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A CONVERGENCE RESULT FOR AN ITERATIVE METHOD FOR THE EQUATIONS OF A STATIONARY QUASI-NEWTONIAN FLOW WITH TEMPERATURE DEPENDENT VISCOSITY (*)

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Abstract — *We study a system of equations describing the stationary and incompressible flow of a quasi-Newtonian fluid with temperature dependent viscosity and with a viscous heating An algorithm wich decouples the calculation of the temperature T and the velocity and the pressure (v, p) is presented It consists in solving iteratively a problem with a nonlinear Stokes's operator for v and p and the Poisson's equation with right-hand side in L^1 for T We prove, using the method of pseudomonotonicity and under a regularity assumption of Meyers type that the mapping defined by this scheme is a contraction for sufficiently small data © Elsevier, Paris*

Résumé — *On étudie un système modélisant l'écoulement d'un fluide quasi-Newtonien stationnaire incompressible avec une viscosité dépendant de la température et en tenant compte des effets d'échauffement visqueux On présente un algorithme découplant le calcul du couple vitesse-pression et de la température il s'agit de résoudre itérativement un problème concernant un opérateur de Stokes non linéaire en vitesse et pression, à température donnée, puis une équation de Poisson à second membre L^1 en température, à vitesse donnée On montre à l'aide de la méthode de pseudo-monotonie et sous une hypothèse de régularité de type Meyers que l'application définie par ce schéma est contractante pour des données suffisamment petites © Elsevier, Paris*

1. INTRODUCTION

We consider equations describing the incompressible quasi-Newtonian fluid flow with temperature dependant viscosity. Existence for such problem of a weak solution has been recently proved by Baranger and Mikelic, (see [3]), using Schauder fixed point theorem; uniqueness of this solution was left as an open problem.

In numerical simulations one usually uses an iterative decoupled algorithm: here, it will consist in solving iteratively a problem with a nonlinear Stokes problem for v and p and the Poisson's equation with right-hand side in L^1 for T.

We prove in this paper, for small data and under a Meyers's type regularity property of the r-Stokesian operator, that this simple algorithm is convergent to the unique weak solution of the problem. In fact, we prove that the operator defined from the iterative method is a contraction and use Banach fixed-point theorem.

Some similar problems, but in the simpler case of two scalar elliptic equations coupling the Laplacian and the heat equation, have been studied by Howinson *et al.* (see [7]) with uniqueness result for sufficiently small data and sufficiently regular solution, (see also [4]). We will adapt the functional framework and some ideas from [3] in proving existence.

Let us consider a bounded domain Ω in $\mathbb{R}^N$, $N = 2$ or 3, with a regular boundary Γ, and an incompressible quasi-Newtonian fluid flowing in Ω, with temperature dependent viscosity and with a viscous heating. We consider

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the steady case and neglect inertia effects. T being the temperature, v the velocity and p the pressure of the fluid, we consider the following problem $(\mathscr{P})$, (see [3] for a derivation of the model from the basic principles of continuum mechanics):

$$(\mathscr{P})\begin{cases} -\operatorname{div}[\mu(T,|D(v)|)D(v)]+\nabla p = f & \text{in } \Omega, \\ \operatorname{div} v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma, \\ -k\,\Delta T+\rho c_p(T)\, v\,\nabla T = \mu(T,|D(v)|)\,|D(v)|^2 & \text{in } \Omega, \\ T = \tau_0 & \text{on } \Gamma. \end{cases}$$

where $D(u)=\frac{1}{2}(\nabla u+\nabla u^T)$, $c_p(\,.\,)$ is a bounded continuous function on $\mathbb{R}$, k is a positive constant, ρ is the constant density of the fluide,

$$\tau_0 \in L^\infty(\Gamma)\cap \bigcap_{q<\frac{N}{N-1}} W^{1-1/q,q}(\Gamma);\quad \tau_0 > C_0 > 0 \text{ (a.e.) on } \Gamma \tag{1.1}$$

This is more realistic than the assumption: $\tau_0 \in H^{1/2}(\Gamma)$, (see [3]).

Furthermore, this assumption on the boundary data ensures the existence of an extension of τ_0, which we will denote by $\bar{\tau}_0$, such that: $\bar{\tau}_0 \in W^{1,q}(\Omega)$, $\forall q < N'$, owing to the isomorphism between $W^{1-1/q,q}(\Gamma)$ and $W^{1,q}(\Omega)/\ker\gamma$, γ being the trace operator on Γ (see [1], Theorem 7.53).

μ is supposed continuous on $\mathbb{R}^2$ and satisfies the following properties: $\forall s_1, s_2 \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^{N^2}_{sym}$

$$|\mu(s_1,|\xi|)-\mu(s_2,|\xi|)| \leq K_1\,\beta(|s_1-s_2|)\,|\xi|^{r-2},\quad 1<r\leq 2, \tag{1.2}$$

$$\text{where}: \beta \in C_b(\mathbb{R}),\quad \beta \geq 0 \text{ and } \beta(0)=0, \tag{1.3}$$

$$[\mu(s,|\xi|)\,\xi-\mu(s,|\eta|)\,\eta]:(\xi-\eta) \geq K_2|\xi-\eta|^2\{|\xi|+|\eta|\}^{r-2}, \tag{1.4}$$

$$\forall s \in \mathbb{R},\ \forall \xi,\eta \in \mathbb{R}^{N^2}_{sym},$$

$$|[\mu(s,|\xi_1|)\,\xi_1-\mu(s,|\xi_2|)\,\xi_2]:\eta| \leq K_3|\eta|\,|\xi_1-\xi_2|^{r-1}, \tag{1.5}$$

$$\forall \xi_1,\xi_2,\eta \in \mathbb{R}^{N^2}_{sym}.$$

We remark that a classical exemple of viscosity is the product of an Arrhenius law: $\lambda(T)=C\exp\frac{K}{T}$ and a power law $\nu(|D(v)|)=\nu_0|D(v)|^{r-2}$ (see [2]), the above conditions being satisfied in that case.

Now, for studying problem $(\mathscr{P})$, we define the following functional spaces: For the velocity v, since we have to solve a r-Stokes monotone problem:

$$V_r=\{v \in [W_0^{1,r}(\Omega)]^N/\operatorname{div} v=0 \text{ in } \Omega\} \tag{1.6}$$

and for the temperature T, since we have a Poisson equation with a right-hand side in $L^1(\Omega)$:

$$W_N=\bigcap_{1\leq q<\frac{N}{N-1}} W_0^{1,q}(\Omega) \tag{1.7}$$

We say that (v, T), with $v \in V_r$, $T - \bar{\tau}_0 \in W_N$, $T > C_0$ (a.e.) in Ω, $f \in L^{r'}(\Omega)$, is a weak solution of problem $(\mathscr{P})$ if:

$$\int_\Omega \mu(T, |D(v)|)\, D(v) : D(\varphi) = \int_\Omega f\varphi, \quad \forall \varphi \in V_r\,; \tag{1.8}$$

$$k\int_\Omega \nabla T \nabla \xi - \rho \int_\Omega v C_p(T)\, \nabla \xi = \int_\Omega \mu(T, |D(v)|)\, |D(v)|^2\, \xi\,, \tag{1.9}$$

$$\forall \xi \in W_0^{1,\infty}(\Omega), \quad \text{where } C_p(T) = \int_0^T c_p(s)\, ds\,.$$

2. THE FIXED POINT ALGORITHM

We introduce the following decoupled algorithm:

We start by $T^0 = \bar{\tau}_0$, and $(v^0, p^0) =$ the solution in $V_r \times L^{r'}(\Omega)$ of the Stokes problem, (see [12]):

$$\begin{cases} -\operatorname{div}\,[\mu(\bar{\tau}_0, |D(v^0)|)\, D(v^0)] + \nabla p^0 = f & \text{in } \Omega \\ \operatorname{div} v^0 = 0 & \text{in } \Omega \\ v^0 = 0 & \text{on } \Gamma\,. \end{cases}$$

For T^n, v^n, p^n given, we search for T^{n+1}, v^{n+1}, p^{n+1} weak solutions in $W_N \times V_r \times L^{r'}(\Omega)$ of the following homogeneous problem:

$$(\mathscr{P}_{n+1}) \begin{cases} -\operatorname{div}\,[\mu(T^n + \bar{\tau}_0, |D(v^{n+1})|)\, D(v^{n+1})] + \nabla p^{n+1} = f & \text{in } \Omega \\ -k\Delta(T^{n+1} + \bar{\tau}_0) + \rho c_p(T^{n+1} + \bar{\tau}_0)\, v^{n+1}\, \nabla (T^{n+1} + \bar{\tau}_0) & \\ \quad = \mu(T^n + \bar{\tau}_0, |D(v^{n+1})|)\, |D(v^{n+1})|^2 & \text{in } \Omega \end{cases}$$

We define, from this algorithm, the following fixed point operator:

$$\Phi : V_r \times W_N \to V_r \times W_N$$

$$(u, T_u) \mapsto (v, T_v) = \Phi(u, T_u) \text{ solution of :}$$

$$\begin{cases} -\operatorname{div}\,[\mu(T_u + \bar{\tau}_0, |D(v)|)\, D(v)] + \nabla p_v = f \text{ in } \Omega, \quad \text{and :} \\ -k\Delta(T_v + \bar{\tau}_0) + \rho c_p(T_v + \bar{\tau}_0)\, v \nabla (T_v + \bar{\tau}_0) = \mu(T_u + \bar{\tau}_0, |D(v)|)\, |D(v)|^2 \text{ in } \Omega\,. \end{cases} \tag{2.1}$$

where $p_v \in L^{r'}(\Omega)$ is the pressure associated to v and is unique up to a constant.

In order to prove that Φ is a contracting mapping and hence, to state a convergence theorem for the algorithm $(\mathscr{P}_{n+1})$, we describe a Meyers's type regularity property of the r-Stokesian operator used in the first step of $(\mathscr{P}_{n+1})$, i.e. solution of the r-Stokes problem:

$$(\mathscr{S}_r) \begin{cases} -\operatorname{div}\,[\mu_r(T, |D(v)|)\, D(v)] + \nabla p = f & \text{in } \Omega \\ \operatorname{div} v = 0 & \text{in } \Omega \\ v = 0 & \text{on } \Gamma\,, \end{cases}$$

where $\mu_r(\,.\,,.\,) := \mu(\,.\,,.\,)$ satisfies assumptions (1.2)-(1.5). We can formulate this property as follows:

There exists $\gamma^* > r$ such that: for $f \in L^{\gamma}(\Omega)$ with $\frac{1}{\gamma} = \frac{1}{\gamma^*} + \frac{1}{N}$, we have, for each v solution of the r-Stokes problem $(\mathcal{S}_r)$:

$$D(v) \in L^p(\Omega), \quad \forall r < p \leqslant \gamma^*, \quad \text{and} \quad \|D(v)\|_{L^{\gamma^*}(\Omega)} \leqslant C\|f\|_{L^{\gamma}(\Omega)} \tag{2.2}$$

the constant C depending only on the data.

Such a regularity result has been proved in [11] for second order equation. See [13] for the case of the r-Stokesian operator.

For technical reason, we introduce:

$$\gamma_0 = \begin{cases} r & \text{if} \quad N = 2 \\ \dfrac{3(r-1)}{2r-3}\, r & \text{if} \quad N = 3 . \end{cases} \tag{2.3}$$

We can state:

THEOREM 2.1: *Assume (1.1)-(1.5), $\frac{N}{2} < r \leqslant 2$, and that the exponent γ^* in (2.2) satisfies: $\gamma^* > \gamma_0$, where γ_0 is given by (2.3). Then there exists a constant $\bar{C}$, depending only on the data, such that: if $\|f\|_{L^{\gamma}(\Omega)} \leqslant \bar{C}$, with $\frac{1}{\gamma} = \frac{1}{\gamma^*} + \frac{1}{N}$, then the fixed point interation is a contraction.*

COROLLARY 2.1: *Under the previous assumptions, Problem $(\mathcal{P})$ has a unique weak solution and the fixed point algorithm $(\mathcal{P}_n)$ is convergent.*

3. PROOF OF THEOREM 2.1

The proof is based on four propositions:

PROPOSITION 3.1: *Under the assumptions of theorem 2.1, the fixed point operator Φ is well defined.*

Proof: Let us prove existence and uniqueness of a weak solution of $(\mathcal{P}_{n+1})$:

The solution v^{n+1} of the r-Stokes problem in $(\mathcal{P}_{n+1})$ exists in V_r, is unique owing to the assumptions (1.2)-(1.4); and there exists a corresponding pressure p^{n+1} unique up to a constant, in $L^{r'}(\Omega)$ (see [12]).

Furthermore, we obtain easily, taking v^{n+1} as a test-function in the first equation of $(\mathcal{P}_{n+1})$, using (1.4) and the Poincaré's inequality:

$$\|D(v^{n+1})\|_{L^r(\Omega)^{N^2}} \leqslant \left(\frac{C(\Omega)}{K_2}\right)^{\frac{1}{r-1}} \|f\|^{\frac{1}{r-1}}_{L^{\bar{r}}(\Omega)} = C(\Omega, r, f) . \tag{3.1}$$

In the second equation in $(\mathcal{P}_{n+1})$, the right-hand side is in $L^1(\Omega)$ since v is in V_r and since μ satisfies (1.2), (1.3). So we do not have a sufficient regularity for using the classical variational formulatin for this problem. Adapting an idea of [3], we decompose this equation in two simpler ones:

Firstly:

$$\begin{cases} -k\,\Delta T_1^{n+1} &= \mu(T^n + \bar{\tau}_0, |D(v^{n+1})|)\, |D(v^{n+1})|^2 \text{ in } \Omega . \\ \quad T_1^{n+1} &= 0 \text{ on } \Gamma . \end{cases} \tag{3.2}$$

Then, we can apply the results on Poisson's equation with right-hand side in L^1, (see for example [5]) and we obtain existence and uniqueness of a solution to (3 2)

$$T_1^{n+1} \in W_0^{1,q}(\Omega), \quad \forall 1 \leq q < \frac{N}{N-1} = N',$$

and we have the estimate $\|T_1^{n+1}\|_{W_0^{1,q}(\Omega)} \leq C(\Omega, N, r, \tau_0), \ \forall 1 \leq q < N'$

In fact, for $N = 3$, we can use some results from [10] (see Theorem 12 1) to get that the solution of (3 2) lies in $W_0^{1,N}(\Omega)$

Indeed, using the first equation of $(\mathscr{P}_{n+1})$, we can write formally the right hand side of (3 2) as follows

$$\operatorname{div}\{[\mu(T^n + \bar{\tau}_0, |D(v^{n+1})|)\, D(v^{n+1}) - p^{n+1} I]\, v^{n+1}\} + fv^{n+1},$$

where

$$[\mu(T^n + \bar{\tau}_0, |D(v^{n+1})|)\, D(v^{n+1}) - p^{n+1} I]\, v^{n+1} \in L^N \quad \text{and} \quad fv^{n+1} \in W^{-1,N}(\Omega)$$

This can be easily seen using Holder's inequality with exponents $p = \frac{(N-1)r}{N(r-1)}$, $p' = \frac{(N-1)r}{N-r}$, (Note that $p > 1$ for $r < N$) Indeed, we obtain, with (1 2)-(1 3)

$$\int_\Omega |\mu(T^n + \bar{\tau}_0, |D(v^{n+1})|)\, D(v^{n+1})\, v^{n+1}|^N \leq C \int_\Omega \{|D(v^{n+1})|^{r-1} |v^{n+1}|\}^N$$

$$\leq \|D(v^{n+1})\|_{L^r}^{N(r-1)} \|v^{n+1}\|_{L^{r^*}}^N$$

$\leq \|D(v^{n+1})\|_{L^r}^{rN}$, by Poincaré's inequality and Sobolev Imbedding Theorem, $\leq C(\Omega, r, f)$, by (3 1)

For fv^{n+1}, it is easy to see that $\forall \varphi \in W_0^{1,N}(\Omega)\ (\subset L^p(\Omega), \forall p < \infty)$,

$$\int_\Omega fv^{n+1} \varphi \leq C \|f\|_{L^{r'}} \|v^{n+1}\|_{L^{r^*}} \|\varphi\|_{L^{r'}\left(\frac{r^*}{r}\right)}$$

Secondly

$$\begin{cases} -k\Delta(T_2^{n+1} + \bar{\tau}_0) + \rho c_p(T_1^{n+1} + T_2^{n+1} + \bar{\tau}_0)\, v^{n+1} \nabla(T_1^{n+1} + T_2^{n+1} + \bar{\tau}_0) &= 0 \text{ in } \Omega \\ T_2^{n+1} &= 0 \text{ on } \Gamma \end{cases} \tag{3 3}$$

We have, since c_p is bounded $\forall T \in H^1(\Omega)$,

$$\left|\int_\Omega v^{n+1} c_p(T_1^{n+1} + \bar{\tau}_0 + T)\, \nabla(T_1^{n+1} + \bar{\tau}_0 + T)\right| \leq C \|v^{n+1}\|_{L^{r^*}} \|T_1^{n+1} + \bar{\tau}_0 + T\|_{W^{1,(r^*)'}}$$

$$\leq C \|v^{n+1}\|_{L^{r^*}} \|T_1^{n+1} + \bar{\tau}_0 + T\|_{W_{N'}} \text{ since } (r^*)' = \frac{Nr}{Nr - N + r} < \frac{N}{N-1}, \text{ for } r > \frac{N}{2},$$

and:

$$\forall \varphi \in H_0^1(\Omega), \quad \left| \int_\Omega v^{n+1} T \nabla \varphi \right| \leqslant C \|\varphi\|_{H_0^1} \|v^{n+1}\|_{L^{r^*}} \|T\|_{L^2\left(\frac{r^*}{2}\right)'},$$

$$\leqslant C \|\varphi\|_{H_0^1} \|v^{n+1}\|_{W^{1,r}} \|T\|_{H^1}, \text{ since } 2\left(\frac{r^*}{2}\right)' = \frac{6r}{5r-6} < 2^* = 6, \text{ for } r > \frac{N}{2},$$

this for $N = 3$; obtaining a same estimate for $N = 2$ being more easy due to Sobolev Imbedding Theorem.

Then, we can apply results of pseudomonotone operators theory, (see [9]), to get existence and uniqueness of a solution T_2^{n+1} in $H_0^1(\Omega)$ to problem (3.3) and that: $\|T_2^{n+1}\|_{H_0^1(\Omega)} \leqslant C$, where C depends only on the coefficients of the equation and the data. So, by (3.1), C depends only on the data.

Note that if $c_p(T^{n+1} + \bar{\tau}_0)$ is replaced by $c_p(T^n + \bar{\tau}_0)$ in the algorithm, then we can deduce existence and uniqueness of a solution of (3.3) in $H_0^1(\Omega)$ directly from the results of linear elliptic equations with unbounded coefficients (see [8]) since the coefficient v^{n+1} satisfies: $\|v^{n+1}\|_{L^{p/2}(\Omega)} \leqslant C < +\infty$, with $p = 2r > N$.

Finally, taking: $T^{n+1} = T_1^{n+1} + T_2^{n+1}$, we obtain a unique weak solution of $(\mathscr{P}_{n+1})$, which satisfies:

$$\|T^{n+1}\|_{W_N} \leqslant C(\Omega, N, r, \tau_0) . \tag{3.4}$$

We conclude that the mapping Φ is well defined. ∎

PROPOSITION 3.2: *If the iterative method converges to (v_0, T_0), then $(v_0, T_0 + \bar{\tau}_0)$ is a weak solution of $(\mathscr{P})$.*

Proof: From the estimates (3.1) and (3.4), we deduce that there exists a subsequence, still denoted by the same symbol, such that:

— firstly: $v^n \to v_0$ in V_r weak. So, by Rellich's Compactness Theorem,

$$v^n \to v_0 \text{ in } L^p(\Omega) \text{ strong, for } 1 \leqslant p < r^* = \frac{Nr}{N-r} \quad \text{if } r < N , \tag{3.5}$$

for all $p < \infty$, if $r = N$

— secondly: $T^n \to T_0$ in $W_0^{1,q}(\Omega)$ weak, $\forall 1 \leqslant q < N'$. Then:

$$T^n \to T_0 \text{ in } L^m(\Omega) \text{ strong for } 1 \leqslant m < (N')^* = \frac{N}{N-2}, \text{ if } N = 3 , \tag{3.6}$$

for all $m < \infty$ if $N = 2$, and $T^n \to T_0$ a.e. in Ω.

Let us now show that: $(v^n) \xrightarrow[n \to \infty]{} v_0$ in V_r strong:

We have by (1.8):

$$\int_\Omega \mu(T^n + \bar{\tau}_0, |D(v^{n+1})|) D(v^{n+1}) : D(\psi) = \int_\Omega f\psi, \quad \forall \psi \in V_r, \tag{3.7}$$

and taking: $\psi = \varphi - v^{n+1}$:

$$\int_\Omega \mu(T^n + \bar{\tau}_0, |D(v^{n+1})|) D(v^{n+1}) : D(\varphi - v^{n+1}) = \int_\Omega f(\varphi - v^{n+1}) . \tag{3.8}$$

But (1.4) gives:

$$\int_\Omega [\mu(T^n + \bar{\tau}_0, |D(\varphi)|) D(\varphi) - \mu(T^n + \bar{\tau}_0, |D(v^{n+1})|) D(v^{n+1})] : D(\varphi - v^{n+1}) \geq 0 .$$

Then:

$$\int_\Omega \mu(T^n + \bar{\tau}_0, |D(\varphi)|) D(\varphi) : D(\varphi - v^{n+1}) \geq \int_\Omega f(\varphi - v^{n+1}) . \tag{3.9}$$

Then, passing to the limit in this inequality, using the continuity of μ, the a.e. convergence of T^n to T_0 and the weak convergence of v^n to v_0, we get:

$$\int_\Omega \mu(T_0 + \bar{\tau}_0, |D(\varphi)|) D(\varphi) : D(\varphi - v_0) \geq \int_\Omega f(\varphi - v_0) . \tag{3.10}$$

Now, by a usual procedure from Minty's lemma (taking first $\varphi = v_0 + \alpha\psi$, with $\alpha > 0$, in (3.10), then letting $\alpha \to 0$, and taking $\psi = -\varphi$), we obtain:

$$\int_\Omega \mu(T_0 + \bar{\tau}_0, |D(v_0)|) D(v_0) : D(\varphi) = \int_\Omega f\varphi, \quad \forall \varphi \in V_r . \tag{3.12}$$

So in particular: $\int_\Omega \mu(T_0 + \bar{\tau}_0, |D(v_0)|)|D(v_0)|^2 = \int_\Omega f v_0$; and, with (3.7):

$$\int_\Omega \mu(T^n + \bar{\tau}_0, |D(v^{n+1})|)\, |D(v^{n+1})|^2 - \int_\Omega \mu(T_0 + \bar{\tau}_0, |D(v_0)|)\, |D(v_0)|^2$$

$$= \left| \int_\Omega f(v^{n+1} - v_0) \right| \xrightarrow[n \to \infty]{} 0 . \tag{3.13}$$

Furthermore, we have:

$$\int_\Omega [\mu(T^n + \bar{\tau}_0, |D(v_0)|) D(v_0) - \mu(T^n + \bar{\tau}_0, |D(v^{n+1})|) D(v^{n+1})] : D(v_0 - v^{n+1})$$

$$= \int_\Omega [\mu(T^n + \bar{\tau}_0, |D(v_0)|) - \mu(T_0 + \bar{\tau}_0, |D(v_0)|)] D(v_0) : D(v_0 - v^{n+1})$$

$$+ \int_\Omega \mu(T_0 + \bar{\tau}_0, |D(v_0)|) D(v_0) : D(v_0 - v^{n+1})$$

$$- \int_\Omega \mu(T^n + \bar{\tau}_0, |D(v^{n+1})|) D(v^{n+1}) : D(v_0 - v^{n+1})$$

$$= \int_\Omega [\mu(T^n + \bar{\tau}_0, |D(v_0)|) - \mu(T_0 + \bar{\tau}_0, |D(v_0)|)] D(v_0) : D(v_0 - v^{n+1}) ,$$

by (3.7) and (3.12). This, with condition (1.4), gives:

$$K_2 \int_\Omega |D(v_0 - v^{n+1})|^2 \{ |D(v_0)| + |D(v^{n+1})| \}^{r-2}$$

$$\leqslant \left| \int_\Omega [\mu(T^n + \bar{\tau}_0, |D(v_0)|) - \mu(T_0 + \bar{\tau}_0, |D(v_0)|)] \, D(v_0) : D(v_0 - v^{n+1}) \right|$$

$$\leqslant K_1 \int_\Omega \beta(|T^n - T_0|) \, |D(v_0)|^{r-1} |D(v_0 - v^{n+1})|, \text{ by } (1.2)$$

$$\leqslant C \| D(v_0 - v^{n+1}) \|_{(L^r(\Omega))^{N^2}} \left[\int_\Omega \beta(|T^n - T_0|)^{r'} |D(v_0)|^r \right]^{\frac{1}{r'}}, \tag{3.14}$$

by Holder's inequality (with $\frac{1}{r'} + \frac{1}{r} = 1$). But, we have (for $r < 2$):

$$\int_\Omega |D(v_0 - v^{n+1})|^r$$

$$= \int_\Omega |D(v_0 - v^{n+1})|^r \{ |D(v_0)| + |D(v^{n+1})| \}^{\frac{r-2}{2} r} \{ |D(v_0)| + |D(v^{n+1})| \}^{\frac{2-r}{2} r},$$

$$\leqslant \left[\int_\Omega |D(v_0 - v^{n+1})|^2 \{ |D(v_0)| + |D(v^{n+1})| \}^{r-2} \right]^{\frac{r}{2}} \left[\int_\Omega \{ |D(v_0)| + |D(v^{n+1})| \}^r \right]^{\frac{2-r}{2}}.$$

and since:

$$\int_\Omega \{ |D(v_0)| + |D(v^{n+1})| \}^r \leqslant 2^{r-1} (\| D(v_0) \|_{L^r}^r + \| D(v^{n+1}) \|_{L^r}^r) \leqslant C,$$

by (3.1), then we get (for $r \leqslant 2$):

$$\| D(v_0 - v^{n+1}) \|_{(L^r(\Omega))^{N^2}}^r \leqslant C \left[\int_\Omega |D(v_0 - v^{n+1})|^2 \{ |D(v_0)| + |D(v^{n+1})| \}^{r-2} \right]^{\frac{r}{2}}.$$

This gives:

$$\| D(v_0 - v^{n+1}) \|_{(L^r(\Omega))^{N^2}}^2$$

$$\leqslant C \int_\Omega |D(v_0 - v^{n+1})|^2 \{ |D(v_0)| + |D(v^{n+1})| \}^{r-2}, \tag{3.15}$$

$$\leqslant C \| D(v_0 - v^{n+1}) \|_{(L^r(\Omega))^{N^2}} \left[\int_\Omega \beta(|T^n - T_0|)^{r'} |D(v_0)|^r \right]^{\frac{1}{r'}}, \text{ by } (3.14).$$

Therefore:

$$\| D(v_0 - v^{n+1}) \|_{(L^r(\Omega))^{N^2}} \leqslant C \left[\int_\Omega \beta(|T^n - T_0|)^{r'} |D(v_0)|^r \right]^{\frac{1}{r'}}. \tag{3.16}$$

Since β is bounded, then we have:

$$\forall n, \quad |\beta(|T^n - T_0|)|^{r'} |D(v_0)|^r \leqslant C|D(v_0)|^r := g \text{ a.e. in } \Omega, \text{ with } g \in L^1(\Omega).$$

Then, using Lebesgue's Dominated Convergence Theorem and the continuity of β (we have: $T^n \to T_0$ a.e.), we deduce from (3.16): $\| D(v_0 - v^{n+1}) \|_{L^r(\Omega)^{N^2}} \xrightarrow[n \to \infty]{} 0$. Consequently,

$$(v^n) \xrightarrow[n \to \infty]{} v_0 \quad \text{in} \quad V_r \text{ strong}. \tag{3.17}$$

For (T^n), we have, by (3.13): $\forall \xi \in W_0^{1,\infty}(\Omega)$,

$$\int_\Omega \mu(T^n + \bar{\tau}_0, |D(v^{n+1})|)\, |D(v^{n+1})|^2 \xi \xrightarrow[n \to \infty]{} \int_\Omega \mu(T_0 + \bar{\tau}_0, |D(v_0)|)\, |D(v_0)|^2 \xi,$$

and by (3.17) and (3.6):

$$\int_\Omega v^{n+1} C_p(T^{n+1} + \bar{\tau}_0) \nabla \xi \xrightarrow[n \to \infty]{} \int_\Omega v_0 C_p(T_0 + \bar{\tau}_0) \nabla \xi.$$

Indeed:

$$\rho \int_\Omega \{ v^{n+1} C_p(T^{n+1} + \bar{\tau}_0) - v_0 C_p(T_0 + \bar{\tau}_0) \} \nabla \xi$$

$$\leqslant C \left\{ \int_\Omega |v^{n+1} - v_0|\, |C_p(T^{n+1} + \bar{\tau}_0)| + \int_\Omega |v_0|\, |C_p(T^{n+1} + \bar{\tau}_0) - C_p(T_0 + \bar{\tau}_0)| \right\}$$

$$\leqslant C\{ \| v^{n+1} - v_0 \|_{L^r} \| T^{n+1} + \bar{\tau}_0 \|_{L^{r'}} + \| v_0 \|_{L^r} \| T^{n+1} - T_0 \|_{L^{r'}} \} \xrightarrow[n \to \infty]{} 0,$$

by (3.6) and (3.17), since we have: $r' = \frac{r}{r-1} < \frac{N}{N-2}$, for $r > \frac{N}{2}$.

Furthermore, by (3.6) and the Sobolev Imbedding: $W_0^{1,\infty}(\Omega) \subset W_0^{1,q'}(\Omega)$, $\forall q < \frac{N}{N-1}$, we have: $\forall \xi \in W_0^{1,\infty}(\Omega)$,

$$k \int_\Omega \nabla(T^{n+1} + \bar{\tau}_0) \nabla \xi \xrightarrow[n \to \infty]{} k \int_\Omega \nabla(T_0 + \bar{\tau}_0) \nabla \xi.$$

So, by uniqueness of the limit, we obtain:

$$k \int_\Omega \nabla(T_0 + \bar{\tau}_0) \nabla \xi - \rho \int_\Omega v_0 C_p(T_0 + \bar{\tau}_0) \nabla \xi$$

$$= \int_\Omega \mu(T_0 + \bar{\tau}_0, |D(v_0)|)\, |D(v_0)|^2 \xi; \quad \forall \xi \in W_0^{1,\infty}(\Omega). \tag{3.18}$$

Furthermore, the assumption on τ_0 implies that the limit $T_0 + \bar{\tau}_0 \geqslant C_0 > 0$ a.e. in Ω, (see [3], [6]).

This, (3.12) and (3.18) imply that $(v_0, T_0 + \bar{\tau}_0)$ is a weak solution of $(\mathscr{P})$.

There exists a corresponding pressure p_0 in $L^{r'}(\Omega)$, convergence of (v^n) giving that of (p^n) in $W^{-1,r'}(\Omega)$. ■

In the sequel, for simplicity, we will take $c_p(T) = 1$, this function being of secondary importance in the obtaining of the following estimates, since it is bounded.

PROPOSITION 3.3: *Under the assumptions of theorem 2.1, the velocities satisfy the following estimate:*

$$\|D(v_1 - v_2)\|_{L^r} \leqslant C\|f\|_{L^{r'}(\Omega)}^{\frac{r}{r'}} \|T_{u_1} - T_{u_2}\|_{W_N},$$

where: $(v_1, T_{v_1}) = \phi(u_1, T_{u_1})$ *and* $(v_2, T_{v_2}) = \phi(u_2, T_{u_2})$, *C depending only on the data:* Ω, N, r, τ_0, f.

Proof: We easily get from the definition of ϕ :

$$\int_\Omega [\mu(T_{u_1} + \bar{\tau}_0, |D(v_1)|)\, D(v_1) - \mu(T_{u_1} + \bar{\tau}_0, |D(v_2)|)\, D(v_2)] : D(v_1 - v_2)$$

$$= - \int_\Omega [\mu(T_{u_1} + \bar{\tau}_0, |D(v_2)|) - \mu(T_{u_2} + \bar{\tau}_0, |D(v_2)|)]\, D(v_2) : D(v_1 - v_2)\,. \tag{3.19}$$

Therefore, by (1.4) and (1.2):

$$K_2 \int_\Omega |D(v_1 - v_2)|^2 \{|D(v_1)| + |D(v_2)|\}^{r-2}$$

$$\leqslant K_1 \int_\Omega \beta(|T_{u_1} - T_{u_2}|)\, |D(v_2)|^{r-1} |D(v_1 - v_2)|\,,$$

$$\leqslant K_1 \|D(v_1 - v_2)\|_{L^r} \left(\int_\Omega \beta(|T_{u_1} - T_{u_2}|)^{r'} |D(v_2)|^r \right)^{\frac{1}{r'}}.$$

Then, similarly as in estimate (3.16), we obtain:

$$\|D(v_1 - v_2)\|_{L^r(\Omega)^{N^2}} \leqslant C \left(\int_\Omega |T_{u_1} - T_{u_2}|^{r'} |D(v_2)|^r \right)^{\frac{1}{r'}}. \tag{3.20}$$

And, by the Meyers's regularity property of the r-Stokes problem, using Hölder's inequality, we obtain:

$$\|D(v_1 - v_2)\|_{L^r(\Omega)^{N^2}} \leqslant C \|D(v_2)\|_{L^{\gamma^*}(\Omega)^{N^2}}^{\frac{r}{r'}} \|T_{u_1} - T_{u_2}\|_{L^{\frac{r'\gamma^*}{\gamma^* - r}}(\Omega)}\,.$$

Hence, by (2.2):

$$\|D(v_1 - v_2)\|_{L^r(\Omega)^{N^2}} \leqslant C \|f\|_{L^{r'}(\Omega)}^{\frac{r}{r'}} \|T_{u_1} - T_{u_2}\|_{L^{\frac{r'\gamma^*}{\gamma^* - r}}(\Omega)}\,. \tag{3.21}$$

Then, in order to have an estimate of $\|T_{u_1} - T_{u_2}\|_{L^{\frac{r'\gamma^*}{\gamma^* - r}}(\Omega)}$ with $r > \frac{N}{2}$, we need to add, for $N = 3$, the following regularity assumption: $\gamma^* > \gamma_0$, where $\gamma_0 = \frac{N(r-1)}{2r - N} r$, which is a necessary and sufficient condition to have: $\frac{r'\gamma^*}{\gamma^* - r} < \frac{N}{N-2}$. This, with (3.21) gives Proposition 3.3. ■

Remark 3.1: The method used in the previous step does not allow us to prove Proposition 3.3 in the case $r > 2$, under a natural assumption on μ, that is:

$$[\mu(s, |\xi|)\,\xi - \mu(s, |\eta|)\,\eta] : (\xi - \eta) \geqslant K_4 |\xi - \eta|^r .$$

Indeed, (3.19) and (1.2) would give:

$$K_4 \int_\Omega |D(v_1 - v_2)|^r \leqslant K_1 \int_\Omega \beta(|T_{u_1} - T_{u_2}|)\, |D(v_2)|^{r-1} |D(v_1 - v_2)| ,$$

$$\leqslant C \|D(v_1 - v_2)\|_{L^r} \left(\int_\Omega |T_{u_1} - T_{u_2}|^{r'} |D(v_2)|^r \right)^{\frac{1}{r'}} .$$

So:

$$\|D(v_1 - v_2)\|_{L^r(\Omega)}^{r-1} \leqslant C \left(\int_\Omega |T_{u_1} - T_{u_2}|^{r'} |D(v_2)|^r \right)^{\frac{1}{r'}}$$

Finally, we would get, by Hölder's inequality and for $\gamma^* > \gamma_0$,

$$\|D(v_1 - v_2)\|_{L^r(\Omega)} \leqslant C \|f\|_{L^\gamma(\Omega)} \|T_{u_1} - T_{u_2}\|_{W_N}^{\frac{1}{r-1}} .$$

Because of the exponent $\frac{1}{r-1} < 1$, for $r > 2$, we can not deduce from this estimate that Φ is a contracting mapping in that case.

PROPOSITION 3.4: *Under the assumptions of theorem 2.1, the temperatures satisfy the following estimate:*

$$\|T_{v_1} - T_{v_2}\|_{W_N} \leqslant C\{\|f\|_{L^\gamma(\Omega)}^{2\frac{r}{r'}} + \|f\|_{L^\gamma(\Omega)}^{\frac{r}{r'}}\}\ \|T_{u_1} - T_{u_2}\|_{W_N}$$

$$+ C \|f\|_{L^\gamma(\Omega)} \|T_{v_1} - T_{v_2}\|_{W_N} ,$$

where the constant C depends only of the data: Ω, N, r, τ_0, f.

Proof: $(T_{v_1} - T_{v_2})$ is a solution of the equation:

$$-k\Delta(T_{v_1} - T_{v_2}) = \{\mu(T_{u_1} + \bar{\tau}_0, |D(v_1)|)\, |D(v_1)|^2 - \mu(T_{u_2} + \bar{\tau}_0, |D(v_2)|)\, |D(v_2)|^2\}$$

$$- \rho\{v_1 \nabla(T_{v_1} + \bar{\tau}_0) - v_2(\nabla T_{v_2} + \bar{\tau}_0)\} . \tag{3.22}$$

We get, from the definition of ϕ :

$$\int_\Omega \{\mu(T_{u_1} + \bar{\tau}_0, |D(v_1)|)\, |D(v_1)|^2 - \mu(T_{u_2} + \bar{\tau}_0, |D(v_2)|)\, |D(v_2)|^2\}$$

$$= \int_\Omega \mu(T_{u_2} + \bar{\tau}_0, |D(v_2)|)\, D(v_2) : D(v_1 - v_2)\,.$$

Then:

$$\left| \int_\Omega \{\mu(T_{u_1} + \bar{\tau}_0, |D(v_1)|)\, |D(v_1)|^2 - \mu(T_{u_2} + \bar{\tau}_0\,, |D(v_2)|)\, |D(v_2)|^2\} \right|$$

$$\leqslant C \int_\Omega |D(v_2)|^{r-1} |D(v_1 - v_2)|, \text{ by } (1.2) - (1.3)\,,$$

$$\leqslant C \| D(v_2) \|_{L^r(\Omega)^{N^2}}^{\frac{r}{r'}} \| D(v_1 - v_2) \|_{L^r(\Omega)^{N^2}}\,,$$

$$\leqslant \|f\|_{L^{r'}(\Omega)}^{2\frac{r}{r'}} \| T_{u_1} - T_{u_2} \|_{W_N}, \text{ by } (2.2) \text{ and Proposition 3.3}\,. \tag{3.23}$$

Furthermore,

$$\rho \left| \int_\Omega v_1\, \nabla(T_{v_1} + \bar{\tau}_0) - v_2\, \nabla(T_{v_2} + \bar{\tau}_0) \right|$$

$$\leqslant \rho \int_\Omega |(v_1 - v_2)\, \nabla(T_{v_1} + \bar{\tau}_0)| + \rho \int_\Omega |v_2(\nabla T_{v_1} - \nabla T_{v_2})|$$

$$\leqslant C \| v_1 - v_2 \| L^{Nr/N-r} \| \nabla(T_{v_1} + \bar{\tau}_0) \| L^{Nr/Nr-N+r}$$

$$+ C \| v_2 \| L^{Nr/N-r} \| \nabla T_{v_1} - \nabla T_{v_2} \| L^{Nr/Nr-N+r} \text{ (for } r < N)\,;$$

$$\leqslant C \| D(v_1 - v_2) \|_{L^r} \| T_{v_1} + \bar{\tau}_0 \|_{W_N} + C \| D(v_2) \|_{L^r} \| T_{v_1} - T_{v_2} \|_{W_N}\,,$$

by Poincaré's inequality and Sobolev imbedding theorem (Recall that: $\frac{Nr}{Nr - N + r} < \frac{N}{N-1}$, for $r > \frac{N}{2}$). Then, by Proposition 3.3, estimates (2.2) and (3.4), we obtain:

$$\rho \left| \int_\Omega \{v_1\, \nabla(T_{v_1} + \bar{\tau}_0) - v_2\, \nabla(T_{v_2} + \bar{\tau}_0)\} \right|$$

$$\leqslant C \|f\|_{L^{r'}(\Omega)}^{\frac{r}{r'}} \| T_{u_1} - T_{u_2} \|_{W_N} + C \|f\|_{L^{r'}(\Omega)} \| T_{v_1} - T_{v_2} \|_{W_N} \tag{3.24}$$

Then, (3.22)-(3.24) imply that: $T_{v_1} - T_{v_2}$ is a solution of the equation: $-\Delta(T_{v_1} - T_{v_2}) = F$, where $F \in L^1(\Omega)$ and consequently the following estimate holds (see [5]):

$$\| T_{v_1} - T_{v_2} \| W^{1,q}(\Omega) \leq C \| F \|_{L^1(\Omega)}, \quad \forall q < \frac{N}{N-1}.$$

This, with estimates (3.23) and (3.24) gives Proposition 3.4.

End of proof of Theorem 2.1: We can now deduce that there exists a closed ball B_R nonempty in $V_r \times W_N$ such that: $\phi(B_R) \subset B_R$; and ϕ is a contracting mapping on B_R, for $r > \frac{N}{2}$ and $\|f\|_{L^\gamma}$ sufficiently small:

By the definition of v^0 and T^0, we can easily choose $R > 0$ such that: $\| D(v^0) \|_{L^r(\Omega)} + \| \tau_0 \|_{L^\infty(\Gamma)} \leq R$, and consequently $(v^0, T^0) \in B_R$.

Our aim is to prove that there exists δ, $0 < \delta < 1$, such that:

$$\| (v_1, T_{v_1}) - (v_2, T_{v_2}) \|_{V_r \times W_N} \leq \delta \| (u_1, T_{u_1}) - (u_2, T_{u_2}) \|_{V_r \times W_N}.$$

Using Proposition 3.4, we obtain that if $\|f\|_{L^\gamma(\Omega)}$ is sufficiently small, that is: $C \max \{ \|f\|^{2r/r'}_{L^\gamma(\Omega)}, \|f\|_{L^\gamma(\Omega)}, \|f\|^{r/r'}_{L^\gamma(\Omega)} \} < \bar{\delta} < \frac{1}{2}$, then:

$$(1 - \bar{\delta}) \| T_{v_1} - T_{v_2} \|_{W_N} \leq \bar{\delta} \| T_{u_1} - T_{u_2} \|_{W_N}.$$

Finally, taking: $\delta = \dfrac{\bar{\delta}}{1 - \bar{\delta}}$, we get:

$$\| T_{v_1} - T_{v_2} \|_{W_N} \leq \delta \| T_{u_1} - T_{u_2} \|_{W_N}, \quad \text{with } 0 < \delta < 1. \tag{3.25}$$

Analogousely, in proposition 3.3, if f is sufficiently small, then:

$$\| D(v_1 - v_2) \|_{L^r} \leq \delta \| T_{u_1} - T_{u_2} \|_{W_N} \tag{3.26}$$

Finally, (3.25) and (3.26) imply that ϕ is a contraction mapping, for $r > \frac{N}{2}$, f sufficiently small and, for $N = 3$, v sufficiently regular: $D(v) \in L^{\gamma^*}$; $\gamma^* > \gamma_0$. This gives Theorem 2.1.

Then, under the above assumptions, we can apply the Banach fixed-point theorem to get that ϕ admits a unique fixed point (v_0, T_0) in B_R. Furthermore, there exists a corresponding pressure p unique up to a constant. Then, the algorithm $(\mathscr{P}_n)$ converges to this solution. Since, a solution of $(\mathscr{P})$ corresponds to a fixed point of ϕ, then, using Proposition 3.2, we obtain that $(v_0, T_0 + \bar{\tau}_0)$ is the unique weak solution of problem $(\mathscr{P})$. Therefore, Corollary 2.1 is proved.

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