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EXPANDED MIXED FINITE ELEMENT METHODS FOR LINEAR SECOND-ORDER ELLIPTIC PROBLEMS, I (*)

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Abstract — *We develop a new mixed formulation for the numerical solution of second-order elliptic problems This new formulation expands the standard mixed formulation in the sense that three variables are explicitly treated the scalar unknown, its gradient, and its flux (the coefficient times the gradient) Based on this formulation, mixed finite element approximations of the second-order elliptic problems are considered Optimal order error estimates in the L^p- and H^{-s}-norms are obtained for the mixed approximations. Various implementation techniques for solving the systems of algebraic equations are discussed A postprocessing method for improving the scalar variable is analyzed, and superconvergent estimates in the L^p-norm are derived The mixed formulation is suitable for the case where the coefficient of differential equations is a small tensor and does not need to be inverted © Elsevier, Paris*

Résumé — *L'objet de cet article est l'écriture d'une nouvelle formulation mixte relative aux problèmes elliptiques d'ordre deux, et l'implémentation de méthodes d'éléments finis mixtes pour la détermination de solutions approchées On donne alors des estimations d'erreurs en norme L^p et H^{-s} Enfin, on construit une méthode pour laquelle des résultats de superconvergence en norme L^p sont obtenus © Elsevier, Paris*

1. INTRODUCTION

Mixed finite element methods have been found to be very useful, for solving flow equations ([17], [18]), along with other applications. For example, when the governing equations that describe two-phase flow in a petroleum reservoir are written in a fractional flow formulation (i.e., in terms of a global pressure and a saturation), mixed methods can be used to solve the pressure equation very efficiently and accurately. However, mixed finite element methods have not yet achieved application in groundwater hydrology. For petroleum reservoirs total flux-type boundary conditions are conveniently imposed and easily incorporated in the mixted finite element formulation. But, for groundwater reservoirs often complex boundary conditions involving combinations of individual fluid fluxes and pressures must be specified, and it is sometimes impractical to express them in terms of the total quantities [8], [33]. Consequently, two-pressure formulations are commonly used by hydrologists ([8], [33]), since the complex individual boundary conditions can easily be handled. However, the coefficient in the two-pressure formulation may tend to zero because of low permeability, so that its reciprocal is not readily usable as in standard mixed finite element methods ([4], [5], [6], [14], [27], [28], [30]). Therefore, a direct application of mixed methods to a two-pressure formulation is usually not practical.

This is the first paper of a series in which we develop and analyze a new mixed formulation for the numerical solution of second-order elliptic problems. This new formulation expands the standard mixed formulation in the sense that three variables are explicitly treated; i.e., the scalar unknown, its gradient, and its flux (the coefficient times the gradient). It applies directly to the two-pressure formulation mentioned above, so that it can treat individual boundary conditions. Also, it is suitable for the case where the coefficient of differential equations is a small tensor and does not need to be inverted. As a result, it works for the differential problems with small diffusion or low

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permeability terms. The other advantage we have found so far with this new formulation is that it leads to optimal error estimates for certain nonlinear elliptic problems while the standard mixed formulation gives only suboptimal error estimates [12]. A detailed analysis for nonlinear problems is given in the second paper of the series [13].

In the next section, we propose the expanded mixed formulation for a fairly general second-order elliptic problem with the variable tensor coefficient. Then we show that this formulation applies to all existing mixed finite elements. In §3, we analyze the continuous problem and prove that the new formulation has a unique solution and is equivalent to the original differential problem. In §4 and §5, we deal with the expanded mixed finite element method. It is demonstrated that the discrete formulation has a unique solution and gives optimal error estimates in the L^p and H^{-s}. Then, in §6 we analyze a postprocessing method for improving the accuracy of the approximation of the original scalar variable and derive superconvergent estimates in L^p. Finally, in §7 we discuss some implementation strategies for solving the system of algebraic equations produced by the expanded mixed method, including preconditionel iterative methods, alternating-direction iterative methods, hybridization methods, etc. Numerical examples are presented in the second paper.

We end this section with a remark that the idea of using an expanded mixed formulation has been used in elasticity (see [24] and the references therein). However, the setting for the present problem is different from that of elasticity problems. Specifically, a combination of the spaces L^2 and H^1 is used in the elasticity problems, while the spaces L^2 and $H(\operatorname{div})$ are used here. In the author's opinion, the analysis for second-order elliptic problems is more elegant and difficult.

2. EXPANDED MIXED FORMULATION

Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2$ or 3, with the boundary $\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$, $\Gamma_1 \cap \Gamma_2 = \phi$. We consider the elliptic problem

$$-\nabla \cdot (a\nabla u - bu + c) + du = f \quad \text{in } \Omega, \tag{2.1a}$$

$$u = -g \quad \text{on } \Gamma_1, \tag{2.1b}$$

$$(a\nabla u - bu - c) \cdot \nu = 0 \quad \text{on } \Gamma_2, \tag{2.1c}$$

where $a(x)$ is a tensor, $b(x)$ and $c(x)$ are vectors, $d(x)$ is a function, $f(x) \in L^2(\Omega)$, $g(x) \in H^{3/2}(\Gamma_1)$ ($H^k(\Omega) = W^{k,2}(\Omega)$ is the Sobolev space of k differentiable functions in $L^2(\Omega)$ with the norm $\|\cdot\|_k$; we omit k when it is zero), and ν is the outer unit normal to the domain. Let

$$H(\operatorname{div};\Omega) = \{v \in (L^2(\Omega))^n : \nabla \cdot v \in L^2(\Omega)\},$$

$$V = \{v \in H(\operatorname{div};\Omega) : v \cdot \nu = 0 \text{ on } \Gamma_2\},$$

$$W = L^2(\Omega),$$

$$\Lambda = (L^2(\Omega))^n,$$

and let $(\cdot,\cdot)_{\mathscr{S}}$ denote the $L^2(\mathscr{S})$ inner product (we omit $\mathscr{S}$ if $\mathscr{S} = \Omega$). Then (2.1) is formulated in the following expanded mixed form for $(\sigma, \lambda, u) \in V \times \Lambda \times W$:

$$(a\lambda, \mu) - (\sigma, \mu) + (bu, \mu) = (c, \mu), \quad \forall \mu \in \Lambda, \tag{2.2a}$$

$$(\lambda, v) - (u, \nabla \cdot v) = (g, v \cdot \nu)_{\Gamma_1}, \quad \forall v \in V, \tag{2.2b}$$

$$(\nabla \cdot \sigma, w) + (du, w) = (f, w), \quad \forall w \in W. \tag{2.2c}$$

To define a finite element method, we need a partition $\mathscr{E}_h$ of Ω into elements E, say, simplexes, rectangular parallelepipeds, and/or prisms, where only edges or faces on $\partial\Omega$ may be curved. In $\mathscr{E}_h$, we also need that adjacent elements completely share their common edge or face; let $\partial\mathscr{E}_h$ denote the set of all *interior* edges ($n=2$) or faces ($n=3$) e of $\mathscr{E}_h$. We tacitly assume that $\partial\mathscr{E}_h \neq \Phi$. Finally, each exterior edge or face has either Dirichlet or Neumann conditions imposed on it but not both [1].

Since mixed finite element spaces are finite dimensional and defined locally on each element, let, for each $E \in \mathscr{E}_h$, $V_h(E) \times W_h(E)$ denote one of the mixed finite element spaces introduced in [4], [5], [6], [14], [21], [27], [28], and [30] for second-order elliptic problems. Then we define

$$\Lambda_h = \{\mu \in \Lambda : \mu|_E \in V_h(E) \text{ for each } E \in \mathscr{E}_h\},$$

$$V_h = \{v \in V : v|_E \in V_h(E) \text{ for each } E \in \mathscr{E}_h\},$$

$$W_h = \{w \in W : w|_E \in W_h(E) \text{ for each } E \in \mathscr{E}_h\}.$$

The expanded mixte finite element method for (2.1) is to find $(\sigma_h, \lambda_h, u_h) \in V_h \times \Lambda_h \times W_h$ such that

$$(a\lambda_h, \mu) - (\sigma_h, \mu) + (bu_h, \mu) = (c, \mu), \quad \forall \mu \in \Lambda_h, \tag{2.3a}$$

$$(\lambda_h, v) - (u_h, \nabla \cdot v) = (g, v \cdot \nu)_{\Gamma_1}, \quad \forall v \in V_h, \tag{2.3b}$$

$$(\nabla \cdot \sigma_h, w) + (du_h, w) = (f, w), \quad \forall w \in W_h. \tag{2.3c}$$

We shall show that standard stability and convergence results hold for (2.2) and (2.3) in the next two sections if the usual regularity assumption on the solution of (2.1) is made. Note taht if a is positive definite, λ_h can be eliminated element by element from (2.3) to obtain a traditional system involving only u_h and σ_h where a is inverted. However, as mentioned in the introduction, we keep λ_h in (2.3) since this formulation applies to the case in which a is a tensor and may tend to zero. Also, while an extra unknown is introduced in (2.3), the computational cost for solving (2.3) is almost the same as that for solving the traditional mixed method system, as shown in §7. Finally, we shall see that the expanded formulations (2.2) and (2.3) are not a trivial extension of the traditional one from the discussions carried out in the next two sections.

3. CONTINUOUS PROBLEM

To fix ideas we carry out an analysis for the model problem

$$Lu = -\nabla \cdot (a \nabla u) = f \quad \text{in } \Omega, \tag{3.1a}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{3.1b}$$

The extension of the analysis below to the general problem (2.1) is straightforward [1], [20]. As mentioned in the previous section, we want to show that the standard stability and convergence results hold for the expanded mixed method if the usual regularity assumptions on the solution of (3.1) are made; the less regular case will be discussed in a later work. Hence, we assume that $a(x)$ is a uniformly positive definite, bounded tensor:

$$(a\mu, \mu) \geq \alpha \|\mu\|^2, \quad \forall \mu \in \Lambda, \tag{3.2}$$

and that, if $f \in H^s(\Omega)$, then $u \in H^{s+2}(\Omega) \cap H_0^1(\Omega)$ and there is a constant C such that

$$\|u\|_{s+2} \leq C\|f\|_s, \tag{3.3}$$

which will be used in the H^{-s}-error analysis later. For problem (3.1), (2.2) reduces to the following form for $(\sigma, \lambda, u) \in V \times \Lambda \times W$:

$$(a\lambda, \mu) - (\sigma, \mu) = 0, \quad \forall \mu \in \Lambda, \tag{3.4a}$$

$$(\lambda, v) - (u, \nabla \cdot v) = 0, \quad \forall v \in V, \tag{3.4b}$$

$$(\nabla \cdot \sigma, w) = (f, w), \quad \forall w \in W, \tag{3.4c}$$

where now $V = H(\operatorname{div}; \Omega)$.

For a mathematical analysis of (3.4), let $U = W \times \Lambda$ with the usual product norm $\|\tau\|_U^2 = \|w\|^2 + \|\mu\|^2$, $\tau = (w, \mu) \in U$, and introduce the bilinear forms $a(\,.\,,.\,) : U \times U \to \mathbb{R}$ and $b(\,.\,,.\,) : U \times V \to \mathbb{R}$:

$$a(\chi, \tau) = (a\lambda, \mu), \qquad \chi = (u, \lambda), \quad \tau = (w, \mu) \in U, \tag{3.5}$$

$$b(\tau, v) = (w, \nabla \cdot v) - (\mu, v), \qquad \tau = (w, \mu) \in U, \quad v \in V. \tag{3.6}$$

Then, (3.4) can be written in the standard form for $(\chi, \sigma) \in U \times V$ such that

$$a(\chi, \tau) + b(\tau, \sigma) = F(\tau), \quad \forall \tau \in U, \tag{3.7a}$$

$$b(\chi, v) = 0, \quad \forall v \in V, \tag{3.7b}$$

where the continuous form $F(\tau)$ on U is defined by

$$F(\tau) = (f, w), \quad \tau = (w, \mu) \in U.$$

Finally, we define the notation

$$Z = \{\tau \in U : b(\tau, v) = 0, \forall v \in V\}.$$

We are now ready to prove the next results.

LEMMA 3.1: *Let $\tau = (w, \mu) \in U$. Then, $\tau \in Z$ if and only if $w \in H_0^1(\Omega)$ and $\mu = -\nabla w$.*

Proof: First, let $\tau = (w, \mu) \in U$ such that $w \in H_0^1(\Omega)$ and $\mu = -\nabla w$. Then, for all $v \in V$,

$$b(\tau, v) = (w, \nabla \cdot v) - (\mu, v) = -(\nabla w, v) - (\mu, v) = 0,$$

so that $\tau \in Z$.

Next, let $\tau = (w, \mu) \in Z$. Define $v \in V$ with $v_1 = \phi \in D(\bar{\Omega})$, the restriction of the functions infinitely differentiable and with compact support in $\mathbb{R}^n$ to Ω, and $v_i = 0$, $i = 2, ..., n$. Then, by the definition of Z,

$$(w, \partial\phi/\partial x_1) = (\mu_1, \phi), \quad \forall \phi \in D(\bar{\Omega}).$$

Since $H^1(\Omega) = \overline{D(\bar{\Omega})}$, the closure of $D(\bar{\Omega})$, this implies that $\mu_1 = -\partial w/\partial x_1$. Similarly, $\mu_i = -\partial w/\partial x_i$, $i = 2, ..., n$; consequently, $\mu = -\nabla w$. Therefore, by the definition of Z and Green's formula, we have

$$(w, v \cdot \nu)_{\partial\Omega} = 0, \quad \forall v \in V.$$

Since $w|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$ and the mapping: $v \mapsto v \cdot \nu|_{\partial\Omega}$ defined on V is onto $H^{1/2}(\partial\Omega)$, the equation above implies that $w|_{\partial\Omega} = 0$; i.e., $w \in H_0^1(\Omega)$. □

LEMMA 3.2: *The mapping* $\tau = (w, \mu) \in Z \mapsto \|\mu\|$ *defines a norm in* Z *equivalent to the original norm* $\|\tau\|_U$; *i.e., there is constant* $\alpha_1 > 0$ *such that*

$$\alpha_1 \|\tau\|_U \leqslant \|\mu\| \leqslant \|\tau\|_U . \tag{3.8}$$

Proof: Let $\tau = (w, \mu) \in Z$. Then, by Lemma 3.1, $w \in H_0^1(\Omega)$ and $\mu = -\nabla w$. Hence,

$$\|\tau\|_U^2 = \|w\|^2 + \|\mu\|^2 \leqslant C\|\nabla w\|^2 + \|\mu\|^2 \leqslant C\|\mu\|^2 ,$$

from which (3.8) follows, since the second inequality is obvious. □

LEMMA 3.3: *There is a constant* $\beta > 0$ *such that*

$$a(\tau, \tau) \geqslant \beta \|\tau\|_U^2, \quad \tau \in Z ; \tag{3.9}$$

i.e., $a(\,.\,,.\,)$ *is Z-elliptic, and*

$$\sup_{\tau \in U} b(\tau, v)/\|\tau\|_U \geqslant \|v\|, \quad \forall v \in V . \tag{3.10}$$

Proof: Let $\tau = (w, \mu) \in Z$. Then, by (3.2) and (3.8),

$$a(\tau, \tau) = (a\mu, \mu) \geqslant \alpha \|\mu\|^2 \geqslant \alpha\alpha_1^2 \|\tau\|_U^2 ,$$

which implies (3.9) with $\beta = \alpha\alpha_1^2$. Next, for all $v \in V$,

$$\sup_{\tau \in U} b(\tau, v)/\|\tau\|_U \geqslant \sup_{(0,\mu) \in U} b((0, \mu), v)/\|(0, \mu)\|_U = \sup_{\mu \in \Lambda} (\mu, v)/\|\mu\| = \|v\| ,$$

so (3.10) is true. □

THEOREM 3.4: *The problem (3.7) (and thus (3.4)) has at most one solution.*

Proof: Let $f = 0$. From (3.7) and (3.9) we see that $\chi = 0$. Hence, (3.7a) becomes

$$b(\tau, \sigma) = 0, \quad \forall \tau \in U ,$$

which, together with (3.10), implies that $\sigma = 0$. □

The following result characterizes the relation between the solutions of (3.1) and (3.7).

THEOREM 3.5: *If* $(\chi, \sigma) \in U \times V$ *is the solution of (3.7) with* $\chi = (u, \lambda)$, *then* $u \in H_0^1(\Omega)$ *is the solution of (3.1) with* $\lambda = -\nabla u$. *Conversely, if* $u \in H_0^1(\Omega)$ *is the solution of (3.1), then (3.7) has the solution* $(\chi, \sigma) \in U \times V$ *with* $\chi = (u, \lambda)$, $\lambda = -\nabla u$, *and* $\sigma = -a\nabla u$.

Proof: First, let $(\chi, \sigma) \in U \times V$ be the solution of (3.7) with $\chi = (u, \lambda)$. Then (3.7b) implies that $\chi \in Z$ so that, by Lemma 3.1, $u \in H_0^1(\Omega)$ and $\lambda = -\nabla u$. Hence, for all $w \in H_0^1(\Omega)$ and $\mu = -\nabla w$, it follows from Lemma 3.1 that

$$a(\chi, \tau) = F(\tau), \quad \forall \tau = (w, \mu) \in Z ;$$

i.e.,

$$(a\nabla u, \nabla w) = (f, w), \quad \forall w \in H_0^1(\Omega) .$$

Hence, u is the solution of (3.1).

Next, we assume that $u \in H_0^1(\Omega)$ is the solution of (3.1). Set $\chi = (u, \lambda)$ with $\lambda = -\nabla u$ and $\sigma = -a\nabla u$. Then it follows from Lemma 3.1 that $\chi \in Z$ so that (3.7b) holds. Thus it remains to prove (3.7a). For each $\tau \in U$ with $\tau = (w, \mu)$,

$$\begin{aligned} a(\chi, \tau) + b(\tau, \sigma) &= (a\lambda, \mu) + (w, \nabla \cdot \sigma) - (\mu, \sigma) \\ &= (w, -\nabla \cdot (a\nabla u)) \\ &= (f, w), \quad \forall w \in W, \end{aligned}$$

which yields (3.7a). □

4. H^{-s}-CONVERGENCE ANALYSIS

In this section, we derive error estimates for the expanded mixed method (2.3). We focus on the Brezzi-Douglas-Marini mixed triangular space [6] if $n = 2$ and on the Brezzi-Douglas-Durán-Fortin mixed simplicial space [4] if $n = 3$. Other mixed finite element families can be treated in the same way. For each $E \in \mathscr{E}_h$, the Brezzi-Douglas-Marini mixed triangular space [6] or the Brezzi Douglas-Durán-Fortin mixed simplicial space [4] is defined by

$$V_h(E) = (P_k(E))^n,$$

$$W_h(E) = P_{k-1}(E),$$

where $P_k(E)$ is the restriction of the set of all polynomials of total degree not bigger than $k \geqslant 1$ to E. Note that this space is the most natural choice for the expanded mixed method (2.3) from the standard finite element point of view. We again consider the problem (3.1). In this case, (2.3) is formulated for $(\sigma_h, \lambda_h, u_h) \in V_h \times \Lambda_h \times W_h$ as

$$(a\lambda_h, \mu) - (\sigma_h, \mu) = 0, \qquad \forall \mu \in \Lambda_h, \tag{4.1a}$$

$$(\lambda_h, v) - (u_h, \nabla \cdot v) = 0, \qquad \forall v \in V_h, \tag{4.1b}$$

$$(\nabla \cdot \sigma_h, w) = (f, w), \qquad \forall w \in W_h. \tag{4.1c}$$

The error analysis below makes use of three projection operators. The first operator is, if $n = 2$, the Brezzi-Douglas-Marini operator [6] or, if $n = 3$, the Brezzi-Douglas-Durán-Fortin operator [4] $\Pi_h : (H(\Omega))^n \to V_h$; Π_h satisfies

$$\|v - \Pi_h v\| \leqslant C\|v\|_r h^r, \qquad 1 \leqslant r \leqslant k+1, \tag{4.2}$$

$$(\nabla \cdot (v - \Pi_h v), w) = 0, \qquad \forall w \in W_h. \tag{4.3}$$

The other two operators are the standard L^2-projections P_h and R_h onto W_h and Λ_h, respectively:

$$(w - P_h w, \nabla \cdot v) = 0, \qquad \forall w \in W, v \in V_h, \tag{4.4}$$

$$(\mu - R_h \mu, \tau) = 0, \qquad \forall \mu \in \Lambda, \tau \in \Lambda_h. \tag{4.5}$$

They have the approximation properties

$$\|w - P_h w\|_{-s} \leqslant C\|w\|_r h^{r+s}, \qquad 0 \leqslant s, r \leqslant k, \tag{4.6}$$

$$\|\mu - R_h \mu\|_{-s} \leqslant C\|\mu\|_r h^{r+s}, \qquad 0 \leqslant s, r \leqslant k+1. \tag{4.7}$$

Now, let

$$\alpha_h = \lambda - \lambda_h, \quad \beta_h = R_h \lambda - \lambda_h ,$$

$$d_h = \sigma - \sigma_h, \quad e_h = \Pi_h \sigma - \sigma_h ,$$

$$z_h = P_h u - u_h .$$

Then subtract (4.1) from (3.4) and apply (4.4) and (4.5) to obtain the error equations

$$(a\alpha_h, \mu) - (\mu, d_h) = 0, \qquad \forall \mu \in \Lambda_h , \tag{4.8a}$$

$$(z_h, \nabla \cdot v) - (\beta_h, v) = 0, \qquad \forall v \in V_h , \tag{4.8b}$$

$$(w, \nabla \cdot d_h) = 0 , \qquad \forall w \in W_h . \tag{4.8c}$$

LEMMA 4.1: *There is a constant $C > 0$ independent of h such that*

$$\| \nabla \cdot d_h \| \leqslant C \| \nabla \cdot \sigma \|_r h^r, \quad 1 \leqslant r \leqslant k . \tag{4.9}$$

Proof: It follows from (4.3) that

$$(w, \nabla \cdot e_h) = (w, \nabla \cdot d_h) = 0, \quad \forall w \in W_h ,$$

so that $\nabla \cdot e_h = 0$. Hence,

$$\| \nabla \cdot d_h \| = \| \nabla \cdot (\sigma - \Pi_h \sigma) \| \leqslant C \| \nabla \cdot \sigma \|_r h^r, \quad 1 \leqslant r \leqslant k . \quad \square$$

LEMMA 4.2: *There is a constant $C > 0$ independent of h such that*

$$\| \alpha_h \| \leqslant C(\| \lambda \|_r + \| \sigma \|_r) h^r, \quad 1 \leqslant r \leqslant k + 1 . \tag{4.10}$$

Proof: Take $\mu = \beta_h$ in (4.8a) and $v = e_h$ in (4.8b) to see that

$$\begin{aligned}
(a\beta_h, \beta_h) &= (a(R_h \lambda - \lambda), \beta_h) + (a\alpha_h, \beta_h) \\
&= (a(R_h \lambda - \lambda), \beta_h) + (\beta_h, d_h) \\
&= (a(R_h \lambda - \lambda), \beta_h) + (z_h, \nabla \cdot e_h) + (\beta_h, \sigma - \Pi_h \sigma) \\
&= (a(R_h \lambda - \lambda), \beta_h) + (\beta_h, \sigma - \Pi_h \sigma) ,
\end{aligned}$$

so that

$$\| \beta_h \| \leqslant C(\| R_h \lambda - \lambda \| + \| \sigma - \Pi_h \sigma \|) .$$

Thus, by (4.2) and (4.7), we see that

$$\begin{aligned}
\| \alpha_h \| &\leqslant \| \beta_h \| + \| R_h \lambda - \lambda \| \\
&\leqslant C(\| \lambda \|_r + \| \sigma \|_r) h^r, \quad 1 \leqslant r \leqslant k + 1 . \quad \square
\end{aligned}$$

LEMMA 4.3: *There is a constant $C > 0$ independent of h such that*

$$\| d_h \| \leqslant C(\| \lambda \|_r + \| \sigma \|_r) h^r, \quad 1 \leqslant r \leqslant k + 1 . \tag{4.11}$$

Proof: Choose $\mu = e_h$ in (4.8a) to get

$$(e_h, e_h) = (\Pi_h \sigma - \sigma, e_h) + (d_h, e_h)$$
$$= (\Pi_h \sigma - \sigma, e_h) + (a\alpha_h, e_h) .$$

Consequently,

$$\| e_h \| \leqslant C(\| \Pi_h \sigma - \sigma \| + \| \alpha_h \|) ,$$

which, together with (4.2) and (4.10), yields the desired result (4.11). □

We now turn to the analysis of the errors z_h, α_h, and d_h in $H^{-s}(\Omega)$, $s \geqslant 0$. Our analysis follows the argument described in [20]. Note that (4.8b) is still satisfied when β_h is replaced by α_h by (4.5).

LEMMA 4.4: *For $s \geqslant 0$, we have*

$$\| z_h \|_{-s} \leqslant C((\| \alpha_h \| + \| d_h \|) h^{\min\{s+1, k+1\}} + \| \nabla . d_h \| h^{\min\{s+2, k\}}) . \tag{4.12}$$

Proof: Let $\xi \in H^s(\Omega)$ and let $\phi \in H^{s+2}(\Omega) \cap H_0^1(\Omega)$ such that $L^* \phi = \xi$. Then, by (4.3) and (4.8), we see that

$$(z_h, \xi) = (z_h, -\nabla . (a \nabla \phi)) = (z_h, -\nabla . (\Pi_h(a \nabla \phi))) = -(\alpha_h, \Pi_h(a \nabla \phi))$$
$$= -(\alpha_h, a \nabla \phi) + (\alpha_h, a \nabla \phi - \Pi_h(a \nabla \phi))$$
$$= -(a\alpha_h, \nabla \phi - R_h(\nabla \phi)) - (a\alpha_h, R_h(\nabla \phi)) + (\alpha_h, a \nabla \phi - \Pi_h(a \nabla \phi))$$
$$= -(a\alpha_h, \nabla \phi - R_h(\nabla \phi)) - (R_h(\nabla \phi), d_h) + (\alpha_h, a \nabla \phi - \Pi_h(a \nabla \phi))$$
$$= -(a\alpha_h, \nabla \phi - R_h(\nabla \phi)) + (\nabla . d_h, \phi - P_h \phi)$$
$$+ (d_h, \nabla \phi - R_h(\nabla \phi)) + (\alpha_h, a \nabla \phi - \Pi_h(a \nabla \phi)) ,$$

so that

$$|(z_h, \xi)| \leqslant C \{ \| \alpha_h \| \, \| \nabla \phi - R_h(\nabla \phi) \| + \| \nabla . d_h \| \, \| \phi - P_h \phi \|$$
$$+ \| d_h \| \, \| \nabla \phi - R_h(\nabla \phi) \| + \| \alpha_h \| \, \| a \nabla \phi - \Pi_h(a \nabla \phi) \| \} .$$

Hence, by (4.2), (4.6), (4.7), and (3.3), we obtain the desired inequality (4.12). □

LEMMA 4.5: *For $s \geqslant 0$, we have*

$$\| \nabla . d_h \|_{-s} \leqslant C \| \nabla . d_h \| h^{\min\{s, k\}} . \tag{4.13}$$

Proof: Let $\xi \in H^s(\Omega)$. By (4.8c), we see that

$$(\nabla . d_h, \xi) = (\nabla . d_h, \xi - w), \quad \forall w \in W_h .$$

Thus, (4.13) follows from (4.6). □

LEMMA 4.6: *For $s \geqslant 0$, we have*

$$\| d_h \|_{-s} \leqslant C((\| d_h \| + \| \alpha_h \|) h^{\min\{s, k+1\}} + \| z_h \|_{-s+1}) . \tag{4.14}$$

Proof: Let $\xi \in (H^s(\Omega))^n$. By (4.3) and (4.8), we find that

$$\begin{aligned}(d_h, \xi) &= (d_h, \xi - R_h \xi) + (a\alpha_h, R_h \xi) \\ &= (d_h, \xi - R_h \xi) - (a\alpha_h, \xi - R_h \xi) + (\alpha_h, a\xi) \\ &= (d_h, \xi - R_h \xi) - (a\alpha_h, \xi - R_h \xi) + (\alpha_h, a\xi - \Pi_h(a\xi)) + (z_h, \nabla \cdot (\Pi_h(a\xi))) \\ &= (d_h, \xi - R_h \xi) - (a\alpha_h, \xi - R_h \xi) + (\alpha_h, a\xi - \Pi_h(a\xi)) + (z_h, \nabla \cdot (a\xi)),\end{aligned}$$

so it follows from (4.2) and (4.7) that

$$|(d_h, \xi)| \leqslant \{ \|d_h\| \, \|\xi\|_s h^{\min\{s, k+1\}} + \|\alpha_h\| \, \|\xi\|_s h^{\min\{s, k+1\}} + \|z_h\|_{-s+1} \|\nabla \cdot (a\xi)\|_{s-1} \}.$$

Therefore, by the regularity result (3.3), we have the error bound (4.14). □

The same argument can be used to demonstrate the lemma below.

LEMMA 4.7: *For $s \geqslant 0$, we have*

$$\|\alpha_h\|_{-s} \leqslant C((\|d_h\| + \|\alpha_h\|) h^{\min\{s, k+1\}} + \|z_h\|_{-s+1}). \tag{4.15}$$

We now collect the above results in the following theorem.

THEOREM 4.8: *Let $(\sigma, \lambda, u) \in V \times \Lambda \times W$ and $(\sigma_h, \lambda_h, u_h) \in V_h \times \Lambda_h \times W_h$ be solutions to (3.4) and (4.1), respectively. Then,*

$$\|u - u_h\|_{-s} \leqslant C\|f\|_{r-2} h^{r+s}, \qquad 2 \leqslant r \leqslant k+2, 0 \leqslant s \leqslant k-2, \tag{4.16}$$

$$\|\lambda - \lambda_h\|_{-s} \leqslant C\|f\|_{r-1} h^{r+s}, \qquad 1 \leqslant r \leqslant k+1, 0 \leqslant s \leqslant k-1, \tag{4.17}$$

$$\|\sigma - \sigma_h\|_{-s} \leqslant C\|f\|_{r-1} h^{r+s}, \qquad 1 \leqslant r \leqslant k+1, 0 \leqslant s \leqslant k-1, \tag{4.18}$$

$$\|\nabla \cdot (\sigma - \sigma_h)\|_{-s} \leqslant C\|f\|_r h^{r+s}, \qquad 0 \leqslant r, s \leqslant k, \tag{4.19}$$

$$\|u_h - P_h u\| \leqslant C\|f\|_k h^{\min\{k+2, 2k\}}. \tag{4.20}$$

The proof of this theorem is completed from Lemmas 4.1-4.7 and the elliptic regularity result (3.3). The error result (4.20) is needed for the analysis of a postprocessing scheme given in §6. When $k = 1$, we need the requirement $\|f\|_1$. As shown in [32], if $f \in W_h$, we need only $\|f\|$ in the right side of (4.20). The same remark applies later.

We end this section with the discussion of existence and uniqueness of the solution to (4.1). Since it is a system of linear algebraic equations, it suffices to establish the uniqueness. For this we assume that $f = 0$. Take $w = \nabla \cdot \sigma_h$ in (4.1c) to have $(\nabla \cdot \sigma_h, \nabla \cdot \sigma_h) = 0$, so that $\nabla \cdot \sigma_h = 0$. Next, take $\mu = \lambda_h$ in (4.1a) and $v = \sigma_h$ in (4.1b) to yield $(a\lambda_h, \lambda_h) = 0$, so that $\lambda_h = 0$. Also, choose $\mu = \sigma_h$ in (4.1a) to see that $(\sigma_h, \sigma_h) = 0$, and $\sigma_h = 0$. finally, as in Lemma 4.4 we see that

$$\|u_h\| \leqslant C(\|\lambda_h\| h + \|\sigma_h\| h + \|\nabla \cdot d_h\|),$$

which means that $u_h = 0$. Therefore, the uniqueness has been demonstrated.

5. L^p-CONVERGENCE ANALYSIS

We derived the optimal error estimates in the H^{-s}-norm for the mixed finite element method (4.1) in the last section. For the completeness of the error analysis, we now obtain error estimates in the L^p-norm through an adaptation of Durán's arguments [22]. To use his arguments requires that we make some appropriate assumptions. We consider the planar case and assume that the coefficient a is constant. The error estimates below still hold for the cases of three space variables and variable coefficient if we apply more sophisticated arguments, such as those of Gastaldi and Nochetto [25].

We need the following approximation properties:

$$\|v - \Pi_h v\|_{0,p} \leqslant C\|v\|_{r,p} h^r, \qquad 1/p < r \leqslant k+1,\ 1 \leqslant p \leqslant \infty, \tag{5.1}$$

$$\|w - P_h w\|_{0,p} \leqslant C\|w\|_{r,p} h^r, \qquad 0 \leqslant r \leqslant k,\ 1 \leqslant p \leqslant \infty, \tag{5.2}$$

$$\|\mu - R_h \mu\|_{0,p} \leqslant C\|\mu\|_{r,p} h^r, \qquad 0 \leqslant r \leqslant k+1,\ 1 \leqslant p \leqslant \infty. \tag{5.3}$$

Also, let D_h be the L^2-projection onto the space $\bar{V}_h$ [22] of the divergence-free vectors:

$$\bar{V}_h = \{v \in V_h : \nabla \cdot v = 0\}.$$

It has the stability properties

$$\|D_h v\|_{0,p} \leqslant C_p \|v\|_{0,p}, \quad 2 \leqslant p < \infty, \tag{5.4}$$

$$\|D_h v\|_{0,\infty} \leqslant C(\|v\|_{0,\infty} + |\log h|\ \|\nabla \cdot v\|_{-1,\infty}), \tag{5.5}$$

where $C_p = Cp$. It follows from (5.5) that

$$\|D_h v\|_{0,\infty} \leqslant C|\log h|\ \|v\|_{0,\infty}, \tag{5.6}$$

and that, by duality,

$$\|D_h v\|_{0,1} \leqslant C|\log h|\ \|v\|_{0,1}. \tag{5.7}$$

LEMMA 5.1: *We have*

$$\sigma_h - \Pi_h \sigma = D_h(\sigma - \Pi_h \sigma), \tag{5.8}$$

$$R_h \lambda - \lambda_h = R_h(a^{-1}(\sigma - \sigma_h)). \tag{5.9}$$

Proof: It follows from (4.8a) and (4.8b) that

$$(\sigma - \sigma_h, v) = 0, \quad \forall v \in \bar{V}_h;$$

i.e.,

$$((\sigma - \Pi_h \sigma) + (\Pi_h \sigma - \sigma_h), v) = 0, \quad \forall v \in \bar{V}_h.$$

Thus, (5.8) holds. Now, by (4.5) and (4.8a), we see that

$$((R_h\lambda - \lambda_h) - a^{-1}(\sigma - \sigma_h), a\mu) = 0, \quad \forall \mu \in \Lambda_h,$$

which implies (5.9). □

THEOREM 5.2: *We have, for* $2 \leq p < \infty$ *and* $1 \leq r \leq k+1$,

$$\|\sigma - \sigma_h\|_{0,p} \leq C_p \|\sigma\|_{r,p} h^r, \tag{5.10a}$$

$$\|\sigma - \sigma_h\|_{0,\infty} \leq C(\|\sigma\|_{r,\infty} + |\log h| \, \|f\|_{r-1,\infty}) h^r, \tag{5.10b}$$

$$\|\sigma - \sigma_h\|_{0,1} \leq C|\log h| \, \|\sigma\|_{r,1} h^r, \tag{5.10c}$$

$$\|\lambda - \lambda_h\|_{0,p} \leq C_p \|\lambda\|_{r,p} h^r, \tag{5.11a}$$

$$\|\lambda - \lambda_h\|_{0,\infty} \leq C(\|\lambda\|_{r,\infty} + |\log h| \, \|f\|_{r-1,\infty}) h^r, \tag{5.11b}$$

$$\|\lambda - \lambda_h\|_{0,1} \leq C|\log h| \, \|\lambda\|_{r,1} h^r. \tag{5.11c}$$

Proof: The estimates (5.10a) and (5.10c) follow from (5.8), (5.4), (5.7), and (5.1). To see that (5.10b) is true, note that

$$\nabla \cdot (\sigma - \Pi_h \sigma) = f - P_h f, \tag{5.12}$$

and, by duality,

$$\|f - P_h f\|_{-1,\infty} \leq Ch \|f - P_h f\|_{0,\infty}, \tag{5.13}$$

so that, by (5.5) and (5.8),

$$\begin{aligned}
\|\sigma - \sigma_h\|_{0,\infty} &\leq \|\sigma - \Pi_h \sigma\|_{0,\infty} + \|\Pi_h \sigma - \sigma_h\|_{0,\infty} \\
&\leq C(\|\sigma - \Pi_h \sigma\|_{0,\infty} + |\log h| \, \|\nabla \cdot (\sigma - \Pi_h \sigma)\|_{-1,\infty}) \\
&\leq C(\|\sigma - \Pi_h \sigma\|_{0,\infty} + |\log h| \, h \|f - P_h f\|_{0,\infty}),
\end{aligned}$$

which, together with (5.1) and (5.2), implies (5.10b). Finally, (5.11) follows from (5.3), (5.9), and (5.10). □

LEMMA 5.3: *For* $k \geq 1$ *and* $2 \leq p < \infty$, *we have*

$$\|z_h\|_{0,p} \leq C_p((\|\lambda - \lambda_h\|_{0,p} + \|\sigma - \sigma_h\|_{0,p}) h + \|\nabla \cdot (\sigma - \sigma_h)\|_{0,p} h^{\min\{2,k\}}). \tag{5.14}$$

Proof: Let $\xi \in L^q(\Omega)$, where $1/p + 1/q = 1$, and let $\phi \in W_0^{1,q}(\Omega)$ satisfy $L^* \phi = \xi$. Then, as in Lemma 4.4, we see that

$$\begin{aligned}
(z_h, \xi) &= -(a(\lambda - \lambda_h), \nabla\phi - R_h(\nabla\phi)) + (\lambda - \lambda_h, a\nabla\phi - R_h(a\nabla\phi)) \\
&\quad + (\nabla \cdot (\sigma - \sigma_h), \phi - P_h \phi) + (\sigma - \sigma_h, \nabla\phi - R_h(\nabla\phi)) \\
&\leq C\{\|\lambda - \lambda_h\|_{0,p} \|\phi\|_{2,q} h + \|\nabla \cdot (\sigma - \sigma_h)\|_{0,p} \|\phi\|_{2,q} h^{\min\{2,k\}} + \|\sigma - \sigma_h\|_{0,p} \|\phi\|_{2,q} h\},
\end{aligned}$$

which, together with an elliptic regularity result for (3.1), yields (5.14). □

THEOREM 5.4: *The following results hold:*

$$\|u-u_h\|_{0,p} \leqslant \begin{cases} C_p^2\|u\|_{2,p}\,h, & \text{if } k=1\,, \\ C_p^2\|u\|_{r,p}\,h^r, & 2\leqslant r\leqslant k, \text{ if } k>1\,, \end{cases} \tag{5.15}$$

$$\|u-u_h\|_{0,\infty} \leqslant C|\log h|^2\,h^r\|u\|_{r,\infty}\,, \quad 2\leqslant r\leqslant k\,. \tag{5.16}$$

Proof: Note that, by (5.14), (5.10a), (5.11a), (5.12), and (5.2),

$$\|z_h\|_{0,p} \leqslant C_p^2\|\sigma\|_{r,p}\,h^{r+1-\delta_{1k}}, \quad 1\leqslant r\leqslant k+1\,, \tag{5.17}$$

where δ_{1k} is the Kronecker symbol. Thus, (5.15) follows from (5.2) and this inequality. Also, by (5.17) and the inverse estimate

$$\|z_h\|_{0,\infty} \leqslant Ch^{-2}\|z_h\|_{0,p}\,, \quad 2\leqslant p<\infty\,,$$

we have

$$\|z_h\|_{0,\infty} \leqslant C_p^2\,h^{-2/p+r+1-\delta_{1k}}\|\sigma\|_{r,p}, \quad 1\leqslant r\leqslant k+1\,. \tag{5.18}$$

Since $C_p = Cp$, take $p=|\log h|$ in (5.18) to yield

$$\|z_h\|_{0,\infty} \leqslant C|\log h|^2\,h^{r+1-\delta_{1k}}\|\sigma\|_{r,\infty}\,,$$

which, together with (5.2), implies (5.16). □

We close this section with a remark that all of the results in this section can be extended to the case $1<p<2$ since (5.4) can be replaced by the following inequality [22] in this case:

$$\|D_h v\|_{0,p} \leqslant C_p\|v\|_{0,p} + C_p^2\|\nabla\cdot v\|_{-1,p}, \quad 1<p<2\,,$$

where $C_p = C/(p-1)$.

6. POSTPROCESSING AND SUPERCONVERGENCE

From the error analysis carried out in the previous two sections we see that λ_h and σ_h are more accurate approximations than u_h. In this section, we consider a postprocessing scheme which leads to a new, more accurate approximation to the solution u than u_h. The present scheme is an extension to the expanded mixed method (4.1) of the postprocessing procedure originally developed in [9], [11], and [32] for the traditional mixed method. Another postprocessing procedure, proposed in [2], differs from our scheme. First, the construction of their scheme is ad hoc in the sense that different mixed finite element familites need different constructions; our scheme is applicable to all mixed families. Second, their construction depends on Lagrange multipliers defined over edges or faces, while ours does not. Finally, the present scheme can be implemented more efficiently.

Let

$$W_h^* = \{w\in W : w|_E \in P_{k+1}(E) \text{ for each } E\in\mathscr{E}_h\}\,.$$

Then the postprocessing method is defined for $u_h^*\in W_h^*$ as the solution of the system

$$(u_h^*, 1)_E = (u_h, 1)_E\,, \qquad \forall E\in\mathscr{E}_h\,, \tag{6.1a}$$

$$(a\,\nabla u_h^*, \nabla w)_E = (f, w)_E - (\sigma_h\cdot\nu_E, w)_{\partial E}, \qquad \forall w\in P_{k+1}(E), E\in\mathscr{E}_h\,, \tag{6.1b}$$

where (σ_h, u_h) satisfies (4.1) and ν_E is the exterior unit normal to E.

THEOREM 6.1: *Let u_h^* be defined by (6.1). Then*

$$\|u - u_h^*\| \leq C\|u\|_{k+2} h^{\min\{k+2, 2k\}}, \quad k \geq 1 . \tag{6.2}$$

Proof: For each $E \in \mathscr{E}_h$, let P_E denote the L^2-projection onto $P_0(E)$. Note that $P_0(E) \subset W_h(E)$, so that it follows from (4.1c) that

$$(f, w)_E - (\sigma_h \cdot \nu_E, w)_{\partial E} = 0, \quad \forall w \in P_0(E), E \in \mathscr{E}_h .$$

Therefore, system (6.1) has a unique solution u_h^*.

From (3.1) and Theorem 3.5 we see that

$$(a\nabla u, \nabla w)_E = (f, w)_E - (\sigma \cdot \nu_E, w)_{\partial E}, \quad \forall w \in P_{k+1}(E), E \in \mathscr{E}_h . \tag{6.3}$$

So, we have the error equation

$$(a\nabla(u - u_h^*), \nabla w)_E = ((\sigma_h - \sigma) \cdot \nu_E, w)_{\partial E}, \quad \forall w \in P_{k+1}(E), E \in \mathscr{E}_h . \tag{6.4}$$

Choose $\tilde{u} \in P_{k+1}(E)$, shift u to $\tilde{u}$, and take $w = \tilde{u} - u_h^*$ to obtain

$$\begin{aligned}(a\nabla(\tilde{u} - u_h^*), \nabla(\tilde{u} - u_h^*))_E &= (a\nabla(\tilde{u} - u), \nabla(\tilde{u} - u_h^*))_E \\ &\quad + ((\sigma_h - \sigma) \cdot \nu_E, \tilde{u} - u_h^*)_{\partial E}, \quad \forall E \in \mathscr{E}_h .\end{aligned} \tag{6.5}$$

Then, it follows from (3.2) that there is a constant C such that

$$\begin{aligned}\|\nabla(\tilde{u} - u_h^*)\|_E^2 \leq C\{ &\|\nabla(\tilde{u} - u)\|_E \|\nabla(\tilde{u} - u_h^*)\|_E \\ &+ \|h_E^{1/2}(\sigma_h - \sigma) \cdot \nu_E\|_{\partial E} \|h_E^{-1/2}(\tilde{u} - u_h^*)\|_{\partial E}\} .\end{aligned} \tag{6.6}$$

Now, properly choose $\tilde{u}$ to approximate u on each E that $P_E(\tilde{u} - u) = 0$, and apply a scaling argument to have

$$\|h_E^{-1/2}(\tilde{u} - u_h^*)\|_{\partial E} \leq C\|\nabla(\tilde{u} - u_h^*)\|_E . \tag{6.7}$$

Using (6.7), (6.6) becomes

$$\|\nabla(\tilde{u} - u_h^*)\|_E \leq C(\|\nabla(\tilde{u} - u)\|_E + \|h_E^{1/2}(\sigma_h - \sigma) \cdot \nu_E\|_{\partial E}) . \tag{6.8}$$

Again, it follows from a simple scaling argument that

$$\|w\|_E \leq Ch_E\|\nabla w\|_E, \quad \forall w \in (I - P_E) P_{k+1}(E) ,$$

where I is the identity operator. Consequently, by (6.8), we find that

$$\begin{aligned}\|\tilde{u} - u_h^*\|_E \leq Ch_E(&\|\nabla(\tilde{u} - u)\|_E + \|h_E^{1/2}(\sigma_h - \sigma) \cdot \nu_E\|_{\partial E}) \\ &+ \|P_E(\tilde{u} - u_h^*)\|_E .\end{aligned} \tag{6.9}$$

Hence, it remains to estimate $\|P_E(\tilde{u} - u_h^*)\|_E$. Since P_E is bounded and $P_E(\tilde{u} - u) = 0$, we have

$$\|P_E(\tilde{u} - u_h^*)\|_E \leq \|P_E(u - u_h^*)\|_E . \tag{6.10}$$

Also, since $P_E P_h|_E = P_E$ by the definition of P_E, it follows from (6.1a) and (6.10) that

$$\|P_E(\tilde{u} - u_h^*)\|_E \leq \|P_h u - u_h\|_E . \tag{6.11}$$

Therefore, combine (6.9), (6.10), and (6.11) to observe that

$$\|\tilde{u} - u_h^*\|_E \leq \|P_h u - u_h\|_E + Ch_E(\|\nabla(\tilde{u} - u)\|_E + \|h_E^{1/2}(\sigma_h - \sigma) \cdot \nu_E\|_{\partial E}) ; \tag{6.12}$$

then sum on E to obtain

$$\|\tilde{u} - u_h^*\| \leq C\Big\{ h^{k+2}\|u\|_{k+2} + \|P_h u - u_h\| + h \sum_{E \in \mathscr{E}_h} \|h_E^{1/2}(\sigma - \Pi_h \sigma) \cdot \nu_E\|_{\partial E}$$

$$+ h(\|\sigma - \Pi_h \sigma\| + \|\sigma - \sigma_h\|)\Big\} .$$

Finally, the desired result (6.2) follows from (6.12), (4.20), (4.18), the approximation property of Π_h, and the triangle inequality. □

As in §5, we now consider the L^p-error estimates for the scheme (6.1). For $1 \leq p \leq 2$, the following results follow immediately from (6.2).

THEOREM 6.2: *Let u_h^* be defined by (6.1). Then*

$$\|u - u_h^*\|_{0,p} \leq C\|u\|_{k+2} h^{\min\{k+2, 2k\}}, \quad k \geq 1, 1 \leq p \leq 2 .$$

Next, we concentrate on another interesting and useful situation: $k = 0$. This case, in fact, corresponds to a postprocessing scheme for improving the lowest order Raviart-Thomas-Nedelec mixed method solution on triangles or simplices, the most commonly-used case in practical computation. When $k = 0$, W_h^* becomes

$$W_h^* = \{w \in W : w|_E \in P_1(E) \text{ for each } E \in \mathscr{E}_h\} . \tag{6.13}$$

THEOREM 6.3: *Let u_h^* be the solution of (6.1) with W_h^* given in (6.13). Then, if $u \in W^{2,p}(\Omega)$,*

$$\|u - u_h^*\|_{0,p} \leq C_p\|u\|_{2,p} h^2, \quad 2 < p < \infty , \tag{6.14a}$$

$$\|u - u_h^*\|_{0,\infty} \leq C|\log h|\, h^2\|u\|_{2,\infty} . \tag{6.14b}$$

Proof: Let W_h^{**} be the space of continuous functions in $\bar{\Omega}$, which are piecewise linear polynominals and vanish on $\partial\Omega$, and let $\tilde{u}$ be the interpolation of u in W_h^{**}. Then, for each $E \in \mathscr{E}_h$, by (3.1) and (6.1) we see that

$$C\|\nabla(\tilde{u} - u_h^*)\|_{0,p,E}^p \leq (a\nabla(\tilde{u} - u_h^*), \nabla(\tilde{u} - u_h^*)\,|\nabla(\tilde{u} - u_h^*)|^{p-2})_E$$

$$= (a\nabla(\tilde{u} - u), \nabla(\tilde{u} - u_h^*)\,|\nabla(\tilde{u} - u_h^*)|^{p-2})_E$$

$$+ ((\sigma_h - \sigma) \cdot \nu_E, \nabla(\tilde{u} - u_h^*)\,|\nabla(\tilde{u} - u_h^*)|^{p-2})_{\partial E} .$$

Thus, using the same techniques as in Theorem 6.1, we obtain

$$\|\nabla(\tilde{u} - u_h^*)\|_{0,p,E} \leq C(\|\nabla(\tilde{u} - u)\|_{0,p,E} + \|\sigma - \sigma_h\|_{0,p,E}) . \tag{6.15}$$

To estimate $\|\tilde{u} - u_h^*\|_{0,p}$, we apply a standard duality argument. Let w be determined by

$$\begin{aligned} -\Delta w &= \operatorname{sign}(\tilde{u} - u_h^*)\,|\tilde{u} - u_h^*|^{p-1} && \text{in } \Omega\,, \\ w &= 0 && \text{on } \partial\Omega\,. \end{aligned}$$

Then, with $q = p/(p-1)$, we seet that

$$\|w\|_{2,q} \leqslant C\|\Delta w\|_{0,q} \leqslant C\|\tilde{u} - u_h^*\|_{0,p}^{p-1}\,. \tag{6.16}$$

Let w_h be the interpolation of w in the space W_h^{**}. It can be shown in the same way as in (6.12) that

$$\begin{aligned} \|\tilde{u} - u_h^*\|_{0,p,E}^p &= (\tilde{u} - u_h^*, -\Delta w)_E \\ &= (\nabla(\tilde{u} - u_h^*), \nabla(w - w_h))_E - (\tilde{u} - u_h^*, \nabla(w - w_h)\cdot \nu_E)_{\partial E} \\ &\leqslant Ch\|\nabla(\tilde{u} - u_h^*)\|_{0,p,E}\|w\|_{2,q}\,, \end{aligned}$$

which, together with (6.15) and (6.16), yields

$$\|\tilde{u} - u_h^*\|_{0,p} \leqslant Ch(\|\nabla(\tilde{u} - u)\|_{0,p} + \|\sigma - \sigma_h\|_{0,p})\,.$$

Consequently, the result (6.14a) follows from the definition of W_h^{**}, (5.10a), and the triangle inequality. Finally, (6.14b) can be shown in the same manner as (5.16). □

7. IMPLEMENTATION

Let α, β, and γ denote the degrees of freedom of the solution functions λ_h, σ_h, and u_h, respectively. Then the algebraic system associated with the mixed method (4.1) takes the form

$$A\alpha - C\beta = 0\,, \tag{7.1a}$$

$$B\gamma - C^T\alpha = 0\,, \tag{7.1b}$$

$$B^T\beta = F\,, \tag{7.1c}$$

where A, B, and C are the coefficient matrices of appropriate dimensions, and F is the vector associated with the right-band side of (4.1c). In this section, we discuss several implementation techniques for solving (7.1).

7.1. Traditional approach

If (3.2) is satisfied, then (7.1) can be inverted to the algebraic system arising from the traditional mixed finite element method as follows. Since the degrees of freedom for λ_h are internal to a single element, A has a simple block diagonal structure with each block corresponding to one element. Thus, an *a priori* inversion of A element by element leads to

$$C^T A^{-1} C\beta - B\gamma = 0\,, \tag{7.2a}$$

$$B^T\gamma = F\,. \tag{7.2b}$$

This is a sparse linear system for σ_h and u_h, where $C^T A^{-1} C$ is symmetric and positive definite. We may solve this system and then recover λ_h (if needed) through $\alpha = A^{-1}\beta$ by means of a simple element-by-element postprocess. However, when (3.2) is not satisfied, the following implementation techniques are suggested.

7.2. Preconditioned iterative methods

Note that, using the bilinear forms $a(.,.)$ and $b(.,.)$, the mixed method (4.1) can be written for $(\chi_h, \sigma_h) \in U_h \times V_h$ as

$$a(\chi_h, \tau) + b(\tau, \sigma_h) = F(\tau), \qquad \forall \tau \in U_h, \tag{7.3a}$$

$$b(\chi_h, v) = 0, \qquad \forall v \in V_h, \tag{7.3b}$$

where $\chi_h = (u_h, \lambda_h)$ and $U_h = W_h \times \Lambda_h$. The coefficient matrix of system (7.3), given by

$$\begin{pmatrix} M & N \\ N^T & 0 \end{pmatrix},$$

is symmetric, nonsingular, and indefinite. Namely, the algebraic system associated with (7.3) is given by

$$M\xi + N\beta = F, \tag{7.4a}$$

$$N^T \xi = 0, \tag{7.4b}$$

where ξ is the degrees of freedom of χ_h. Thus, the minimum residual iterative method [7], [29] can be used to solve this system. Since one of the condition numbers associated with M and N increases as the discretization is refined and the convergence is too slow, a direct application of the minimum residual method is usually not practical. Therefore, to speed up the convergence, preconditioned versions of this method have been suggested [23], [31]. For completeness, we will consider a block diagonal preconditioner for the system (7.4).

Let the dimensions of U_h and V_h be n and m, respectively, and let $L \in \mathbb{R}^{n \times n}$ and $S \in \mathbb{R}^{m \times m}$ be nonsingular matrices. Then the system (7.4) is equivalent to the system

$$L^{-1} M L^{-T} \zeta + L^{-1} N S^{-1} \psi = L^{-1} F, \tag{7.5a}$$

$$(L^{-1} N S^{-1})^T \zeta = 0, \tag{7.5b}$$

where $\zeta = L^T \xi$ and $\psi = S\beta$. The system (7.5) has the same structure as (7.4). The minimum residual method applied to (7.5) converges faster if L and S are appropriately chosen. The matrices L and S should have the property that linear systems with coefficient matrices given by LL^T or $S^T S$ can be solved by a fast solver. This requirement is necessary since such linear systems have to be solved once in each iteration of the preconditioned minimum residual method. One example of the choices for L and S are that $L = I$, the identity matrix, and S should be chosen such that $S^T S$ is a preconditioner for $N^T N$. $S^T S$ can be obtained from the incomplete Cholesky factorization of $N^T N$ [31], for example.

7.3. Alternating direction iterative techniques

Uzawa and Arrow-Hurwitz alternating-direction iterative methods have been developed for solving the algebraic equations arising from traditional mixed methods [4], [5], [15], [16], [19]. We now describe similar iterative techniques for solving (7.1). We limit ourselves here to the Uzawa-type algorithms for the Raviart-Thomas spaces on rectangles; the Arrow-Hurwitz-type algorithms and other mixed finite element families can be treated analogously.

The Uzawa iterative techniques are based on a virtual parabolic problem introduced by adding the virtual time derivative of γ to (7.1c) and initiating the resulting evolution by an initial guess for γ. Thus, we consider the system

$$A\alpha - C\beta = 0, \qquad t \geq 0, \tag{7.6a}$$

$$B\gamma - C^T \alpha = 0, \qquad t \geq 0, \tag{7.6b}$$

$$D \frac{\partial \gamma}{\partial t} + B^T \beta = F, \qquad t \geq 0, \tag{7.6c}$$

$$\gamma(0) = \gamma^0, \tag{7.6d}$$

where the choice of D is somewhat arbitrary, though it should be symmetric and positive definite. The system (7.6) corresponds to an expanded mixed finite element method for the initial value problem

$$d \frac{\partial u}{\partial t} - \nabla \cdot (a \nabla u) = f,$$

for some coefficient d. Let now the domain Ω be a rectangle and $\mathscr{E}_h$ be a partition of Ω into subrectangles. Then, if the Raviart-Thomas space on rectangles is used in (4.1), it is easy to see that (7.6) splits into equations of the form

$$\begin{aligned} &A_1 \alpha_1 - C_1 \beta_1 = 0, && A_2 \alpha_2 - C_2 \beta_2 = 0, \\ &B_1 \gamma - C_1^T \alpha_1 = 0, && B_2 \gamma - C_2^T \alpha_2 = 0, \\ &D \frac{\partial \gamma}{\partial t} + B_1^T \beta_1 + B_2^T \beta_2 = F, \\ &\gamma(0) = \gamma^0, \end{aligned}$$

where the α_1, β_1-parameters and α_2, β_2-parameters are ordered in an x_1-orientation and an x_2-orientation, respectively.

The Uzawa iterative algorithm is described as follows. Let γ^0 be given arbitrarily and determine α^0 and β^0 (only β_2^0 need to be computed to initiate the iteration) by the system

$$\begin{aligned} &A_1 \alpha_1^0 - C_1 \beta_1^0 = 0, && A_2 \alpha_2^0 - C_2 \beta_2^0 = 0, \\ &B_1 \gamma^0 - C_1^T \alpha_1^0 = 0, && B_2 \gamma^0 - C_2^T \alpha_2^0 = 0. \end{aligned}$$

The general step splits into the following x_1-sweep and x_2-sweep:

$$A_1 \alpha_1^{n+1/2} - C_1 \beta_1^{n+1/2} = 0,$$

$$B_1 \gamma^{n+1/2} - C_1^T \alpha_1^{n+1/2} = 0,$$

$$D \frac{\gamma^{n+1/2} - \gamma^n}{t^n} + B_1^T \beta_1^{n+1/2} + B_2^T \beta_2^n = F,$$

$$A_2 \alpha_2^{n+1/2} - C_2 \beta_2^{n+1/2} = 0,$$

$$B_2 \gamma^{n+1/2} - C_2^T \alpha_2^{n+1/2} = 0,$$

and

$$A_2\,\alpha_2^{n+1} - C_2\,\beta_2^{n+1} = 0\,,$$

$$B_2\,\gamma^{n+1} - C_2^T\,\alpha_2^{n+1} = 0\,,$$

$$D\,\frac{\gamma^{n+1} - \gamma^{n+1/2}}{t^n} + B_1^T\,\beta_1^{n+1/2} + B_2^T\,\beta_2^{n+1} = F\,,$$

$$A_1\,\alpha_1^{n+1} - C_1\,\beta_1^{n+1} = 0\,,$$

$$B_1\,\gamma^{n+1} - C_1^T\,\alpha_1^{n+1} = 0\,.$$

Note that $\alpha_2^{n+1/2}$, $\beta_2^{n+1/2}$, α_1^{n+1}, and β_1^{n+1} do not enter into the evolution; they need not be calculated at all, though it is probably a good idea to compute them to be consistent with the final γ upon termination of the iteration.

No spectral analysis has been made yet for this iteration. However, on the basis of experience with the traditional mixed methods, we conjecture that the Uzawa iterative algorithm converges rapidly when the parameters t^n are properly chosen. A complete spectral analysis for this iteration is for future work.

7.4. Hybridization

Note that the normal component of the members in V_h is continuous across the interior boundaries in $\partial\mathscr{E}_h$. Following [2], we relax this constraint on V_h by letting $V_h = \Lambda_h$, and introduce the Lagrange multipliers to enforce the required continuity on Λ_h:

$$L_h = \left\{ \mu \in L^2\left(\bigcup_{e \in \partial\mathscr{E}_h} e \right) : \mu|_e \in V_h \cdot \nu|_e \text{ for each } e \in \partial\mathscr{E}_h \right\}.$$

Then the unconstrained expanded mixed method is to find $(\sigma_h, \lambda_h, u_h, l_h) \in \Lambda_h \times \Lambda_h \times W_h \times L_h$ such that

(7.7a) $$(a\lambda_h, \mu) - (\sigma_h, \mu) = 0\,, \qquad \forall \mu \in \Lambda_h\,,$$

(7.7b) $$(\lambda_h, v) + \sum_{E \in \mathscr{E}_h} ((l_h, v \cdot \nu_E)_{\partial E} - (u_h, \nabla \cdot v)_E) = 0, \qquad \forall v \in \Lambda_h\,,$$

(7.7c) $$\sum_{E \in \mathscr{E}_h} (\nabla \cdot \sigma_h, w) = (f, w)\,, \qquad \forall w \in W_h\,,$$

(7.7d) $$\sum_{E \in \mathscr{E}_h} (\sigma_h \cdot \nu_E, v)_{\partial E} = 0\,, \qquad \forall v \in L_h\,.$$

As a result of (7.7d), the solution function σ_h generated from (7.7) coincides with that produced by (4.1). Hence the triple $(\sigma_h, \lambda_h, u_h)$ from (7.7) is the same as that from (4.1). That is why we use the same notation as before. The system associated with (7.7) is given by

(7.8a) $$A\alpha - \tilde{C}\beta = 0\,,$$

(7.8b) $$\tilde{C}\alpha - \tilde{B}\gamma + K\epsilon = 0\,,$$

(7.8c) $$\tilde{B}^T\beta = F\,,$$

(7.8d) $$K^T\beta = 0\,,$$

where ϵ is the degrees of freedom of the solution l_h. Now note that $\tilde{C}$ is symmetric and positive definite. An elimination of β element-by-element gives the following new system:

$$\tilde{C}\alpha - \tilde{B}\gamma + K\varepsilon = 0\,, \tag{7.9a}$$

$$\tilde{B}^T\tilde{C}^{-1}A\alpha = F\,, \tag{7.9b}$$

$$K^T\tilde{C}^{-1}A\alpha = 0\,. \tag{7.9c}$$

The system (7.9) has the same number of unknowns as the system generated by the hybridization of the standard mixed methods. We can solve this system for α, γ, and ϵ in the manner that follows and then recover β (if needed) through $\beta = \tilde{C}^{-1}A\alpha$. Multiply (7.9a) with $\tilde{C}^{-1}$ to have the new system

$$I\alpha - \tilde{C}^{-1}\tilde{B}\gamma + \tilde{C}^{-1}K\epsilon = 0\,, \tag{7.10a}$$

$$\tilde{B}^T\tilde{C}^{-1}A\alpha = F\,, \tag{7.10b}$$

$$K^T\tilde{C}^{-1}A\alpha = 0\,. \tag{7.10c}$$

Let the matrix

$$\mathscr{A} = \begin{pmatrix} I & -\tilde{C}^{-1}\tilde{B} & \tilde{C}^{-1}K \\ -\tilde{B}^T\tilde{C}^{-1}A & 0 & 0 \\ K^T\tilde{C}^{-1}A & 0 & 0 \end{pmatrix},$$

and let the inner product on $\Lambda_h \times W_h \times L_h$ be

$$\left[\begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] = (Au, x) + (v, y) + (w, z)\,.$$

THEOREM 7.1: *The matrix $\mathscr{A}$ is symmetric with respect to the inner product* $[\,.\,,\,.\,]$.

Proof: Note that

$$\begin{aligned}
&\left[\mathscr{A}\begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] \\
&= (Au - A\tilde{C}^{-1}\tilde{B}v + A\tilde{C}^{-1}Kw, x) - (\tilde{B}^T\tilde{C}^{-1}Au, y) + (K^T\tilde{C}^{-1}Au, z) \\
&= (Au, x - \tilde{C}^{-1}\tilde{B}y + \tilde{C}^{-1}Kz) - (v, \tilde{B}^T\tilde{C}^{-1}Ax) + (w, K^T\tilde{C}^{-1}Ax) \\
&= \left[\begin{pmatrix} u \\ v \\ w \end{pmatrix}, \mathscr{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right]. \quad \square
\end{aligned}$$

From this theorem we see that the iterative techniques developed for the conventional mixed method (*cf.* [3]) apply to the expanded mixed method.

7.5. Other options

Traditional mixed methods can be efficiently implemented by exploiting their equivalence with certain nonconforming Galerkin methods [2], [1], [10]. Again, on the basis of experience with the traditional mixed methods, we think that the expanded mixed methods (2.3) and (4.1) may be rewritten equivalently in standard finite elements methods or even in finite difference methods. This will be explored in the second paper. Also, numerical results to illustrate our theoretical results will be presented there.

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