda) \, e_\iota = \int_{Y^*(x)} (A^1 + \lambda B^1) \nabla_y \hat{w}_\iota(\lambda) \, dy + \int_{T(x)} A^0 \nabla_y \hat{w}_\iota(\lambda) \, dy - \int_{Y^*(x)} B^1 \nabla_y (\theta^b_\iota - \theta^a_\iota) \, dy . \tag{4.24}$$

We now prove that, for all $\lambda > 0$,

$$\lambda \hat{w}_\iota(\lambda) + \theta^a_\iota = \theta^{c^\lambda}_\iota . \tag{4.25}$$

In fact,

$$\begin{aligned} &\int_{Y^*(x)} ((A^1 + \lambda B^1) \nabla_y (\lambda \hat{w}_\iota(\lambda) + \theta^a_\iota), \nabla_y \varphi) \, dy \\ &\quad + \int_{T(x)} (A^0 \nabla_y (\lambda \hat{w}_\iota(\lambda) + \theta^a_\iota), \nabla_y \varphi) \, dy \\ &= \lambda \left[\int_{Y^*(x)} ((A^1 + \lambda B^1) \nabla_y \hat{w}_\iota(\lambda), \nabla_y \varphi) \, dy + \int_{T(x)} (A^0 \nabla_y \hat{w}_\iota(\lambda), \nabla_y \varphi) \, dy \right] \\ &\quad + \int_{Y^*(x)} ((A^1 + \lambda B^1) \nabla_y \theta^a_\iota, \nabla_y \varphi) \, dy + \int_{T(x)} (A^0 \nabla_y \theta^a_\iota, \nabla_y \varphi) \, dy . \end{aligned} \tag{4.26}$$

Using (4.23), (4.14) and (4.9), the right-hand side of (4.26) becomes successively

$$= \lambda\left[\int_{Y^*(x)} (B^1 \nabla_y(\theta_\iota^b - \theta_\iota^a), \nabla_y \varphi)\, dy\right] + \int_{Y^*(x)} ((A^1 + \lambda B^1)\nabla_y \theta_\iota^a, \nabla_y \varphi)\, dy$$

$$+ \int_{T(x)} (A^0 \nabla_y \theta_\iota^a, \nabla_y \varphi)\, dy .$$

$$= \lambda \int_{Y^*(x)} (B^1 \nabla_y \theta_\iota^b, \nabla_y \varphi)\, dy + \int_{Y^*(x)} (A^1 \nabla_y \theta_\iota^a, \nabla_y \varphi)\, dy$$

$$+ \int_{T(x)} (A^0 \nabla_y \theta_\iota^a, \nabla_y \varphi)\, dy .$$

$$= - \lambda \int_{Y^*(x)} (B^1 \nabla_y \varphi, e_\iota)\, dy - \int_{Y^*(x)} (A^1 \nabla_y \varphi, e_\iota)\, dy - \int_{T(x)} (A^0 \nabla_y \varphi, e_\iota)\, dy .$$

$$= - \int_{Y^*(x)} ((A^1 + \lambda B^1)\nabla_y \varphi, e_\iota)\, dy - \int_{T(x)} (A^0 \nabla_y, e_\iota)\, dy .$$

Then $\lambda \hat{w}_\iota(\lambda) + \theta_\iota^a$ satisfies problem (4.12), and we obtain the equality (4.25).

Using (4.25), equality (4.24) turns

$$\lambda \hat{K}(\lambda)\, e_\iota = \int_{Y^*(x)} (A^1 + \lambda B^1)\nabla_y(\theta_\iota^{c^\lambda} - \theta_\iota^a)\, dy + \int_{T(x)} A^0 \nabla_y(\theta_\iota^{c^\lambda} - \theta_\iota^a)\, dy$$

$$- \lambda \int_{Y^*(x)} B^1 \nabla_y(\theta_\iota^{c^\lambda} - \theta_\iota^a)\, dy$$

$$= \int_{Y^*(x)} (A^1 + \lambda B^1)\nabla_y \theta_\iota^{c^\lambda}\, dy + \int_{T(x)} A^0 \nabla_y \theta_\iota^{c^\lambda}\, dy$$

$$- \int_{Y^*(x)} A^1 \nabla_y \theta_\iota^a\, dy$$

$$- \int_{T(x)} A^0 \nabla_y \theta_\iota^a\, dy - \lambda \int_{Y^*(x)} B^1 \nabla_y \theta_\iota^b\, dy$$

$$= \int_{Y^*(x)} (A^1 + \lambda B^1)\nabla_y(\theta_\iota^{c^\lambda} - y_\iota)\, dy + \int_{T(x)} A^0 \nabla_y(\theta_\iota^{c^\lambda} - y_\iota)\, dy$$

$$- \int_{Y^*(x)} A^1 \nabla_y(\theta_\iota^a - y_\iota)\, dy - \int_{T(x)} A^0 \nabla_y(\theta_\iota^a - y_\iota)\, dy$$

$$- \lambda \int_{Y^*(x)} B^1 \nabla_y(\theta_\iota^b - y_\iota)\, dy = C(\lambda)\, e_\iota - A e_\iota - \lambda B e_\iota ,$$

which completes the proof. □

4.2. Layered media

We consider, for the sake of simplicity, the case of an isotropic layered medium in the two-dimensional case, defined by the matrices:

$$A^{\varepsilon} = [\alpha^1 \chi^{\varepsilon} + \alpha^0(1 - \chi^{\varepsilon})]\, I\,,$$

$$B^{\varepsilon}_{\delta} = [\beta^1 \chi^{\varepsilon} + \delta(1 - \chi^{\varepsilon})]\, I\,,$$

$$\chi^{\varepsilon}(x) = \chi^{\varepsilon}(x_1)\,.$$

Then we have

$$C^{\varepsilon}_{\delta}(\lambda) = [(\alpha^1 + \lambda\beta^1)\chi^{\varepsilon} + (\alpha^0 + \lambda\delta)(1 - \chi^{\varepsilon})]\, I\,.$$

The H-limlits of A^{ε}, B^{ε}_{δ} and $C^{\varepsilon}_{\delta}(\lambda)$ are (see [13])

$$A = \begin{pmatrix} \dfrac{\alpha^1 \alpha^0}{\alpha^0 \theta + \alpha^1(1 - \theta)} & 0 \\ 0 & \alpha^1 \theta + \alpha^0(1 - \theta) \end{pmatrix} \tag{4.27}$$

$$B_{\delta} = \begin{pmatrix} \dfrac{\beta^1 \delta}{\delta\theta + \beta^1(1 - \theta)} & 0 \\ 0 & \beta^1 \theta + \delta(1 - \theta) \end{pmatrix} \tag{4.28}$$

(4.29)
$$C_{\delta}(\lambda) = \begin{pmatrix} \dfrac{(\alpha^1 + \lambda\beta^1)(\alpha^0 + \lambda\delta)}{(\alpha^0 + \lambda\delta)\,\theta + (\alpha^1 + \lambda\beta^1)(1 - \theta)} & 0 \\ 0 & (\alpha^1 + \lambda\beta^1)\,\theta + (\alpha^0 + \lambda\delta)(1 - \theta) \end{pmatrix}$$

The limits of B_{δ} and $C_{\delta}(\lambda)$, as $\delta \to 0$, are

$$B = \begin{pmatrix} 0 & 0 \\ 0 & \beta^1 \theta \end{pmatrix} \tag{4.30}$$

$$C(\lambda) = \begin{pmatrix} \dfrac{(\alpha^1 + \lambda\beta^1)\,\alpha^0}{\alpha^0 \theta + (\alpha^1 + \lambda\beta)(1 - \theta)} & 0 \\ 0 & (\alpha^1 + \lambda\beta^1)\,\theta + \alpha^0(1 - \theta) \end{pmatrix} \tag{4.31}$$

All the coefficients $\hat{K}_{ij}(\lambda)$ of $\hat{K}(\lambda)$ vanish, exception made for the coefficient $\hat{K}_{11}(\lambda)$ that satisfies:

$$\lambda \hat{K}_{11}(\lambda) = C_{11}(\lambda) - A_{11}\,. \tag{4.32}$$

This means that the memory effect only apears in the direction of the layers.
Introducing the notations:

$$a = \alpha^1 \alpha^0$$

$$b = \alpha^0 \beta^1$$

$$c = \alpha^0 \theta + \alpha^1 (1 - \theta)$$

$$d = \beta^1 (1 - \theta),$$

we obtain

$$\lambda \hat{K}_{11} = \frac{a + \lambda b}{c + \lambda d} - \frac{a}{c}. \tag{4.33}$$

The inverse Laplace transform of (4.33) gives us, finally:

$$K_{11}(t) = \left(\frac{\alpha^0}{1 - \theta} - \frac{\alpha^1 \alpha^0}{\alpha^0 \theta + \alpha^1 (1 - \theta)} \right) e^{-t \frac{\alpha^0 \theta + \alpha^1 (1 - \theta)}{\beta^1 (1 - \theta)}}. \tag{4.34}$$

Note that when $\beta \to 0$ we have $K_{11} \to 0$, which is an expected result since the viscoelastic term, respresented by the matrix B, vanishes.

APPENDIX

Here we recall some definitions and results of H-convergence, not only as they were developed by Murat and Tartar in [13] and [16], but also considering their extensions to the complex valued case. The proofs corresponding to the extended results are analogous to the ones presented in the real case.

If a and b are two real numbers such that $0 < a < b < +\infty$, we define $M(a, b; \Omega; \mathbb{R})$ and $M(a, b; \Omega; \mathbb{C})$ as follows

$$M(a, b; \Omega; \mathbb{R}) = \Big\{ A \in [L^\infty(\Omega; \mathbb{R})]^{N \times N} : (A\xi, \xi) \geq a|\xi|^2,$$
$$(A^{-1}\eta, \eta) \geq \frac{1}{b} |\eta|^2, \quad \text{a.e. in } \Omega, \forall \xi, \eta \in \mathbb{R}^N \Big\},$$

$$M(a, b; \Omega; \mathbb{C}) = \Big\{ A \in [L^\infty(\Omega; \mathbb{C})]^{N \times N} : \Re(A\xi, \xi) \geq a|\xi|^2,$$
$$\Re(A^{-1}\eta, \eta) \geq \frac{1}{b} |\eta|^2, \quad a.e. \text{ } in \text{ } \Omega, \forall \xi, \eta \in \mathbb{C}^N \Big\}.$$

We have $M(a, b ; \Omega ; \mathbb{R}) \subset M(a, b ; \Omega ; \mathbb{C})$ and, if $A \in M(a, b ; \Omega ; \mathbb{C})$,

$$|A\xi| \leq b|\xi|, \quad |A^{-1}\eta| \leq \frac{1}{a}|\eta|, \quad \forall \xi, \eta \in \mathbb{C}^N .$$

Let ε denote any sequence of real numbers converging to zero.

DEFINITION A.1: *A sequence A^ε in $M(a, b ; \Omega ; \mathbb{R})$ H-converges to an element $A \in M(a, b ; \Omega ; \mathbb{R})$ if for any open $\omega \subset \Omega$, and for any real valued $f \in H^{-1}(\omega)$ (i.e., for any f in the dual $H^{-1}(\omega)$ of the real space $H_0^1(\omega)$, The solution u^ε to*

$$\text{(A.1)} \qquad \begin{cases} -\operatorname{div}(A^\varepsilon \nabla u^\varepsilon) = f & \text{in } \omega , \\ u^\varepsilon \in H_0^1(\omega) , \end{cases}$$

satisfies, as ε tends to zero,

$$\text{(A.2)} \qquad \begin{cases} u^\varepsilon \rightharpoonup u, & \text{weakly in } H_0^1(\omega) , \\ A^\varepsilon \nabla u^\varepsilon \rightharpoonup A\nabla u, & \text{weakly in } [L^2(\omega)]^{N \times N} , \end{cases}$$

where u is the unique solution to

$$\begin{cases} -\operatorname{div}(A\nabla u) = f, & \text{in } \omega , \\ u \in H_0^1(\omega) . \end{cases}$$

Analogously we say that a sequence A^ε in $M(a, b ; \Omega ; \mathbb{C})$ H-converges to an element $A \in M(a, b ; \Omega ; \mathbb{C})$ if for any open $\omega \subset \Omega$, and for any complex valued $f \in H^{-1}(\omega)$ (i.e., for any f in the dual $H^{-1}(\omega)$ of the complex space $H_0^1(\omega)$), the solution of problem (A.1) satisfies the convergences (A.2).

The H-convergence in $M(a, b ; \Omega ; \mathbb{R})$ implies the H-convergence in $M(a, b ; \Omega ; \mathbb{C})$. Moreover it is possible to prove that there exists a metrizable topology in $M(a, b ; \Omega ; \mathbb{C})$ for which the H-convergence is the convergence of a sequence, and for which $M(a, b ; \Omega ; \mathbb{R})$ is closed (see [12]).

We denote by $A^\varepsilon \xrightarrow{H} A$ the H-convergence of a sequence A^ε to A.

The main H-convergence results that will be used in this paper are the following:

THEOREM A.2: *For any sequence A^ε in $M(a, b ; \Omega ; \mathbb{C})$, there exists a subsequence, still denoted A^ε, and an element A of $M(a, b ; \Omega ; \mathbb{C})$ such that A^ε H-converges to A. The same compactness result holds in $M(a, b ; \Omega ; \mathbb{R})$, since it is a closed subset of $M(a, b ; \Omega ; \mathbb{C})$. Furthermore, if the matrices A^ε in $M(a, b ; \Omega ; \mathbb{R})$ are symmetric, then A is also symmetric.*

THEOREM A.3: *A sequence A^ε in $M(a, b ; \Omega ; \mathbb{C})$ H-converges to $A \in M(a, b ; \Omega ; \mathbb{C})$ if and only if for any open $\omega \subset \Omega$,*

$$\begin{cases} u^\varepsilon \in H^1(\omega), f^\varepsilon \in H^{-1}(\omega), \\ -\operatorname{div}(A^\varepsilon \nabla u^\varepsilon) = f^\varepsilon, \quad \text{in } \omega, \\ u^\varepsilon \rightharpoonup u, \quad \text{weakly in } H^1(\omega), \\ f^\varepsilon \to f, \quad \text{strongly in } H^{-1}(\omega), \end{cases}$$

implies that

$$A^\varepsilon \nabla u^\varepsilon \rightharpoonup A \nabla u, \quad \text{weakly in } [L^2(\omega)]^N .$$

Therefore

$$-\operatorname{div}(A \nabla u) = f .$$

The following stability result is an adaptation of the one already presented in [10] and follows the main ideas of Boccardo and Murat [2]:

THEOREM A.4: *Let A^ε and B^ε be two H-converging sequences in $M(a, b ; \Omega ; \mathbb{C})$ and $M(a', b' ; \Omega ; \mathbb{C})$, respectively. Denote by A and B their respective H-limits and assume that the following estimate holds for some constant γ, independent of ε :*

$$|A_\varepsilon(x) - B_\varepsilon(x)| \leq \gamma, \quad \text{a.e. in } x \in \Omega . \tag{A.3}$$

Then

$$|A(x) - B(x)| \leq \gamma \left(\frac{bb'}{aa'} \right)^{\frac{1}{2}}, \quad \text{a.e. in } x \in \Omega . \tag{A.4}$$

Proof: For $f, g \in H^{-1}(\Omega)$ let u^ε and v^ε be the solutions, in $H_0^1(\Omega)$, to the equations

$$-\operatorname{div}(A^\varepsilon \nabla u^\varepsilon) = f \quad \text{and} \quad -\operatorname{div}(B^\varepsilon \nabla v^\varepsilon) = g, \quad \text{in } \Omega . \tag{A.5}$$

By the definition of the H-convergence, as ε tends to zero, $u^\varepsilon \rightharpoonup u$ and $v^\varepsilon \rightharpoonup v$, weakly in $H_0^1(\Omega)$, where u and v are the solutions to

$$-\operatorname{div}(A \nabla u) = f \quad \text{and} \quad -\operatorname{div}(B \nabla v) = g, \quad \text{in } \Omega . \tag{A.6}$$

For any positive $\phi \in \mathscr{D}(\Omega)$, define Γ^{ε} as

$$\Gamma^{\varepsilon} = \int_{\Omega} ((A^{\varepsilon} - B^{\varepsilon}) \nabla u^{\varepsilon}, \nabla v^{\varepsilon}) \phi \, dx \tag{A.7}$$

$$= \int_{\Omega} (A^{\varepsilon} \nabla u^{\varepsilon}, \nabla v^{\varepsilon}) \phi \, dx - \int_{\Omega} (B^{\varepsilon} \nabla u^{\varepsilon}, \nabla v^{\varepsilon}) \phi \, dx .$$

Integrating (A.7) by parts and passing to the limit, as ε tends to zero, in the resulting equation leads to

$$\Gamma^{\varepsilon} \to \Gamma = \int_{\Omega} ((A - B) \nabla u, \nabla v) \phi \, dx . \tag{A.8}$$

From (A.3) and (A.7),

$$|\Gamma^{\varepsilon}| \leqslant \int_{\Omega} |A^{\varepsilon} - B^{\varepsilon}| \, |\nabla u^{\varepsilon}| \, |\nabla v^{\varepsilon}| \, \phi \, dx \leqslant \gamma \int_{\Omega} |\nabla u^{\varepsilon}| \, |\nabla v^{\varepsilon}| \, \phi \, dx . \tag{A.9}$$

Since

$$|\nabla u^{\varepsilon}|^2 \leqslant \frac{1}{a} \Re(A^{\varepsilon} \nabla u^{\varepsilon}, \nabla u^{\varepsilon}), \quad |\nabla v^{\varepsilon}|^2 \leqslant \frac{1}{a'} \Re(B^{\varepsilon} \nabla v^{\varepsilon}, \nabla v^{\varepsilon}) ,$$

relation (A.9) implies, with the help of Hölder's inequality, that

(A.10)

$$|\Gamma^{\varepsilon}| \leqslant \gamma \frac{1}{\sqrt{aa'}} \left(\int_{\Omega} \Re(A^{\varepsilon} \nabla u^{\varepsilon}, \nabla u^{\varepsilon}) \phi \, dx \right)^{\frac{1}{2}} \left(\int_{\Omega} \Re(B^{\varepsilon} \nabla v^{\varepsilon}, \nabla v^{\varepsilon}) \phi \, dx \right)^{\frac{1}{2}} .$$

The limit of the right-hand side of (A.10) is once more computed through integration by parts. By (A.8) and since $A \in M(a, b ; \Omega ; \mathbb{C})$ and $B \in M(a', b' ; \Omega ; \mathbb{C})$, we obtain

$$|\Gamma^{\varepsilon}| \leqslant \gamma \sqrt{\frac{bb'}{aa'}} \left(\int_{\Omega} \phi |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{\Omega} \phi |\nabla v|^2 \, dx \right)^{\frac{1}{2}} . \tag{A.11}$$

Choosing $f = - \operatorname{div} (A \nabla [(\lambda, x) \psi(x)])$, $g = - \operatorname{div} (B \nabla [(\mu, x) \psi(x)])$, where $\psi = 1$ on the support of ϕ, and λ, μ are two arbitrary elements of $\mathbb{C}^N$, yields

$$\left| \int_\Omega ((A - B)\, \lambda, \mu)\, \phi \, dx \right| \leqslant \gamma \sqrt{\frac{bb'}{aa'}}\, |\lambda|\, |\mu| \int_\Omega \phi \, dx\,. \tag{A12}$$

Estimate (A.4) immediately follows from (A.12) and the arbitrary character of ϕ. □

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