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NUMERICAL APPROXIMATION OF STIFF TRANSMISSION PROBLEMS BY MIXED FINITE ELEMENT METHODS (*)

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Abstract — *We are interested in the approximation of stiff transmission problems by the means of mixed finite element methods We prove a result of uniform convergence in ε, the small parameter, as the discretization's step tends to 0, showing thus the robustness of the method The discrete problem is then numerically solved* via *hybrid methods, since the ill-conditioning of the matrix makes the standard Uzawa's algorithm impracticable The numerical results ascertain an optimal rate of convergence for both the stress tensor and the anti-plane displacement So, contrarily to primal methods, the mixed ones avoid the locking phenomenon over any regular triangulation © Elsevier, Paris*

Résumé — *On s'intéresse à l'approximation des problèmes de transmission raide par méthodes d'éléments finis mixtes On établit un résultat de convergence uniforme en ε, le petit paramètre quand le pas de discrétisation tend vers 0, prouvant ainsi la robustesse de la méthode Le problème discret est ensuite résolu numériquement* via *une méthode hybride, car le mauvais conditionnement de la matrice rend impraticable l'algorithme standard d'Uzawa Les résultats numériques montrent une convergence optimale en $O(h)$ pour le tenseur des contraintes et pour le déplacement anti-plane En conclusion, contrairement aux méthodes primales conformes, les méthodes mixtes échappent au verrouillage numérique, et ceci indépendamment de triangulation © Elsevier, Paris*

1. INTRODUCTION

In this paper we study stiff transmission problems and their solution by Raviart-Thomas mixed finite element methods. A stiff transmission problem is a parameter-dependent problem which is best described by the following physical problem: an elastic body onto which is grafted a thin shell of thickness ε and whose stiffness is an increasing linear function of $1/\varepsilon$. One clearly sees that, when ε tends towards zero, the thin shell acts like a stiffener.

This kind of problem has already been solved using primal finite element methods, as in [4] and [21]; in all instances, locking has been exposed. Briefly, the locking phenomenon can be described as a loss of convergence which stems from the approximation scheme: although mathematical convergence is secured and computer accuracy is adequate, the approximation does not square with the expected solution. A definition of locking is given in [4] and a criterion for avoiding locking is established in [11].

In previous cases, primal finite element methods have been employed. Mesh choice is then very important and constraining altogether: the accuracy of the approximation for small values of the parameter ε strongly depends on the structure of the mesh, see [21]. Essentially, standard primal methods give place to numerical locking on arbitrary meshes.

Mixed methods weaken the continuity constraint at the internal edges of the triangulation by introducing an additional field, the stress tensor. Actually, this is a side effect of the main purpose of mixed methods, which is to take into account the equilibrium constraint. As a result, the behavior of the computed solution is independent of the mesh structure. Moreover, we shall prove that the mixed methods are free of locking.

Let us note that the final system is never symmetric definite positive since it corresponds to the solution of a saddle-point problem. For our model problem, the usual Uzawa's algorithm is impracticable because the resulting matrix is ill-conditioned for small values of the parameter ε.

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To counter this drawback, we shall make use of *mixed hybrid methods,* which dualize the coupling constraints in order to obtain a symmetric definite positive system. The underlying idea of mixed hybrid methods is to find a field which satisfies the equilibrium conditions only at the element level (see [22]). For low-order methods or in particular instances, this field can be worked out by hand. However, such a field can be computed systematically by using the results established in [7].

The numerical approximation of the stiff transmission problem is surprising: it is bereft of locking, at least for the lowest-order method. This is an important result since it means that we can apply mixed methods to solve efficiently locking-prone problems.

An outline of the paper is as follows: in Section 2, we introduce our model problem as well as the Ventcel problem, formally obtained by letting ε to go to 0, and we give a summary of theoretical results. This is essentially a recall of some results of [19]. In Section 3 we give some numerical results, showing how locking occurs when using a primal finite element method on an arbitrary regular triangulation. The last section is devoted to the study of mixed finite element methods for the stiff transmission problem. Firstly, a uniform convergence result is established, which shows that the Raviart-Thomas method is robust in the sense of [4], and this for any order $k \in \mathbb{N}$. Secondly, the system obtained after discretization is hybridized. We obtain in this manner a linear system whose matrix is symmetric definite positive and whose unknowns are the multipliers introduced by the mixed hybrid method. Next, we recover by a local post-processing both the approximations of the displacement and of the stress tensor. Numerical results are also presented for the lowest-order method: they clearly show an optimal rate of convergence for any type of mesh structure.

2. STIFF TRANSMISSION PROBLEMS

2.1. Physical motivation

To fix the frame of our study, we consider a linear anti-plane displacement elasticity problem. Of course, this type of problem is not the most meaningful in structural mechanics. However, the terminology linked to it will help us to illustrate the main features of the problem.

The general situation of the elasticity problem studied here is described as follows. Let Ω^+ be an elastic body (an open set) whose boundary is denoted by $\partial\Omega^+$; let us suppose that $\partial\Omega^+ = \Sigma \cup \Gamma^+$ where both Σ and Γ^+ are of non-vanishing measure. This elastic body has a thin shell, noted Ω_ε^-, grafted onto Σ; let us precise that ε is a physical parameter which will tend to 0, characterizing the thickness of the thin shell. The boundary Γ^+ is clamped.

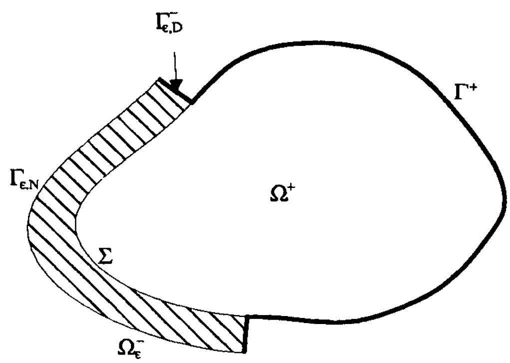

Figure 1. — Elastic body Ω^+ with a thin shell Ω_ε^-.

Both bodies Ω^+ and Ω_ε^- behave according to a linear and isotropic law which we suppose to be characterized by their respective Lamé coefficients λ, μ and λ/ε, μ/ε. It is thus obvious that the thin shell Ω_ε^- behaves itself as a tightener on Σ, see *figure 1*. We denote by Ω_ε the interior of $\bar{\Omega}^+ \cup \bar{\Omega}_\varepsilon^-$. For an anti-plane two-dimensional problem, the displacement is described by a scalar function u^ε verifying:

$$(1) \qquad \begin{cases} -\Delta u^\varepsilon = f & \text{in } \Omega^+ \\ -\dfrac{1}{\varepsilon}\Delta u^\varepsilon = f & \text{in } \Omega_\varepsilon^- \\ u^\varepsilon = 0 & \text{on } \Gamma_{\varepsilon,D} = \Gamma^+ \cup \Gamma_{\varepsilon,D}^- \\ \partial_\nu u^\varepsilon = 0 & \text{on } \Gamma_{\varepsilon,N} \\ (u^\varepsilon)^+ = (u^\varepsilon)^- & \text{on } \Sigma \\ (\partial_\nu u^\varepsilon)^+ = \dfrac{1}{\varepsilon}(\partial_\nu u^\varepsilon)^- & \text{on } \Sigma\,, \end{cases}$$

where $\mathbf{v}$ denotes the unit normal to $\Gamma_{\varepsilon,N}$ outwardly directed with respect to Ω_ε^- and the unit normal to Σ outwardly directed with respect to Ω^+. The superscripts + and – in the above transmission conditions indicate that the trace on Σ is worked out from the value of u^ε in Ω^+ and Ω_ε^- respectively.

Problem (1) stems from various physical phenomena. It models, for example, the way heat spreads out in a body Ω^+ which is either partially or completely covered with a highly heat-conductive thin shell Ω_ε^-; the quantity u^ε is then the temperature. However, it is the study of an electric current scattering in a plate bordered by a highly conductive rod which, in steady-state, constitutes the most typical example modeled by (1). This model is also a good approximation of wave scattering problems by a scatterer covered with a penetrable thin shell, provided we limit ourselves to the main part of the operator of the partial differential equation set on Ω^+ (see [15] and [6]).

2.2. Variational formulation

We denote by $|\xi|$ the euclidean norm of the vector ξ of $\mathbb{R}^2$ and by $\xi \cdot \eta$ the scalar product of the vectors ξ and η of $\mathbb{R}^2$ identified with column vectors. Problem (1) fits into the following setting. The thin shell Ω_ε^- is described by:

$$\Omega_\varepsilon^- = \{\mathbf{x} \in \mathbb{R}^2, \mathbf{x} = \mathbf{m} + t\mathbf{v}(\mathbf{m})\,;\, \mathbf{m} \in \Sigma \text{ and } 0 < t < \varepsilon h(\mathbf{m})\}\,,$$

$$\Gamma_{\varepsilon,N} = \{\mathbf{x} \in \mathbb{R}^2, \mathbf{x} = \mathbf{m} + \varepsilon h(\mathbf{m})\,\mathbf{v}(\mathbf{m})\,;\, \mathbf{m} \in \Sigma\}\,,$$

where h is a smooth real-valued function defined on Σ such that there exists h_*, $h^* > 0$ with $h_* < h(\mathbf{m}) < h^*$. One obviously has $\Gamma_{\varepsilon,D} = \partial\Omega_\varepsilon \backslash \Gamma_{\varepsilon,N}$. We also introduce $V_\varepsilon = \{v \in H^1(\Omega_\varepsilon);\, v = 0 \text{ on } \Gamma_{\varepsilon,D}\}$ and the bilinear forms defined on $V_\varepsilon \times V_\varepsilon$ by:

$$a^+(u,v) = \int_{\Omega^+} \mathbf{grad}\, u \,.\, \mathbf{grad}\, v \; d\Omega^+,$$

$$a_\varepsilon^-(u,v) = \int_{\Omega_\varepsilon^-} \mathbf{grad}\, u \,.\, \mathbf{grad}\, v \; d\Omega_\varepsilon^- \,.$$

It has the following variational formulation:

(2)
$$\begin{cases} u^\varepsilon \in V_\varepsilon, \quad \forall v \in V_\varepsilon \\ \frac{1}{\varepsilon} a_\varepsilon^-(u^\varepsilon, v) + a^+(u^\varepsilon, v) = \int_{\Omega_\varepsilon} fv \, d\Omega_\varepsilon, \end{cases}$$

with f a given function of $L^\infty(\mathcal{U})$ and $\mathcal{U}$ an open neighbourhood of $\bar{\Omega}^+$. The Lax-Milgram lemma insures that (2) has a unique solution.

This problem depends singularly on ε when ε tends towards zero: actually, two difficulties appear. Firstly, the thickness of the thin shell tends towards zero and secondly, the stiffness coefficients tend towards infinity.

To highlight the locking phenomenon, from now on we focus on a *sample case* which bears all the main features of the general case, thoroughly treated in [21]. We shall therefore suppose that Σ is straight, which is an important case when numerically solving the problem (2), and that the thickness of the thin shell is constant, that is $h \equiv 1$. The following geometry is a typical case of a straight interface (see *fig. 2*): we set $\Omega^+ =]0, 1[\times]0, 1[$ and $\Omega_\varepsilon^- =]0, 1[\times] - \varepsilon, 0[$.

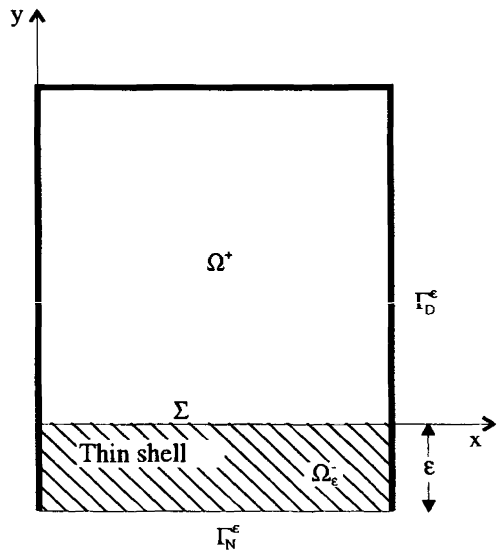

Figure 2. — Domain with straight interface.

By using the scale change $\tilde{y} = y/\varepsilon$, $\tilde{v}(x, \tilde{y}) = v(x, y)$ for $-\varepsilon < y < 0$, the thin domain Ω_ε^- becomes $\Omega^- =]0, 1[\times] - 1, 0[$ and the space V_ε turns into $V = \{v \in H^1(\Omega); v = 0 \text{ on } \Gamma_D\}$, where Ω is the interior of $\bar{\Omega}^+ \cup \bar{\Omega}^-$. The remaining notations may be found on *figure 3*.

To simplify the notation, from now on v will denote either a function v or $\tilde{v}$, obtained by the previous scale change, while the solution $\tilde{u}^\varepsilon$ on the new domain Ω will simply be denoted by u^ε. Problem (2) can thus be written under the following form, which has a simple and explicit dependency on ε:

(3)
$$\begin{cases} u^\varepsilon \in V, \quad \forall v \in V \\ \frac{1}{\varepsilon^2} a_y^-(u^\varepsilon, v) + a_x^-(u^\varepsilon, v) + a^+(u^\varepsilon, v) = \int_{\Omega^+} f^- v \, d\Omega^+ + \varepsilon \int_{\Omega} f^- v \, d\Omega^-, \end{cases}$$

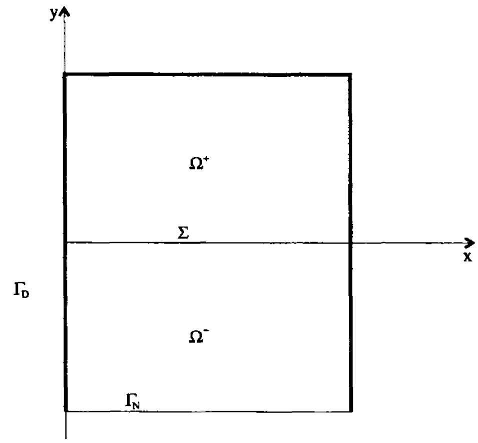

Figure 3. — Domain after change of variable.

where $f^+ = f_{|\Omega^+}$ and f^- is the function obtained from $f_{|\Omega_\varepsilon^-}$ through the previous scale change and where:

$$a_x^-(u, v) = \int_{\Omega^-} \partial_x u \, \partial_x v \, d\Omega^- ,$$

$$a_y^-(u, v) = \int_{\Omega^-} \partial_y u \, \partial_y v \, d\Omega^- .$$

Let us also introduce the notation:

$$a_\varepsilon(u, v) = a^+(u, v) + a_x^-(u, v) + \frac{1}{\varepsilon^2} a_y^-(u, v) .$$

Problem (3) fits into the general setting of parameter dependent problems, studied in [4] and equally in [11]. The same general framework is also used for the analysis of arch modelisation problems (see [2]). The Stokes system, in the case of weakly compressible fluids and, more generally, saddle-point problems arising from a penalized constraint (see [4], [8], [16], [17] and so on) use the same formulation, although with different properties.

The sample problem (3) may actually be considered *via* a domain decomposition procedure as a first order approximation of problem (1) by neglecting the curvature terms.

2.3. The Ventcel problem. Convergence of u^ε to u^0

An approach for solving (3) consists in identifying the limit u^0 of u^ε when ε vanishes. It verifies a non-standard boundary value problem, called a *Ventcel* problem (see [1] and [19]) and standing for a first-order approximation of the stiff transmission problem (3).

To improve the approximation, a series expansion of u^0 into powers of ε may be obtained through an asymptotic analysis (*cf.* [9], [13] and [18]). However, the computations and the model obtained can be very complex. We shall therefore restrict ourselves to the Ventcel problem, formally obtained from (3) by letting ε to go to 0:

$$\text{(4)} \qquad \begin{cases} u^0 \in V_0, \quad \forall v \in V_0 \\ a^+(u^0, v) + a_x^-(u^0, v) = \int_{\Omega^+} f^+ v \, d\Omega^+ , \end{cases}$$

where

$$\text{(5)} \qquad V_0 = \{ v \in V; \partial_y v = 0 \text{ in } \Omega^- \} .$$

It comes in a standard manner that u^0 also verifies the strong variational problem:

$$\text{(6)} \qquad \begin{cases} -\operatorname{div}(\mathbf{grad}\, u^0) = f & \text{in } \Omega^+ \\ u^0 = 0 & \text{on } \Gamma^+ \\ \mathbf{grad}\, u^0 \,.\, \mathbf{v} - \partial_s^2 u^0 = 0 & \text{on } \Sigma . \end{cases}$$

This formulation involves on Σ an operator whose order is greater or equal to the one of the operator involved in the partial derivative equation on Ω^+. This type of non-standard boundary value problem has been studied by Lemrabet [19]. Let us note that the same type of boundary conditions appears in absorbing boundary conditions of wave-scattering problems (see [5], [14]) and, more generally, in multi-layer structure problems (see [12], [18]).

For $k \in \mathbb{N}$, we denote by $\| \,.\, \|_{k, \Omega^+}$ and by $\| \,.\, \|_{k, \Omega^-}$ the respective norms equipping the Sobolev spaces $H^k(\Omega^+)$ and $H^k(\Omega^-)$. Let us endow the space V with the norm $\| \,.\, \|_V = (\| \,.\, \|_{1,\Omega^+}^2 + \| \,.\, \|_{1,\Omega^-}^2)^{1/2}$. From now on, we shall note $f_\varepsilon = (f^+, \varepsilon f^-) \in L^2(\Omega^+) \times L^2(\Omega^-)$.

Let us recall the following result, initially established in [19]:

THEOREM 2.1: *We have the strong convergence results:*

$$\lim_{\varepsilon \to 0} u^\varepsilon = u^0 \textit{ in } V ,$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \partial_y u^\varepsilon = 0 \textit{ in } L^2(\Omega^-) . \quad \square$$

3. NUMERICAL LOCKING

We show in this section the numerical results obtained for our sample problem when employing a primal $\mathscr{P}_1$-conforming finite element method. Let $\mathcal{T}_h^+$, respectively $\mathcal{T}_h^-$ be regular mesh partitions (in the sense of Ciarlet [12]) of Ω^+, respectively of Ω^- into triangles of diameter no greater than h, compatible on the interface Σ. From now on, we shall note $\mathcal{T}_h = \mathcal{T}_h^+ \cup \mathcal{T}_h^-$ and we denote by $\mathscr{P}_1$ the space of polynomials of degree at most 1. We are now able to introduce the finite dimensional space

$$V_h = \{ v_h \in V; \forall T \in \mathcal{T}_h, v_{h|T} \in \mathscr{P}_1 \} .$$

We seek to approximate u^ε by $\bar{u}_h^\varepsilon \in V_h$ when ε is very small and we need to do so with some degree of accuracy. We proceed as follows: using a standard finite element method, we solve the approximate transmission problem. The initial data is chosen so that u^ε, solution of the continuous problem, can be worked out explicitly.

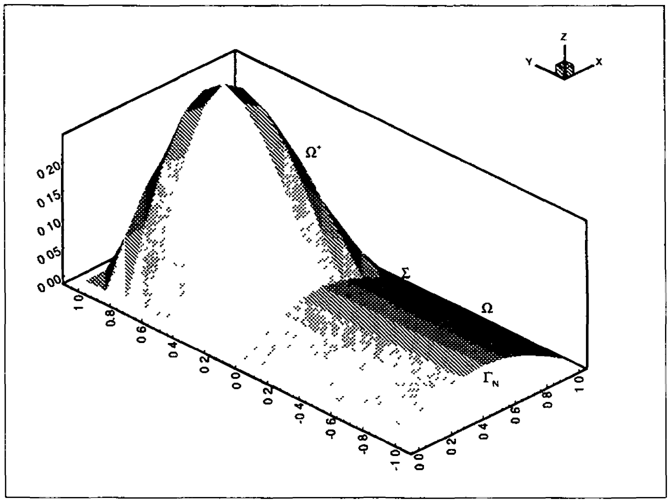

Figure 4. — Exact solution to the transmission problem ($\varepsilon = 10^{-6}$)

Let us note that the discretization of problem (3) by a standard finite element method leads to an ill-conditioned problem. More specifically, even when theoretical convergence is guaranteed, the round off error linked to the floating-point arithmetic may generate a solution without any link with the exact solution. In order to focus just on the locking phenomenon, we will keep to the range of ε ($\varepsilon \geqslant 10^{-6}$) where this degeneracy does not occur.

We now visualise the exact solution of the transmission problem (see *fig. 4*), computed for $f^+(x, y) = \pi^2 \sin(\pi x)\, y$, $f^-(x, y) = 0$ and $\varepsilon = 10^{-6}$, by an exact analytical formula.

It is obvious from *figure 4* that the exact solution has a particular geometric behavior: its value along the normal to Σ is constant over Ω^-. This confirms the result of the previous section, that is $\partial_y u^\varepsilon \to 0$ in Ω^- when $\varepsilon \to 0$.

3.1. Locking on unstructured meshes

The numerical results of this paragraph are obtained over the unstructured mesh given in *figure 5*.

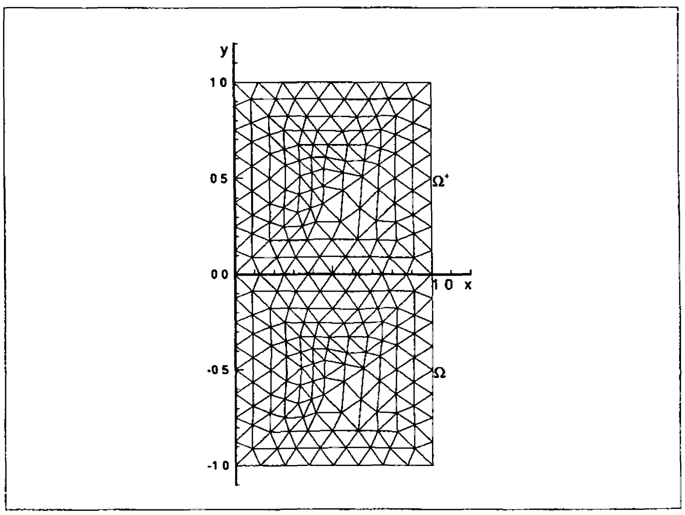

Figure 5. — Unstructured mesh.

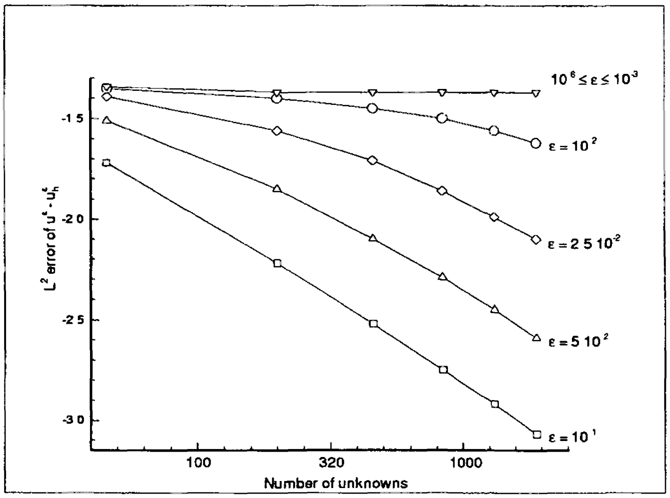

Figure 6. — Rate of convergence of $\bar{u}_h^\varepsilon$ in $L^2(\Omega)$-norm.

The well-known technique of mesh refinement, when applied bluntly to our model problem, does not lead to any improvement of the accuracy of the approximation (see also [4]). Indeed, *fig. 6* and *fig. 7* show the rates of convergence of $u^\varepsilon - \bar{u}_h^\varepsilon$ for the $L^2(\Omega)$-norm, respectively for the energy norm of the problem. We precise that the energy norm is defined for any $v \in V$ by $|v|_\varepsilon = \sqrt{a_\varepsilon(v, v)}$.

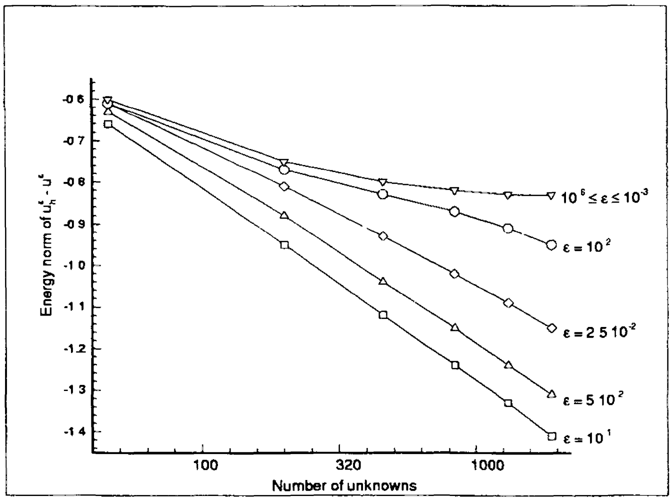

Figure 7. — Rate of convergence of $\bar{u}_h^\varepsilon$ in energy norm.

As ε decreases, one can see that the convergence rates deteriorate from $O(N^{-1}) = O(h^2)$ in *figure 6*, respectively $O(N^{-1/2}) = O(h)$ in *figure 7* (for $\varepsilon = 10^{-1}$) to $O(1)$ in both cases (for $\varepsilon = 10^{-6}$). Here, N denotes the number of unknowns of the final system. We also note that, when ε runs from 10^{-3} to 10^{-6}, the corresponding curves cannot be distinguished: the values of the error have identical significant digits.

So, when using an arbitrary triangulation, the method presents *complete locking* of order $O(N^{1/2}) = O(h^{-1})$ and therefore is not robust (see [4]).

Perhaps the most glaring feature of the approximation corresponding to the top curve of *figure 6,* where no convergence is ensured, is that it vanishes on Ω^- (see also *fig. 8*): the exact solution is just not approached. It is a clear manifestation of the locking phenomenon. In *figure 8* we have represented the discrete solution computed by the Lagrange method on the mesh of *figure 5,* corresponding to $\varepsilon = 10^{-6}$.

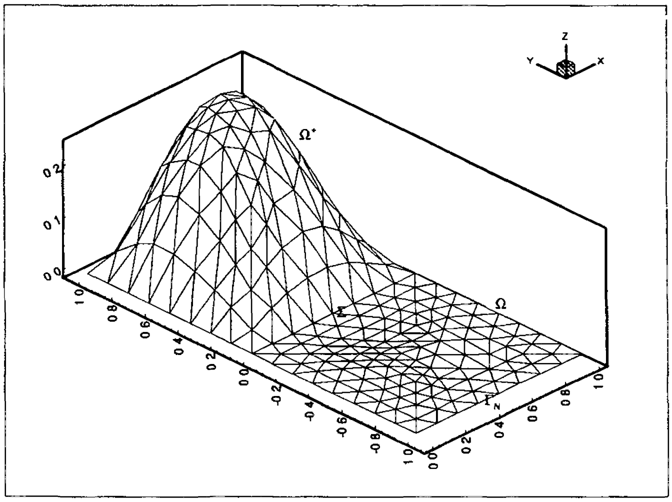

Figure 8. — Three dimensional representation of $\bar{u}_h^\varepsilon$ ($\varepsilon = 10^{-6}$)

Note that the solution does not show any sign of numerical instability: it does not blow-out or oscillate quickly. Actually, the restriction of the approximation $\bar{u}_h^\varepsilon$ to Ω^+ is the solution of a standard Laplace problem with an homogeneous Dirichlet boundary condition, with no connection whatsoever with problem (1).

3.2. Why locking?

To answer this question, we need to recall that the exact solution u^ε of (3) tends to u^0 when $\varepsilon \to 0$, with $u^0 \in V_0$ verifying (4). Its approximation $\bar{u}_h^0$ by the $\mathscr{P}_1$-conforming method satisfies the following discrete problem:

$$\begin{cases} \bar{u}_h^0 \in V_{0h}, \quad \forall v_h \in V_{0h} \\ a^+(\bar{u}_h^0, v_h) + a_x^-(\bar{u}_h^0, v_h) = \displaystyle\int_{\Omega^+} f^+ v_h \, d\Omega^+ , \end{cases}$$

where

$$V_{0h} = \{ v_h \in V_h ;\ \partial_y v_h = 0 \text{ in } \Omega^- \} .$$

If V_0 is well-approximated by the discrete kernel V_{0h}, the results of [11] then yield the uniform convergence in ε and h of $\bar{u}_h^\varepsilon$ towards u^0.

There exists a class of meshes, called *adapted* meshes, for which the above property can be proved using the Lagrange interpolation operator (see [21]). These meshes adapt their structure to the behavior of u^ε: all the segments linking Σ to Γ_N are parallel to the y-axis (a typical example is a *uniform* mesh). The discretization over adapted meshes provides optimal rate of convergence $O(N^{-1/2})$.

On the contrary, the approximation of the kernel V_0 does not hold on unstructured meshes, which yield the poor convergence of the previous paragraph Indeed, when considering meshes with no *vertical* interior edge on the domain Ω^-, the constraint $\partial_y v_h = 0$ implies $v_h = 0$ on Ω^-, this is the case for the mesh of *figure 5* This aspect of locking has been more comprehensively researched in [21] In the next section of this paper, we will focus on mixed finite element methods to counter locking

4 MIXED METHODS

4.1. The dual mixed formulation

In order to write the mixed formulation of (2), we use the standard technique which we describe below The anti plane displacement u^ε satisfies the equation

(7)
$$- \operatorname{div} (A_\varepsilon \,\mathbf{grad}\, u^\varepsilon) = f_\varepsilon \text{ in } \Omega ,$$

where A_ε is the identity matrix on Ω^+ and $A_\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1/\varepsilon^2 \end{bmatrix}$ on Ω We then introduce as unknown

(8)
$$\mathbf{p}^\varepsilon = A_\varepsilon \,\mathbf{grad}\, u^\varepsilon$$

and express variationally this equation as well as the equilibrium relation

(9)
$$\operatorname{div} \mathbf{p}^\varepsilon + f_\varepsilon = 0 \text{ in } \Omega$$

Equality (8) shows that $\mathbf{p}^\varepsilon \in H(\operatorname{div}, \Omega)$, where $H(\operatorname{div}, \Omega)$ is the sub-space of $(L^2(\Omega))^2$-fields with $L^2(\Omega)$-divergence (see [22])

Let us introduce the sub-space W of $H(\operatorname{div}, \Omega)$ defined by

$$W = \{\mathbf{q} \in H(\operatorname{div}, \Omega),\ \mathbf{q} \,.\, \mathbf{v} = 0 \text{ on } \Gamma_N\} ,$$

the partial normal traces being defined in $H^{\ 1/2}(\Gamma_N)$, which is the dual space of $H^{1/2}_{00}(\Gamma_N)$, we call $H^{1/2}_{00}(\Gamma_N)$ the space of traces on Γ_N of functions from $H^1(\Omega)$ which vanish on $\partial\Omega\backslash\Gamma_N$ Now, the dual mixed formulation of (2) amounts to solving the following problem

(10)
$$\begin{cases} (\mathbf{p}^\varepsilon, u^\varepsilon) \in W \times L^2(\Omega), \quad \forall (\mathbf{q}, v) \in W \times L^2(\Omega) \\ \displaystyle\int_\Omega A_\varepsilon^{-1} \mathbf{p}^\varepsilon \,.\, \mathbf{q} \, d\Omega + \int_\Omega u^\varepsilon \operatorname{div} \mathbf{q} \, d\Omega = 0 , \\ \displaystyle\int_\Omega v \operatorname{div} \mathbf{p}^\varepsilon \, d\Omega = - \int_\Omega f_\varepsilon v \, d\Omega \end{cases}$$

Let $\mathcal{T}_h$ be a regular triangulation of Ω introduced in Section 3 For any $k \in \mathbb{N}$, we denote by $\mathcal{P}_k$ and $\tilde{\mathcal{P}}_k$ the space of polynomials of degree at most k, respectively the space of homogeneous polynomials of degree k Let us recall the definition of the Raviart-Thomas polynomial space of order k

$$RT_k = (\mathcal{P}_k)^2 \oplus \mathbf{r}\tilde{\mathcal{P}}_k,$$

where $\mathbf{x} \mapsto \mathbf{r}(\mathbf{x})$ is the vector field of coordinates (x, y) of point $\mathbf{x}$. We also introduce the following decomposition of the Raviart-Thomas space (see [7]):

$$RT_k = \{\mathbf{q} \in (\mathscr{P}_k)^2;\ \operatorname{div} \mathbf{q} = 0\} \oplus \mathbf{r}\mathscr{P}_k, \tag{11}$$

which will be needed for the numerical solving of the system obtained after hybridization. For any $k \in \mathbb{N}$, one can now define the following finite dimensional spaces:

$$W_h = \{\mathbf{q}_h \in H(\operatorname{div}; \Omega);\ \forall T \in \mathscr{T}_h, \mathbf{q}_{h|T} \in RT_k, \mathbf{q}_h \cdot \mathbf{v} = 0 \text{ on } \Gamma_N\},$$

$$M_h = \{v_h \in L^2(\Omega);\ \forall T \in \mathscr{T}_h, v_{h|T} \in \mathscr{P}_k\}$$

and write down the mixed discrete formulation of (2) for the Raviart-Thomas method of order k:

$$\begin{cases} (\mathbf{p}_h^\varepsilon, u_h^\varepsilon) \in W_h \times M_h, \quad \forall (\mathbf{q}_h, v_h) \in W_h \times M_h \\ \displaystyle\int_\Omega A_\varepsilon^{-1} \mathbf{p}_h^\varepsilon \cdot \mathbf{q}_h \, d\Omega + \int_\Omega u_h^\varepsilon \operatorname{div} \mathbf{q}_h \, d\Omega = 0, \\ \displaystyle\int_\Omega v_h \operatorname{div} \mathbf{p}_h^\varepsilon \, d\Omega = -\int_\Omega f_\varepsilon v_h \, d\Omega. \end{cases} \tag{12}$$

4.2. Existence and uniqueness

For the sake of clarity, let us introduce further notation. We note the bilinear forms which describe the continuous mixed problem by $c_\varepsilon(\cdot, \cdot)$ and $b(\cdot, \cdot)$, where

$$\forall \mathbf{p}, \mathbf{q} \in W, \quad c_\varepsilon(\mathbf{p}, \mathbf{q}) = \int_\Omega A_\varepsilon^{-1} \mathbf{p} \cdot \mathbf{q} \, d\Omega,$$

$$\forall v \in V, \quad \forall \mathbf{q} \in W, \quad b(v, \mathbf{q}) = \int_\Omega v \operatorname{div} \mathbf{q} \, d\Omega.$$

Since $W_h \subset W$ and $M_h \subset L^2(\Omega)$, they are obviously well-defined in the discrete case too. The space $L^2(\Omega)$ is equipped with its classical norm $\|\cdot\|_{0,\Omega}$, while W is endowed with the following weighted norm:

$$\|\mathbf{q}\|_{W,\varepsilon} = (\|\mathbf{q}\|_{0,\Omega,\varepsilon}^2 + \|\operatorname{div} \mathbf{q}\|_{0,\Omega}^2)^{1/2},$$

where for any $\mathbf{q} = (q_1, q_2) \in W$ we set:

$$\|\mathbf{q}\|_{0,\Omega,\varepsilon}^2 = \int_\Omega A_\varepsilon^{-1} \mathbf{q} \cdot \mathbf{q} \, d\Omega = \int_{\Omega^+} \mathbf{q} \cdot \mathbf{q} \, d\Omega^+ + \int_{\Omega^-} [(q_1)^2 + (\varepsilon q_2)^2] \, d\Omega^-. \tag{13}$$

By the means of the Babuška-Brezzi theory (see [22], [8]), the existence and the uniqueness of the solution of the continuous problem (10) are quite obvious. Therefore, we are only interested in the discrete case. Denoting the discrete kernel of $b(\cdot, \cdot)$ by V_h, one can easily see that

$$V_h = \{\mathbf{q}_h \in W_h;\ \operatorname{div} \mathbf{q}_h = 0\}.$$

Next, we remark that the following two conditions hold:

$$\forall \mathbf{q}_h \in V_h, \quad c_\varepsilon(\mathbf{q}_h, \mathbf{q}_h) \geqslant \alpha \| \mathbf{q}_h \|^2_{W, \varepsilon}, \tag{14}$$

$$\inf_{v_h \in M_h} \sup_{\mathbf{q}_h \in W_h} \frac{|b(v_h, \mathbf{q}_h)|}{\| v_h \|_{0, \Omega} \| \mathbf{q}_h \|_{W, \varepsilon}} \geqslant \beta, \tag{15}$$

with strictly positive constante α, β independent on ε or h (in fact, $\alpha = 1$). The relation (15) is the well-known discrete *inf-sup* condition of Babuška-Brezzi, which assures the compatibility between the finite dimensional spaces W_h and M_h. It results easily from standard properties of mixed formulations of 2nd order elliptic problems (see [8], [22] for instance). So, we can now make use of the Babuška-Brezzi theory once again, in order to deduce the existence and uniqueness of the solution of (12).

4.3. Uniform convergence

Let us first remark that $\mathbf{p}^\varepsilon$ is not uniformly bounded in $H^1(\Omega^+ \cup \Omega^-)$ (*cf.* [20]). So, the direct application of classical error estimates given by the mixed methods theory does not yield a uniform bound of the error. In order to elude this difficulty, we shall make use of the limit problem (4). See also [10] for further details and equally for a different approach, *via* nonconforming methods.

More precisely, we establish here the uniform convergence of $(\mathbf{p}_h^\varepsilon, u_h^\varepsilon)$, the unique solution of (12), towards $(\mathbf{p}^\varepsilon, u^\varepsilon)$. To do that, we shall first study the weak limit of $(\mathbf{p}_h^\varepsilon, u_h^\varepsilon)$ as $\varepsilon \to 0$ and $h \to 0$. We need the lemmas below:

LEMMA 4.1: *The next statements hold:*

$$\begin{array}{ll} \exists z \in L^2(\Omega), & \lim\limits_{\substack{\varepsilon \to 0 \\ h \to 0}} u_h^\varepsilon = z \textit{ weakly in } L^2(\Omega), \\ \exists \mathbf{r}^+ \in H(\mathrm{div}; \Omega^+), & \lim\limits_{\substack{\varepsilon \to 0 \\ h \to 0}} \mathbf{p}_h^\varepsilon = \mathbf{r}^+ \textit{ weakly in } H(\mathrm{div}; \Omega^+), \\ \exists r_1^- \in L^2(\Omega^-), & \lim\limits_{\substack{\varepsilon \to 0 \\ h \to 0}} p_{h,1}^\varepsilon = r_1^- \textit{ weakly in } L^2(\Omega^-), \\ \exists r_2^- \in L^2(\Omega^-), & \lim\limits_{\substack{\varepsilon \to 0 \\ h \to 0}} \varepsilon p_{h,2}^\varepsilon = r_2^- \textit{ weakly in } L^2(\Omega^-), \end{array}$$

at least for a sub-sequence of (ε, h).

Proof: Using the first equation of (12), we have:

$$c_\varepsilon(\mathbf{p}_h^\varepsilon, \mathbf{p}_h^\varepsilon) = -b(u_h^\varepsilon, \mathbf{p}_h^\varepsilon) = -\int_\Omega f_\varepsilon u_h^\varepsilon \, d\Omega,$$

which translates into $\| \mathbf{p}_h^\varepsilon \|^2_{0, \Omega, \varepsilon} \leqslant \| f_\varepsilon \|_{0, \Omega} \| u_h^\varepsilon \|_{0, \Omega} \leqslant c \| u_h^\varepsilon \|_{0, \Omega}$, with c a positive constant independent of both ε and h. Next, the *inf-sup* condition (15) gives:

$$\beta \| u_h^\varepsilon \|_{0, \Omega} \leqslant \sup_{\mathbf{q}_h \in W_h} \frac{|b(u_h^\varepsilon, \mathbf{q}_h)|}{\| \mathbf{q}_h \|_{W, \varepsilon}} = \sup_{\mathbf{q}_h \in W_h} \frac{|c_\varepsilon(\mathbf{p}_h^\varepsilon, \mathbf{q}_h)|}{\| \mathbf{q}_h \|_{W, \varepsilon}} \leqslant \| \mathbf{p}_h^\varepsilon \|_{0, \Omega, \varepsilon},$$

so one immediately gets that $\| u_h^\varepsilon \|_{0, \Omega} \leqslant c$ and $\| \mathbf{p}_h^\varepsilon \|_{0, \Omega, \varepsilon} \leqslant c$. Let us also note that

$$\mathrm{div}\, \mathbf{p}_h^\varepsilon = -P_h f_\varepsilon \quad \text{a.e. in } \Omega^+,$$

with P_h the $L^2(\Omega)$-orthogonal projection from $L^2(\Omega)$ to M_h, so that we have:

$$\| \mathbf{p}_h^\varepsilon \|_{w,\varepsilon} \leqslant c .$$

Recalling the definition (13) of the norm $\| \cdot \|_{0,\Omega,\varepsilon}$, we obtain the weak convergence results announced previously, at least for a sub-sequence of (ε, h). $\square$

Let us now introduce the spaces

$$W^+ = \{ \mathbf{q} \in H(\operatorname{div}; \Omega^+); \mathbf{q} \cdot \mathbf{v} = 0 \text{ on } \Sigma \} ,$$

$$W^- = \{ \mathbf{q} \in H(\operatorname{div}; \Omega^-); \mathbf{q} \cdot \mathbf{v} = 0 \text{ on } \Sigma \cup \Gamma_N \}$$

and let us also denote by W_h^+ and W_h^- the corresponding discrete sub-spaces:

$$W_h^+ = \{ \mathbf{q}_h \in W^+; \forall T \in \mathcal{T}_h^+ , \mathbf{q}_{h|T} \in RT_k \} ,$$

$$W_h^- = \{ \mathbf{q}_h \in W^-; \forall T \in \mathcal{T}_h^- , \mathbf{q}_{h|T} \in RT_k \} .$$

With the help of the equilibrium interpolation operator (see [22] for more details), one can prove that the functions of W^+ and W^- are correctly approached by those of W_h^+ and W_h^- respectively. We are now able to establish:

LEMMA 4.2: *We have the following relations:*

$$z \in V_0, \quad \partial_v z = 0 \text{ on } \Gamma_N,$$

$$\mathbf{r}^+ = \mathbf{grad}\, z \quad \text{a.e. in } \Omega^+,$$

$$r_1^- = \partial_x z \quad \text{a.e. in } \Omega^- .$$

Proof: Let $\mathbf{q}$ be any given element of W^+. There exists then a sequence $\mathbf{q}_h \in W_h^+$ such that $\lim_{h \to 0} \mathbf{q}_h = \mathbf{q}$, strongly in $H(\operatorname{div}; \Omega^+)$. Defining $\tilde{\mathbf{q}}_h$ by $\tilde{\mathbf{q}}_h = \mathbf{q}_h$ in Ω^+ and $\tilde{\mathbf{q}}_h = 0$ in Ω^-, we clearly have that $\tilde{\mathbf{q}}_h \in W_h$. So, considering the test-function $\tilde{\mathbf{q}}_h$ in the first equation of (12), one gets:

$$\int_{\Omega^+} \mathbf{p}_h^\varepsilon \cdot \mathbf{q}_h \, d\Omega^+ + \int_{\Omega^+} u_h^\varepsilon \operatorname{div} \mathbf{q}_h \, d\Omega^+ = 0 . \tag{16}$$

Next, passing to the limit when $\varepsilon \to 0$, $h \to 0$ in (16) gives

$$\forall \mathbf{q} \in W^+, \quad \int_{\Omega^+} \mathbf{r}^+ \cdot \mathbf{q} \, d\Omega^+ + \int_{\Omega^+} z \operatorname{div} \mathbf{q} \, d\Omega^+ = 0 .$$

This relation implies that $\mathbf{r}^+ = \operatorname{grad} z$ in Ω^+ in the sense of distributions; since $\mathbf{r}^+ \in (L^2(\Omega^+))^2$, it comes that $\mathbf{r}^+ = \operatorname{grad} z$ a.e. in Ω^+. By the means of the Green's formula, we equally obtain that $z = 0$ on $\partial\Omega^+ \cap \Gamma_D$.

A similar argument in the domain Ω^- and a passage to the limit in the relation

$$\forall \mathbf{q}_h \in W_h^- , \quad \int_{\Omega^-} A_\varepsilon^{-1} \mathbf{p}_h^\varepsilon \cdot \mathbf{q}_h \, d\Omega^- + \int_{\Omega^-} u_h^\varepsilon \operatorname{div} \mathbf{q}_h \, d\Omega^- = 0$$

yields the following statement:

$$\forall \mathbf{q} \in W^-, \quad \int_{\Omega^-} r_1^- \, q_1 \, d\Omega^- + \int_{\Omega^-} z \operatorname{div} \mathbf{q} \, d\Omega^- = 0 .$$

With the notation $\mathbf{r}^- = (r_1^-, 0)$, we deduce by a similar reasoning as above that:

$$\mathbf{r}^- = \mathbf{grad}\, z \text{ a.e. in } \Omega^-, \quad z = 0 \text{ on } \partial\Omega^- \cap \Gamma_D, \quad \partial_\nu z = 0 \text{ on } \Gamma_N .$$

We already know that $z_{|\Omega^+} \in H^1(\Omega^+)$, $z_{|\Omega^-} \in H^1(\Omega^-)$; we still have to show that $z \in V_0$.

Since the functions of W_h approach correctly those of W, for any $\mathbf{q} \in W$ there exists a sequence $(\mathbf{q}_h)_{h>0} \subset W_h$ such that $\lim_{h \to 0} \mathbf{q}_h = \mathbf{q}$ strongly in $H(\operatorname{div}; \Omega)$. So, when ε and h tend to zero we obtain from the first equation of (12):

$$\forall \mathbf{q} \in W, \quad \int_{\Omega^+} \operatorname{grad} z \cdot \mathbf{q} \, d\Omega^+ + \int_{\Omega^-} \partial_x z q_1 \, d\Omega^- + \int_{\Omega^+} z \operatorname{div} \mathbf{q} \, d\Omega^+ + \int_{\Omega^-} z \operatorname{div} \mathbf{q} \, d\Omega^- = 0 .$$

Thus Green's formula on Ω^+ and Ω^- respectively (and taking $\mathbf{q} \in W$ with $\mathbf{q} \cdot \mathbf{v} \in L^2(\partial\Omega)$, for instance) directly leads to:

$$\int_{\Sigma} \mathbf{q} \cdot \mathbf{v}(z^+ - z^-) \, ds = 0 .$$

As a consequence, $z^+ = z^-$ on Σ and so $z \in V_0$. This concludes the proof. □

LEMMA 4.3: *We have:*

$$z = u^0 ,$$

where u^0 is the solution of the Ventcel problem (4).

Proof: Let us consider any $v \in V_0$. Since $\mathbf{p}_h^\varepsilon \in W_h \subset H(\operatorname{div}; \Omega)$, we have that:

$$\int_{\Sigma} v((\mathbf{p}_h^\varepsilon)^+ \cdot \mathbf{v} - (\mathbf{p}_h^\varepsilon)^- \cdot \mathbf{v}) \, ds = 0 .$$

Once again, the Green's formula gives:

$$\int_{\Omega^+} \mathbf{p}_h^\varepsilon \cdot \mathbf{grad}\, v \, d\Omega^+ + \int_{\Omega^+} v \operatorname{div} \mathbf{p}_h^\varepsilon \, d\Omega^+ + \int_{\Omega^-} \mathbf{p}_{h,1}^\varepsilon \, \partial_x v \, d\Omega^- + \int_{\Omega^-} v \operatorname{div} \mathbf{p}^h \, d\Omega^- = 0 .$$

When ε and h both tend towards zero, we obtain in fact the variational formulation of (4):

$$\forall v \in V_0, \quad \int_{\Omega^+} \operatorname{grad} z \cdot \operatorname{grad} v \, d\Omega^+ - \int_{\Omega^+} f^+ v \, d\Omega^+ + \int_{\Omega^-} \partial_x z \, \partial_x v \, d\Omega^- = 0 .$$

Since (4) has a unique solution u^0, we clearly get that $z = u^0$. As a consequence, the weak convergences established in Lemma 4.1 hold for all the sequence (ε, h). □

We are now able to prove the following theorem, which is the key part of this paragraph.

THEOREM 4.4: *For $(f^+, f^-) \in H^1(\Omega^+) \times H^1(\Omega^-)$, we have:*

$$\lim_{h \to 0} \sup_{\varepsilon \in]0,1[} \left(\| \mathbf{p}^\varepsilon - \mathbf{p}_h^\varepsilon \|_{W,\varepsilon} + \| u^\varepsilon - u_h^\varepsilon \|_{0,\Omega} \right) = 0 .$$

Proof: Let us first notice that, because of the uniqueness of the limit of weakly convergent sub-sequences of $(\mathbf{p}_h^\varepsilon, u_h^\varepsilon)$ (according to Lemma 4.2 and Lemma 4.3), the convergences established in Lemma 4.1 hold for the whole sequence (ε, h). Using the definition of $\| \cdot \|_{0,\Omega,\varepsilon}$ and the formulation of the mixed problem, we can next write:

$$\| \mathbf{p}^\varepsilon - \mathbf{p}_h^\varepsilon \|_{0,\Omega,\varepsilon}^2 = \int_\Omega f_\varepsilon (u^\varepsilon + u_h^\varepsilon)\, d\Omega - 2 \int_\Omega A_\varepsilon^{-1} \mathbf{p}^\varepsilon \cdot \mathbf{p}_h^\varepsilon \, d\Omega .$$

Passing to the limit as $\varepsilon \to 0$, $h \to 0$, the results of the preceding lemmas lead to:

$$\lim_{\substack{\varepsilon \to 0 \\ h \to 0}} \| \mathbf{p}^\varepsilon - \mathbf{p}_h^\varepsilon \|_{0,\Omega,\varepsilon} = 0 .$$

Since

$$\| \operatorname{div} (\mathbf{p}^\varepsilon - \mathbf{p}_h^\varepsilon) \|_{0,\Omega} = \| f_\varepsilon - P_h f_\varepsilon \|_{0,\Omega} ,$$

by taking $(f^+, f^-) \in H^1(\Omega^+) \times H^1(\Omega^-)$ we easily obtain:

$$\lim_{\substack{\varepsilon \to 0 \\ h \to 0}} \| \mathbf{p}^\varepsilon - \mathbf{p}_h^\varepsilon \|_{W,\varepsilon} = 0 . \tag{17}$$

Concerning the primal unknown u^ε, we get by classical estimates of mixed methods theory that

$$\| u^\varepsilon - u_h^\varepsilon \|_{0,\Omega} \leqslant \left(1 + \frac{1}{\beta} \right) \inf_{v_h \in M_h} \| u^\varepsilon - v_h \|_{0,\Omega} + \frac{1}{\beta} \| \mathbf{p}^\varepsilon - \mathbf{p}_h^\varepsilon \|_{0,\Omega,\varepsilon} ,$$

and since u^ε is bounded in $H^1(\Omega)$ the above relation implies:

$$\lim_{\substack{\varepsilon \to 0 \\ h \to 0}} \| u^\varepsilon - u_h^\varepsilon \|_{0,\Omega} = 0 . \tag{18}$$

Let now fix any $\varepsilon_0 \in \,]0, 1]$. Classical error's estimates for mixed formulations (see [8], [22]) then give, for any $\varepsilon \in [\varepsilon_0, 1]$:

$$\begin{aligned} \| \mathbf{p}^\varepsilon - \mathbf{p}_h^\varepsilon \|_{W,\varepsilon} + \| u^\varepsilon - u_h^\varepsilon \|_{0,\Omega} &\leqslant c \left(\inf_{\mathbf{q}_h \in W_h} \| \mathbf{p}^\varepsilon - \mathbf{q}_h \|_{W,\varepsilon} + \inf_{v_h \in M_h} \| u^\varepsilon - v_h \|_{0,\Omega} \right) \\ &\leqslant ch(|\mathbf{p}^\varepsilon|_{1,\Omega^+} + |\mathbf{p}^\varepsilon|_{1,\Omega^-} + |u^\varepsilon|_{1,\Omega}) , \end{aligned}$$

with c a constant independent on ε or h. The last inequality can be obtained, for instance, by the means of equilibrium, respectively Lagrange interpolation. Using now that $\mathbf{p}^\varepsilon = A_\varepsilon \operatorname{grad} u^\varepsilon$ and that u^ε is uniformly bounded in $H^2(\Omega^+ \cup \Omega^-)$, *cf.* [20], it comes that:

$$|\mathbf{p}^\varepsilon|_{1,\Omega^+} + |\mathbf{p}^\varepsilon|_{1,\Omega^-} + |u^\varepsilon|_{1,\Omega} \leqslant \frac{1}{\varepsilon_0^2} \| u^\varepsilon \|_{2,\Omega} \leqslant c(\varepsilon_0) .$$

Therefore, we have:

$$\lim_{h \to 0} \sup_{\varepsilon \in [\varepsilon_0, 1]} (\| \mathbf{p}^\varepsilon - \mathbf{p}_h^\varepsilon \|_{W,\varepsilon} + \| u^\varepsilon - u_h^\varepsilon \|_{0,\Omega}) = 0 .$$

Together with (17) and (18), this yields the announced result. □

REMARK 4 1 This result gives the uniform convergence of the mixed method The method is thus robust in the way described by Babuška and Suri in [4] Moreover, one can see that the proof is valid for any order $k \in \mathbb{N}$ of the method □

REMARK 4 2 Taking into account Theorem 2 1 and Theorem 4 4, one can see that we have also proved the strong convergence of $(\mathbf{p}_h^\varepsilon, u_h^\varepsilon)$ towards $(\mathbf{p}^0, u^0)$, for the norms $\| \cdot \|_{0\,\Omega\,\varepsilon}$ and $\| \cdot \|_{0\,\Omega}$ respectively

$$\lim_{\substack{\varepsilon \to 0 \\ h \to 0}} \left(\| \mathbf{p}^0 - \mathbf{p}_h^\varepsilon \|_{0\,\Omega\,\varepsilon} + \| u^0 - u_h^\varepsilon \|_{0\,\Omega} \right) = 0$$

We recall that $\mathbf{p}^0$ is given by the relations below

$$\mathbf{p}^0 = \mathbf{grad}\, u^0 \text{ in } \Omega^+, \quad \mathbf{p}^0 = (\partial_x u^0, 0) \text{ in } \Omega^-$$

Notice that $\mathbf{p}^0 \notin H(\operatorname{div}, \Omega)$, since there is no continuity of the normal traces on Σ, this explains why the above convergence of $\mathbf{p}_h^\varepsilon$ takes place only for a weighted norm $\| \cdot \|_{0\,\Omega\,\varepsilon}$ of $(L^2(\Omega))^2$ □

4.4. Mixed hybrid method

The mixed formulation (12) leads to a linear system whose matrix is not definite positive A direct resolution by the means of Uzawa's algorithm cannot be applied, since the resulting matrix is ill conditioned for small values of ε A solution is then to introduce Lagrangian multipliers, in order to relax the continuity constraint contained in the definition of W_h, see [8] or [22] This idea has also been analyzed in [3]

To this end, we need further notation For any $T \in \mathcal{T}_h$, we denote by T' any edge of the triangle T, $\mathcal{E}_h$ then denotes the set-wise non-disjoint reunion of all edges of $\mathcal{T}_h$ Making an abuse of notation, we will write that

$$L^2(\mathcal{E}_h) = \prod_{T' \in \mathcal{E}_h} L^2(T')$$

We can define L_h and $W_{h\,-1}$ by

$$L_h = \{ \lambda_h \in L^2(\mathcal{E}_h), \forall T' \in \mathcal{E}_h, \lambda_{h|T'} \in \mathcal{P}_k \text{ and } \lambda_h = 0 \text{ on } \Gamma_D \},$$

$$W_{h\,-1} = \{ \mathbf{q}_h \in (L^2(\Omega))^2, \forall T \in \mathcal{T}_h, \mathbf{q}_{h|T} \in RT_k \}$$

The key property of $W_{h\,-1}$ is that there is no coupling constraint on its elements The new multipliers λ_h^ε now join the formulation and describe the coupling of the normal component of $\mathbf{p}_h^\varepsilon$ on the internal edges

$$(19) \quad \begin{cases} \forall T \in \mathcal{T}_h, \quad \forall \mathbf{q}_T \in RT_k(T), \\ \displaystyle\int_T A_\varepsilon^{-1} \mathbf{p}_h^\varepsilon \cdot \mathbf{q}_T \, dT + \int_T u_h^\varepsilon \operatorname{div} \mathbf{q}_T \, dT - \sum_{T' \in T} \int_{T'} \lambda_h^\varepsilon \mathbf{q}_T \cdot \mathbf{v} \, dT' = 0, \\ \forall T \in \mathcal{T}_h, \quad \forall v_T \in \mathcal{P}_k(T), \quad \displaystyle\int_T v_T \operatorname{div} \mathbf{p}_h^\varepsilon \, dT = -\int_T f_\varepsilon v_T \, dT, \\ \forall \mu_h \in L_h, \quad \displaystyle\sum_{T \in \mathcal{T}_h} \sum_{T' \in T} \int_{T'} \mu_h \mathbf{p}_h^\varepsilon \cdot \mathbf{v} \, dT' = 0 \end{cases}$$

Recalling the results established in [7], the decomposition of the Raviart-Thomas polynomial space (11) allows us to choose test-functions $\mathbf{q}_T$ such that $\operatorname{div} \mathbf{q}_T = 0$ on $T \in \mathcal{T}_h$. This, in turn, leads to a symmetric definite positive system which is assembled from elementary matrices computed at the element level. More specifically, this method can be viewed as a non-conforming pseudo-method of order $k + 1$ (see [3] or [7] for further details).

4.5. Numerical results

We solve (19) through the lowest-order mixed method, that is for $k = 0$, over the mesh of *figure 5*. The degrees of freedom of the final system are the nodal values at the mid-point of the internal edges of the mesh. On each edge submitted to a Dirichlet constraint, the value of λ_h^ε is set to zero.

The multipliers λ_h^ε approximate the values of u^ε at the mid-point of the edges of the triangulation. A simple post-process enables us to recover $\mathbf{p}_h^\varepsilon$ and u_h^ε. This is a local post-process, i.e. it is applied at the element level.

For practical reasons, we choose to show only the approximated values of u_h^ε at the vertices of the triangulation, obtained after a second post-process. The corresponding three-dimensional representation (for $\varepsilon = 10^{-6}$ and for the same charges (f^+, f^-) as in *fig. 4* and *fig. 8*) is given below by *figure 9*.

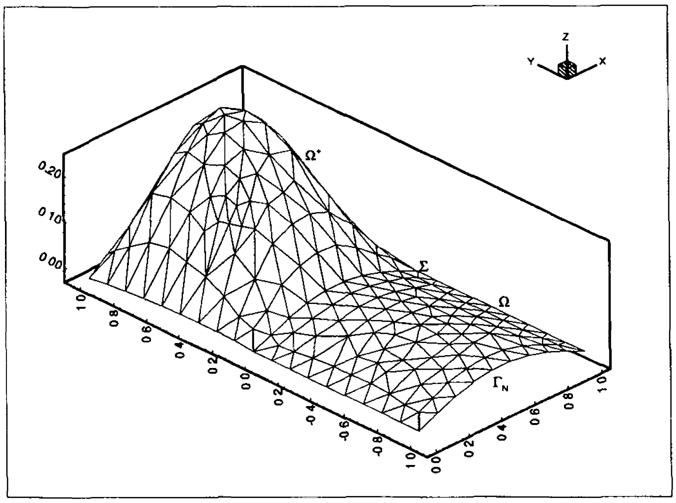

Figure 9. — Three dimensional representation of u_h^ε ($\varepsilon = 10^{-6}$).

The *figure 10* gives the rate of convergence of $u_h^\varepsilon - u^\varepsilon$ when computed over an arbitrary mesh of the type given in *figure 5*, while *figure 11* represents the rate of convergence of $\mathbf{p}_h^\varepsilon - \mathbf{p}^\varepsilon$ for the energy norm $\| \cdot \|_{W, \varepsilon}$. In both cases, the rates of convergence are optimal: $O(N^{-1/2}) = O(h)$. Moreover, we obtain identical curves for ε between 10^{-1} and 10^{-6}: indeed, the $L^2(\Omega)$-error on u^ε as well as the W-error on $\mathbf{p}^\varepsilon$ present very little sensitiveness to the variation of the parameter ε, which finally give the same significant digits of their logarithms.

As a conclusion, the lowest-order mixed method is *loking-free* over any regular triangulation.

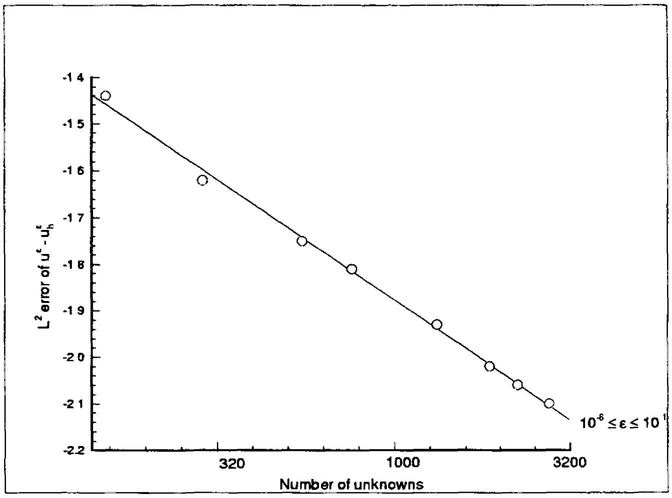

Figure 10. — Convergence rates for u_h^ε in $L^2(\Omega)$-norm.

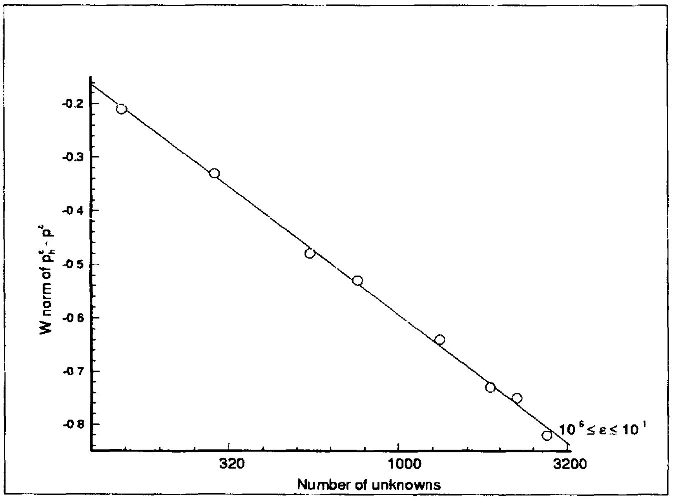

Figure 11. — Convergence rates for p_h^ε in W-norm.

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