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RESOLUTION OF THE MAXWELL EQUATIONS IN A DOMAIN WITH REENTRANT CORNERS (*)

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Abstract — *In the case when the computational domain is a polygon with reentrant corners, we give a decomposition of the solution of Maxwell's equations into the sum of a regular part and a singular part It is proved that the space to which the singular part belongs is spanned by the solutions of a steady state problem The precise regularity of the solution is given depending on the angle of the reentrant corners The mathematical decomposition is then used to introduce an algorithm for the numerical resolution of Maxwell's equations in presence of reentrant corners This paper is a continuation of the work exposed in [3] The same methodology can be applied to the Helmholtz equation or to the Lamé system as well © Elsevier, Paris*

Résumé — *Lorsque le domaine de calcul est un polygone non convexe, c'est-à-dire avec un ou plusieurs coins rentrants, nous donnons une décomposition de la solution des équations de Maxwell en une partie régulière et une partie singulière Nous prouvons que l'espace des parties singulières est engendré par les solutions d'un problème stationnaire simple La régularité exacte de la solution est déterminée en fonction de l'angle aux coins rentrants Cette décomposition mathématique permet alors de construire un algorithme de résolution numérique des équations de Maxwell dans un polygone non convexe Cet article est la suite de la note [3] Cette méthodologie peut également s'appliquer à l'équation de Helmholtz ou au système de Lamé © Elsevier, Paris*

1. INTRODUCTION

The resolution of the steady-state or time-dependent Maxwell equations in a bounded domain has become classical thanks to finite difference methods in rectangular domains or finite element methods conforming in $H(\text{curl})$ [33] or mixed, conforming in $H(\text{curl}, \text{div})$ ([5], [14]) in more complicated geometries. However, when the boundary is not regular and when the domain is not convex, that is in presence of reentrant corners, the mesh needs to be refined drastically in the neighborhood of the reentrant corners in order to get an acceptable numerical solution (see [7], [18] among others for a study of this approach). Another method consists in using special singular shape functions (see for instance [21], [27]). It is however generally accepted that grid refinement is a better approach, except in some special cases ([20], [25]). Let us finally mention the more recent approach called the method of auxiliary mapping which deals with elliptic boundary value problems with singularities ([8], [10] or [34]).

In this work, we are going to study this problem in a bounded domain of $\mathbb{R}^2$. Physically, this can describe a 3D problem in which the electromagnetic field is independent of one of the three space variables (x, y, z), which we assume to be z: in this case, we are working in a plane perpendicular to the Oz axis. This happens for example in an infinite cylinder of axis Oz, when the electromagnetic field is independent of z.

In this paper we shall introduce several methods to solve numerically the Maxwell equations in domains with reentrant corners as well in their steady-state as in their time-dependent form, with a perfectly conducting boundary condition. More precisely, following the work of Grisvard for the Laplace problem and the wave equation [26], we shall introduce a decomposition of the $L^2(\Omega)$ space, from which we shall obtain a decomposition of the solution of the Maxwell equations in a "regular" part and a "singular" part. Then we shall show how to calculate the singular part in order to reduce the problem to the numerical computation of the regular part of the electromagnetic field which can be done with a usual method.

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Concerning the study of the singularities of the wave equation as well as the Maxwell equations in a unbounded domain, we refer the reader to the work of Gérard and Lebeau [22] and Lafitte ([29], [30]), who deal with the problem of the diffraction of a wave incident to a curved corner with perfectly conducting as well as mixed (of impedance type) boundary conditions. In the case of a conical geometry which allows to use polar coordinates, we refer the reader to the work of Cessenat ([11, 12]) who solves problems linked to the Helmholtz equation in polar coordinates with the Sommerfeld radiation condition. These studies obviously yield a useful basis for the treatment of the steady-state problem, at a given non vanishing frequency, comparable to the one we propose hereafter for the time-dependent problem. However, the methodology we apply here on the time-dependent Maxwell equations can be straightforwardly extended to the Lamé system, or to the Helmholtz equation by substituting $\mathbb{C}$ for $\mathbb{R}$.

This paper is organized as follows. In Section 2, we introduce the notations and useful properties of some functional spaces. In Section 3, the model problems (steady-state and time-dependent) are presented. Then, the orthogonal space decompositions are introduced in Section 4, from which the decomposition of the solution into a regular and a singular part is obtained. Section 5 is devoted to the computation of the solution: we first present a determination of a basis of the singular part by using several formulations, and then the resolution of the time-dependent regular part. Finally, concluding remarks and perspectives are given in Section 6. For the sake of simplicity, we restrict ourselves in these Sections to the case of a single reentrant corner, and to the boundary condition $\mathbf{u} \cdot \tau = 0$. The case when $\mathbf{u} \cdot \nu = 0$ on the boundary is postponed to appendix A, and the general case of several reentrant corners is addressed in appendix B.

2. NOTATIONS AND PROPERTIES OF SOME FUNCTIONAL SPACES

Let Ω be a connected and simply connected polygon of $\mathbb{R}^2$ with a boundary Γ for which all the angles at the vertices have a value not greater than π, except for one reentrant corner whose angle is $\frac{\pi}{\alpha}$ with $1/2 < \alpha < 1$ (see *fig.* 1). We denote by Ω^c an open angular sector in the neighborhood of the reentrant corner and by Γ^c its boundary. We call Ω^e the open subdomain such that $\Omega^c \cap \Omega^e = \Phi$ and $\overline{\Omega^c} \cup \overline{\Omega^e} = \overline{\Omega}$, and Γ^e its boundary. Finally, we call $\mathcal{B}$ the boundary $\Gamma^c \cap \Gamma^e$ and we decompose Γ^c (respectively Γ^e) in $\Gamma^c = \mathcal{B} \cup \tilde{\Gamma}^c$ (resp. $(\Gamma^e = \mathcal{B} \cup \tilde{\Gamma}^e)$.

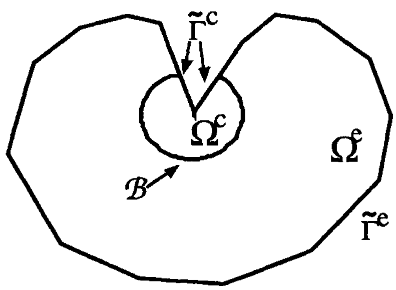

Figure 1. — Shape of the domain Ω.

As we are working here in a domain of $\mathbb{R}^2$, there exists a scalar curl operator which maps $\mathbb{R}^2$-valued functions into $\mathbb{R}$-valued functions and a vector curl operator **curl** which maps $\mathbb{R}$-valued functions into $\mathbb{R}^2$-valued functions. In order to avoid confusions, we shall write in bold face the functions and operators having vector values. The extension to $\mathbb{C}$-valued functions yields similar results on the Helmholtz equation. We shall denote by

$$H(\text{curl}, \Omega) = \{\mathbf{v} \in L^2(\Omega)^2, \text{curl } \mathbf{v} = \partial_x v_y - \partial_y v_x \in L^2(\Omega)\},$$

and

$$L_0^2(\Omega) = \left\{ f \in L^2(\Omega), \int_\Omega f\, d\mathbf{x} = 0 \right\}, H_0(\mathrm{curl}, \Omega) = \{ \mathbf{v} \in H(\mathrm{curl}, \Omega), \mathbf{v} \cdot \tau = 0 \text{ on } \Gamma \} .$$

For a function f, we have

$$\mathbf{curl}\, f = \begin{pmatrix} \partial_y f \\ - \partial_x f \end{pmatrix} \tag{1}$$

and so $\mathbf{curl}\, f \in L^2(\Omega)^2$ if and only if $\nabla f \in L^2(\Omega)^2$. Hence the space that we could denote by $H(\mathbf{curl}, \Omega)$ as above is identical to $H^1(\Omega)$. On the other hand, if $\nu = (\nu_x, \nu_y)$ is the outgoing normal vector at any point of the domain (except the corners) we denote by $\tau = (\nu_y, -\nu_x)$ the associated tangent vector.

We shall need the following functional spaces:

$$H(\mathrm{div}\, 0\, ; \Omega) = \{ \mathbf{v} \in L^2(\Omega)^2, \mathrm{div}\, \mathbf{v} = 0 \}, \text{ as well as } \quad V = \{ \mathbf{v} \in H_0(\mathrm{curl}, \Omega), \mathrm{div}\, \mathbf{v} = 0 \}$$

the Hilbert space endowed with the canonical scalar product of $H(\mathrm{curl}, \Omega)$. And also Φ the space of stream functions:

$$\Phi = \left\{ \varphi \in H^1(\Omega), \Delta\varphi \in L^2(\Omega), \frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \Gamma \right\} .$$

It can be easily checked that

LEMMA 2.1: *We have the following vector space isomorphisms:*
1. *The* curl *operator defines an isomorphism from* V *onto* $L_0^2(\Omega)$.
2. *The* **curl** *operator defines an isomorphism from* $\Phi/\mathbb{R}$ *onto* V.
3. *The* Δ *operator defines an isomorphism from* $\Phi/\mathbb{R}$ *onto* $L_0^2(\Omega)$.

In the case when the boundary Γ is of class C^2, or in the case when the domain Ω is convex with a Lipschitz continuous boundary Γ, the space V is included in $H^1(\Omega)^2$ (see for example Girault-Raviart [23]) and the space Φ is included in $H^2(\Omega)$ (see for example Grisvard [26]). This is not true anymore in presence of reentrant corners. Hence we need to introduce the regularized subspaces of V and Φ :

$$V_R = V \cap H^1(\Omega)^2 = \{ \mathbf{v} \in H^1(\Omega)^2, \mathrm{div}\, \mathbf{v} = 0, \mathbf{v} \cdot \tau = 0 \text{ on } \Gamma \}$$

and

$$\Phi_R = \Phi \cap H^2(\Omega) = \left\{ \varphi \in H^2(\Omega), \frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \Gamma \right\} .$$

3. THE MODEL PROBLEMS

3.1. The steady-state problem

Given a function $f \in L_0^2(\Omega)$, we consider the following problem:
Find $\mathbf{u} \in H(\mathrm{curl}, \Omega)$ *such that:*

$$\mathrm{curl}\, \mathbf{u} = f \quad \text{in } \Omega \tag{2}$$

$$\mathrm{div}\, \mathbf{u} = 0 \quad \text{in } \Omega \tag{3}$$

$$\mathbf{u} \cdot \tau = 0 \quad \text{on } \Gamma \tag{4}$$

PROPOSITION 3.1: *Take* $f \in L_0^2(\Omega)$. *Then problem (2)-(4) admits a unique solution* $\mathbf{u} \in H(\mathrm{curl}, \Omega)$.

Proof: Let us use the associated stream function. Due to lemma 2.1, every function $\mathbf{u} \in V$ is associated to one and only one function $\phi \in \Phi/\mathbb{R}$ such that $\mathbf{curl}\,\phi = \mathbf{u}$, and we have $\mathrm{curl}\,\mathbf{u} = \mathrm{curl}\,\mathbf{curl}\,\phi = -\Delta\phi$. Problem (2)-(4) is therefore equivalent to the following problem:

$$-\Delta\phi = f \quad \text{in } \Omega$$

$$\frac{\partial\phi}{\partial\nu} = 0 \quad \text{on } \Gamma .$$

This is a Laplace problem with a Neumann boundary condition, which, as the compatibility condition $\int_\Omega f\,d\mathbf{x} = 0$ is fulfilled, admits a unique solution ϕ in $H^1(\Omega)/\mathbb{R}$. ∎

Remark 3.1: In this section, $\mathbf{u}$ stands for the electric field, thus (3) corresponds to the Coulomb equation with a zero right-band side. Nevertheless, the more general problem, whith $g \in L^2(\Omega)$, in which (3) is replaced by

$$\mathrm{div}\,\mathbf{u} = g \text{ in } \Omega ,$$

can be brought back to the previous problem by letting $\mathbf{w} = \mathbf{u} - \nabla\psi$, ψ being the unique element of $H_0^1(\Omega)$ verifying $\Delta\psi = g$. The function $\mathbf{w}$ then satisfies indeed problem (2)-(4) and ψ verifies a Laplace problem which has been studied exhaustively by Grisvard [26]. ∎

3.2. The time-dependent problem

Given a function $\mathbf{f}(t) \in L^2([0,T]\,;L^2(\Omega)^2)$ such that $\mathrm{div}\,\mathbf{f} = 0$ and two functions $\mathbf{u}_0 \in V$ and $\mathbf{u}_1 \in H(\mathrm{div}\,0\,;\Omega)$ which do not depend on time, we consider now the following problem:

Find $\mathbf{u}(t) \in L^2([0,T]\,;H_0(\mathrm{curl},\Omega))$, $\partial\mathbf{u}/\partial t(t) \in L^2([0,T]\,;H(\mathrm{div}\,0\,;\Omega))$ *such that*

$$\frac{\partial^2\mathbf{u}}{\partial t^2} + \mathbf{curl}\,\mathrm{curl}\,\mathbf{u} = \mathbf{f} \text{ in } \Omega \tag{5}$$

$$\mathrm{div}\,\mathbf{u} = 0 \text{ in } \Omega \tag{6}$$

$$\mathbf{u}\,.\,\tau = 0 \text{ on } \Gamma \tag{7}$$

with the initial conditions

$$\mathbf{u}(0) = \mathbf{u}_0 \tag{8}$$

$$\frac{\partial\mathbf{u}}{\partial t}(0) = \mathbf{u}_1 . \tag{9}$$

These equations can be written in variational form:

Find $\mathbf{u}(t) \in L^2([0,T]\,;H_0(\mathrm{curl},\Omega))$ *such that*

$$\frac{d^2}{dt^2}\int_\Omega \mathbf{u}\,.\,\mathbf{v}\,d\mathbf{x} + \int_\Omega \mathrm{curl}\,\mathbf{u}\,\mathrm{curl}\,\mathbf{v}\,d\mathbf{x} = \int_\Omega \mathbf{f}\,.\,\mathbf{v}\,d\mathbf{x}, \quad \forall\mathbf{v} \in H_0(\mathrm{curl},\Omega) \tag{10}$$

$$\mathrm{div}\,\mathbf{u} = 0 \text{ in } \Omega \tag{11}$$

with the initial conditions

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \frac{\partial\mathbf{u}}{\partial t}(0) = \mathbf{u}_1 . \tag{12}$$

PROPOSITION 3.2: *Let* $\mathbf{f} \in L^2([0, T]; \mathbf{H}(\operatorname{div} 0; \Omega))$, $\mathbf{u}_0 \in V$ *and* $\mathbf{u}_1 \in H(\operatorname{div} 0; \Omega)$. *Then problem (10)-(12) admits a unique solution* $\mathbf{u}$ *such that* $\mathbf{u} \in C^0([0, T]; V) \cap C^1([0, T]; H(\operatorname{div} 0; \Omega))$.

Proof: Apply the variational theory of Lions-Magenes [31], Tome 1, p. 286. ■

Remark 3.2: As in the case of the steady-state problem, the more general problem where (6) is replaced, for $g \in C^2([0, T]; L^2(\Omega))$, by

$$\operatorname{div} \mathbf{u} = g \text{ in } \Omega,$$

and the compatibility condition $\operatorname{div} \mathbf{f} = 0$ is replaced by

$$\frac{\partial^2 g}{\partial t^2} - \operatorname{div} \mathbf{f} = 0 \text{ in } \Omega$$

can be brought back to a problem of type (5)-(9). Indeed, taking $\mathbf{w} = \mathbf{u} - \nabla\psi$ where ψ is the unique element of $H_0^1(\Omega)$ such that $\Delta\psi = g$, $\mathbf{w}$ belongs to V and verifies

$$\frac{\partial^2 \mathbf{w}}{\partial t^2} + \mathbf{curl} \operatorname{curl} \mathbf{w} = \mathbf{f} - \frac{\partial^2 \nabla\psi}{\partial t^2} \text{ in } \Omega$$

with $\operatorname{div}\left(\mathbf{f} - \dfrac{\partial^2 \nabla\psi}{\partial t^2}\right) = 0.$ ■

4. DECOMPOSITION OF THE SOLUTION INTO A REGULAR PART AND A SINGULAR PART

4.1. Space decomposition

LEMMA 4.1: *The* L^2*-norm of the* curl *defines on* V *a norm which is equivalent to the canonical norm of* $H(\operatorname{curl}, \Omega)$.

Proof: It is clear that for all $\mathbf{v} \in V$ we have

$$\|\operatorname{curl} \mathbf{v}\|_{L^2} \leqslant \|\mathbf{v}\|_{L^2} + \|\operatorname{curl} \mathbf{v}\|_{L^2}.$$

For the other inequality, we associate to any $\mathbf{v} \in V$ its stream function $\varphi \in \Phi/\mathbb{R}$ such that $\mathbf{curl}\, \varphi = \mathbf{v}$, and we have $\operatorname{curl} \mathbf{v} = \operatorname{curl} \mathbf{curl}\, \varphi = -\Delta\varphi$. It results, multiplying by φ and using a Green formula, that

$$\int_\Omega \nabla\varphi \cdot \nabla\varphi \, d\mathbf{x} = \int_\Omega \operatorname{curl} \mathbf{v}\varphi \, d\mathbf{x} \leqslant \|\operatorname{curl} \mathbf{v}\|_{L^2} \|\varphi\|_{L^2},$$

which yields, using the norm equivalence, $\varphi \mapsto \|\nabla\varphi\|_{L^2}$ and $\varphi \mapsto \|\varphi\|_{H^1}$ in $H^1(\Omega)/\mathbb{R}$, see theorem 1.9, chapter 1 of [23]: $\exists C_1 > 0$ such that

$$\|\nabla\varphi\|_{L^2} \leqslant C_1 \|\operatorname{curl} \mathbf{v}\|_{L^2}.$$

Then as

$$\|\nabla\varphi\|_{L^2} = \|\mathbf{curl}\, \varphi\|_{L^2} = \|\mathbf{v}\|_{L^2}$$

we finally get that

$$\|\mathbf{v}\|_{L^2} + \|\operatorname{curl} \mathbf{v}\|_{L^2} \leqslant (1 + C_1)\|\operatorname{curl} \mathbf{v}\|_{L^2}.$$

■

Remark 4.1: A similar result has been proved by Grisvard [26] for the space $\Phi/\mathbb{R}$. Here, the norm $\varphi \mapsto \|\Delta\varphi\|_{L^2}$ is equivalent to the canonical norm on $\Phi/\mathbb{R}$, i.e. $\varphi \mapsto (\|\varphi\|_{H^1} + \|\Delta\varphi\|_{L^2})^{\frac{1}{2}}$. ∎

COROLLARY 4.1: *Endowing $\Phi/\mathbb{R}$ with the L^2-norm of the laplacian and V with the L^2-norm of the curl, the isomorphisms defined by lemma 2.1 preserve orthogonality.*

We shall assume in the sequel that V and $\Phi/\mathbb{R}$ are endowed with those norms.

DEFINITION 4.1: *We shall denote by* curl V_R *the image of the space* V_R *by the* curl *operator.*

LEMMA 4.2: *The space V_R and* curl V_R *are closed in V and $L_0^2(\Omega)$ respectively.*

Proof: Thanks to a result of Costabel [16] (see also Moussaoui [32]), we have for all $\mathbf{v} \in V_R$,

$$\|\mathbf{v}\|_{L^2}^2 + \|\operatorname{curl} \mathbf{v}\|_{L^2}^2 = \|\mathbf{v}\|_{H^1}^2 .$$

The claimed closure properties then result of the completeness of $H^1(\Omega)$ for its canonical norm. ∎

DEFINITION 4.2: *We shall denote by $\Delta\Phi_R$ the image of Φ_R by the Laplace operator and we let* $N = (\operatorname{curl} V_R)^{\perp}$.

LEMMA 4.3: *The space* curl V_R *is identical to $\Delta\Phi_R$ and the space N is of dimension 1. We have the direct orthogonal sum*

$$L_0^2(\Omega) = \operatorname{curl} V_R \overset{\perp}{\oplus} N .$$

Proof: Let $\mathbf{v} \in V_R$. An element $\varphi \in \Phi$ can be associated to it using isomorphism 2 of lemma 2.1. As $\mathbf{v} \in H^1(\Omega)^2$ and $\mathbf{v} = \mathbf{curl}\, \varphi$, we have $\nabla\varphi \in H^1(\Omega)^2$ and so $\varphi \in H^2(\Omega)$, which means that $\varphi \in \Phi_R$. We then have by definition $\operatorname{curl} \mathbf{v} = \Delta\varphi$, hence curl V_R is included in $\Delta\Phi_R$. In the same way, to $\varphi \in \Phi_R$, we can associate $\mathbf{v} \in V_R$ to show the converse inclusion. As $\Delta\Phi_R = \operatorname{curl} V_R$ is closed, we have by denoting $N = (\operatorname{curl} V_R)^{\perp}$ the following orthogonal decomposition

$$\begin{aligned} L_0^2(\Omega) &= \Delta\Phi_R \overset{\perp}{\oplus} N , \\ &= \operatorname{curl} V_R \overset{\perp}{\oplus} N . \end{aligned}$$

By definition, for $p \in N$ we have

$$\int_\Omega \Delta\varphi\, p\, d\mathbf{x} = 0 \quad \forall \varphi \in \Phi_R .$$

It follows that $\Delta p = 0$ in the sense of distributions. On the other hand, we can write a double Green formula (see theorem 1.5.3 of [26]) and define the trace of $\frac{\partial p}{\partial \nu}$ on each segment Γ_j of the boundary Γ in the space $(H_{00}^{\frac{3}{2}}(\Gamma_j))'$. We write in a "condensed" (and abusive) manner that $\frac{\partial p}{\partial \nu}$ is in $H^{-\frac{3}{2}}(\Gamma)$, and as $\frac{\partial \varphi}{\partial \nu} = 0$ we also find that $\frac{\partial p}{\partial \nu} = 0$ on the boundary. Finally, N is the vector space of functions $p \in L_0^2(\Omega)$ such that

$$\begin{aligned} \Delta p &= 0 \text{ in } \Omega , \\ \frac{\partial p}{\partial \nu} &= 0 \text{ on } \Gamma . \end{aligned}$$

It has been proved in Grisvard [26], theorem 2.3.7, that N is a one dimensional vector space. ∎

Remark 4.2: According to lemma 2.3.2 (i) and to theorem 2.3.3 of [26], p has to satisfy some compatibility conditions at the corners. We shall not describe these conditions here, knowing that they are automatically satisfied by the local expressions of p (cf. infra, theorem 4.2). ∎

The lemmas that we have proved so far allow us to state an orthogonal decomposition of vector fields into singular and regular parts.

DEFINITION 4.3: *We shall denote by* V_S *(resp.* Φ_S*) the reciprocal image of* N *by the* curl *(resp. the Laplace) operator, i.e.* $V_S = \text{curl}^{-1} N$ *and* $\Phi_S = \Delta^{-1} N$.

THEOREM 4.1: *We have the following decompositions into direct orthogonal sums:*

$$V = V_R \overset{\perp}{\oplus} V_S ,$$

$$\Phi = \Phi_R \overset{\perp}{\oplus} \Phi_S .$$

Remark 4.3: The properties we give here for the spaces V and V_R in this section have their equivalent for the spaces Φ and Φ_R. These properties have been proved by Grisvard [26] in his study of the singularities of the Laplace problem. ■

Given $\mathbf{u} = \mathbf{u}_R + \mathbf{u}_S$ a solution in V of (2)-(4) or of (5)-(7), we shall call regular part of the solution $\mathbf{u}_R \in V_R$, and singular part $\mathbf{u}_S \in V_S$.

4.2. Regularity of the solution

In Ω^c, we can use polar coordinates (r, θ) centered on the reentrant corner, with $0 < r \leqslant R$, $0 \leqslant \theta \leqslant \frac{\pi}{\alpha}$. We have:

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}, \quad \mathbf{curl}\, f = \begin{pmatrix} \frac{1}{r}\frac{\partial f}{\partial \theta} \\ -\frac{\partial f}{\partial r} \end{pmatrix}$$

$$\text{curl}\, \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(r\, v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}, \quad \text{div}\, \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(r\, v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} .$$

According to a classical result that can be found for example in [26], if a function g regular outside the reentrant corner is identical to $r^\beta z(\theta)$ in Ω^c, with z regular and $\beta \in \mathbb{R} \backslash \mathbb{Z}$, then

$$g \in H^s(\Omega) \quad \text{if} \quad s < \beta + 1, \text{ and}$$

$$g \notin H^s(\Omega) \quad \text{if} \quad s \geqslant \beta + 1 .$$

THEOREM 4.2: *A function* $\mathbf{u}$ *of* $V \backslash V_R$ *belongs to* $H^{\alpha - \varepsilon}(\Omega)^2$ *for all* $\varepsilon > 0$ *and does not belong to* $H^\alpha(\Omega)^2$, *where* $\frac{\pi}{\alpha}$ *is the value of the angle at the rentrant corner* $\left(\frac{1}{2} < \alpha < 1\right)$.

Remark 4.4: This result precises the general regularity result $\mathbf{u} \in H^{1/2}(\Omega)^2$, obtained by Costabel [15] in any polyhedra. ■

Proof: Such a $\mathbf{u}$ can, according to theorem 4.1, be decomposed into two parts, one being in $H^1(\Omega)^2$, the other (non zero) part being in the less regular vector space V_S. According to lemma 2.1, V_S is of dimension 1.

According to lemma 2.3.4 of [26], the functions of N are regular outside a neighborhood of the corners. If we call $\mathbf{v}_S$ an element of V_S, there exists $p_S \in N$ such that curl $\mathbf{v}_S = p_S$. As moreover div $\mathbf{v}_S = 0$, we deduce that $\Delta \mathbf{v}_S = \mathbf{curl}\, p_S$. p_S being regular outside corners, this is also true for $\mathbf{v}_S$. The regularity of $\mathbf{u}$ will hence be that of any element of V_S in the neighborhood of these corners.

By definition of V_S, it is natural to start by studying the behavior of the functions of N near the corners For that, we consider first the neighborhood of the reentrant corner For all the other corners, it will suffice to substitute α' to α, where $\frac{\pi}{\alpha'}$ is the value of the angle at the considered vertex (in particular, we always have $\alpha' > 1$)

So we are looking for the functions S^0 solution of

Find $S^0 \in L^2(\Omega^c)$ non vanishing such that

$$\Delta S^0 = 0 \text{ in } \Omega^c , \tag{13}$$

$$\frac{\partial S^0}{\partial \nu} = 0 \text{ on } \tilde{\Gamma}^c , \tag{14}$$

S^0 not belonging to $H^1(\Omega^c)$ Note that the solutions of (13)-(14) form a vector space Using the method of separation of variables (mathematically justified in [26]) we find that all the terms of the sequence

$$(r^{n\alpha} \cos(n\alpha\theta))_{n \in \mathbb{Z}}$$

are solution of (13) (14) As $r^{n\alpha} \cos(n\alpha\theta) \in L^2(\Omega^c)$ if and only if $n \geqslant -1$ and as it belongs to $H^1(\Omega^c)$ for $n \geqslant 0$, S^0 can be written

$$S^0(r, \theta) = \sum_{n \geqslant \ 1} A_n r^{n\alpha} \cos(n\alpha\theta), \quad \text{with } A_{-1} \neq 0 \tag{15}$$

As N is a one dimensional vector space, the coefficients $(A_n)_{n \geqslant -1}$ are all related when S^0 is actually considered to be the restriction of an element of N (see subsection 5 1) The functions of V_S can be deduced from those of N by a lifting with the operator curl Thus, knowing S^0, we define $\mathbf{S}^1$ a local singular lifting of S^0, i e

Find $\mathbf{S}^1 \in L^2(\Omega^c)^2$ non vanishing such that

$$\operatorname{curl} \mathbf{S}^1 = S^0 \text{ in } \Omega^c , \tag{16}$$

$$\operatorname{div} \mathbf{S}^1 = 0 \text{ in } \Omega^c , \tag{17}$$

$$\mathbf{S}^1 \cdot \tau = 0 \text{ on } \tilde{\Gamma}^c , \tag{18}$$

$\mathbf{S}^1$ not belonging to $H^1(\Omega^c)^2$ Note that the solutions of (16)-(18) form an affine space, the associated vector space being the curl free functions verifying (17) and (18) A particular solution of (16) (18) is

$$\sum_{n \geqslant -1} A_n r^{n\alpha + 1} \begin{pmatrix} \dfrac{n\alpha}{4\,n\alpha + 4} \sin(n\alpha\theta) \\ \dfrac{n\alpha + 2}{4\,n\alpha + 4} \cos(n\alpha\theta) \end{pmatrix}$$

This particular solution belongs to $H^1(\Omega^c)^2$ The homogeneous solutions are themselves the terms of the sequence

$$\left(r^{n\alpha - 1} \begin{pmatrix} \sin(n\alpha\theta) \\ \cos(n\alpha\theta) \end{pmatrix} \right)_{n \in \mathbb{Z}}$$

Here both components belong to $L^2(\Omega^c)$ if and only if $n \geqslant 1$, and they belong to $H^1(\Omega^c)$ for $n \geqslant 2$ Hence

$$\mathbf{S}^1(r, \theta) = \sum_{n \geqslant 1} B_n r^{n\alpha - 1} \begin{pmatrix} \sin(n\alpha\theta) \\ \cos(n\alpha\theta) \end{pmatrix} + \sum_{n \geqslant -1} A_n r^{n\alpha + 1} \begin{pmatrix} \dfrac{n\alpha}{4\,n\alpha + 4} \sin(n\alpha\theta) \\ \dfrac{n\alpha + 2}{4\,n\alpha + 4} \cos(n\alpha\theta) \end{pmatrix}, \text{ with } B_1 \neq 0 \tag{19}$$

Thus, the least regular term, corresponding to $n = 1$, can be written $r^{\alpha - 1} z(\theta)$ it belongs to $H^s(\Omega^c)$ for $s < \alpha$ and does not belong to $H^\alpha(\Omega^c)$

In a neighborhood of the other corners, we notice easily that the functions of N can be expressed locally as

$$\sum_{n \geq 0} A'_n r'^{n \alpha} \cos(n'\alpha'\theta')$$

As $\alpha' > 1$, we must notice here that the condition according to which the function belongs to L^2 implies that the sum starts at $n' = 0$ In particular, the solution belongs to H^1

Concerning the local behavior of the functions of V_S, i e the lifting (16)-(18), we verify first that each component of the particular solution is in H^2 As for the homogeneous part, it can be written

$$\sum_{n \geq 1} B'_n r'^{n \alpha - 1} \begin{pmatrix} \sin(n'\alpha'\theta') \\ \cos(n'\alpha'\theta') \end{pmatrix}$$

Hence each of its component is locally in H^1

In short, the regularity of the functions belonging to V_S is the one at the neighborhood of the reentrant corner, this means that they belong to $H^s(\Omega)$ for $s < \alpha$ but not to $H^\alpha(\Omega)$ ■

Remark 4 5 S^0 belongs to $L^2(\Omega^c)$ by definition, which means that $\int_{\Omega^c} (S^0)^2 \, d\mathbf{x} < \infty$ After some algebra, we obtain

$$\frac{\pi}{2\alpha}\left[\frac{1}{2 - 2\alpha}(A_{-1})^2 R^{2 - 2\alpha} + \left\{A_{-1} A_1 + \frac{A_0^2}{2}\right\} R^2 + \sum_{n \geq 1} \frac{1}{2 + 2n\alpha} A_n^2 R^{2 + 2n\alpha}\right] < \infty$$

In particular, we deduce that for $\varepsilon > 0$, there exists $C(\varepsilon)$ such that,

$$\forall (r, \theta) \in \,]\varepsilon, R - \varepsilon[\, \times [0, \tfrac{\pi}{\alpha}], \ |S^0(r, \theta)| < C(\varepsilon)$$

In the same way, as $\mathbf{S}^1$ belongs to $L^2(\Omega^c)^2$, we obtain, for $\varepsilon > 0$, the existence of $C'(\varepsilon)$ such that,

$$\forall (r, \theta) \in \,]\varepsilon, R - \varepsilon[\, \times [0, \tfrac{\pi}{\alpha}], \ \|\mathbf{S}^1(r, \theta)\| < C'(\varepsilon)$$

■

COROLLARY 4 2 *Let $\phi \in \Phi$ be the antecedent (defined up to a constant) of* $\mathbf{u}$ *by the isomorphism* **curl**, $\mathbf{u} \in V \backslash V_R$ *Then, ϕ belongs to $H^{1 + \alpha - \varepsilon}(\Omega)$ for all $\varepsilon > 0$ and does not belong to $H^{1 + \alpha}(\Omega)$, where $\frac{\pi}{\alpha}$ is the value of the angle at the reentrant corner* $\left(\frac{1}{2} < \alpha < 1\right)$

Proof We deduce from theorem 4 1 that $\phi = \phi_R + \phi_S$, with $\phi_R \in \Phi_R$ and $\phi_S \in \Phi_S$ Moreover, ϕ_S is solution of the following problem, for $p_S \in N$

Find $\phi_S \in H^1(\Omega)$ such that

$$- \Delta \phi_S = p_S \text{ in } \Omega \,, \tag{20}$$

$$\frac{\partial \phi_S}{\partial \nu} = 0 \text{ on } \Gamma \tag{21}$$

According to remark 2 4 6 of [26], $\phi_S \in H^{1 + \alpha - \varepsilon}(\Omega)$, for all $\varepsilon > 0$, and $\phi_S \notin H^{1 + \alpha}(\Omega)$ ■

Remark 4.6: This result precises corollary 23.5 of Dauge [19], in the case we are interested in: in this corollary, it is proven that there exists a non negative constant δ_N depending only upon Ω such that $\phi \in H^{3/2 + \delta_N}(\Omega)$, for the Laplace problem with a Neumann boundary condition on a polygonal open domain with right-hand-side in $L^2(\Omega)$. ∎

COROLLARY 4.3: *All stream functions of Φ belong to $C^0(\Omega)$.*

Remark 4.7: In the case when the domain is in $\mathbb{R}^2$, recall that $H^1(\Omega) \not\subset C^0(\Omega)$. ∎

An explicit expression of ϕ_S can be obtained in a neighborhood of the reentrant corner. Indeed, the solution S^2 of

Find $S^2 \in H^1(\Omega^c)$ non vanishing such that

$$\frac{\partial S^2}{\partial \nu} = 0 \text{ on } \tilde{\Gamma}^c , \tag{22}$$

and, equivalently,

$$\text{either } \mathbf{curl}\, S^2 = \mathbf{S}^1 \text{ in } \Omega^c , \tag{23}$$

$$\text{or } \quad - \Delta S^2 = S^0 \text{ in } \Omega^c , \tag{24}$$

and S^2 not belonging to $H^2(\Omega^c)$, is of the form

$$S^2(r, \theta) = - \sum_{n \geq 1} \frac{B_n}{n\alpha} r^{n\alpha} \cos(n\alpha\theta) - \sum_{n \geq -1} \frac{A_n}{4\, n\alpha + 4} r^{n\alpha + 2} \cos(n\alpha\theta) . \tag{25}$$

Remark 4.8: S^2 belongs to $H^1(\Omega^c)$ by definition. Hence, for $\varepsilon > 0$, there exists $C''(\varepsilon)$ such that,

$$\forall (r, \theta) \in \,]\varepsilon, R - \varepsilon[\times [0, \tfrac{\pi}{\alpha}], \; |S^2(r, \theta)| < C''(\varepsilon) .$$

The fact that its gradient is bounded is a direct consequence of (23). ∎

5. COMPUTATION OF THE SOLUTION

5.1. Determination of a basis of N

The space N being of dimension 1, we only need to exhibit a non vanishing element of N. We shall denote it by p_S. We recall that $\mathscr{B}$ stands for the arc of circle of radius R being in the domain Ω. The computation of p_S uses the method called "Dirichlet-to-Neumann (DtN)" by Keller and Givoli [28]. This method, developed initially in order to bring a problem posed on an infinite domain back to a bounded domain for numerical purposes, has then been extended to handle singularities at reentrant corners (see [24]). We find here a particular case of the theory of Steklov-Poincare operators (see Agoshkov [1]). The method can be split up into three steps:

1. Analytical computation of the singular local solution in the neighborhood of the reentrant corner.
2. Determination of the Dirichlet-to-Neumann operator in order to obtain, with the help of the transmission conditions, the boundary condition for the outer problem on $\mathscr{B}$.
3. Numerical resolution. First, of the outer problem, whose solution is then exactly the restriction of the solution of the initial problem. Then, numerical reconstruction of the solution in the neighborhood of the reentrant corner, using again the transmission conditions.

Remark 5.1: This method offers a double avantage. First, it yields an explicit expression of p_S (see (31)) in the neighborhood of the reentrant corner. On the other hand, p_S being smooth enough away from this corner, a variational formulation can be used to find it there (see (36)-(38)). Finally, the explicit knowledge of p_S will enable us to preserve the orthogonality in V and Φ between the regular and singular parts of $\mathbf{u}$ and ϕ, which is not the case if we regularize "locally", i.e., if we substract from p_S the term $A_{-1} r^{-\alpha} \cos(\alpha\theta)\, \eta(r)$, where η is a regular cut-off function (cf. [26], theorem 2.4.3). ■

More precisely, in order to determine a basis of N, we are looking for p_S a non vanishing solution of:

Find $p_S \in L_0^2(\Omega)$ such that

$$\Delta p_S = 0 \text{ in } \Omega\,, \tag{26}$$

$$\frac{\Delta p_S}{\partial \nu} = 0 \text{ on } \Gamma\,. \tag{27}$$

The restriction of p_S to Ω^c, denoted by p_S^c, verifies in particular (13)-(14). We have previously computed (see (15)) a family of local solutions S^0 in $L^2(\Omega^c)$: $S^0(r, \theta) = \sum_{n \geqslant -1} A_n r^{n\alpha} \cos(n\alpha\theta)$. This will enable us to complete point 1 (computation at the neighborhood of the reentrant corner). Indeed, we shall express each of the A_n as a function of the trace of S^0 on $\mathscr{B}$, by using the orthogonality of $\theta \mapsto \cos(m\alpha\theta)$ for the different $m \geqslant 0$. Thus, by integrating $S^0(R, \theta) \cos(m\alpha\theta)$ from 0 to π/α in θ, we obtain:

$$m \geqslant 2 \quad \int_0^{\pi/\alpha} \left\{ \sum_{n \geqslant -1} A_n R^{n\alpha} \cos(n\alpha\theta) \right\} \cos(m\alpha\theta)\, d\theta = \frac{\pi}{2\,\alpha} R^{m\alpha} A_m\,, \tag{28}$$

$$m = 1, -1 \quad \int_0^{\pi/\alpha} \left\{ \sum_{n \geqslant -1} A_n R^{n\alpha} \cos(n\alpha\theta) \right\} \cos(\alpha\theta)\, d\theta = \frac{\pi}{2\,\alpha} (R^{-\alpha} A_{-1} + R^{\alpha} A_1)\,, \tag{29}$$

$$m = 0 \quad \int_0^{\pi/\alpha} \left\{ \sum_{n \geqslant -1} A_n R^{n\alpha} \cos(n\alpha\theta) \right\} d\theta = \frac{\pi}{\alpha} A_0\,. \tag{30}$$

p_S^c can hence be written as

$$\sum_{n \geqslant -1} A_n r^{n\alpha} \cos(n\alpha\theta)\,, \tag{31}$$

with, for $n \geqslant 2$:

$$A_n = \frac{2\,\alpha}{\pi} R^{-n\alpha} \int_0^{\pi/\alpha} p_S^c(R, \theta) \cos(n\alpha\theta)\, d\theta\,. \tag{32}$$

The value of A_0 is given by (30). However, the value of either A_{-1} or A_1 is undetermined, as we can not solve (29). To overcome this problem, we simply add the relationship

$$\int_0^{\pi/\alpha} \frac{\partial p_S^c}{\partial \nu}(R, \theta) \cos(\alpha\theta)\, d\theta = \frac{\pi}{2\,R}(-R^{-\alpha} A_{-1} + R^{\alpha} A_1)\,. \tag{33}$$

This, together with (29), removes the indetermination. Therefore, we can choose to express A_1 as a function of A_{-1}, which does not vanish by definition:

$$A_1 = \frac{2\,\alpha}{\pi} R^{-\alpha} \int_0^{\pi/\alpha} p_S^c(R, \theta) \cos(\alpha\theta)\, d\theta - R^{-2\alpha} A_{-1}\,. \tag{34}$$

Let us now proceed to point 2. We define the Dirichlet-to-Neumann operator T: $p^c_{S|\mathscr{B}} \mapsto \frac{\partial p^c_S}{\partial \nu^c}\Big|_{\mathscr{B}}$, from $H^{1/2}(\mathscr{B})$ to $(H^{1/2}_{00}(\mathscr{B}))'$. So we take the trace on $\mathscr{B}$, that is at $r = R$, of the normal derivative of p^c_S, and by injecting the expressions (32) and (34) of the A_n, we obtain:

$$T(p^c_{S|\mathscr{B}}) = \frac{2\,\alpha^2}{\pi R} \sum_{n \geqslant 1} n \left\{ \int_0^{\pi/\alpha} p^c_S(R, \theta') \cos(n\alpha\theta')\, d\theta' \right\} \cos(n\alpha\theta) - 2\,\alpha \frac{A_{-1}}{R^{\alpha+1}} \cos(\alpha\theta)\,. \tag{35}$$

As a function of $p^c_{S|\mathscr{B}}$ only, The operator T such as it is defined above is not univalent, because the value of either A_{-1} or A_1 is undetermined (cf. (29)): here again, the relationship (33) removes the undetermination.

Concerning point 3, let p^e_S be the restriction of p_S on Ω^e. Let us show that the transmission conditions $p^e_{S|\mathscr{B}} = p^c_{S|\mathscr{B}}$ and $\frac{\partial p^e_S}{\partial \nu^c}\Big|_{\mathscr{B}} = \frac{\partial p^c_S}{\partial \nu^c}\Big|_{\mathscr{B}}$ given by Agoshkov [1] for the H^1 case are still valid for p_S in $L^2(\Omega)$ only.

We have, for all $z \in \mathscr{D}(\Omega)$:

$$\int_\Omega z\,\Delta p_S\, d\mathbf{x} = 0\,.$$

Taking either $z \in \mathscr{D}(\Omega^c)$, or $z \in \mathscr{D}(\Omega^e)$, we get

$$\Delta p^c_S = 0 \text{ in } \Omega^c, \quad \Delta p^e_S = 0 \text{ in } \Omega^e\,.$$

On the other hand, the boundary condition $\frac{\partial p_S}{\partial \nu} = 0$ is verified in $H^{-3/2}(\Gamma)$ (more precisely, according to [26], on a product space, each space being defined on a segment Γ_j of the polygonal boundary Γ, and equal to $(H^{3/2}_{00}(\Gamma_j))'$). From there on, we find immediately that

$$\frac{\partial p^c_S}{\partial \nu^c} = 0 \text{ in } (H^{3/2}_{00}(\tilde{\Gamma}^c))', \quad \frac{\partial p^e_S}{\partial \nu^e} = 0 \text{ in } (\mathrm{H}^{3/2}_{00}(\tilde{\Gamma}^e))'.$$

Concerning the transmission conditions, applying the integration by parts formula of theorem 1.5.3 of [26] yields, for all $\varphi \in \Phi_R$:

$$\begin{aligned}
0 &= \int_\Omega \varphi\,\Delta p_S\, d\mathbf{x} \\
&= \int_{\Omega^c} \varphi\,\Delta p^c_S\, d\mathbf{x} + \int_{\Omega^e} \varphi\,\Delta p^e_S\, d\mathbf{x} \\
&= \int_{\Omega^c} \Delta\varphi p^c_S\, d\mathbf{x} + \left\langle \frac{\partial p^c_S}{\partial \nu^c}, \varphi \right\rangle_{H^{3/2}_{00}(\mathscr{B})', H^{3/2}_{00}(\mathscr{B})} - \left\langle p^c_S, \frac{\partial \varphi}{\partial \nu^c} \right\rangle_{H^{1/2}_{00}(\mathscr{B})', H^{1/2}_{00}(\mathscr{B})} \\
&\quad + \int_{\Omega^e} \Delta\varphi p^e_S\, d\mathbf{x} + \left\langle \frac{\partial p^e_S}{\partial \nu^e}, \varphi \right\rangle_{H^{3/2}_{00}(\mathscr{B})', H^{3/2}_{00}(\mathscr{B})} - \left\langle p^e_S, \frac{\partial \varphi}{\partial \nu^e} \right\rangle_{H^{1/2}_{00}(\mathscr{B})', H^{1/2}_{00}(\mathscr{B})} \\
&= \left\langle \frac{\partial p^c_S}{\partial \nu^c} - \frac{\partial p^e_S}{\partial \nu^c}, \varphi \right\rangle_{H^{3/2}_{00}(\mathscr{B})', H^{3/2}_{00}(\mathscr{B})} - \left\langle p^c_S - p^e_S, \frac{\partial \varphi}{\partial \nu^c} \right\rangle_{H^{1/2}_{00}(\mathscr{B})', H^{1/2}_{00}(\mathscr{B})},
\end{aligned}$$

because p_S is orthogonal by definition in the sense of the canonical inner product on $L^2(\Omega)$ to all functions of $\Delta\Phi_R$

As the mappings $\varphi \mapsto \varphi_{|\mathscr{B}}$ and $\varphi \mapsto \frac{\partial \varphi}{\partial \nu^c}\Big|_{\mathscr{B}}$ are from Φ_R onto $H^{1/2}_{00}(\mathscr{B})$ and $H^{3/2}_{00}(\mathscr{B})$ respectively (see theorem 1 4 6 of [26]), we finally get the desired transmission conditions

Using them we can find the boundary condition verified by p_S^e on $\mathscr{B}$ In addition, we know from theorem 4 2 that p_S^e is H^1 regular Thus, the outer problem can be written

Find $p_S^e \in H^1(\Omega^e)/\mathbb{R}$ *such that*

$$\Delta p_S^e = 0 \quad \text{in } \Omega^e , \tag{36}$$

$$\frac{\partial p_S^e}{\partial \nu^e} = 0 \quad \text{on } \tilde{\Gamma}^e , \tag{37}$$

$$\frac{\partial p_S^e}{\partial \nu^e} + T(p_S^e) = 0 \quad \text{on } \mathscr{B} \tag{38}$$

Noticing that $\int_{\mathscr{B}} p_S^e(\sigma)\, d\sigma$ is nothing but $\int_0^{\pi/\alpha} p_S^e(\theta)\, R\, d\theta$, we can write these equations in variational form

$$\int_{\Omega^e} \nabla p_S^e \,.\, \nabla q \, d\mathbf{x} + R\int_0^{\pi/\alpha} T_1(p_S^e)\, q\, d\theta = 2\,\alpha \frac{A_{-1}}{R^\alpha}\int_0^{\pi/\alpha} \cos(\alpha\theta)\, q\, d\theta \quad \forall q \in H^1(\Omega^e)/\mathbb{R} , \tag{39}$$

where $T_1(\,.\,)$ stands for the first term of the right-hand-side of (35) We verify that the bilinear form $(p, q) \mapsto R\int_0^{\pi/\alpha} T_1(p)\, q\, d\theta$ is symmetric and positive Indeed

$$R\int_0^{\pi/\alpha} T_1(p)\, p\, d\theta = \frac{2\,\alpha^2}{\pi}\sum_{n \geqslant 1} n\left\{\int_0^{\pi/\alpha} p\cos(n\alpha\theta)\, d\theta\right\}^2$$

So, A_{-1} being fixed, this problem is well posed, as the term $\int_{\Omega^e} \nabla p \,.\, \nabla q\, d\mathbf{x}$ is coercive on $H^1(\Omega^e)/\mathbb{R}$ p_S^e being known, we use the transmission condition $p^c_{S|\mathscr{B}} = p^e_{S|\mathscr{B}}$ in order to determine the $(A_n)_{n \geqslant 0}$ with the help of (30), (32) and (34), and, as a consequence, p_S^c This procedure enables us to build p_S

Remark 5 2 According to what we have seen, $A_{-1} = 0$ corresponds to $p_S = 0$ For, if A_{-1} is equal to zero, the right-hand-side of (39) is also vanishing, which implies that p_S^e is zero as well On the other hand, when A_{-1} is equal to zero, p_S^c belongs to $H^1(\Omega^c)$ As moreover, on the one hand, $\frac{\partial p_S^c}{\partial \nu^c}\Big|_{\mathscr{B}} = 0$ (transmission condition), and on the other hand, p_S^c verifies (13)-(14), we deduce that p_S^c is also zero Conversely, if $p_S = 0$, $A_{-1} = 0$ straightforwardly The choice of A_{-1} induces a unique element p_S of the one dimensional vector space of solutions ∎

5.2. Determination of a basis of V_S

In the same way as in the previous section, we can find a basis of the space $V_S = \text{curl}^{-1} N$, for a given $p_S \in N$, by exhibiting a non vanishing element of V_S, which we shall call $\mathbf{v}_S$, solution of

Find $\mathbf{v}_S \in H(\text{curl}, \Omega)$ *such that*

$$\text{curl}\, \mathbf{v}_S = p_S \quad \text{in } \Omega, \tag{40}$$

$$\text{div}\, \mathbf{v}_S = 0 \quad \text{in } \Omega, \tag{41}$$

$$\mathbf{v}_S \,.\, \tau = 0 \quad \text{on } \Gamma \tag{42}$$

If we want to use the fact that $\mathbf{v}_S$ belongs to $H(\text{curl}, \Omega)$ in order to solve the above problem, we can for example transform it into a problem in "**curl** curl" An alternative way to find $\mathbf{v}_S$ is to use its stream function The following subsections describe these two methods in detail

5 2 1 *A* **curl** curl *formulation of problem (40) (42)*

Our aim is to find a non vanishing element $\mathbf{v}_S$ of V_S, solution of (40) (42)

PROPOSITION 5 1 *An element* $\mathbf{v}_S$ *of* $H(\text{curl}, \Omega)$ *is solution of (40)-(42) if and only if* $\mathbf{v}_S$ *verifies*

$$\mathbf{curl}\,\text{curl}\,\mathbf{v}_S = \mathbf{curl}\,p_S \quad \text{in } \Omega, \tag{43}$$

$$\text{div}\,\mathbf{v}_S = 0 \quad \text{in } \Omega, \tag{44}$$

$$\mathbf{v}_S \,.\, \tau = 0 \quad \text{on } \Gamma \tag{45}$$

Proof It is obvious that if $\mathbf{v}_S \in H(\text{curl}, \Omega)$ verifies (40) (42), it also verifies (43)-(45) Conversely, let $\mathbf{v}'_S \in H(\text{curl}, \Omega)$ solution of (43)-(45) Then, $(\mathbf{v}'_S - \mathbf{v}_S)$ belongs to $H(\text{curl}, \Omega)$ and verifies

$$\mathbf{curl}\,\text{curl}\,(\mathbf{v}'_S - \mathbf{v}_S) = 0 \quad \text{in } \Omega,$$

$$\text{div}\,(\mathbf{v}'_S - \mathbf{v}_S) = 0 \quad \text{in } \Omega,$$

$$(\mathbf{v}'_S - \mathbf{v}_S)\,.\,\tau = 0 \quad \text{on } \Gamma$$

From the first equation, as Ω is simply connected, we deduce that there exists a constant λ such that $\text{curl}\,(\mathbf{v}'_S - \mathbf{v}_S) = \lambda$ Moreover, the boundary condition allows us to write

$$\lambda|\Omega| = \int_\Omega \text{curl}\,(\mathbf{v}'_S - \mathbf{v}_S)\,d\mathbf{x} = \langle (\mathbf{v}'_S - \mathbf{v}_S)\,.\,\tau, 1\rangle_\Gamma = 0\,,$$

so $\lambda = 0$ We conclude thanks to proposition 3 1 ■

Remark 5 3 Neither **curl** curl $\mathbf{v}_S$ nor **curl** p_S belong to $L^2(\Omega)^2$ (43) is to be taken in the sense of distributions or, more precisely, in the dual space of $H_0(\text{curl}, \Omega)$ Also, as noticed earlier, (43) can be rewritten equivalently, as in a Stokes problem,

$$\Delta\mathbf{v}_S = \mathbf{curl}\,p_S$$

■

(i) We shall start with the global resolution of (43)-(45)

THEOREM 5 1 *Problem (43)-(45) admits a unique solution in* $H(\text{curl}, \Omega)$

Proof Dualize the divergence-free condition and use the theory of Babuska [6] and Brezzi [9] ■

Remark 5 4 The global regularity of the solution in Ω is the one given by theorem 4 2, that is $\mathbf{v}_S \in H^{\alpha-\varepsilon}(\Omega)^2$ for all $\varepsilon > 0$ and $\mathbf{v}_S \notin H^\alpha(\Omega)^2$ ■

(ii) Let us now reformulate this problem using the DtN method A few steps of the reasoning shall remain formal, i e without rigorous proof

In the **curl** curl formulation, the restriction of $\mathbf{v}_S$ to Ω^c, denoted by $\mathbf{v}^c_S$, verifies

$$\mathbf{curl}\,\text{curl}\,\mathbf{v}^c_S = \mathbf{curl}\,p^c_S \text{ in } \Omega^c,$$

$$\text{div}\,\mathbf{v}^c_S = 0 \text{ in } \Omega^c,$$

$$\mathbf{v}^c_S\,.\,\tau^c = 0 \text{ on } \tilde{\Gamma}^c$$

We notice that $\mathbf{v}_S^c$ verifies $\operatorname{curl} \mathbf{v}_S^c = p_S^c + \lambda$, with $\lambda \in \mathbb{R}$ (see the proof of proposition 5.1). So the vector $\mathbf{v}_S^c$ is identical to $\mathbf{S}^1$ (which satisfies (16)-(18)) up to a particular solution $\mathbf{S}_p^1$ which verifies $\operatorname{curl} \mathbf{S}_p^1 = \lambda$, $\operatorname{div} \mathbf{S}_p^1 = 0$ and $\mathbf{S}_p^1 \cdot \tau^c = 0$ on $\tilde{\Gamma}^c$. Let

$$\mathbf{S}_p^1(r, \theta) = \begin{pmatrix} 0 \\ \frac{\lambda r}{2} \end{pmatrix}. \tag{46}$$

So that $\mathbf{v}_S^c$ can be written, according to (19), with $B_1 \neq 0$

$$\mathbf{v}_S^c(r, \theta) = \sum_{n \geq 1} B_n r^{n\alpha - 1} \begin{pmatrix} \sin (n\alpha\theta) \\ \cos (n\alpha\theta) \end{pmatrix} + \sum_{n \geq -1} A_n r^{n\alpha + 1} \begin{pmatrix} \frac{n\alpha}{4n\alpha + 4} \sin (n\alpha\theta) \\ \frac{n\alpha + 2}{4n\alpha + 4} \cos (n\alpha\theta) \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{\lambda r}{2} \end{pmatrix} \tag{47}$$

The $(A_n)_{n \geq -1}$ of the second sum have been computed previously, see (30), (32) and (34). These numbers being known, using once more the orthogonality of $\theta \mapsto \cos (m\alpha\theta)$, we can express λ and each of the $(B_m)_{m \geq 1}$ as a function of the trace of $\mathbf{v}_S^c \cdot \tau^c$ on $\mathscr{B}$:

$$m = 0 \quad \lambda = \frac{2\alpha}{\pi R} \int_0^{\pi/\alpha} \mathbf{v}_S^c(R, \theta) \cdot \tau^c \, d\theta - A_0 , \tag{48}$$

$$m = 1 \quad B_1 = \frac{2\alpha}{\pi R^{\alpha - 1}} \int_0^{\pi/\alpha} \mathbf{v}_S^c(R, \theta) \cdot \tau^c \cos (\alpha\theta) \, d\theta$$

$$- \left(\frac{2 - \alpha}{4 - 4\alpha} A_{-1} R^{2 - 2\alpha} + \frac{2 + \alpha}{4 + 4\alpha} A_1 R^2 \right), \tag{49}$$

$$m \geq 2 \quad B_m = \frac{2\alpha}{\pi R^{m\alpha - 1}} \int_0^{\pi/\alpha} \mathbf{v}_S^c(R, \theta) \cdot \tau^c \cos (m\alpha\theta) \, d\theta - \frac{m\alpha + 2}{4m\alpha + 4} A_m R^2 . \tag{50}$$

This time $\mathcal{T}$ stands for the "DtN" operator defined by $\mathcal{T}$: $\mathbf{v}_S^c \cdot \tau^c_{|\mathscr{B}} \mapsto \operatorname{curl} \mathbf{v}_{S|\mathscr{B}}^c$. In order to build $\mathcal{T}$, from the respective expressions in polar coordinates of p_S^c and $\mathbf{v}_S^c$ (for $r = R$), we notice that the trace on $\mathscr{B}$ of the curl of $\mathbf{v}_S^c$ satisfies

$$\operatorname{curl} \mathbf{v}_{S|\mathscr{B}}^c = p_{S|\mathscr{B}}^c + \lambda \tag{51}$$

and we use (48) to obtain

$$\mathcal{T}(\mathbf{v}_S^c \cdot \tau^c) = \frac{2\alpha}{\pi R} \int_0^{\pi/\alpha} \mathbf{v}_S^c(R, \theta) \cdot \tau^c \, d\theta + p_S^c(R, \theta) - A_0 . \tag{52}$$

We now determine the transmission conditions in order to be able to describe completely the outer problem.

First, we have, integrating by parts, for all $z \in \mathscr{D}(\Omega)$:

$$\int_\Omega z \operatorname{curl} \mathbf{v}_S \, d\mathbf{x} = \int_\Omega \mathbf{curl} \, z \cdot \mathbf{v}_S \, d\mathbf{x} .$$

Cutting each integral in two and integrating those on the right-hand-side by parts yields

$$\int_{\Omega^c} z \operatorname{curl} \mathbf{v}_S^c \, d\mathbf{x} + \int_{\Omega^e} z \operatorname{curl} \mathbf{v}_S^e \, d\mathbf{x} = \int_{\Omega^c} \mathbf{curl}\, z \,.\, \mathbf{v}_S^c \, d\mathbf{x} + \int_{\Omega^e} \mathbf{curl}\, z \,.\, \mathbf{v}_S^e \, d\mathbf{x}\,,$$

$$0 = \langle \mathbf{v}_S^c \,.\, \tau^c, z \rangle_{\Gamma^c} + \langle \mathbf{v}_S^e \,.\, \tau^e, z \rangle_{\Gamma^e}\,.$$

This is true for any function z of $\mathscr{D}(\Omega)$, whence $\mathbf{v}_S^e \,.\, \tau^c_{|\mathscr{B}} = \mathbf{v}_S^c \,.\, \tau^c_{|\mathscr{B}}$.

Remark 5.5: This is simply the necessary and sufficient condition so that, if the pair $(\mathbf{v}_S^c, \mathbf{v}_S^e)$ belongs to $H(\operatorname{curl}, \Omega^c) \times H(\operatorname{curl}, \Omega^e)$, the function $\mathbf{v}_S$ belongs to $H(\operatorname{curl}, \Omega)$. ■

In a second step, we use explicitly (43): for any function $\mathbf{z} \in \mathscr{D}(\Omega)^2$, we have

$$\langle \mathbf{curl} \operatorname{curl} \mathbf{v}_S, \mathbf{z} \rangle_\Omega = \langle \mathbf{curl}\, p_S, \mathbf{z} \rangle_\Omega\,,$$

$$\int_\Omega \operatorname{curl} \mathbf{v}_S \operatorname{curl} \mathbf{z} \, d\mathbf{x} = \int_\Omega p_S \operatorname{curl} \mathbf{z} \, d\mathbf{x},$$

$$\int_{\Omega^c} \operatorname{curl} \mathbf{v}_S^c \operatorname{curl} \mathbf{z} \, d\mathbf{x} + \int_{\Omega^e} \operatorname{curl} \mathbf{v}_S^e \operatorname{curl} \mathbf{z} \, d\mathbf{x} = \int_{\Omega^c} p_S^c \operatorname{curl} \mathbf{z} \, d\mathbf{x} + \int_{\Omega^e} p_S^e \operatorname{curl} \mathbf{z} \, d\mathbf{x}\,.$$

But, we know that $p_S^e \in H^1(\Omega^e)$. On the other hand, as $\operatorname{curl} \mathbf{v}_S^e = p_S^e + \lambda$, $\operatorname{curl} \mathbf{v}_S^e$ also belongs to $H^1(\Omega^e)$. We can integrate the integrals on Ω^e by parts, and we obtain:

$$\int_{\Omega^c} \operatorname{curl} \mathbf{v}_S^c \operatorname{curl} \mathbf{z} \, d\mathbf{x} + \langle \operatorname{curl} \mathbf{v}_S^e, \mathbf{z} \,.\, \tau^e \rangle_{\Gamma^e} = \int_{\Omega^c} p_S^c \operatorname{curl} \mathbf{z} \, d\mathbf{x} + \langle p_S^e, \mathbf{z} \,.\, \tau^e \rangle_{\Gamma^e}\,.$$

Concerning the integrals on Ω^c, we simply know that $\operatorname{curl} \mathbf{v}_S^c = p_S^c + \lambda$. Hence,

$$\int_{\Omega^c} \lambda \operatorname{curl} \mathbf{z} \, d\mathbf{x} + \langle \operatorname{curl} \mathbf{v}_S^e, \mathbf{z} \,.\, \tau^e \rangle_{\Gamma^e} = \langle p_S^e, \mathbf{z} \,.\, \tau^e \rangle_{\Gamma^e}\,.$$

Integrating by parts, we find:

$$\langle \lambda, \mathbf{z} \,.\, \tau^c \rangle_{\Gamma^c} + \langle \operatorname{curl} \mathbf{v}_S^e, \mathbf{z} \,.\, \tau^e \rangle_{\Gamma^e} = \langle p_S^e, \mathbf{z} \,.\, \tau^e \rangle_{\Gamma^e}\,.$$

As $p^c_{S|\mathscr{B}} = p^e_{S|\mathscr{B}}$, we have $p_S^c \in (H_{c0}(\operatorname{curl}, \Omega^c))'$, where

$$H_{c0}(\operatorname{curl}, \Omega^c) = \{\mathbf{w} \in H(\operatorname{curl}, \Omega^c), \mathbf{w} \,.\, \tau^c = 0 \text{ on } \tilde{\Gamma}^c\}\,.$$

As $\operatorname{curl} \mathbf{v}_S^c = p_S^c + \lambda$, we also have $\operatorname{curl} \mathbf{v}_S^c \in (H_{c0}(\operatorname{curl}, \Omega^c))'$, which enables us to define the trace of $\operatorname{curl} \mathbf{v}_S^c$ on $\mathscr{B}$. So we finally obtain

$$\langle \operatorname{curl} \mathbf{v}_S^c - \operatorname{curl} \mathbf{v}_S^e, \mathbf{z} \,.\, \tau^c \rangle_{\mathscr{B}} = 0, \quad \forall \mathbf{z} \in \mathscr{D}(\Omega)^2\,,$$

whence $\operatorname{curl} \mathbf{v}^c_{S|\mathscr{B}} = \operatorname{curl} \mathbf{v}^e_{S|\mathscr{B}}$.

With the latter transmission conditions, we can now write the outer problem:

Find $\mathbf{v}_S^e \in H(\mathrm{curl}, \Omega^e)$ *solution of*

$$\mathbf{curl}\ \mathrm{curl}\ \mathbf{v}_S^e = \mathbf{curl}\ p_S^e \text{ in } \Omega^e, \tag{53}$$

$$\mathrm{div}\ \mathbf{v}_S^e = 0 \text{ in } \Omega^e, \tag{54}$$

$$\mathbf{v}_S^e \cdot \tau^e = 0 \text{ on } \tilde{\Gamma}^e, \tag{55}$$

$$\mathrm{curl}\ \mathbf{v}_S^e + \mathcal{T}(\mathbf{v}_S^e \cdot \tau^e) = 0 \text{ on } \mathcal{B}. \tag{56}$$

Remark 5.6: $\mathbf{curl}\ \mathrm{curl}\ \mathbf{v}_S^e \in L^2(\Omega^e)$, because $p_S^e \in H^1(\Omega^e)$. ■

Denoting by $H_{e0}(\mathrm{curl}, \Omega^e) = \{\mathbf{w} \in H(\mathrm{curl}, \Omega^e), \mathbf{w} \cdot \tau^e = 0 \text{ on } \tilde{\Gamma}^e\}$, we can put the previous problem in variational form:

Find $\mathbf{v}_S^e \in H_{e0}(\mathrm{curl}, \Omega^e)$ *solution of*

$$\int_{\Omega^e} \mathrm{curl}\ \mathbf{v}_S^e\ \mathrm{curl}\ \mathbf{v}\ d\mathbf{x} + R\int_0^{\pi/\alpha} \mathcal{T}_1(\mathbf{v}_S^e \cdot \tau^e)\ \mathbf{v} \cdot \tau^e\ d\theta =$$

$$\int_{\Omega^e} \mathbf{curl}\ p_S^e \cdot \mathbf{v}\ d\mathbf{x} - R\int_0^{\pi/\alpha} (p_S^e(R, \theta) - A_0)\ \mathbf{v} \cdot \tau^e\ d\theta, \quad \forall \mathbf{v} \in H_{e0}(\mathrm{curl}, \Omega^e), \tag{57}$$

$$\mathrm{div}\ \mathbf{v}_S^e = 0 \text{ in } \Omega^e. \tag{58}$$

In particular, we have decomposed the boundary term in (57), considering the embedding of the test functions in the set $H_{e0}(\mathrm{curl}, \Omega^e)$. Here $\mathcal{T}_1(\mathbf{v}_S^e \cdot \tau)$ stands for the first term of the right-hand-side of (52): actually, this is a number. We verify in a straightforward manner that (53)-(56) and (57)-(58) are equivalent.

Remark 5.7: If a function $\mathbf{w}$ of $H_{e0}(\mathrm{curl}, \Omega^e)$ verifies $\mathrm{curl}\ \mathbf{w} = 0$, we must have, integrating by parts, $\int_0^{\pi/\alpha} \mathbf{w} \cdot \tau^e\ d\theta = 0$. On the other hand, we notice that the bilinear form $(\mathbf{u}, \mathbf{v}) \mapsto R\int_0^{\pi/\alpha} \mathcal{T}_1(\mathbf{u} \cdot \tau^e)\ \mathbf{v} \cdot \tau^e\ d\theta$ is symmetric and positive, because:

$$R\int_0^{\pi/\alpha} \mathcal{T}_1(\mathbf{u} \cdot \tau^e)\ \mathbf{u} \cdot \tau^e\ d\theta = \frac{2\alpha}{\pi}\left\{\int_0^{\pi/\alpha} \mathbf{u} \cdot \tau^e\ d\theta\right\}^2.$$

■

THEOREM 5.2: *The problem (53)-(56) has a unique solution in* $H(\mathrm{curl}, \Omega^e)$.

Proof: This amounts to showing that (57)-(58) admits a unique solution in $H(\mathrm{curl}, \Omega^e)$. For that, we dualize the divergence-free condition. We introduce in this case the space $H_{e0}^1(\Omega^e) = \{q \in H^1(\Omega^e), q = 0 \text{ on } \tilde{\Gamma}^e\}$. According to the previous remark, there exists a compatibility condition for any element q of $H_{e0}^1(\Omega^e)$, that can be written $\int_0^{\pi/\alpha} \nabla q \cdot \tau^e\ d\theta = 0$.

We solve:

Find $(\mathbf{v}_S^e, q_m) \in H_{e0}(\mathrm{curl}, \Omega^e) \times H^1_{e0}(\Omega^e)$ *solution of*

$$\int_{\Omega^e} \mathrm{curl}\, \mathbf{v}_S^e \,\mathrm{curl}\, \mathbf{v}\, d\mathbf{x} + R\int_0^{\pi/\alpha} \mathcal{T}_1(\mathbf{v}_S^e \cdot \tau^e)\, \mathbf{v} \cdot \tau^e\, d\theta + \int_{\Omega^e} \nabla q_m \cdot \mathbf{v}\, d\mathbf{x} =$$

$$\int_{\Omega^e} \mathbf{curl}\, p_S^e \cdot \mathbf{v}\, d\mathbf{x} - R\int_0^{\pi/\alpha} (p_S^e(R,\theta) - A_0)\, \mathbf{v} \cdot \tau^e\, d\theta, \forall \mathbf{v} \in H_{e0}(\mathrm{curl}, \Omega^e), \tag{59}$$

$$\int_{\Omega^e} \nabla q \cdot \mathbf{v}_S^e\, d\mathbf{x} = R\int_0^{\pi/\alpha} q \mathbf{v}_S^e \cdot \nu^e\, d\theta, \forall q \in H^1_{e0}(\Omega^e)\,. \tag{60}$$

The right-hand-side of (60) is formal, as $\mathbf{v}_S^e \cdot \nu^e_{|\mathcal{B}}$ might not belong to the dual of $H^{1/2}_{00}(\mathcal{B})$ as we only have $\mathbf{v}_S^e \in H_{e0}(\mathrm{curl}, \Omega^e)$. In order to prove the existence and uniqueness of the solution (59)-(60), we start by proving that the bilinear form $a^e : (\mathbf{u}, \mathbf{v}) \mapsto \int_{\Omega^e} \mathrm{curl}\, \mathbf{u}\, \mathrm{curl}\, \mathbf{v}\, d\mathbf{x} + R\int_0^{\pi/\alpha} \mathcal{T}_1(\mathbf{u} \cdot \tau^e)\, \mathbf{v} \cdot \tau^e\, d\theta$ is W^e-elliptic on the subspace W^e of $H_{e0}(\mathrm{curl}, \Omega^e)$ defined by

$$W^e = \left\{ \mathbf{w} \in H_{e0}(\mathrm{curl}, \Omega^e), \forall q \in H^1_{e0}(\Omega^e), \int_{\Omega^e} \nabla q \cdot \mathbf{w}\, d\mathbf{x} = 0 \right\}$$

$$= \{ \mathbf{w} \in H(\mathrm{curl}, \Omega^e), \mathrm{div}\, \mathbf{w} = 0, \mathbf{w} \cdot \tau^e = 0 \text{ on } \tilde{\Gamma}^e, \mathbf{w} \cdot \nu^e = 0 \text{ on } \mathcal{B} \}\,.$$

We have to prove that there exists $C > 0$ such that, for all $\mathbf{w} \in W^e$, $\|\mathbf{w}\|^2_{L^2} \leqslant C a^e(\mathbf{w}, \mathbf{w})$. If this is not the case, there exists a sequence $(\mathbf{w}_n)_n$ of W^e such that $\|\mathbf{w}_n\|_{L^2} = 1$ for all n and such that $a^e(\mathbf{w}_n, \mathbf{w}_n) \to 0$. In this case, according to a compact embedding result proved in [13], there exists a subsequence still called $(\mathbf{w}_n)_n$ which converges in $L^2(\Omega^e)^2$, to a limit $\mathbf{w}$.

Remark 5.8: Notice that here the boundary condition is $\mathbf{w} \cdot \tau^e = 0$ on $\tilde{\Gamma}^e$ and $\mathbf{w} \cdot \nu^e = 0$ on $\mathcal{B}$, which does not permit to apply Weber's results [35] which hold for a boundary condition of the same kind on the whole boundary.

Therefore $(\mathbf{w}_n)_n$ converges in $H(\mathrm{curl}, \Omega^e)$ to $\mathbf{w}$, with

$$\|\mathbf{w}\|_{L^2} = 1, \mathrm{curl}\, \mathbf{w} = 0, \int_0^{\pi/\alpha} \mathbf{w} \cdot \tau^e\, d\theta = 0 \quad \text{and} \quad \mathbf{w} \cdot \tau^e = 0 \text{ on } \tilde{\Gamma}^e\,.$$

So, there exists $q \in H^1_{e0}(\Omega^e)$ (the compatibility condition is automatically fulfilled) such that $\mathbf{w} = \nabla q$. Passing to the limit in W^e, we finally find:

$$\|\mathbf{w}\|^2_{L^2} = \int_{\Omega^e} \mathbf{w} \cdot \nabla q\, d\mathbf{x} = 0\,.$$

We come to a contradiction, which proves the W^e-ellipticity.

In a second step, the inf-sup condition is straighforward to verify, which induces the existence and uniqueness of the solution in $H_{e0}(\mathrm{curl}, \Omega^e) \times H^1_{e0}(\Omega^e)$.

Then, we notice that

$$H^1_{e0}(\Omega^e) = H^1_0(\Omega^e) \overset{\perp}{\oplus} \mathcal{H}(\Omega^e)\,,$$

for the $(q, q') \mapsto \int_{\Omega^e} \nabla q \cdot \nabla q' \, d\mathbf{x}$ scalar product, with

$$\mathscr{H}(\Omega^e) = \{q' \in H^1_{e0}(\Omega^e), \Delta q' = 0\}.$$

We decompose q_m into $q_0 + q_\Delta$, with $q_0 \in H^1_0(\Omega^e)$ and $q_\Delta \in \mathscr{H}(\Omega^e)$. As $\nabla q_0 \in H_{e0}(\mathrm{curl}, \Omega^e)$ we can use it as a test function in (59), whence $q_0 = 0$.

In (59), if we now use $\mathbf{v} = \nabla q'$, for q' in $\mathscr{H}(\Omega^e)$, we notice that:

$$R \int_0^{\pi/\alpha} \{\mathscr{T}_1(\mathbf{v}^e_S \cdot \tau^e) + p^e_S(R, \theta) - A_0\} \nabla q' \cdot \tau^e \, d\theta + \int_{\Omega^e} \nabla q_\Delta \cdot \nabla q' \, d\mathbf{x} = \int_{\Omega^e} \mathbf{curl}\, p^e_S \cdot \nabla q' \, d\mathbf{x},$$

$$\text{or} \quad R \int_0^{\pi/\alpha} \{\mathscr{T}_1(\mathbf{v}^e_S \cdot \tau^e) + 2\, p^e_S(R, \theta) - A_0\} \nabla q' \cdot \tau^e \, d\theta + R \int_0^{\pi/\alpha} q_\Delta \nabla q' \cdot \nu^e \, d\theta = 0.$$

As $\nabla q' \cdot \tau^e_{|\mathscr{B}}$ and $\nabla q' \cdot \nu^e_{|\mathscr{B}}$ are independent, we deduce in particular that $q_\Delta = 0$ on $\mathscr{B}$, which amounts to saying that q_Δ also belongs to $H^1_0(\Omega^e)$, and so $q_\Delta = 0$.

Finally $q_m = 0$, which proves the existence and uniqueness of the solution of (57)-(58) in $H_{e0}(\mathrm{curl}, \Omega^e)$, and as a consequence the existence and uniqueness of the solution of (53)-(56) in $H(\mathrm{curl}, \Omega^e)$. So the problem is well-posed in $H(\mathrm{curl}, \Omega^e)$. ■

Remark 5.9: We have seen that p_S is determined by A_{-1} : with this method, this is also the case for $\mathbf{v}_S$. Indeed, A_{-1} being fixed, we deduce p_S and hence in particular the $(A_n)_{n \geq 0}$. Using this we can compute $\mathbf{v}^e_S$ solution of (59)-(60). Finally, using the formulas (48)-(50), we find $\mathbf{v}^c_S$. ■

Remark 5.10: In the stationnary case (2)-(4) we can compute the solution directly using one of these methods. Indeed, recall that $\mathbf{u}_S = c\mathbf{v}_S$, as the dimension of N is 1. Also, we have, by orthogonality,

$$\int_\Omega f \,\mathrm{curl}\, \mathbf{v}_S \, d\mathbf{x} = \int_\Omega \mathrm{curl}\, \mathbf{u} \,\mathrm{curl}\, \mathbf{v}_S \, d\mathbf{x} = c \int_\Omega (\mathrm{curl}\, \mathbf{v}_S)^2 \, d\mathbf{x} = c \,\|p_S\|^2_{L^2},$$

which yields the value of the constant c. Then, the regular part $\mathbf{u}_R$ can be determined with the help of a usual finite element method ([33], [5] or [14]). On the other hand, due to the steep gradients, we might need to refine the mesh considerably at the reentrant corner if the global method (43)-(45) is chosen. ■

5.2.2. *Computation of* $\mathbf{v}_S$ *using its stream function*

In order to solve (40)-(42), we shall now use isomorphism 2 of lemma 2.1 between $\Phi/\mathbb{R}$ and V (through the **curl** operator). Indeed, to $\mathbf{v}_S$ we can associate $\phi_S \in H^1(\Omega)/\mathbb{R}$ such that

$$-\Delta \phi_S = p_S \text{ in } \Omega, \tag{61}$$

$$\frac{\partial \phi_S}{\partial \nu} = 0 \text{ on } \Gamma. \tag{62}$$

As ϕ_S is sufficiently smooth (i.e. of regularity H^1), this problem is equivalent to the variational formulation:

Find $\phi_S \in H^1(\Omega)/\mathbb{R}$ *such that*

$$\int_\Omega \nabla \phi_S \cdot \nabla \psi \, d\mathbf{x} = \int_\Omega p_S \, \psi \, d\mathbf{x}, \quad \forall \psi \in H^1(\Omega)/\mathbb{R}. \tag{63}$$

As with the **curl** curl formulation, we could solve this problem directly with the above variational formulation. However, error estimates in finite element methods rely on H^2 -regularity of the solution which we do not have here. From a computational point of view, the problem will be that the mesh will have to be refined drastically near the reentrant corner in order to get an acceptable solution.

So, we shall make use of the explicit knowledge of ϕ_S near the reentrant corner. Moreover, we know that ϕ_S is of regularity H^2 away from this reentrant corner. We call ϕ_S^c (resp. ϕ_S^e) the restriction of ϕ_S to Ω^c (resp. Ω^e). As ϕ_S^c verifies (22) and (24), we can write it as

$$\phi_S^c(r, \theta) = -\sum_{n \geq 1} \frac{B_n}{n\alpha} r^{n\alpha} \cos(n\alpha\theta) - \sum_{n \geq -1} \frac{A_n}{4n\alpha + 4} r^{n\alpha + 2} \cos(n\alpha\theta).$$

The expression of $(A_n)_{n \geq -1}$ is given by (30), (32) and (34). From there on, we can easily express each of the $(B_m)_{m \geq 1}$ in function of the trace of ϕ_S^c on $\mathscr{B}$. We have:

$$m = 1 \quad B_1 = -\frac{2\alpha^2}{\pi R^\alpha} \int_0^{\pi/\alpha} \phi_S^c(R, \theta) \cos(\alpha\theta)\, d\theta - \left(\frac{\alpha}{4 - 4\alpha} A_{-1} R^{2-2\alpha} + \frac{\alpha}{4 + 4\alpha} A_1 R^2 \right), \tag{64}$$

$$m \geq 2 \quad B_m = -\frac{2m\alpha^2}{\pi R^{m\alpha}} \int_0^{\pi/\alpha} \phi_S^c(R, \theta) \cos(m\alpha\theta)\, d\theta - \frac{m\alpha}{4m\alpha + 4} A_m R^2. \tag{65}$$

The DtN operator, denoted by t, is defined by t: $\phi^c_{S|\mathscr{B}} \mapsto \left.\frac{\partial \phi_S^c}{\partial \nu^c}\right|_{\mathscr{B}}$. Which yields:

$$t(\phi_S^c) = t_1(\phi_S^c) - \frac{1}{2} \int_{r=0}^{r=R} p_S^c(r, \theta)\, dr + \frac{\alpha}{2 - 2\alpha} A_{-1} R^{1-\alpha} \cos(\alpha\theta), \quad \text{and} \tag{66}$$

$$t_1(\phi_S^c) = \frac{2\alpha^2}{\pi R} \sum_{n \geq 1} n \left\{ \int_0^{\pi/\alpha} \phi_S^c(R, \theta') \cos(n\alpha\theta')\, d\theta' \right\} \cos(n\alpha\theta). \tag{67}$$

Remark 5.11: Notice that the operators t_1 and T_1 are identical. ■

The function ϕ_S^e is a solution of the following problem:

Find $\phi_S^e \in H^1(\Omega^e)/\mathbb{R}$ such that

$$-\Delta\phi_S^e = p_S^e \text{ in } \Omega^e, \tag{68}$$

$$\frac{\partial \phi_S^e}{\partial \nu^e} = 0 \text{ on } \tilde{\Gamma}^e, \tag{69}$$

$$\frac{\partial \phi_S^e}{\partial \nu^e} + t(\phi_S^e) = 0 \text{ on } \mathscr{B}. \tag{70}$$

Remark 5.12: As $\frac{\partial \phi_S^c}{\partial \nu^c} = 0$ on $\tilde{\Gamma}^c$, we have $\left.\frac{\partial \phi_S^e}{\partial \nu^c}\right|_{\mathscr{B}} \in H_{00}^{1/2}(\mathscr{B})$. We then deduce from (69) and from the transmission condition on the normal derivatives that $\frac{\partial \phi_S^e}{\partial \nu^e} \in H^{1/2}(\Gamma^e)$. ■

As ϕ_S^e is sufficiently smooth (i e in H^1), this problem is equivalent to the variational formulation
Find $\phi_S^e \in H^1(\Omega^e)/\mathbb{R}$ *such that*

$$\int_{\Omega^e} \nabla\phi_S^e \cdot \nabla\psi \, d\mathbf{x} + R\int_0^{\pi/\alpha} t_1(\phi_S^e)\,\psi\, d\theta =$$

$$\int_{\Omega^e} p_S^e\,\psi\, d\mathbf{x} + \frac{1}{2}R\int_0^{\pi/\alpha}\left\{\int_{r-0}^{r=R} p_S^c(r,\theta)\,dr\right\}\psi(R,\theta)\,d\theta$$

$$-\frac{\alpha}{2-2\alpha}A_{-1}R^{2-\alpha}\int_0^{\pi/\alpha}\cos(\alpha\theta)\,\psi(R,\theta)\,d\theta,\quad \forall\psi\in H^1(\Omega^e)/\mathbb{R} \tag{71}$$

It is important to notice that the bilinear symmetric operators of (39) and (71) (associated respectively to p_S^e and ϕ_S^e) are identical As a consequence, there exists one and only one solution of the above problem

Remark 5 13 In the steady-state case (2)-(4) we can compute directly the solution by using one of these two methods ■

Remark 5 14 As Ω^e is a curved polygon and as $\dfrac{\partial\phi_S^e}{\partial\nu^e} \in H^{1/2}(\Gamma^e)$, according to Costabel-Dauge [17], section 4 d, we know that $\phi_S^e \in H^2(\Omega^e)$ We find again here the result giving the regularity of ϕ_S^e On the other hand, we have seen that p_S is determined by A_{-1} this is again true for ϕ_S Indeed, A_{-1} being fixed, we get p_S and hence in particular the $(A_n)_{n\geq 0}$ Therefore, we can compute ϕ_S^e solution of (71) Finally, using formulas (64)-(65), this yields ϕ_S^c ■

5.3. Resolution of the time dependent variational problem

We start by semi-discretizing the variational problem (10) in space In addition to the classical test functions given by the choice of the finite element method and which belong to a space denoted by V_R^h, we also use the test function $\mathbf{v}_S$, the antecedent of p_S by the curl operator, which we assume being known exactly The test functions are hence being chosen in $V_R^h \oplus V_S$ We assume that $V_R^h \subset V_R$ thus, (11) is automatically satisfied We denote by P_h the projection on V_R^h in the sense of the $L^2(\Omega)^2$ inner product, i e for an element $\mathbf{v}\in V$,

$$\int_\Omega \mathbf{v}\cdot\mathbf{v}_R^h\,d\mathbf{x} = \int_\Omega P_h\mathbf{v}\cdot\mathbf{v}_R^h\,d\mathbf{x}\quad \forall\mathbf{v}_R^h\in V_R^h$$

Remark 5 15 The projection P_h verifies $\lim_{h\to 0}\|\mathbf{v}_S - P_h\mathbf{v}_S\|_{L^2} = 0$ ■

Now, if we write $\mathbf{u}^h = \mathbf{u}_R^h + c(t)\,\mathbf{v}_S$ with $\mathbf{u}_R^h \in V_R^h$, the variational formulation becomes, taking $\mathbf{v}^h = \mathbf{v}_S - P_h\mathbf{v}_S$ as a test function and using the orthogonality of $\operatorname{curl}\mathbf{u}_R^h$ and $\operatorname{curl}\mathbf{v}_S$

$$c''(t)\int_\Omega(\mathbf{v}_S - P_h\mathbf{v}_S)^2\,d\mathbf{x} + c(t)\int_\Omega p_S^2\,d\mathbf{x} = \int_\Omega \operatorname{curl}\mathbf{u}_R^h \operatorname{curl}P_h\mathbf{v}_S\,d\mathbf{x} + \int_\Omega \mathbf{f}^h\cdot(\mathbf{v}_S - P_h\mathbf{v}_S)\,d\mathbf{x} \tag{72}$$

The function $c(t)$ is obtained by solving this ordinary differential equation According to the regularity of $\mathbf{v}_S$, an estimation of the coefficient of $c''(t)$ is

$$\|\mathbf{v}_S - P_h\mathbf{v}_S\|_{L^2}^2 \leq C_\varepsilon h^{2\alpha-2\varepsilon},\quad \forall\varepsilon>0 \tag{73}$$

From this we can conclude that this differential equation is not stiff, and that it can be solved by a classical time discretization scheme

In order to compute $\mathbf{u}_R^h$, we write down the classical variational formulation for $\mathbf{v}^h \in V_R^h$:
Find $\mathbf{u}_R^h \in V_R^h$ *such that*

$$\frac{d^2}{dt^2}\int_\Omega \mathbf{u}_R^h \cdot \mathbf{v}_R^h \, d\mathbf{x} + \int_\Omega \operatorname{curl} \mathbf{u}_R^h \operatorname{curl} \mathbf{v}_R^h \, d\mathbf{x} = \int_\Omega \mathbf{f}^h \cdot \mathbf{v}_R^h \, d\mathbf{x} - c''(t) \int_\Omega \mathbf{v}_S \cdot \mathbf{v}_R^h \, d\mathbf{x}, \quad \forall \mathbf{v}_R^h \in V_R^h . \tag{74}$$

This formulation only involves the regular part of the fields, and can hence be solved by a usual finite element method.

6. CONCLUSION

In this paper, we have presented several methods to solve numerically the Maxwell equations with perfectly conducting boundary condition, in two-dimensional polygonal domains with reentrant corners. A mathematical theory has been developed which supports these methods, and some numerical results have already been obtained ([2]). The more general case of mixed boundary condition is considered in ([13]) and will be presented in a companion paper.

APPENDICES

A. CASE WHEN $\mathbf{u} \cdot \nu = 0$ ON THE BOUNDARY

We now focuse on the magnetic field, still denoted by $\mathbf{u}$.

We consider here the resolution of the steady-state and time-dependent problems with a boundary condition of the type

$$\mathbf{u} \cdot \nu = 0 \text{ on } \Gamma .$$

Hereafter, we mainly emphasize differences with the case $\mathbf{u} \cdot \tau = 0$ on Γ.

First, we shall need the spaces:

$$H_0(\operatorname{div} 0 \,;\, \Omega) = \{\mathbf{v} \in L^2(\Omega)^2, \operatorname{div} \mathbf{v} = 0, \mathbf{v} \cdot \nu = 0 \text{ on } \Gamma\},$$

as well as

$$\overline{V} = \{\mathbf{v} \in H(\operatorname{curl}, \Omega), \operatorname{div} \mathbf{v} = 0, \mathbf{v} \cdot \nu = 0 \text{ on } \Gamma\}$$

endowed with the canonical inner product of $H(\operatorname{curl}, \Omega)$, and the space of stream functions:

$$\overline{\Phi} = \{\varphi \in H_0^1(\Omega), \Delta\varphi \in L^2(\Omega)\}.$$

Here we followed remark 2.3 of [23], chapter 1, where it is being noticed that for any function φ of $H^1(\Omega)$, $\nabla\varphi \cdot \tau = 0$ on Γ is equivalent to $\varphi = \lambda$ on the boundary, λ being a constant.

LEMMA A.1: *We have the following vector space isomorphisms:*

1. *The* curl *operator defines an isomorphism from* $\overline{V}$ onto $L^2(\Omega)$.
2. *The* **curl** *operator defines an isomorphism from* $\overline{\Phi}$ *onto* $\overline{V}$.
3. *The* Δ *operator defines an isomorphism from* $\overline{\Phi}$ *onto* $L^2(\Omega)$.

As for the spaces V and Φ, we know that when the boundary Γ is of class C^2, or when the domain Ω is convex with a Lipschitz boundary Γ, the space $\overline{V}$ is included in $H^1(\Omega)^2$ according to [23], and the space $\overline{\Phi}$ is included in $H^2(\Omega)$ (see [26]). This is no longer true in presence of reentrant corners. So we shall need the regularized spaces of $\overline{V}$ and $\overline{\Phi}$:

$$\overline{V}_R = \{\mathbf{v} \in H^1(\Omega)^2, \operatorname{div} \mathbf{v} = 0, \mathbf{v} \,.\, \nu = 0 \text{ on } \Gamma\}$$

and

$$\overline{\Phi}_R = H^2(\Omega) \cap H^1_0(\Omega)\,.$$

A.1. The model problems with $\mathbf{u} \,.\, \nu = 0$ on the boundary

In this case the steady-state problem is defined by equations (2)-(3), (4) being replaced by $\mathbf{u} \,.\, \nu = 0$ on Γ.

The time dependent problem must be formulated slightly differently than with the other boundary condition. Given a function $f(t)$ which belongs to $L^2([0, T]\,;L^2(\Omega))$ and two functions $\mathbf{u}_0 \in \overline{V}$ and $\mathbf{u}_1 \in H_0(\operatorname{div} 0\,;\Omega)$ independent of t, we are interested in the following problem:

Find $\mathbf{u}(t) \in L^2([0, T]\,; H(\text{curl}, \Omega))$, $\partial \mathbf{u}/\partial t(t) \in L^2([0, T]\,; H_0(\operatorname{div} 0\,;\Omega))$ *such that*

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} + \mathbf{curl}\, \text{curl}\, \mathbf{u} = \mathbf{curl}\, f \text{ in } \Omega \tag{75}$$

$$\operatorname{div} \mathbf{u} = 0 \text{ in } \Omega \tag{76}$$

$$\mathbf{u} \,.\, \nu = 0 \text{ on } \Gamma \tag{77}$$

$$\text{curl}\, \mathbf{u} - f = 0 \text{ on } \Gamma \tag{78}$$

with the initial conditions

$$\mathbf{u}(0) = \mathbf{u}_0 \tag{79}$$

$$\frac{\partial \mathbf{u}}{\partial t}(0) = \mathbf{u}_1\,. \tag{80}$$

Remark A.1: The boundary condition (78) appears naturally when solving mathematically Maxwell's equations. It is in fact a "physical" condition and is to be taken in some "weak" sense. It is only prescribed to ensure the equivalence between (75)-(78) and the following variational formulation. ■

These equations can be written in variational form as follows:

Find $\mathbf{u}_0(t) \in L^2([0, T]\,; H(\text{curl}, \Omega))$ *such that*

$$\frac{d^2}{dt^2}\int_\Omega \mathbf{u} \,.\, \mathbf{v}\, dx + \int_\Omega \text{curl}\, \mathbf{u}\, \text{curl}\, \mathbf{v}\, dx = \int_\Omega f\, \text{curl}\, \mathbf{v}\, d\mathbf{x} \quad \forall \mathbf{v} \in H(\text{curl}, \Omega) \tag{81}$$

$$\operatorname{div} \mathbf{u} = 0 \text{ in } \Omega \tag{82}$$

$$\mathbf{u} \,.\, \nu = 0 \text{ on } \Gamma \tag{83}$$

with the initial conditions

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \frac{\partial \mathbf{u}}{\partial t}(0) = \mathbf{u}_1\,. \tag{84}$$

PROPOSITION A.1: *Let* $f \in L^2(0, T\,; L^2(\Omega))$, $\mathbf{u}_0 \in \overline{V}$ *and* $\mathbf{u}_1 \in H_0(\operatorname{div} 0\,;\Omega)$. *Then problem (81)-(84) admits a unique solution* $\mathbf{u}$ *such that* $\mathbf{u} \in C^0([0, T]\,; \overline{V}) \cap C^1([0, T]\,; H_0(\operatorname{div} 0\,;\Omega))$, *provided it stands for the magnetic field.*

Proof: Apply the variational theory of Lions-Magenes [31], Tome 1, p. 286, to the complete Maxwell system of equations. ∎

The results that we obtained in the case $\mathbf{u} \cdot \tau = 0$ on Γ still remain valid (see [4] for details). Let us only give here the local expressions of the singular functions.

Let us denote by $\overline{V}_S = \operatorname{curl}^{-1} \bar{N}$, where $\bar{N}$ is the orthogonal of $\Delta\overline{\Phi}_R$ in $L^2(\Omega)$. Then $\bar{p}_S$ can locally be expressed as $\bar{S}^0 \in L^2(\Omega^c)$ solution of

$$\Delta \bar{S}^0 = 0 \text{ in } \Omega^c , \tag{85}$$

$$\bar{S}^0 = 0 \text{ on } \tilde{\Gamma}^c . \tag{86}$$

Using the method of separation of variables we find that all the terms of the sequence

$$(r^{n\alpha} \sin (n\alpha\theta))_{n \in \mathbb{Z}}$$

are solution of (85)-(86). As $r^{n\alpha} \sin (n\alpha\theta) \in L^2(\Omega^c)$ if and only if $n \geqslant -1$ and as it belongs to $H^1(\Omega^c)$ as soon as $n \geqslant 0$, $\bar{S}^0$ can be written

$$\bar{p}_S(r, \theta) = \sum_{n \geqslant -1} \bar{A}_n r^{n\alpha} \sin (n\alpha\theta), \quad \text{with} \quad \bar{A}_{-1} \neq 0 . \tag{87}$$

The basis function $\bar{\mathbf{u}}_S$ of $\overline{V}_S$ is locally equivalent to $\bar{\mathbf{S}}^1$ solution of

$$\operatorname{curl} \bar{\mathbf{S}}^1 = \bar{S}^0 \text{ in } \Omega^c , \tag{88}$$

$$\operatorname{div} \bar{\mathbf{S}}^1 = 0 \text{ in } \Omega^c , \tag{89}$$

$$\bar{\mathbf{S}}^1 \cdot \nu = 0 \text{ in } \tilde{\Gamma}^c . \tag{90}$$

Computing $\bar{\mathbf{S}}^1$ as before yields

$$\bar{\mathbf{S}}^1(r, \theta) = \sum_{n \geqslant 1} \bar{B}_n r^{n\alpha - 1} \begin{pmatrix} \cos (n\alpha\theta) \\ -\sin (n\alpha\theta) \end{pmatrix} + \sum_{n \geqslant -1} \bar{A}_n r^{n\alpha + 1} \begin{pmatrix} -\dfrac{n\alpha}{4\, n\alpha + 4} \cos (n\alpha\theta) \\ \dfrac{n\alpha + 2}{4\, n\alpha + 4} \sin (n\alpha\theta) \end{pmatrix}, \quad \text{with } \bar{B}_1 \neq 0 . \tag{91}$$

The stream function $\bar{\phi}_S$ of $\bar{\mathbf{u}}_S$ is locally identical to some $\bar{S}^2$ such that

$$\bar{S}^2(r, \theta) = 0 \text{ on } \tilde{\Gamma}^c , \tag{92}$$

and

$$\begin{aligned} &\text{either } \mathbf{curl}\, \bar{S}^2 = \bar{\mathbf{S}}^1 \text{ in } \Omega^c , \\ &\text{or} \qquad -\Delta \bar{S}^2 = \bar{S}^0 \text{ in } \Omega^c . \end{aligned} \tag{93}$$

$\bar{S}^2$ is of the form

$$\bar{S}^2(r, \theta) = \sum_{n \geqslant 1} \frac{\bar{B}_n}{n\alpha} r^{n\alpha} \sin (n\alpha\theta) - \sum_{n \geqslant -1} \frac{\bar{A}_n}{4\, n\alpha + 4} r^{n\alpha + 2} \sin (n\alpha\theta) . \tag{95}$$

The $\bar{A}_n$ are computed as functions of the trace of $\bar{S}^0$ on $\mathcal{B}$, using the orthogonality of $\theta \mapsto \sin(m\alpha\theta)$ for the different $m \geq 1$. Integrating $\bar{S}^0(R, \theta) \sin(m\alpha\theta)$ from 0 to π/α in θ, we obtain

$$\bar{A}_1 = \frac{2\alpha}{\pi} R^{-\alpha} \int_0^{\pi/\alpha} \bar{S}^0(R, \theta) \sin(\alpha\theta)\, d\theta + R^{-2\alpha} \bar{A}_{-1}, \tag{96}$$

$$m \geq 2 \quad \bar{A}_m = \frac{2\alpha}{\pi} R^{-m\alpha} \int_0^{\pi/\alpha} \bar{S}^0(R, \theta) \sin(m\alpha\theta)\, d\theta. \tag{97}$$

Finally, the $\bar{B}_m$ are computed as functions of the $\bar{A}_m$ and of the trace of $\bar{S}^2$ on $\mathcal{B}$ in the same way, leading to

$$\bar{B}_1 = \frac{2\alpha^2}{\pi} R^{-\alpha} \int_0^{\pi/\alpha} \bar{S}^2(R, \theta) \sin(\alpha\theta)\, d\theta - \frac{\alpha}{4 - 4\alpha} \bar{A}_{-1} R^{2-2\alpha} + \frac{\alpha}{4 + 4\alpha} \bar{A}_1 R^2, \tag{98}$$

$$m \geq 2 \quad \bar{B}_m = \frac{2m\alpha^2}{\pi} R^{-m\alpha} \int_0^{\pi/\alpha} \bar{S}^2(R, \theta) \sin(m\alpha\theta)\, d\theta + \frac{m\alpha}{4m\alpha + 4} \bar{A}_m R^2. \tag{99}$$

B. CASE OF SEVERAL REENTRANT CORNERS

We assume here that Ω is a connected and simply connected polygon of $\mathbb{R}^2$ with a boundary Γ whose angles at the vertices have a value less or equal to π, except for K reentrant corners, where the angle is $\frac{\pi}{\alpha_i}$, with $1/2 < \alpha_i < 1$, for $1 \leq i \leq K$. For each reentrant corner, we denote by Ω_i^c an open angular sector in its neighborhood and by Γ_i^c its boundary. We denote by Ω^e the open set such that $\left(\bigcup_{i=1}^{i=K} \Omega_i^c\right) \cap \Omega^e = \Phi$ and $\left(\bigcup_{i=1}^{i=K} \overline{\Omega_i^c}\right) \cup \overline{\Omega^e} = \overline{\Omega}$, of boundary Γ^e. For $1 \leq i \leq K$, we call $\mathcal{B}_i$ the boundary $\Gamma_i^c \cap \Gamma^e$, and we decompose Γ_i^c into $\Gamma_i^c = \mathcal{B}_i \cup \tilde{\Gamma}_i^c$. Finally, we decompose Γ^e into $\Gamma^e = \bigcup_{i=1}^{i=K} \mathcal{B}_i \cup \tilde{\Gamma}^e$.

We consider the steady-state problem (2)-(4) and the time dependent problem (5)-(7) on such a domain.

Remark B.1: The case where the boundary condition $\mathbf{u} \cdot \tau = 0$ is replaced by $\mathbf{u} \cdot \nu = 0$ can be dealt with in a similar manner. ■

B.1. Computation of the solution

The first difference with the case of a single reentrant corner appears in the following statement (cf. lemma 4.3).

DEFINITION B.1: *We shall denote by $\Delta\Phi_R$ the image of Φ_R by the Laplace operator and we let* $N = (\operatorname{curl} V_R)^{\perp}$.

LEMMA B.1: *The space* curl V_R *is identical to $\Delta\Phi_R$ and the space N is of dimension K. We have the direct orthogonal sum* $L_0^2(\Omega) = \operatorname{curl} V_R \overset{\perp}{\oplus} N$.

Proof: The proof is the same as for lemma 4.3. In the case of a domain with K reentrant corners, Grisvard [26] has proved that the space N of functions $p \in L_0^2(\Omega)$ such that

$$\Delta p = 0 \text{ in } \Omega,$$

$$\frac{\partial p}{\partial \nu} = 0 \text{ on } \Gamma,$$

is a vector space of dimension K. ■

Remark B.2: As in the case of a unique reentrant corner, there exist some compatibility conditions which are automatically satisfied by the local expressions of p in the neighborhood of corners. ■

Consequently, the space $V_S = \mathrm{curl}^{-1} N$ is also of dimension K. When we write $\mathbf{u} = \mathbf{u}_R + \mathbf{u}_S$, where $\mathbf{u}$ is the solution of (2)-(4) or of (5)-(7), with $\mathbf{u}_R \in V_R$, and $\mathbf{u}_S \in V_S$, $\mathbf{u}_S$ can be written as

$$\mathbf{u}_S = \sum_{i=1}^{i=K} c_i \, \mathbf{v}_S^i ,$$

where $(\mathbf{v}_S^i)_{1 \leqslant i \leqslant K}$ is a basis of V_S. This brings us naturally to look for this basis.

To that aim, we start by determining a basis $(p_S^i)_{1 \leqslant i \leqslant K}$ of solutions of

Find $p_S^i \in L_0^2(\Omega)$ such that

$$\Delta p_S^i = 0 \text{ in } \Omega ,$$

$$\frac{\partial p_S^i}{\partial \nu} = 0 \text{ on } \Gamma .$$

We use the following strategy: knowing that on Ω_j^c the restriction of p_S^i, denoted by $p_S^{i,j}$, is a solution of

Find $p_S^{i,j} \in L^2(\Omega_j^c)$ such that

$$\Delta p_S^{i,j} = 0 \text{ in } \Omega_j^c ,$$

$$\frac{\partial p_S^{i,j}}{\partial \nu} = 0 \text{ on } \tilde{\Gamma}_j^c ,$$

the general solutions of which are

$$p_S^{i,j} = \sum_{n \geqslant -1} A_n^{i,j} \, r_j^{\,n\alpha_j} \cos(n\alpha_j \, \theta_j) ,$$

we can choose in particular

$$p_S^{i,i} = \sum_{n \geqslant -1} A_n^{i,i} \, r_i^{\,n\alpha_i} \cos(n\alpha_i \, \theta_i) \text{ in } \Omega_i^c, \quad A_{-1}^{i,i} \neq 0 \tag{100}$$

$$p_S^{i,j} = 0 \text{ in } \Omega_j^c, \quad \text{for } j \neq i . \tag{101}$$

We have:

$$A_0^{i,i} = \frac{\alpha_i}{\pi} \int_0^{\pi/\alpha_i} p_S^{i,i}(R_i, \theta_i) \, d\theta_i , \tag{102}$$

$$A_1^{i,i} = \frac{2\,\alpha_i}{\pi} R_i^{-\alpha_i} \int_0^{\pi/\alpha_i} p_S^{i,i}(R_i, \theta_i) \cos(\alpha_i \, \theta_i) \, d\theta_i - R_i^{-2\alpha_i} A_{-1}^{i,i} , \tag{103}$$

$$m \geqslant 2 \quad A_m^{i,i} = \frac{2\,\alpha_i}{\pi} R_i^{-m\alpha_i} \int_0^{\pi/\alpha_i} p_S^{i,i}(R_i, \theta_i) \cos(m\alpha_i \, \theta_i) \, d\theta_i . \tag{104}$$

This yields, with the help of the transmission conditions on the boundary, the restriction of p_S^i to Ω^e, denoted by $p_S^{i,e}$, solution of

Find $p_S^{i,e} \in H^1(\Omega^e)/\mathbb{R}$ *such that*

$$\Delta p_S^{i,e} = 0 \text{ in } \Omega^e \,, \tag{105}$$

$$\frac{\partial p_S^{i,e}}{\partial \nu^e} = 0 \text{ on } \tilde{\Gamma}^e \cup \left(\bigcup_{j \neq i} \mathcal{B}_j \right), \tag{106}$$

$$\frac{\partial p_S^{i,e}}{\partial \nu^e} + T^i(p_S^{i,e}) = 0 \text{ on } \mathcal{B}_i \,. \tag{107}$$

Here, T^i and T_1^i, $1 \leqslant i \leqslant K$, are the DtN operators defined by (35), with R replaced by R_i, α by α_i, etc. We verify that this problem, equivalent to:

Find $p_S^{i,e} \in H^1(\Omega^e)/\mathbb{R}$ *such that*

$$\int_{\Omega^e} \nabla p_S^{i,e} \cdot \nabla q \, d\mathbf{x} + R_i \int_0^{\pi/\alpha_i} T_1^i(p_S^{i,e}) \, q \, d\theta_i = 2 \, \alpha_i \frac{A_{-1}^{i,i}}{R_i^{\alpha_i}} \int_0^{\pi/\alpha_i} \cos(\alpha_i \theta_i) \, q \, d\theta_i \quad \forall q \in H^1(\Omega^e)/\mathbb{R} \,, \tag{108}$$

is well posed in $H^1(\Omega^e)/\mathbb{R}$ and that it consequently admits one and only one solution in this space. Moreover, it is easily checked that the global solution p_S^i is determined in function of $A_{-1}^{i,i}$. Finally, it is clear that the familly of K solutions $(p_S^i)_{1 \leqslant i \leqslant K}$ is free in N by construction.

Hence there remains to determine the $(\mathbf{v}_S^i)_{1 \leqslant i \leqslant K}$ solutions of

Find $\mathbf{v}_S^i \in H(\text{curl}, \Omega)$ *such that*

$$\text{curl } \mathbf{v}_S^i = p_S^i \text{ in } \Omega \,, \tag{109}$$

$$\text{div } \mathbf{v}_S^i = 0 \text{ in } \Omega \,, \tag{110}$$

$$\mathbf{v}_S^i \cdot \tau = 0 \text{ on } \Gamma \,. \tag{111}$$

Remark B.3: A resolution method of this problem based on the **curl** curl formulation can also be used (see [4]). ■

In order to solve (109)-(111), we use the isomorphism **curl** of lemma 2.1 between $\Phi/\mathbb{R}$ and V: to $\mathbf{v}_S^i$ we associate $\phi_S^i \in H^1(\Omega)/\mathbb{R}$ such that

$$-\Delta \phi_S^i = p_S^i \text{ in } \Omega \,, \tag{112}$$

$$\frac{\partial \phi_S^i}{\partial \nu} = 0 \text{ on } \Gamma \,. \tag{113}$$

This problem is equivalent to:

Find $\phi_S^i \in H^1(\Omega)/\mathbb{R}$ *such that*

$$\int_\Omega \nabla \phi_S^i \cdot \nabla \psi \, d\mathbf{x} = \int_\Omega p_S^i \, \psi \, d\mathbf{x}, \quad \forall \psi \in H^1(\Omega)/\mathbb{R} \,. \tag{114}$$

We can, in a second step, make use of the explicit knowledge of ϕ_S^i in the neighborhood of the reentrant corner. We call $\phi_S^{i,j}$(resp. $\phi_S^{i,e}$) the restriction of ϕ_S^i to Ω_j^c (resp. Ω^e). So $\phi_S^{i,j}$ is a solution of

Find $\phi_S^{i,j} \in H^1(\Omega_j^c)$ such that

$$\Delta \phi_S^{i,j} = p_S^{i,j} \text{ in } \Omega_j^c ,$$

$$\frac{\partial \phi_S^{i,j}}{\partial \nu} = 0 \text{ on } \tilde{\Gamma}_j^c .$$

thus we can write it as

$$\phi_S^{i,i} = - \sum_{n \geqslant 1} \frac{B_n^{i,i}}{n\alpha_i} r_i^{\;n\alpha_i} \cos(n\alpha_i \theta_i) - \sum_{n \geqslant -1} \frac{A_n^{i,i}}{4\, n\alpha_i + 4} r_i^{\;n\alpha_i + 2} \cos(n\alpha_i \theta_i) ,$$

$$\phi_S^{i,j} = - \sum_{n \geqslant 1} \frac{B_n^{i,j}}{n\alpha_j} r_j^{\;n\alpha_j} \cos(n\alpha_j \theta_j), \quad \text{for } i \neq j .$$

The $(A_n^{i,i})_{n \geqslant -1}$, $1 \leqslant i \leqslant K$, are those of formulas (102)-(104). The $(B_m^{i,j})_{m \geqslant 1}$, $1 \leqslant i, j \leqslant K$, are given by formulas of the type (64)-(65).

The DtN operators, denoted by t^j, are defined by $t^j : \phi^{i,j}_{S|\mathscr{B}_j} \mapsto \frac{\partial \phi_S^{i,j}}{\partial \nu_j^c}\Big|_{\mathscr{B}_j}$. They correspond to (66)-(67), with R replaced by R_j, etc. Notice that, for $i \neq j$, $t^j(\phi_S^{i,e})$ reduces to $t_1^j(\phi_S^{i,e})$.

The function $\phi_S^{i,e}$ is a solution of the following problem:

Find $\phi_S^{i,e} \in H^1(\Omega^e)/\mathbb{R}$ such that

$$- \Delta \phi_S^{i,e} = p_S^{i,e} \text{ in } \Omega^e , \tag{115}$$

$$\frac{\partial \phi_S^{i,e}}{\partial \nu^e} = 0 \text{ on } \tilde{\Gamma}^e , \tag{116}$$

$$\frac{\partial \phi_S^{i,e}}{\partial \nu^e} + t^j(\phi_S^{i,e}) = 0 \text{ on } \mathscr{B}_j, \quad \text{for } 1 \leqslant j \leqslant K . \tag{117}$$

As $\phi_S^{i,e}$ is sufficiently smooth (i.e. in H^1), this problem is equivalent to the variational formulation:

Find $\phi_S^{i,e} \in H^1(\Omega^e)/\mathbb{R}$ such that

$$\int_{\Omega^e} \nabla \phi_S^{i,e} \cdot \nabla \psi \, d\mathbf{x} + \sum_{j=1}^{K} R_j \int_0^{\pi/\alpha_j} t_1^j(\phi_S^{i,e})\, \psi \, d\theta_j =$$

$$\int_{\Omega^e} p_S^{i,e}\, \psi \, d\mathbf{x} + \frac{1}{2} R_i \int_0^{\pi/\alpha_i} \left\{ \int_{r_i = 0}^{r_i = R_i} p_S^{i,i}(r_i, \theta_i)\, dr_i \right\} \psi(R_i, \theta_i)\, d\theta_i$$

$$- \frac{\alpha_i}{2 - 2\,\alpha_i} A_{-1}^{i,i} R_i^{2 - \alpha_i} \int_0^{\pi/\alpha_i} \cos(\alpha_i \theta_i)\, \psi(R_i, \theta_i)\, d\theta_i, \quad \forall \psi \in H^1(\Omega^e)/\mathbb{R} . \tag{118}$$

Remark B.4: The bilinear symmetric operators of (108) and (118) (respectively for the computation of $p_S^{i,e}$ and $\phi_S^{i,e}$) are still identical, if we notice that $T_1^j = t_1^j$ for $1 \leqslant j \leqslant K$ and that $T_1^j(p_S^{i,e}) = 0$ for $i \neq j$, as $p^{i,e}_{S|\mathscr{B}_j} = p^{i,j}_{S|\mathscr{B}_j} = 0$. ■

B.2. Resolution of the variational time dependent problem

We semi-discretize (10) in space. The test functions are still chosen in $V_R^h \oplus V_S$, with $V_R^h \subset V_R$, whence (11) can be satisfied. We use again P_h, the projection onto V_R^h in the sense of the $L^2(\Omega)$ inner product. Compared to (72)-(74), the difference is that we have in this appendix K functions $(\mathbf{v}_S^j)_{1 \leqslant j \leqslant K}$ instead of only one, so that $\mathbf{u}^h$ can be written

$$\mathbf{u}^h = \mathbf{u}_R^h + \sum_{j=1}^{j=K} c_j(t)\, \mathbf{v}_S^j .$$

First, we take the following K functions: $\mathbf{v}_i^h = \mathbf{v}_S^i - P_h \mathbf{v}_S^i$. We obtain, thanks to the orthogonality of curl $\mathbf{u}_R^h$ and $(\operatorname{curl} \mathbf{v}_S^j)_{1 \leqslant j \leqslant K}$:

$$\sum_{j=1}^{j=K} c_j''(t) \int_\Omega (\mathbf{v}_S^i - P_h \mathbf{v}_S^i) \cdot (\mathbf{v}_S^j - P_h \mathbf{v}_S^j)\, d\mathbf{x} + \sum_{j=1}^{j=K} c_j(t) \int_\Omega p_S^i p_S^j \, d\mathbf{x}$$
$$= \int_\Omega \operatorname{curl} \mathbf{u}_R^h \operatorname{curl} P_h \mathbf{v}_S^i \, d\mathbf{x} + \int_\Omega \mathbf{f}^h \cdot (\mathbf{v}_S^i - P_h \mathbf{v}_S^i)\, d\mathbf{x} \quad \text{for } 1 \leqslant i \leqslant K .$$

This system of equations can be put in matrix form

$$\mathbf{A}\mathbf{c}''(t) + \mathbf{B}\mathbf{c}(t) = \mathbf{f}(t)\,, \tag{119}$$

with the $K \times K$ matrices $\mathbf{A} = (a_{ij})_{1 \leqslant i,j \leqslant K}$ and $\mathbf{B} = (b_{ij})_{1 \leqslant i,j \leqslant K}$, as well as the vectors $\mathbf{c}(t) = (c_j(t))_{1 \leqslant j \leqslant K}$ and $\mathbf{f}(t) = (f_i(t))_{1 \leqslant i \leqslant K}$. Here,

$$a_{ij} = \int_\Omega (\mathbf{v}_S^i - P_h \mathbf{v}_S^i) \cdot (\mathbf{v}_S^j - P_h \mathbf{v}_S^j)\, d\mathbf{x},\ b_{ij} = \int_\Omega p_S^i p_S^j \, d\mathbf{x}, \text{ and}$$

$$f_i(t) = \int_\Omega \operatorname{curl} \mathbf{u}_R^h(t) \operatorname{curl} P_h \mathbf{v}_S^i \, d\mathbf{x} + \int_\Omega \mathbf{f}^h(t) \cdot (\mathbf{v}_S^i - P_h \mathbf{v}_S^i)\, d\mathbf{x} .$$

PROPOSITION B.1: **A** *and* **B** *are two symmetric positive definite matrices. Moreover, we have the following estimation:*

$$|a_{ij}| \leqslant C_\varepsilon h^{\alpha_i + \alpha_j - 2\varepsilon}, \quad \forall \varepsilon > 0 . \tag{120}$$

Proof: Obviously **A** and **B** are symmetric and positive, and **B** is positive definite as $(p_S^i)_{1 \leqslant i \leqslant K}$ is a basis of N. Moreover, the estimation of the elements of **A** is obtained by using the result which specifies the regularity of the functions $(\mathbf{v}_S^i)$.

Then let $\mathbf{y} = (y_j)_{1 \leqslant j \leqslant K}$ be such that $\mathbf{A}\mathbf{y} = 0$. Set $\mathbf{v}_S = \sum_j y_j \mathbf{v}_S^j \in V_S$.

From $\mathbf{A}\mathbf{y} = 0$, we deduce

$$0 = \sum_j y_j \int_\Omega (\mathbf{v}_S^i - P_h \mathbf{v}_S^i) \cdot (\mathbf{v}_S^j - P_h \mathbf{v}_S^j)\, d\mathbf{x}, \text{ for } 1 \leqslant i \leqslant K ,$$

$$\Rightarrow \quad 0 = \int_\Omega (\mathbf{v}_S^i - P_h \mathbf{v}_S^i) \cdot (\mathbf{v}_S - P_h \mathbf{v}_S)\, d\mathbf{x}, \text{ for } 1 \leqslant i \leqslant K ,$$

$$\Rightarrow \quad 0 = \int_\Omega (\mathbf{v}_S - P_h \mathbf{v}_S)^2 \, d\mathbf{x}.$$

Whence finally $\mathbf{v}_S = P_h \mathbf{v}_S \in V_R^h$, that is $\mathbf{v}_S = 0$, or $\mathbf{y} = 0$.

In particular, we deduce that if the value $\min_j \alpha_j$ is reached for a unique j_0, the dominant term is $\bar{a}_{j_0 j_0}$. If there exist $j_0, \ldots, j_l$ which minimise the value of $\min_j \alpha_j$, the dominant terms are among $(a_{jj'})_{j, j' \in \{j_0, \ldots, j_l\}}$. In any case, this linear system of ordinary differential equations is not stiff, and can be solved by a classical time discretization scheme.

Finally, to compute $\mathbf{u}_R^h$, we write the following variational formulation:

Find $\mathbf{u}_R^h \in V_R^h$ *such that*

$$\frac{d^2}{dt^2} \int_\Omega \mathbf{u}_R^h \cdot \mathbf{v}_R^h \, d\mathbf{x} + \int_\Omega \operatorname{curl} \mathbf{u}_R^h \operatorname{curl} \mathbf{v}_R^h \, d\mathbf{x} = \int_\Omega \mathbf{f}^h \cdot \mathbf{v}_R^h \, d\mathbf{x} - \sum_{j=1}^{j=K} c_j''(t) \int_\Omega \mathbf{v}_S^j \cdot \mathbf{v}_R^h \, d\mathbf{x}, \quad \forall \mathbf{v}_R^h \in V_R^h. \tag{121}$$

This formulation, involving only the regular part of the fields, can be solved by a usual finite element method.

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