G. TOSCANI

The grazing collisions asymptotics of the non cut-off Kac equation

RAIRO — Modélisation mathématique et analyse numérique, tome 32, nº 6 (1998), p. 763-772.

<http://www.numdam.org/item?id=M2AN_1998__32_6_763_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 32, n° 6, 1998, p. 763 à 772)

THE GRAZING COLLISIONS ASYMPTOTICS OF THE NON CUT-OFF KAC EQUATION (*)

G. TOSCANI

Abstract. — *We investigate the non cut-off Kac equation when the cross-section is concentrating on grazing collisions. We prove that this process leads to the one-dimensional Fokker-Planck equation, and that, at last under suitable regularity of the initial data, the convergence is uniform in time. © Elsevier, Paris*

Résumé. — *On s'intéresse à l'équation de Kac « non cut-off » lorsque la section transversale se concentre sur les collisions rasantes. On montre que l'on aboutit à l'équation de Fokker-Planck en dimension 1 d'espace, et que sous des hypothyses de régularité de la donnée initiale, il y a convergence uniforme en temps. © Elsevier, Paris*

1. INTRODUCTION

The non cut-off Kac equation [6] is a model for a single molecule motion in a pseudomolecules like-bath confined to move on a straight line. According to this model the numerical density evolution of molecular velocities $v \in \mathbf{R}$ at time $t \geq 0$ is determined by the Boltzmann problem solution

$$\frac{\partial f(v,t)}{\partial t} = \int_{\mathbf{R} \times [-\pi, \pi]} \beta(|\theta|) \, [f(v^*, t) f(w^*, t) - f(v,t) f(w,t)] \, dw \, d\theta \,. \tag{1.1}$$

In (1.1) (v^*, w^*) are the post-collisional velocities, given by

$$v^* = v \cos\theta - w \sin\theta, \quad w^* = v \sin\theta + w \cos\theta \tag{1.2}$$

while the function $\beta(\mu)$ has a singularity of the form $\mu^{-\alpha}$ when $\mu \to 0^+$ and $1 < \alpha < 3$.

The kernel β was introduced by Desvillettes by analogy with the non cutoff kernel of the Boltzmann equation [3] to understand, at least in this "simple" case, if the singularity, destroying at each positive time the memory of the initial density, introduces regularizing effects on the solution. It is a fact that in the classical Kac's caricature of a Maxwell gas [10], where $\beta(\mu) = 1/(2\pi)$, the solution to the initial value problem keeps at best the regularity of the initial density.

One of the reasons of assuming such a conjecture is that an asymptotics of the Boltzmann equation when the cross-section is concentrating on the grazing collisions (these collisions are those that are neglected when the cut-off assumption is made) leads to the Fokker-Planck-Landau equation, which is known to induce regularizing effects (or at least compactness properties, even in the spatially inhomogeneous case [11]).

At time when Desvillettes wrote his paper, few results on this asymptotic equivalence were available [5], [4]. In additions, these papers were only concerned with formal results, proving that the Boltzmann collision operator reduces to the Landau one for a smooth density.

(*) Manuscript received April 17, 1997
This work has been partially supported by I.A.N.-C.N.R., Pavia (Italy).
Department of Mathematics, University of Pavia via Abbiategrasso 209, 27100 Pavia, Italy.

In the last two years, a lot of progresses have been made on the subject, mainly due to Arsen'ev and Buryak [1], Villani [16], [17] and Goudon [9]. In particular, in [17] the author shows how to treat the asymptotics of grazing collisions in a frame consistent with the existence theorems on the Boltzmann equation without cut-off. Villani's technique covers also the Coulomb case, thus giving a first rigorous basis to the work of Degond and Lucquin [4].

Despite the motivations quoted by Desvillettes to introduce the non cut-off Kac equation, the asymptotic problem for this equation remains untouched.

In this note we prove that in the grazing collisions limit, the target equation is the classical Fokker-Planck equation. Moreover, at least when the initial density satisfies some regularity hypotheses, and possesses a sufficiently high number of moments, the solution to the non cut-off Kac equation converges towards the solution to the Fokker-Planck equation uniformly in time, and in a strong sense (related to the regularity of the initial density). In addition, we obtain explicit bounds on the distance in terms of the small parameter involved in the limit procedure.

The results of the present paper depend deeply on the fact that in this case we can consider the Fourier transform of the Boltzmann equation, and we have the possibility of particularly exact computations. The same idea is at the basis of some recent results of Gabetta, Toscani and Wennberg [8], and of Gabetta and Pareschi [7]. In the former paper the authors obtain exponential convergence to equilibrium for Kac equation and for the Boltzmann equation for Maxwell molecules with or without cut-off in a norm equivalent to weak-* convergence of measures. In the latter, convergence to equilibrium for Kac equation with or without cut-off in various Sobolev spaces is discussed. More recently, Carlen, Gabetta and Toscani [2], always using the Fourier transform version of the Boltzmann equation, obtained a sharp bound on the rate of exponential convergence to equilibrium for the same equations with cut-off in a weak norm. These results were then combined, using interpolation inequalities, to obtain the optimal rate of convergence in the strong L^1-norm, as well as various Sobolev norm.

2. THE NON CUT-OFF KAC EQUATION AND ITS FOURIER TRANSFORM

In this section we collect results on the non cut-off Kac equation. Most of these results are published, and we refer to them only shortly. Some detail will be added when necessary. First of all we quote the following

THEOREM 2.1 [6]: *Let $\varphi \geq 0$ be an initial datum such that, for some $r \in \mathbf{N}$,*

$$\int_{\mathbf{R}} (1 + v^{2r} + |\log f_0(v)|)\, \varphi(v)\, dv < +\infty \tag{2.1}$$

and let $\beta(\mu)$ be a cross-section satisfying the following property on $(0, \pi]$

$$\beta_0 \mu^{-\alpha} \leq \beta(\mu) \leq \beta_1 \mu^{-\alpha} \tag{2.2}$$

for some constants $0 < \beta_0 < \beta_1$ and $\alpha \in (1, 3)$. Then, there exists a nonnegative solution $f(v,t) \in L^{\infty}([0, +\infty); L^1(\mathbf{R}))$ to eq. (1.1) with initial datum φ in the following sense:

For all functions $\phi \in W^{2,\infty}(\mathbf{R})$ we have

$$\frac{\partial}{\partial t} \int_{\mathbf{R}} f(v,t)\, \phi(v)\, dv = \int_{\mathbf{R}} \int_{\mathbf{R}} K^{\phi}(v,w) f(v,t) f(w,t)\, dv\, dw \tag{2.3}$$

where

$$K^{\phi}(v,w) = \int_{-\pi}^{\pi} \{\phi(w) - \phi(v)\}\, \beta(|\theta|)\, d\theta\,. \tag{2.4}$$

The solution conserves the mass,

$$\int_{\mathbf{R}} f(v,t)\,dv = \int_{\mathbf{R}} \varphi(v)\,dv\,.$$

Moreover, if in (2.1) $r \geqslant 2$, also the energy is conserved

$$\int_{\mathbf{R}} v^2 f(v,t)\,dv = \int_{\mathbf{R}} v^2\,\varphi(v)\,dv\,,$$

and we can find a constant C_r such that, for all $t \geqslant 0$

$$\int_{\mathbf{R}} (1 + v^{2r}) f(v,t)\,dv < C_r\,. \tag{2.5}$$

Finally, if assumption (2.1) holds with $r \geqslant 2$, for all $\bar{t} > 0$ and all $\nu > 0$ we have

$$f(v,t) \in L^\infty([\bar{t}, +\infty)\,; H^{2r-1/2-\nu}(\mathbf{R}))$$

Desvillettes's theorem shows that the solution to eq. (1.1) gains regularity, and links this gain to the number of moments that are initially finite. The uniqueness of the solution to eq. (1.1) was proved subsequently by Gabetta and Pareschi [7], by means of Tanaka's functional. There, the authors were mainly interested in the problem of convergence to equilibrium in Sobolev spaces. For this reason, they were looking at uniform bounds on Sobolev norms of the solution (these bounds are not available in consequence of Theorem 2.1). Since we will need such type of bounds, let us briefly recall the method, which has been developed first by Lions and Toscani in their proof of the central imit theorem [12].

In the sequel, let $f(v) \geqslant 0$ denote a function with unit mass and energy σ. Let us introduce the convex functionals

$$L_k(f) = \int_{\mathbf{R}} \frac{[f'(v)]^{2k}}{f(v)^{2k-1}}\,dv.\,; \quad k \geqslant 1 \tag{2.6}$$

and

$$J_k(f) = \int_{\mathbf{R}} \frac{[f^{(k)}(v)]^2}{f(v)}\,dv.\,; \quad k \geqslant 1 \tag{2.7}$$

We remark that when $k = 1$ they coincide with Fisher's measure of information. The meaning and relevance of the aforementioned functionals in connection with the propagation of regularity is contained into the next results.

THEOREM 2.2 [12]: *Let $n > 1$. There exists a constant c_n such that, for all densities f with finite energy and $k \leqslant n$*

$$\int_{\mathbf{R}} \left|\frac{f^{(k)}}{f}\right|^p f\,dv \leqslant c_n [L_n(f) + J_n(f)] \tag{2.8}$$

whenever $kp = 2n$.

Theorem 2.2 has a simple consequence.

THEOREM 2.3: *Let $f \geqslant 0$ be a density of bounded energy, such that for some $n \geqslant 1$ $L_n(f)$ and $J_n(f)$ are finite. Then $f \in H^n(\mathbf{R})$, and, for $0 \leqslant k \leqslant n$ we have*

$$\int_{\mathbf{R}} [f^{(k)}]^2 \, dv \leqslant \{c_n [L_n(f) + J_n(f)]\}^{\frac{2k+1}{2n}} \tag{2.9}$$

Proof: Given $k \leqslant n$,

$$\int_{\mathbf{R}} [f^{(k)}]^2 \, dv \leqslant \|f\|_\infty \int_{\mathbf{R}} \left|\frac{f^{(k)}}{f}\right|^2 f \, dv$$

Since $k \leqslant n$, we can apply Hölder inequality and (2.8) to obtain

$$\int_{\mathbf{R}} \left|\frac{f^{(k)}}{f}\right|^2 f \, dv \leqslant \left[\int_{\mathbf{R}} \left|\frac{f^{(k)}}{f}\right|^{2n/k} f \, dv\right]^{k/n} \{c_n [L_n(f) + J_n(f)]\}^{\frac{k}{n}}$$

Finally

$$\|f\|_\infty \leqslant \int_{\mathbf{R}} \left|\frac{f'}{f}\right|^2 f \, dv$$

and the result follows.

The last result we quote deals with the time growth of the functionals L_k and J_k.

THEOREM 2.4 [7]: *Let $f(v, t)$ be the unique solution to eq. (1.1), where $\varphi \geqslant 0$ satisfies (2.1) for some $r \geqslant 2$, and β satisfies (2.2). Then, if for some $k \geqslant 1$ $L_k(f_0)$ and $J_k(f_0)$ are finite, $L_k(f(t))$ and $J_k(f(t))$ are finite for all $t \geqslant 0$, and there exists constants A_k, $B_k \geqslant 0$ such that the following bounds hold*

$$L_1(f(t)) = J_1(f(t)) = L(f(t)) \leqslant L(\varphi) \tag{2.10}$$

$$\begin{aligned} L_k(f(t)) &\leqslant M_k(\varphi) = \max\{L_k(\varphi), A_k [M_{k-1}(\varphi)]^{2k/(2k-2)}\}\,; \qquad k > 1 \\ J_k(f(t)) &\leqslant N_k(\varphi) = \max\left\{J_k(\varphi), B_k \max_{j \leqslant k-1} N_j(\varphi) \, N_{k-j}(\varphi)\right\}; \qquad k > 1 \end{aligned} \tag{2.11}$$

We can now pass to the Fourier transform of eq. (1.1). The Fourier transform of $f(v, t)$ is

$$\hat{f}(\xi, t) = \int_{\mathbf{R}} e^{-iv\xi} f(v, t) \, dv \,. \tag{2.17}$$

Since $v \to e^{-iv\xi}$ lies in $W^{2,\infty}(\mathbf{R}_v)$, it is possible to use eq. (2.3). Then, a simple calculation leads to the following equation for $\hat{f}(\xi, t)$

$$\frac{\partial \hat{f}(\xi, t)}{\partial t} = \int_{-\pi}^{\pi} \beta(|\theta|) \, [\hat{f}(\xi \cos\theta, t) \hat{f}(\xi \sin\theta, t) - \hat{f}(0, t) \hat{f}(\xi, t)] \, d\theta \,. \tag{2.18}$$

which is in certain respects much simpler than the equation for $f(v,t)$, being the collision operator less complicated. The conservation of mass and energy for f translates to

$$\hat{f}(0,t)=1, \qquad \|\hat{f}(\,.\,,t)\|_\infty \leqslant 1$$
$$\hat{f}''_{\xi\xi}(0,t)=-\sigma \quad \|\hat{f}''_{\xi\xi}(\,.\,,t)\|_\infty \leqslant \sigma\,. \tag{2.19}$$

3. THE FOKKER-PLANCK EQUATION AND THE PROPAGATION OF REGULARITY

In this short section we will deal with the one-dimensional in space Fokker-Planck equation

$$\frac{\partial f(v,t)}{\partial t}=\frac{\partial}{\partial v}\left[vf(v,t)+\sigma\frac{\partial f(v,t)}{\partial v}\right]. \tag{3.1}$$

A simple calculation leads to the following equation for the Fourier transform $\hat{f}(\xi,t)$

$$\frac{\partial \hat{f}(\xi,t)}{\partial t}=-\sigma\xi^2\hat{f}(\xi,t)-\xi\hat{f}'_{\xi}(\xi,t)\,. \tag{3.2}$$

Let $\varphi(v)$ be any density on $\mathbf{R}$ with unit mass and energy σ. Let X be any random variable with this density, and let W be any independent Gaussian random variable with density M_σ given by

$$M_\sigma(v)=(2\pi\sigma)^{-n/2}\exp\left\{-\frac{v^2}{2\sigma}\right\}. \tag{3.3}$$

For every $t>0$ define

$$Z_t=e^{-t}X+(1-e^{-2t})^{1/2}W\,. \tag{3.4}$$

Then the random variable Z_t has a density $f(v,t)$ at each t, and it is well-known that $f(v,t)$ is evolved from φ under the action of the adjoint Ornstein-Uhlenbeck semigroup. Therefore $f(v,t)$ satisfies equation (3.1), which can of course be checked directly from the definition.

For any $\alpha>0$, we set $f_\alpha(v)=\alpha^{-1/2}f(\alpha^{-1/2}v)$. Then, $f(v,t)$ is expressed by the convolution formula

$$f(v,t)=\varphi_{\alpha(t)}\star M_{\sigma\beta(t)} \tag{3.5}$$

where $\alpha(t)=e^{-2t}$, $\beta(t)=1-e^{-2t}$.

The corresponding of Theorem 2.2 is a consequence of (3.5) and of the properties of the functionals L_k and J_k. We have

THEOREM 3.1: *Let $f(v,t)$ be the unique solution to eq. (3.1), where the initial value $\varphi\geqslant 0$ has unit mass and energy σ. Then, if for some $k\geqslant 1$ $L_k(\varphi)$ and $J_k(\varphi)$ are finite, $L_k(f(t))$ and $J_k(f(t))$ are finite for all $t\geqslant 0$, and the following bounds hold*

$$\begin{aligned} L_k(f(t)) &\leqslant \max\{e^{2k}L_k(\varphi)\,;\,(1-e^{-2})^{-k}L_k(M_\sigma)\}\\ J_k(f(t)) &\leqslant \max\{e^{2k}J_k(\varphi)\,;\,(1-e^{-2})^{-k}J_k(M_\sigma)\}\,. \end{aligned} \tag{3.6}$$

Proof: Since both φ and M_a have unit mass, by Lemma 2.1 of Lions and Toscani, [12] for any $k\geqslant 1$

$$L_k(\varphi_{\alpha(t)}\star M_{\sigma\beta(t)})\leqslant \min\{L_k(\varphi_{\alpha(t)})\,;\,L_k(M_{\sigma\beta(t)})\}\,.$$

Hence, we can bound $L_k(f(t))$ with $L_k(\varphi_{\alpha(t)})$ in the time interval $t \geq 1$, and with $L_k(M_{\sigma\beta(t)})$ whenever $t > 1$. Since $L_k(f_\alpha) = \alpha^{-k} L_k(f)$, the first bound in (3.6) follows. The same argument can be used for J_k.

4. THE GRAZING COLLISION LIMIT

We can now make precise assumptions on the asymptotics of the grazing collisions, namely in letting the kernel β concentrate on the singularity $\theta = 0$. We will introduce a family of kernels $\{\beta_\epsilon(|\theta|)\}_{\epsilon > 0}$ satisfying the following hypotheses

i) For any given $\epsilon > 0$, β_ϵ satisfies (2.2).

ii) $\beta_\epsilon(|\theta|)\,\theta^2 \in L^1([0, \pi)]$, and

$$\lim_{\epsilon \to 0^+} \int_0^\pi \beta_\epsilon(|\theta|)\, d\theta = 1$$

i) and ii) can be obtained in several ways, for example taking, for $0 < \nu < 1$

$$\beta_\epsilon(|\theta|) = \frac{1-\nu}{\epsilon|\theta|^{2+\nu}} \quad \theta \in (0, \epsilon^{1/(1-\nu)})$$

$$\beta_\epsilon(|\theta|) = \frac{(1-\nu)\,\epsilon}{|\theta|^{2+\nu}} \quad \theta \in (\epsilon^{1/(1-\nu)}, (\pi)) .$$

We prove now our main result.

THEOREM 4.1: *Let $0 \leq \varphi$ be an initial datum with unit mass, zero mean velocity, energy σ, satisfying hypothesis (2.1) with $r \geq 2$. Moreover let $L_3(\varphi)$ and $J_3(\varphi)$ be finite. Let $\{\beta_\epsilon\}$ be a sequence of kernels concentrating to zero and satisfying hypotheses i) and ii).*

Then, for all $t > 0$ and $\delta > 0$ the solution $f_\epsilon(\,.\,, t)$ to the non cut-off Kac equation with initial value φ converges to $f(\,.\,, t)$, solution to the Fokker-Planck equation (3.1) with the same initial value strongly in $W^{3-\delta,2}(\mathbf{R})$. The convergence is uniform in time, and the following bound holds

$$\|f_\epsilon(\,.\,, t) - f(\,.\,, t)\|_{W^{3-\delta,2}} \leq \phi(\epsilon, \delta, \varphi)$$

$$\lim_{\epsilon \to 0^+} \phi(\epsilon, \delta, \varphi) = 0 . \tag{4.1}$$

The function ϕ can be computed explicitly.

Proof: Let $\varphi(v)$ satisfy the hypotheses of the theorem. Then, given $\epsilon > 0$, by Theorem 2.1 we conclude that the non cut-off Kac equation

$$\frac{\partial f_\epsilon(v, t)}{\partial t} = \int_{\mathbf{R} \times [-\pi, \pi]} \beta_\epsilon(|\theta|)\, [f_\epsilon(v^*, t) f_\epsilon(w^*, t) - f_\epsilon(v, t) f_\epsilon(w, t)]\, dw\, d\theta \tag{4.2}$$

has a unique solution $f_\epsilon(v, t)$, and $\hat{f}_\epsilon(\xi, t)$ satisfies

$$\frac{\partial \hat{f}_\epsilon(\xi, t)}{\partial t} = \int_{-\pi}^{\pi} \beta_\epsilon(|\theta|)\, [\hat{f}_\epsilon(\xi \cos\theta, t) \hat{f}_\epsilon(\xi \sin\theta, t) - \hat{f}_\epsilon(0, t) \hat{f}_\epsilon(\xi, t)]\, d\theta . \tag{4.3}$$

Let us expand $\hat{f}_\epsilon(\xi\cos\theta, t)$ and $\hat{f}_\epsilon(\xi\sin\theta, t)$ in Taylor's series of the variable ξ, the former up to the second order, the latter up to the third order. For the sake of simplicity, we will not take care of time dependence, and we will denote by $\hat{f}'$ the derivative of $\hat{f}$ with respect to ξ. By (2.19) we obtain

$$\hat{f}_\epsilon(\xi\cos\theta) = \hat{f}_\epsilon(\xi) + \hat{f}'_\epsilon\xi(\cos\theta - 1) + \hat{f}''_\epsilon(\bar{\xi})\frac{\xi^2(\cos\theta-1)^2}{2}$$

$$\hat{f}_\epsilon(\xi\sin\theta) = 1 - \sigma\frac{\xi^2\sin^2\theta}{2} + \hat{f}'''_\epsilon(\xi^*)\frac{\xi^3\sin^3\theta}{6}. \tag{4.4}$$

Hence, substituting into the collision integral we obtain

$$\frac{\partial\hat{f}_\epsilon(\xi,t)}{\partial t} = -\sigma\xi^2\hat{f}_\epsilon(\xi) - \xi\hat{f}'_\epsilon(\xi) - \left(\frac{1}{2}\int_{-\pi}^{\pi}\beta_\epsilon(|\theta|)\sin^2\theta\, d\theta - 1\right)\xi^2\hat{f}_\epsilon(\xi)$$

$$-\left(\int_{-\pi}^{\pi}\beta_\epsilon(|\theta|)(1-\cos\theta)\,d\theta - 1\right)\xi\hat{f}'_\epsilon(\xi) + \int_{-\pi}^{\pi}\beta_\epsilon(|\theta|)\frac{\sin^3\theta}{6}\,d\theta\,\xi^3\hat{f}_\epsilon(\xi)\hat{f}'''_\epsilon(\xi^*)$$

$$+\int_{-\pi}^{\pi}\beta_\epsilon(|\theta|)\frac{(\cos\theta-1)^2}{2}\,d\theta\,\xi^2\hat{f}''_\epsilon(\bar{\xi}). \tag{4.5}$$

Let us subtract eq. (3.2) from eq. (4.5). We obtain

$$\frac{\partial}{\partial t}(\hat{f}_\epsilon(\xi,t) - \hat{f}(\xi,t)) = -\sigma\xi^2(\hat{f}_\epsilon(\xi,t) - \hat{f}(\xi,t)) - \xi(\hat{f}_\epsilon(\xi,t) - \hat{f}(\xi,t))' + R(\hat{f}_\epsilon). \tag{4.6}$$

In (4.6) we put

$$R(\hat{f}_\epsilon) = -A(\epsilon)\xi^2\hat{f}_\epsilon(\xi) - B(\epsilon)\xi\hat{f}'_\epsilon(\xi) + C(\epsilon)\xi^3\hat{f}_\epsilon(\xi)\hat{f}'''_\epsilon(\xi^*) + D(\epsilon)\xi^2\hat{f}''_\epsilon(\bar{\xi}) \tag{4.7}$$

with obvious meaning of the quantities A, B, C, D. Let us multiply both sides of (4.6) by $2\overline{(\hat{f}_\epsilon(\xi,t) - \hat{f}(\xi,t))}$, and then integrate over $\mathbf{R}$.

$$\frac{\partial}{\partial t}\int_{\mathbf{R}}|\hat{f}_\epsilon(\xi,t) - \hat{f}(\xi,t)|^2\,d\xi = -2\sigma\int_{\mathbf{R}}\xi^2|\hat{f}_\epsilon(\xi,t) - \hat{f}(\xi,t)|^2\,d\xi$$

$$-2\int_{\mathbf{R}}\xi Re[(\hat{f}_\epsilon(\xi,t) - \hat{f}(\xi,t))'\overline{(\hat{f}_\epsilon(\xi,t) - \hat{f}(\xi,t))}]\,d\xi + 2\int_{\mathbf{R}}Re[R(\hat{f}_\epsilon)\overline{(\hat{f}_\epsilon(\xi,t) - \hat{f}(\xi,t))}]\,d\xi. \tag{4.8}$$

Under the hypotheses we made on φ, by Theorems 2.3 and 2.4, we know that the H^3-norms of the solution to both the non cut-off Kac equation and the Fokker-Planck equation are uniformly bounded in time. Moreover, we outline that the Fisher information $L(f(t))$ of the solution to the Fokker-Planck is monotonically decreasing with time if it is bounded initially. This last property has been recently used in [15] to give a proof of the logarithmic Sobolev inequality. These facts, combined with the uniform boundedness of the third moment of the Kac equation, lead to obtain uniform estimates on the last integral into (4.8). In more detail, we have

$$\left|\int_{\mathbf{R}} R(\hat{f}_\epsilon)\,\overline{(\hat{f}_\epsilon(\xi,t)-\hat{f}(\xi,t))}\,d\xi\right| \leq |A(\epsilon)|\left|\int_{\mathbf{R}} \xi^2 \hat{f}_\epsilon(\xi)\,(\hat{f}_\epsilon(\xi,t)-\hat{f}(\xi,t))\,d\xi\right|$$
$$+\,|B(\epsilon)|\left|\int_{\mathbf{R}} \xi \hat{f}'_\epsilon(\xi)\,(\hat{f}_\epsilon(\xi,t)-\hat{f}(\xi,t))\,d\xi\right|$$
$$+\,|C(\epsilon)|\,|\hat{f}'''_\epsilon(\xi^\star)|\left|\int_{\mathbf{R}} \xi^3 \hat{f}_\epsilon(\xi)\,(\hat{f}_\epsilon(\xi,t)-\hat{f}(\xi,t))\,d\xi\right|$$
$$+\,|D(\epsilon)|\,|\hat{f}''_\epsilon(\bar{\xi})|\left|\int_{\mathbf{R}} \xi^2 (\hat{f}_\epsilon(\xi,t)-\hat{f}(\xi,t))\,d\xi\right|. \tag{4.9}$$

By Cauchy-Schwarz inequality and Parseval's formula we get

$$\left|\int_{\mathbf{R}} \xi^2 \hat{f}_\epsilon(\xi)\,(\hat{f}_\epsilon(\xi,t)-\hat{f}(\xi,t))\,d\xi\right| \leq C\,\|f_\epsilon\|_{H^1}\,\|f_\epsilon - f\|_{H^1}. \tag{4.10}$$

Similarly we obtain

$$\left|\int_{\mathbf{R}} \xi \hat{f}'_\epsilon(\xi)\,(\hat{f}_\epsilon(\xi,t)-\hat{f}(\xi,t))\,d\xi\right| \leq \|v^2 f_\epsilon(v)\|_{L^2}\|f_\epsilon - f\|_{H^1} \leq \sigma\,\|f_\epsilon\|_\infty\,\|f_\epsilon - f\|_{H^1} \tag{4.11}$$

$$|\hat{f}'''_\epsilon(\xi^\star)|\left|\int_{\mathbf{R}} \xi^3 \hat{f}_\epsilon(\xi)\,(\hat{f}_\epsilon(\xi,t)-\hat{f}(\xi,t))\,d\xi\right| \leq C_4^{3/4}\,\|f_\epsilon\|_{H^2}\,\|f_\epsilon - f\|_{H^1}. \tag{4.12}$$

Finally

$$|\hat{f}''_\epsilon(\bar{\xi})|\left|\int_{\mathbf{R}} \xi^2 (\hat{f}_\epsilon(\xi,t)-\hat{f}(\xi,t))\,d\xi\right|$$
$$\leq \sigma\left[\int_{\mathbf{R}} (1+\xi^2)\,\xi^4 (\hat{f}_\epsilon(\xi,t)-\hat{f}(\xi,t))^2\,d\xi\right]^{1/2}\left[\int_{\mathbf{R}} \frac{1}{1+\xi^2}\,d\xi\right]^{1/2} \leq \sigma\sqrt{\pi}\,\|f_\epsilon - f\|_{H^3}. \tag{4.13}$$

The quantities on the right sides of (4.10-13) can be bounded uniformly in terms of φ in view of Theorems 2.4 and 3.1. Then, using these bounds into (4.8), and considering that $A(\epsilon)$, $B(\epsilon)$, $C(\epsilon)$ and $D(\epsilon)$ are infinitesimal with ϵ, we obtain the inequality

$$\frac{\partial}{\partial t}\int_{\mathbf{R}} |\hat{f}_\epsilon(\xi,t)-\hat{f}(\xi,t)|^2\,d\xi \leq -2\sigma\int_{\mathbf{R}} \xi^2 |\hat{f}_\epsilon(\xi,t)-\hat{f}(\xi,t)|^2\,d\xi$$
$$-2\int_{\mathbf{R}} \xi Re\left[(\hat{f}_\epsilon(\xi,t)-\hat{f}(\xi,t))'\,\overline{(\hat{f}_\epsilon(\xi,t)-\hat{f}(\xi,t))}\right]d\xi + E(\epsilon)\,G(\varphi) \tag{4.14}$$

where $E(\epsilon)$ depends on the family of kernels, and G depends on φ through its moments and L_3 and J_3. Let us set $g_\epsilon(v,t) = f_\epsilon(v,t) - f(v,t)$. Then, by Parseval's formula, inequality (4.14) can be rewritten in terms of g_ϵ

$$\frac{\partial}{\partial t}\int_{\mathbf{R}} |g_\epsilon^2(v,t)|\, dv \leq -2\,\sigma \int_{\mathbf{R}} \left|\frac{\partial g_\epsilon(v,t)}{\partial v}\right|^2 dv + \int_{\mathbf{R}} |g_\epsilon(v,t)|^2\, dv + E(\epsilon)\, G(\varphi)\,. \tag{4.15}$$

In dimension one, the Nash inequality [13] states that

$$\left(\int_{\mathbf{R}} |f(v)|^2\, dv\right)^3 \leq C \int_{\mathbf{R}} |f'(v)|^2\, dv \left(\int_{\mathbf{R}} |f(v)|\, dv\right)^4. \tag{4.16}$$

Let us apply inequality (4.16) to the first term on the right of (4.15). Since $\|g_\epsilon\|_{L^1} \leq 2$, we obtain

$$-\int_{\mathbf{R}} \left|\frac{\partial g_\epsilon(v,t)}{\partial v}\right|^2 dv \leq -\frac{1}{16\,C}\left(\int_{\mathbf{R}} |g_\epsilon(v,t)|^2\, dv\right)^3. \tag{4.17}$$

Hence, if we set $y(t) = \int_{\mathbf{R}} |g_\epsilon(v,t)|^2\, dv$, $y(t)$ satisfies the inequality

$$\frac{dy}{dt} \leq y - \frac{\sigma}{8\,C} y^3 + E(\epsilon)\, G(\varphi) \tag{4.18}$$

with $y(0) = 0$. Let $z = ye^{-t}$. Then z satisfies

$$\frac{dz}{dt} \leq -\frac{\sigma}{8\,C} e^{2t} z^3 + E(\epsilon)\, G(\varphi)\, e^{-t} \tag{4.19}$$

with $z(0) = 0$. It is immediate to recognize that at any time t, $z(t)$ can not exceed the limit value $z^\star(t)$, solution of the equation

$$-\frac{\sigma}{8\,C} e^{2t} z^3 + E(\epsilon)\, G(\varphi)\, e^{-t} = 0 \tag{4.20}$$

and this implies the bound

$$\int_{\mathbf{R}} |f_\epsilon(v,t) - f(v,t)|^2\, dv \leq \left(\frac{4\,CE(\epsilon)\, G(\varphi)}{\sigma}\right)^{1/3}. \tag{4.21}$$

Hence we proved that $f_\epsilon(v,t)$ converges stronly in $L^2(\mathbf{R})$ to $f(v,t)$, and the convergence is uniform in time. To conclude the proof of the theorem, consider that from (4.20) follows

$$\int_{\mathbf{R}} |\xi|^3 |\hat{f}_\epsilon(\xi,t) - \hat{f}(\xi,t)|^2\, d\xi \leq \|f_\epsilon - \phi\|_{H^3} \left(\frac{4\,CE(\epsilon)\, G(\varphi)}{\sigma}\right)^{1/6} \tag{4.22}$$

which implies convergence in $W^{3/2,\,2}$. Using the same trick, we can prove convergence in $W^{3+3/2+3/4+\cdots,\,2}$, namely the result.

5. CONCLUDING REMARKS

In this paper, we investigated the grazing collision asymptotics of the non cut-off Kac equation, proving the strong convergence to the solution of the one-dimensional Fokker-Planck equation. The result of Theorem 4.1 can be improved, in the sense that the more regular the initial density φ is, the stronger the convergence of f_ϵ is.

It would be desirable to extend these results to the Boltzmann equation for Maxwell molecules, but the passage to higher dimensions encounters several obstacles. Indeed, propagation of regularity and uniform bounds in Sobolev spaces has been proved only for the Boltzmann equation for Maxwell pseudomolecules (with cut-off) [2]. Better results are available in dimension two of the velocity space [14]. Investigation of this case is in progress.

ACKNOWLEDGEMENTS

This work has been written within the activities of the National Group of Mathematical Physics (GNFM) of the National council for Researches (CNR), Project "Applications of Mathematics for Technology and Society". Partial support of the Institute of Numerical Analysis of the CNR is kindly acknowledged.

REFERENCES

[1] A. A. ARSEN'EV and O. E. BURYAK, "On the connection between a solution of the Boltzmann equation and a solution of the Landau-Fokker-Planck equation". *Math. USSR Sbornik,* **69** (2), 465-478 (1991).

[2] E. A. CARLEN, E. GABETTA, G. TOSCANI, "Propagation of smoothness and the rate of exponential convergence to equilibrium for a spatially homogeneous Maxwellian gas", to appear on *Comm. Math. Phys.* (1997).

[3] C. CERCIGNANI, *The Boltzmann equation and its applications.* Springer, New York (1988).

[4] P. DEGOND and B. LUCQUIN-DESREUX, "The Fokker-Planck asymptotics of the Boltzmann collision operator in the Coulomb case", *Math. Mod. Meth. in appl. Sci.,* **2** (2), 167-182 (1992).

[5] L. DESVILLETTES, "On asymptotics of the Boltzmann equation when the collisions become grazing", *Transp. theory and stat. phys.,* **21** (3), 259-276 (1992).

[6] L. DESVILLETTES, "About the regularizing properties of the non-cut-off Kac equation", *Comm. Math. Phys.,* **168** (2), 417-440 (1995).

[7] E. GABETTA, L. PARESCHI, "About the non cut-off Kac equation: Uniqueness and asymptotic behaviour", *Comm. Appl. Nonlinear Anal.,* **4**, 1-20 (1997).

[8] E. GABETTA, G. TOSCANI, B. WENNBERG, "Metrics for probability distributions and the trend to equilibrium for solutions of the Boltzmann equation", *J. Stat. Phys.* **81**, 901-934 (1995).

[9] T. GOUDON, "On the Boltzmann equation and its relations to the Landau-Fokker-Planck equation: influence of grazing collisions". To appear in *C. R. Acad. Sci.,* 1997.

[10] M. KAC, *Probability and related topics in the physical sciences,* New York (1959).

[11] P. L. LIONS, "On Boltzmann and Landau equations", *Phil. Trans. R. Soc. Lond.,* A(**346**), 191-204 (1994).

[12] P. L. LIONS, G. TOSCANI, "A sthrenghtened central limit theorem for smooth densities", *J. Funct. Anal.* **128**, 148-167 (1995).

[13] J. NASH, "Continuity of solutions of parabolic and elliptic equations", *Amer. J. Math.* **80**, 931-957 (1958).

[14] G. TOSCANI, "On regularity and asymptotic behaviour of a spatially homogeneous Maxwell gas". *Rend. Circolo Mat. Palermo,* Serie II, Suppl. **45**, 649-662 (1996).

[15] G. TOSCANI, "Sur l'inégalité logarithmique de Sobolev", To appear in *C. R. Acad. Sci.,* 1997.

[16] C. VILLANI, "On the Landau equation: weak stability, global existence", *Adv. Diff. Eq.* **1** (5), 793-818 (1996).

[17] C. VILLANI, "On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations".