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A POSTERIORI ERROR ESTIMATES FOR NONLINEAR PROBLEMS. L^r-ESTIMATES FOR FINITE ELEMENT DISCRETIZATIONS OF ELLIPTIC EQUATIONS (*)

R. VERFÜRTH

Abstract. — *We extend the general framework of [18] for deriving a posteriori error estimates for approximate solutions of nonlinear elliptic problems such that it also yields L^r-error estimates. The general results are applied to finite element discretizations of scalar quasilinear elliptic pdes of 2nd order and the stationary incompressible Navier-Stokes equations. They immediately yield a posteriori error estimates for an L^r-norm of the error which can easily be computed from the given data of the problem and the computed numerical solution and which give global upper and local lower bounds on the error of the numerical solution. © Elsevier, Paris*

Résumé. — *On modifie le cadre abstrait de [18] tel qu'il est capable de fournir des estimations d'erreur a posteriori aussi en norme L^r. Les résultats généraux sont appliqués aux équations quasilinéaires d'ordre 2 et aux questions de Navier-Stokes stationnaires et incompressible. On obtient des estimations d'erreurs a posteriori qui ne dépend que des données du problème et de la solution numérique et qui fournissent des bornes supérieures globales et inférieures locales pour la norme L^r de l'erreur. © Elsevier, Paris*

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1. INTRODUCTION

In the last 15-20 years a lot of work has been devoted to the development of a posteriori error estimators for finite element discretizations of pdes (*cf.* e.g., the overview in [20]). Most approaches try to bound a $W^{1,r}$-norm of the error. For problems in continuum mechanics, e.g., this corresponds to the strain energy. In some applications, however, one is more interested in L^r-bounds of the error. In fluid mechanics, e.g., this corresponds to the kinetic energy. Only little work has been done concerning this topic; most of it for transient problems (*cf.* [9, 10, 11, 12, 15, 16, 19]).

In [18] we developped a general framework for the a posteriori error estimation of abstract nonlinear problems. When applied to general quasilinear elliptic equations of 2nd order it yields estimates on the $W^{1,r}$-norm of the error. Here, we extend and modify this approach such it also yields upper and lower bounds on the L^r-norm of the error. Recently, a similar, but less general approach was presented in [4] which yields L^2-error estimates for finite element discretizations of nonlinear elliptic pdes. For linear problems, a posteriori error estimates in the maximum-norm are derived in [9, 16] using sharp a priori error estimates for the Green's function.

In Section 2 we consider abstract nonlinear problems of the form

$$F(u) = 0 \tag{1.1}$$

and corresponding discretizations of the form

$$F_h(u_h^0) = 0 \,. \tag{1.2}$$

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Here, $F \in C^1(X, Y^*)$ and $F_h \in C(X_h, Y_h^*)$, $X_h \subset X$ and $Y_h \subset Y$ are finite dimensional subspaces of the Banach spaces X and Y, and $*$ denotes the dual of a Banach space. In applications, X will be a suitable subspace of a $W^{1,r}$-space. In order to obtain L^r-estimates we must enlarge the space X and restrict the space Y. To this end we consider three additional Banach spaces $X_+ \subset X \subset X_-$ and $Y_+ \subset Y$ with continuous and dense imbeddings. We assume that X_- is reflexive. The $+/-$ sign indicates a space with a stronger/weaker topology. In applications, X_- will be a L^r-space. We then prove that, under suitable assumptions, the error $\|u - u_h\|_{X_-}$ is bounded from below and from above bythe residual $\|F(u_h)\|_{Y_+^*}$. Here, u_h is an approximate solution of Problem (1.2). The main assumption is that $DF(u)^* \in \text{Isom}(Y_+, X_-^*)$. For elliptic equations this is an additional condition about elliptic regularity which is stronger than the minimal regularity assumption needed for the formulation of Problem (1.1). This approach is conceptually similar to the one of [17].

In order to bound the residual $\|F(u_h)\|_{Y_+^*}$ we introduce a continuous restriction operator $R_h : Y \to Y_h$, an approximation $\tilde{F}_h \in C(X_h, Y^*)$ of F at u_h, and a finite dimensional space $\tilde{Y}_h \subset Y_+$ which are linked by Condition (2.3). We then show that the residual $\|F(u_h)\|_{Y_+^*}$ is equivalent to $\|\tilde{F}_h(u_h)\|_{\tilde{Y}_h^*}$ which, for the applications, will be much easier to compute. The main difference with [18] is the condition $\tilde{Y}_h \subset Y_+$. For applications, this means that the elements of $\tilde{Y}_h$ must be of class C^κ with a suitable $\kappa \geq 1$. This makes the construction of $\tilde{Y}_h$ more technical. For practical calculations, however, it is important to note that $\tilde{Y}_h$ is only needed for establishing the a posteriori error estimates. Their concrete calculation as well as the discrete problem (1.2) only do require standard C^0-conforming finite element spaces.

In Section 3 we present auxiliary results which help in the concrete construction of R_h, $\tilde{F}_h$, and $\tilde{Y}_h$. The main point is the construction of local cut-off functions, which are of class C^κ, and of a prolongation operator, which associates with a function of a face of a triangulation an extension that is defined on the whole $\mathbb{R}^n$ and that is globally of class C^κ. The techniques of this section are inspired by those of [18]. But, due to the need for global C^κ-continuity, they considerably differ in details.

In Sections 4 and 5 we apply the results of the previous sections to finite element discretizations of quasilinear elliptic equations of 2nd order and of the stationary incompressible Navier-Stokes equations, respectively. We obtain residual a posteriori error estimates which have the same structure as those given in [18] but which have different scaling factors. This is due to the fact that in deriving the error estimates we implicitely use a duality argument. We also derive a posteriori error estimates which are based on the solution of auxiliary local Dirichlet problems. The auxiliary problems are the same as those used in [18]. But, now, the estimators are based on a L^p-norm of the solution of the auxiliary problem, instead of a $W^{1,p}$-norm in [18]. Again, this difference is due to the duality argument mentioned above.

2. ABSTRACT NONLINEAR EQUATIONS

We consider five Banach spaces $X_- \subset X \subset X_+$ and $Y_+ \subset Y$ with continuous and dense injections. The $+/-$ sign indicates a space with a stronger/weaker norm. We assume that X_- is reflexive.

The norm of a Banach space Z is denoted by $\|\,.\,\|_Z$. For any $u \in Z$ and any real number $R > 0$ we set $B_Z(u, R) := \{v \in Z : \|u - v\|_Z < R\}$. $\mathscr{L}(V, W)$ denotes the Banach space of continuous linear maps of the Banach space V into the Banach space W equipped with the operator norm $\|\,.\,\|_{\mathscr{L}(V,W)}$. Isom (V, W) is the open subset of $\mathscr{L}(V, W)$ consisting of all linear homeomorphisms of V into W. $V^* := \mathscr{L}(V, \mathbb{R})$ and $\langle\,.\,,\,.\,\rangle_V$ are the dual space of V and the corresponding duality pairing. Finally, $A^* \in \mathscr{L}(W^*, V^*)$ denotes the adjoint of a given operator $A \in \mathscr{L}(V, W)$.

Let $F \in C^1(X, Y^*)$ be a given continuously differentiable function. The following proposition shows that in a neighborhood of a solution of Equation (1.1) the error measured in the X_--norm is equivalent to the residual measured in the Y_+^*-norm. A similar result is proven in [17].

PROPOSITION 2.1: *Let $u_0 \in X_+$ be a solution of Problem (1.1). Assume that $DF(u_0)^* \in$ Isom (Y_+, X_-^*) and that there are two numbers $R_0 > 0$ and $\beta > 0$ such that*

$$\|[DF(u_0) - DF(u_0 + tw)]\, w\|_{Y_+^*} \leq \beta t \|w\|_{X_+} \|w\|_{X_-} \tag{2.1}$$

for all $w \in B_{X_+}(0, R_0)$, $t \in [0, 1]$. *Set*

$$R := \min \{ R_0, \beta^{-1} \| DF(u_0)^{*-1} \|^{-1}_{\mathscr{L}(X_-^*, Y_+)}, 2\beta^{-1} \| DF(u_0)^* \|_{\mathscr{L}(Y_+, X_-^*)} \} .$$

Then the following error estimate holds for all $u \in B_{X_+}(u_0, R)$:

$$\begin{aligned} &\tfrac{1}{2} \| DF(u_0)^* \|^{-1}_{\mathscr{L}(Y_+, X_-^*)} \| F(u) \|_{Y_+^*} \\ &\leq \| u - u_0 \|_{X_-} \\ &\leq 2 \| DF(u_0)^{*-1} \|_{\mathscr{L}(X_-^*, Y_+)} \| F(u) \|_{Y_+^*} . \end{aligned} \tag{2.2}$$

Proof: Let $u \in B_{X_+}(u_0, R)$. Consider an arbitrary element $w \in X_-^*$ and set $\varphi := DF(u_0)^{*-1} w \in Y_+$. We then have

$$\begin{aligned} \langle u - u_0, w \rangle_{X_-} &= \langle DF(u_0)(u - u_0), \varphi \rangle_{Y_+} \\ &= \langle F(u), \varphi \rangle_{Y_+} \\ &\quad + \int_0^1 \langle [DF(u_0) - DF(u_0 + t(u - u_0))](u - u_0), \varphi \rangle_{Y_+} dt . \end{aligned}$$

Inequality (2.1) and the continuity of $DF(u_0)^{*-1}$ imply that

$$\begin{aligned} &\left| \int_0^1 \langle [DF(u_0) - DF(u_0 + t(u - u_0))](u - u_0), \varphi \rangle_{Y_+} dt \right| \\ &\leq \int_0^1 t\beta \| u - u_0 \|_{X_-} \| u - u_0 \|_{X_+} \| \varphi \|_{Y_+} dt \\ &\leq \tfrac{1}{2} \beta R \| u - u_0 \|_{X_-} \| \varphi \|_{Y_+} \\ &\leq \tfrac{1}{2} \beta \| DF(u_0)^{*-1} \|_{\mathscr{L}(X_-^*, Y_+)} R \| u - u_0 \|_{X_-} \| w \|_{X_-^*} . \end{aligned}$$

Combined with the above representation of $\langle u - u_0, w \rangle_{X_-}$ this yields

$$\begin{aligned} \langle u - u_0, w \rangle_{X_-} &\leq \| F(u) \|_{Y_+^*} \| \varphi \|_{Y_+} + \tfrac{1}{2} \beta \| DF(u_0)^{*-1} \|_{\mathscr{L}(X_-^*, Y_+)} R \| u - u_0 \|_{X_-} \| w \|_{X_-^*} \\ &\leq \left\{ \| DF(u_0)^{*-1} \|_{\mathscr{L}(X_-^*, Y_+)} \| F(u) \|_{Y_+^*} + \tfrac{1}{2} \| u - u_0 \|_{X_-} \right\} \| w \|_{X_-^*} . \end{aligned}$$

Since X_- is reflexive and $w \in X_-^*$ was arbitrary, this implies the upper bound of Estimate (2.2).

In the same way, we obtain

$$\begin{aligned}
&\langle F(u), \varphi \rangle_{Y_+} \\
&= \langle u - u_0, w \rangle_{X_-} - \int_0^1 \langle [DF(u_0) - DF(u_0 + t(u - u_0))] (u - u_0), \varphi \rangle_{Y_+} dt \\
&\leqslant \| u - u_0 \|_{X_-} \| w \|_{X_-^*} + \frac{1}{2} \beta R \| u - u_0 \|_{X_-} \| \varphi \|_{Y_+} \\
&\leqslant \left\{ \| DF(u_0)^* \|_{\mathscr{L}(Y_+, X_-^*)} \| u - u_0 \|_{X_-} + \frac{1}{2} \beta R \| u - u_0 \|_{X_-} \right\} \| \varphi \|_{Y_+} \\
&\leqslant 2 \| DF(u_0)^* \|_{\mathscr{L}(Y_+, X_-^*)} \| u - u_0 \|_{X_-} \| \varphi \|_{Y_+} .
\end{aligned}$$

Since $\varphi \in Y_+$ is arbitrary, this proves the lower bound of Estimate (2.2). □

Remark 2.2: When comparing the above proposition with Proposition 2.1 in [18] we observe two major differences:

(1) Here, we require that the solution u_0 of Problem (1.1) must be contained in X_+, i.e., it must be more regular.
(2) Here, we require that $DF(u_0)^* \in \operatorname{Isom}(Y_+, X_-^*)$ instead of $DF(u_0) \in \operatorname{Isom}(X, Y^*)$. Since the second condition is equivalent to $DF(u_0)^* \in \operatorname{Isom}(Y^{**}, X^*)$, the present condition is more restrictive. For pdes, it amounts in an additional regularity assumption. For the applications of Sections 4 and 5, it will turn out that this additional regularity assumption is more restrictive than the condition $u_0 \in X_+$.

When considering linear problems, i.e., when DF is constant, we may extend $F \in C^1(X, Y^*)$ by continuity to a continuously differentiable map of X_- into Y_+^*. Then the space X_+ is not needed. For nonlinear problems, however, this extension is often impossible or its derivative is no longer Lipschitz continuous. This is the place where the space X_+ comes into play. □

Remark 2.3: In some applications, in particular when X_- is a Hilbert space, a function $w \in X_-^*$ satisfying

$$\| w \|_{X_-^*} = 1 \quad \text{and} \quad \langle u - u_0, w \rangle_{X_-} = \| u - u_0 \| X_-$$

can explicitely be given in terms of u and u_0. Then $\| DF(u_0)^{*-1} \|_{\mathscr{L}(X_-^*, Y_+)}$ can be replaced by $\| DF(u_0)^{*-1} w \|_{Y_+}$. The latter quantity may be estimated numerically by approximately solving a discrete analogue of the corresponding adjoint linearized pde. The other factor appearing in estimate (2.2), namely $\| DF(u_0)^* \|_{\mathscr{L}(Y_+, X_-^*)}$, is much more harmless since it corresponds to a differential operator which is local and the norm of which can more easily be estimated in terms of its coefficients. □

Remark 2.4: The arguments of [18; §3] can be modified such that Proposition 2.1 extends to branches of solutions including, in particular, simple limit and bifurcation points. □

Let $X_h \subset X_+$ and $Y_h \subset Y$ be finite dimensional subspaces and $F_h \in C(X_h, Y_h^*)$ be an approximation of F. Given an approximate solution u_h of problem (1.2), Proposition 2.1 shows that the error $\| u - u_h \|_{X_-}$ is controlled by $\| F(u_h) \|_{Y_+^*}$, i.e., a dual norm of the residual. Since Y_+ is infinite dimensional, the evaluation of this quantity is as difficult as the solution of the original problem (1.2). In order to obtain an approximation of this quantity which is easier to handle, we modify the strategy of [18].

PROPOSITION 2.5: *Let $u_h \in X_h$ be an approximate solution of Problem (1.2). Assume that there are a restriction operator $R_h \in \mathscr{L}(Y, Y_h)$, a finite dimensional subspace $\tilde{Y}_h \subset Y_+$, and an approximation $\tilde{F}_h \in C(X_h, Y^*)$ of F at u_h such that*

$$\| (Id_{Y_+} - R_h)^* \tilde{F}_h(u_h) \|_{Y_+^*} \leqslant c_0 \| \tilde{F}_h(u_h) \|_{\tilde{Y}_h^*}, \tag{2.3}$$

where $\tilde{Y}_h$ is equipped with the norm of Y_+. Then the following estimates hold

$$\|F(u_h)\|_{Y_+^*} \leq c_0 \|\tilde{F}_h(u_h)\|_{\tilde{Y}_h^*} + \|(Id_{Y_+} - R_h)^* [F(u_h) - \tilde{F}_h(u_h)]\|_{Y_+^*} + \|R_h^*[F(u_h) - F_h(u_h)]\|_{Y_+^*} + \|R_h^* F_h(u_h)\|_{Y_+^*} \tag{2.4}$$

and

$$\|\tilde{F}_h(u_h)\|_{\tilde{Y}_h^*} \leq \|F(u_h)\|_{\tilde{Y}_h^*} + \|F(u_h) - \tilde{F}_h(u_h)\|_{\tilde{Y}_h^*} . \tag{2.5}$$

Proof: Consider an arbitrary element $\varphi \in Y_+$ with $\|\varphi\|_{Y_+} = 1$. We then have

$$\begin{aligned}\langle F(u_h), \varphi \rangle_{Y_+} &= \langle \tilde{F}_h(u_h), \varphi - R_h \varphi \rangle_{Y_+} + \langle F(u_h) - \tilde{F}_h(u_h), \varphi - R_h \varphi \rangle_{Y_+} \\ &\quad + \langle F(u_h) - F_h(u_h), R_h \varphi \rangle_{Y_+} + \langle F_h(u_h), R_h \varphi \rangle_{Y_+} \\ &\leq \|(Id_{Y_+} - R_h)^* \tilde{F}_h(u_h)\|_{Y_+^*} + \|(Id_{Y_+} - R_h)^* [F(u_h) - \tilde{F}_h(u_h)]\|_{Y_+^*} \\ &\quad + \|R_h^*[F(u_h) - F_h(u_h)]\|_{Y_+^*} + \|R_h^* F_h(u_h)\|_{Y_+^*} .\end{aligned}$$

Together with Inequality (2.3), this proves Estimate (2.4).
Estimate (2.5) follows from the triangle inequality. □

Remark 2.6: When comparing Proposition 2.5 with the corresponding Proposition 4.1 in [18] we remark that the space $\tilde{Y}_h$ must be contained in Y_+ and that it must be equipped with the corresponding norm. For the applications of Sections 4 and 5 this means that the functions in $\tilde{Y}_h$ must be continuously differentiable across interelement boundaries. This complicates the construction of the local cut-off functions and of the prolongation operator in the next section.

A non-optimal estimate of the third and fourth term on the right-hand side of Estimate (2.4) is given by

$$\begin{aligned}&\|R_h^*[F(u_h) - F_h(u_h)]\|_{Y_+^*} + \|R_h^* F_h(u_h)\|_{Y_+^*} \\ &\leq \|Id_{Y_+}\|_{\mathscr{L}(Y_+, Y)} \|R_h\|_{\mathscr{L}(Y, Y_h)} \{ \|F(u_h) - F_h(u_h)\|_{Y_h^*} + \|F_h(u_h)\|_{Y_h^*} \} .\end{aligned}$$

Here, as in Proposition 4.1 of [18], Y_h is equipped with the norm of Y.

The second terms on the right-hand sides of Inequalities (2.4) and (2.5) measure the quality of the approximation $\tilde{F}_h(u_h)$ to $F(u_h)$. Usually they are higher order perturbations when compared with $\|\tilde{F}_h(u_h)\|_{\tilde{Y}_h^*}$. The third term on the right-hand side of Estimate (2.4) measures the discretization error and can be bounded a priori. The fourth term on the right-hand side of Estimate (2.4) is the residual of the algebraic system (1.2). It must be estimated seperately. □

When combining Propositions 2.1 and 2.5 we obtain a residual a posteriori error estimator. The following proposition together with Proposition 2.1 yields a framework for those a posteriori error estimators which are based on the solution of auxiliary local problems, such as those described in [2, 3, 5].

PROPOSITION 2.7: *Let $u_h \in X_h$ be an approximate solution of Problem (1.2). Assume that there are finite dimensional spaces $\hat{X}_h \subset X_-$ and $\hat{Y}_h \subset Y_+$ and a linear operator $B \in$* Isom *$(\hat{X}_h, \hat{Y}_h^*)$ such that $\tilde{Y}_h \subset \hat{Y}_h$ and*

$$\|\tilde{F}_h(u_h)\|_{\hat{Y}_h^*} \leq c_1 \|\tilde{F}_h(u_h)\|_{\tilde{Y}_h^*} . \tag{2.6}$$

Here, $\hat{X}_h$ is equipped with the norm of X_- and $\hat{Y}_h$ and $\tilde{Y}_h$ are endowed with the norm of Y_+. Let $\hat{u}_h \in \hat{X}_h$ be the unique solution of

$$\langle B\hat{u}_h, \varphi \rangle_{Y_+} = \langle \tilde{F}_h(u_h), \varphi \rangle_{Y_+} \quad \forall \varphi \in \hat{Y}_h . \tag{2.7}$$

Then the following estimates hold

$$\|B\|^{-1}_{\mathscr{L}(\hat{X}_h, \hat{Y}_h^*)} \|\tilde{F}_h(u_h)\|_{\tilde{Y}_h^*} \leq \|\hat{u}_h\|_{\hat{X}_h} \leq c_1 \|B^{-1}\|_{\mathscr{L}(\hat{Y}_h^*, \hat{X}_h)} \|\tilde{F}_h(u_h)\|_{\tilde{Y}_h^*} . \tag{2.8}$$

Proof: Since $B \in \text{Isom}(\hat{X}_h, \hat{Y}_h^*)$, we conclude from Equation (2.7) that

$$\|B\|^{-1}_{\mathscr{L}(\hat{X}_h, \hat{Y}_h^*)} \|\tilde{F}_h(u_h)\|_{\hat{Y}_h^*} \leq \|\hat{u}_h\|_{\hat{X}_h} \leq \|B^{-1}\|_{\mathscr{L}(\hat{Y}_h^*, \hat{X}_h)} \|\tilde{F}_h(u_h)\|_{\hat{Y}_h^*} .$$

Together with Inequality (2.6) this proves the upper bound of Estimate (2.8).

Since $\tilde{Y}_h \subset \hat{Y}_h$ we have

$$\|\tilde{F}_h(u_h)\|_{\tilde{Y}_h^*} \leq \|\tilde{F}_h(u_h)\|_{\hat{Y}_h^*} .$$

Together with the previous estimate of $\|\hat{u}_h\|_{\hat{X}_h}$, this proves the lower bound of estimate (2.8). □

Remark 2.8: When comparing Propotion 2.7 with Proposition 4.3 of [18] we observe the following differences:

(1) Here, the space $\hat{X}_h$ is equipped with the norm of X_- instead of the stronger norm of X.

(2) The space $\hat{Y}_h$ must now be contained in Y_+ and it must be equipped with the corresponding norm. For the applications of Section 4 and 5 this means that its elements must be continuously differentiable across interelement boundaries. □

3. AUXILIARY RESULTS

Let Ω be a bounded, connected, open domain in $\mathbb{R}^n$, $n \geq 2$, with polyhedral boundary Γ. For any open subset ω of Ω with Lipschitz boundary γ, we denote by $W^{k,p}(\omega)$, $k \in \mathbb{N}$, $1 \leq p \leq \infty$, $L^p(\omega) := W^{0,p}(\omega)$, and $L^p(\gamma)$ the usual Sobolev and Lebesgue spaces equipped with the standard norms $\|\cdot\|_{k,p;\omega} := \|\cdot\|_{W^{k,p}(\omega)}$ and $\|\cdot\|_{p;\gamma} := \|\cdot\|_{L^p(\gamma)}$ (*cf.* [1]). If $\omega = \Omega$, we will omit the index ω. We use the same notation for the corresponding norms of vector-valued functions. Let

$$W_0^{1,p}(\Omega) := \{u \in W^{1,p}(\Omega) : u = 0 \text{ on } \Gamma\}$$

and set for $1 < p < \infty$

$$W^{-1,p'}(\Omega) := W_0^{1,p}(\Omega)^* .$$

Here, p' denotes the dual exponent of p defined by $\frac{1}{p} + \frac{1}{p'} = 1$. In what follows, a prime will always denote the dual of a given Lebesgue exponent.

Let $\mathcal{T}_h$, $h > 0$, be a family of partitions of Ω into n-simplices which satisfies the following conditions

(1) (*Admissibility*) Any two simplices in $\mathcal{T}_h$ are either disjoint of share a complete smooth submanifold of their boundaries.

(2) (*Shape regularity*) The ratio h_T/ρ_T is bounded independently of $T \in \mathcal{T}_h$ and $h > 0$.

Here, h_T, ρ_T, and h_E denote the diameter of $T \in \mathcal{T}_h$, the diameter of the largest ball inscribed into T, and the diameter of a face E of T. Note that condition (2) allows the use of locally refined meshes.

For any $T \in \mathcal{T}_h$ denote by $\mathcal{N}(T)$ and $\mathcal{E}(T)$ the set of its vertices and faces, respectively. Set

$$\mathcal{N}_h := \bigcup_{T \in \mathcal{T}_h} \mathcal{N}(T), \quad \mathcal{E}_h := \bigcup_{T \in \mathcal{T}_h} \mathcal{E}(T)$$

and decompose both sets as

$$\mathcal{N}_h = \mathcal{N}_{h,\Omega} \cup \mathcal{N}_{h,\Gamma}, \quad \mathcal{E}_h = \mathcal{E}_{h,\Omega} \cup \mathcal{E}_{h,\Gamma}$$

with

$$\mathcal{N}_{h,\Gamma} := \{x \in \mathcal{N}_h : x \in \Gamma\}, \quad \mathcal{E}_{h,\Gamma} := \{E \in \mathcal{E}_h : E \subset \Gamma\}.$$

For any $T \in \mathcal{T}_h$, $E \in \mathcal{E}_h$, and $x \in \mathcal{N}_h$ we define the following neighborhoods

$$\omega_T := \bigcup_{T \cap T' \in \mathcal{E}_{h,\Omega}} T', \quad \omega_E := \bigcup_{E \in \mathcal{E}(T')} T', \quad \omega_x := \bigcup_{x \in \mathcal{N}(T')} T',$$

$$\tilde{\omega}_T := \bigcup_{T \cap T' \neq \emptyset} T', \quad \tilde{\omega}_E := \bigcup_{E \cap T' \neq \emptyset} T'.$$

With each face $E \in \mathcal{E}_h$ we associate a unit vector n_E orthogonal to E such that n_E is the unit outward normal to Ω if $E \in \mathcal{E}_{h,\Gamma}$. For any piecewise continuous function u and any $E \in \mathcal{E}_{h,\Omega}$ we denote by $[u]_E$ the jump of u across E in direction n_E:

$$[u]_E(x) := \lim_{t \to 0+} u(x + t n_E) - \lim_{t \to 0+} u(x - t n_E) \quad \forall x \in E.$$

Thanks to condition (2) above we may introduce

$$C_{\mathcal{T}} := \sup \left\{ \sqrt{2}\, h_T / \rho_{T'} : T, T' \in \mathcal{T}_h, T \cap T' \in \mathcal{E}_h, h > 0 \right\}. \tag{3.1}$$

For $k, l \in \mathbb{N}$ we define

$$S_h^{k,-1} := \{\varphi : \Omega \to \mathbb{R} : \varphi|_T \in \mathbb{P}_k \quad \forall T \in \mathcal{T}_h\},$$

$$S_h^{k,l} := S^{k,-1} \cap C^l(\Omega),$$

$$S_{h,0}^{k,0} := S_h^{k,0} \cap W_0^{1,2}(\Omega).$$

Here, $\mathbb{P}_k$, $k \geq 0$, is the space of polynomials of degree at most k. Moreover, we denote by $\pi_{k,S}$ the L^2-projection of $L^1(S)$ onto $\mathbb{P}_{k|S}$.

We denote by $I_h : L^1(\Omega) \to S_{h,0}^{1,0}$ the quasi-interpolation operator of Clément [7] which is defined as follows. Given $x \in \mathcal{N}_{h,\Omega}$, denote by $\pi_x : L^1(\omega_x) \to \mathbb{P}_1$ the $L^2(\omega_x)$-projection, i.e.

$$\int_{\omega_x} p(y)\, \pi_x u(y)\, dy = \int_{\omega_x} p(y)\, u(y)\, dy \quad \forall p \in \mathbb{P}_1, \quad u \in L^1(\omega_x).$$

Then I_h is uniquely defined by the conditions

$$I_h u(x) = \pi_x u(x) \qquad \forall x \in \mathcal{N}_{h,\Omega},$$
$$I_h u(x) = 0 \qquad \forall x \in \mathcal{N}_{h,\Gamma}.$$

I_h satisfies the following approximation properties for all $T \in \mathcal{T}_h$, $E \in \mathcal{E}_{h,\Omega}$, $1 \leq p < \infty$ (*cf.* [7] and Exercise 3.2.3 in [6]):

$$\| u - I_h u \|_{k,p;T} \leq c_{I1} h_T^{l-k} \| u \|_{l,p;\tilde{\omega}_T} \quad \forall 0 \leq k \leq l \leq 2, u \in W^{l,p}(\tilde{\omega}_T), \tag{3.2}$$

$$\| u - I_h u \|_{p;E} \leq c_{I2} h_E^{l-1/p} \| u \|_{l,p;\tilde{\omega}_E} \quad \forall 1 \leq l \leq 2, u \in W^{l,p}(\tilde{\omega}_E). \tag{3.3}$$

The constants c_{I1} and c_{I2} only depend on $\sup_{h>0} \sup_{T \in \mathcal{T}_h} h_T / \rho_T$.

Denote by $\hat{T} := \left\{ \hat{x} \in \mathbb{R}^n : \sum_{i=1}^n \hat{x}_i \leq 1, \hat{x}_j \geq 0, 1 \leq j \leq n \right\}$ the reference simplex and set $\hat{E} := \hat{T} \cap \{ \hat{x} \in \mathbb{R}^n : \hat{x}_n = 0 \}$. Let $\psi_{\hat{T}}, \psi_{\hat{E}} \in C^\kappa(\mathbb{R}^n)$ be two functions which satisfy the following conditions:

$$\begin{aligned}
&0 \leq \psi_{\hat{T}}(\hat{x}) \leq 1 \quad \forall \hat{x} \in \hat{T}, \\
&\max_{\hat{x} \in \hat{T}} \psi_{\hat{T}}(\hat{x}) = 1, \\
&\nabla^l \psi_{\hat{T}} = 0 \quad \text{on } \partial \hat{T} \quad \forall 0 \leq l \leq \kappa, \\
&0 \leq \psi_{\hat{E}}(\hat{x}) \leq 1 \quad \forall \hat{x} \in \hat{T}, \\
&\max_{\hat{x} \in \hat{E}} \psi_{\hat{E}}(\hat{x}) = 1, \\
&\nabla^l \psi_{\hat{E}} = 0 \quad \text{on } \partial \hat{T} \backslash \hat{E} \quad \forall 0 \leq l \leq \kappa.
\end{aligned} \tag{3.4}$$

Here, $\kappa > 0$ is an arbitrary integer which is kept fixed in what follows. Set

$$C_{\mathcal{E},\kappa} := \sup \{ | \nabla^l \psi_{\hat{E}}(y) | : 0 \leq l \leq \kappa, y \in B_{\mathbb{R}^n}(0, C_{\mathcal{T}}) \}, \tag{3.5}$$

where $C_{\mathcal{T}}$ is given by Equation (3.1). Finally, $V_{\hat{T}} \in C^\kappa(\hat{T})$ and $V_{\hat{E}} \in C^\kappa(\hat{E})$ are two arbitrary finite dimensional spaces which are kept fixed throughout this section.

Example 3.1: Set

$$\varphi_{\hat{T}}(\hat{x}) := \begin{cases} \left\{ (n+1)^{n+1} \left[1 - \sum_{i=1}^n \hat{x}_i \right] \prod_{j=1}^n \hat{x}_j \right\}^{\kappa+1}, & \forall \hat{x} \in \hat{T}. \\ 0, & \forall \hat{x} \notin \hat{T} \end{cases}$$

and

$$\varphi_{\hat{E}}(\hat{x}) := \left\{ n^n \left[1 - \sum_{i=1}^n \hat{x}_i \right] \prod_{j=1}^{n-1} \hat{x}_j \right\}^{\kappa+1} \quad \forall \hat{x} \in \mathbb{R}^n.$$

These functions satisfy Conditions (3.4). The constant $C_{\mathcal{E},\kappa}$ behaves like $C_{\mathcal{T}}^{n(\kappa+1)}$. □

Let $T \in \mathcal{T}_h$ be an arbitrary n-simplex and $E \in \mathcal{E}(T)$ be a face of T. There is an invertible affine mapping $F_T: \hat{T} \to T$, $\hat{x} \to x := F_T(\hat{x}) = b_T + B_T \hat{x}$ such that $\hat{T}$ is mapped onto T and $\hat{E}$ is mapped onto E. Denote by B'_T the matrix, which is obtained from B_T by discarding its last column, and set $\beta_T := \det(B'^t_T B'_T)^{1/2}$. β_T is the Gram determinant of the transformation $F_E: \hat{E} \to E$ induced by F_T. Note that (*cf.* Theorem 3.1.3 in [6])

$$|||B_T||| \leq h_T/\rho_{\hat{T}}, \quad |||B_T^{-1}||| \leq h_{\hat{T}}/\rho_T, \tag{3.6}$$

where $|||\,.\,|||$ denotes the spectral norm in $\mathbb{R}^{n \times n}$.

For $T \in \mathcal{T}_h$ and $E = T_1 \cap T_2 \in \mathcal{E}_{h,\Omega}$, T_1, $T_2 \in \mathcal{T}_h$, we set

$$\psi_T(x) := \begin{cases} \psi_{\hat{T}} \circ F_T^{-1}(x) & , \quad \forall x \in T, \\ 0 & , \quad \forall x \notin T, \end{cases}$$

$$\psi_E(x) := \begin{cases} \psi_{\hat{E}} \circ F_{T_1}^{-1}(x) \cdot \psi_{\hat{E}} \circ F_{T_2}^{-1}(x) & , \quad \forall x \in \omega_E, \\ 0 & , \quad \forall x \notin \omega_E, \end{cases} \tag{3.7}$$

$$V_T := \{\hat{u} \circ F_T^{-1} : \hat{u} \in V_{\hat{T}}\},$$

$$V_E := \{\hat{\sigma} \circ F_E^{-1} : \hat{\sigma} \in V_{\hat{E}}\}.$$

Note, that $\psi_E \in C^\kappa(\mathbb{R}^n)$ and supp $\psi_E \subset \omega_E$.

For $E \in \mathcal{E}_{h,\Omega}$, we finally define a continuation operator $P : L^\infty(E) \to L^\infty(\mathbb{R}^n)$ as follows. Denote by $x_E = (x_{E1}, ..., x_{En})$ an Euclidean coordinate system such that E is contained in the set $\{x_{En} = 0\}$. Set $x'_E := (x_{E1}, ..., x_{E(n-1)})$ and define

$$P\sigma(x_E) := \begin{cases} \psi_E(x'_E, 0)\, \sigma(x_{E'}, 0) & , \quad \text{if } (x'_E, 0) \in E, \\ 0 & , \quad \text{if } (x'_E, 0) \notin E. \end{cases} \tag{3.8}$$

Note, that $P_E \sigma \in C^\kappa(\mathbb{R}^n)$ if $\sigma \in C^\kappa(E)$.

Remark 3.2: In Section 5 of [18] the cut-off function ψ_E is defined as the piecewise pull-back of $\psi_{\hat{E}}$. Similarly, the continuation operator P is the piecewise pull-back of a continuation operator $\hat{P} : L^\infty(\hat{E}) \to L^\infty(\hat{T})$. This construction is easier to analyze since all estimates can be done on the reference element. On the other hand, it does not yield C^1-continuity across the interface E.

The above construction of P can also be interpreted as follows: Multiply $\sigma \in L^\infty(E)$ with the cut-off function ψ_E restricted to E, extend the product by zero to a function in $L^\infty(\mathbb{R}^{n-1})$, and identify the result in a canonical way with a function in $L^\infty(\mathbb{R}^n)$. □

PROPOSITION 3.3: *There are constants $c_1, ..., c_4$, which only depend on the spaces $V_{\hat{T}}$ and $V_{\hat{E}}$, the number p, and the quantity $\sup_{h>0} \sup_{T \in \mathcal{T}_h} h_T/\rho_T$, such that the following estimates hold for all $T \in \mathcal{T}_h$, $E \in \mathcal{E}_{h,\Omega}$, $0 \leq l \leq \kappa$, $u \in V_T$, and $\sigma \in V_E$:*

$$c_1 \| u \|_{0,p;T} \leq \sup_{v \in V_T \setminus \{0\}} \frac{\int_T u \psi_T v}{\| v \|_{0,p';T}} \leq \| u \|_{0,p;T}, \tag{3.9}$$

$$c_2 \| \sigma \|_{p;E} \leq \sup_{\tau \in V_E \setminus \{0\}} \frac{\int_E \sigma\, \psi_E P\tau}{\| \tau \|_{p';E}} \leq \| \sigma \|_{p;E}, \tag{3.10}$$

$$\| \nabla^l(\psi_T u) \|_{0,p';T} \leq c_3 h_T^{-l} \| u \|_{0,p';T}, \tag{3.11}$$

$$\| \nabla^l(\psi_E P\sigma) \|_{0,p';\omega_E} \leq c_4 h_E^{\frac{1}{p'}-l} \| \sigma \|_{p';E}. \tag{3.12}$$

Proof: The upper bounds of Inequalities (3.9) and (3.10) follow from Hölder's inequality, the definition of P, and the observation that $0 \leq \psi_T \leq 1$ on T and $0 \leq \psi_E \leq 1$ on E.

The lower bounds of Inequalities (3.9) and (3.10) and Estimate (3.11) are proven in the usual way by transforming to $\hat{T}$ and $\hat{E}$, resp., using the equivalence of norms on finite dimensional spaces there, and transforming back to T and E, respectively.

In order to prove Estimate (3.12), let $E = T_1 \cap T_2$ with $T_1, T_2 \in \mathcal{T}_h$. Then by Leibniz's rule for differentiation, $\| \nabla^l (\psi_E P\sigma) \|_{0,p';T_1}$ is bounded by a linear combination of terms of the form

$$\| \nabla^{l_1} (\psi_{\hat{E}} \circ F_{T_2}^{-1}) \|_{0,\infty;T_1} \| \nabla^{l_2} (\psi_{\hat{E}} \circ F_{T_1}^{-1}) \|_{0,\infty;T_1} \| \nabla^{l_3} (P\sigma) \|_{0,p';T_1}$$

with $l_1, l_2, l_3 \geq 0$ and $l_1 + l_2 + l_3 = l$. Using Inequality (3.6), the first factor can be bounded by

$$\begin{aligned} \| \nabla^{l_1} (\psi_{\hat{E}} \circ F_{T_2}^{-1}) \|_{0,\infty;T_1} &\leq \left|\left|\left| B_{T_2}^{-1} \right|\right|\right|^{l_1} \| (\nabla^{l_1} \psi_{\hat{E}}) \circ F_{T_2}^{-1} \|_{0,\infty;T_1} \\ &\leq (h_{\hat{T}}/\rho_{T_2})^{l_1} \| (\nabla^{l_1} \psi_{\hat{E}}) \circ F_{T_2}^{-1} \|_{0,\infty;T_1} . \end{aligned}$$

Since $F_{T_2}^{-1}(T_1) \subset B_{\mathbb{R}^n}(0, C_T)$ we have from (3.5)

$$\| (\nabla^{l_1} \psi_{\hat{E}}) \circ F_{T_2}^{-1} \|_{0,\infty;T_1} \leq C_{\mathcal{E},\kappa} .$$

Transforming to the reference element and using (3.5) and (3.6) we obtain

$$\| \nabla^{l_2} (\psi_{\hat{E}} \circ F_{T_1}^{-1}) \|_{0,\infty;T_1} \leq (h_{\hat{T}}/\rho_{T_1})^{l_2} \| \nabla^{l_2} \psi_{\hat{E}} \|_{0,\infty;\hat{T}} \leq C_{\mathcal{E},\kappa} (h_{\hat{T}}/\rho_{T_1})^{l_2} .$$

Since $P\sigma$ is constant in the direction n_E and vanishes on the complement of E with respect to the hyperplane passing through E, we conclude from the shape regularity of $\mathcal{T}_h$ that

$$\| \nabla^{l_3} (P\sigma) \|_{0,p';T_1} \leq c h_E^{1/p'} \| \nabla^{l_3} (P\sigma) \|_{p';E} .$$

Transforming to $\hat{E}$, using the equivalence of norms on $V_{\hat{E}}$, and transforming back to E, we get

$$\| \nabla^{l_3} (P\sigma) \|_{p';E} \leq c h_E^{-l_3} \| \sigma \|_{p';E} .$$

Collecting the above estimates and using the shape regularity of $\mathcal{T}_h$ once more, we conclude that

$$\| \nabla^l (\psi_E P\sigma) \|_{0,p';T_1} \leq c h_E^{1/p' - l} \| \sigma \|_{p';E}$$

with a constant c which only depends on κ and $\sup_{h>0} \sup_{T \in \mathcal{T}_h} h_T/\rho_T$. The same arguments yield a similar estimate for $\| \nabla^l (\psi_E P\sigma) \|_{0,p';T_2}$ and thus establish Estimate (3.12). □

Remark 3.4: The estimates of Proposition 3.3 also hold for "slightly curved" simplices. More precisely, assume that the transformation F_T is no longer affine, but that it still is a diffeomorphism. Let $A_T : \hat{T} \to \mathbb{R}^n$ be the invertible affine mapping which is uniquely determined by the condition that $A_T^{-1} \circ F_T$ leaves the vertices of $\hat{T}$ invariant. Denote by α_T the Gram determinant of the transformationof $\hat{E}$ induced by A_T. A perturbation argument then shows that the estimates of Proposition 3.3 remain valid, provided

$$\| \, |||I - DF_T^{-1} DA_T||| \, \|_{0,\infty;\hat{T}}, \quad \| \, |||I - DA_T^{-1} DF_T||| \, \|_{0,\infty;\hat{T}},$$

$$\| 1 - |\det DF_T|^{-1} |\det DA_T| \, \|_{0,\infty;\hat{T}}, \quad \| 1 - \beta_T^{-1} \alpha_T \|_{0,\infty;\hat{T}}$$

are smaller than a positive threshold which only depends on the constants in the corresponding estimates on $\hat{T}$. Note that the faces and vertices of a curved element T are defined as the images under F_T of the faces respective vertices of $\hat{T}$.

Partitions into generalized n-cubes may be treated in the same way. More precisely, denote by $\hat{Q} := [0, 1]^n$ the reference n-cube and by $\hat{S} := \hat{Q} \cap \{\hat{x} \in \mathbb{R}^n : \hat{x}_n = 0\}$ its reference face. Assume that for every element $Q \in \mathcal{T}_h$ and every complete, smooth $(n-1)$-dimensional submanifold S of ∂Q there exists a diffeomorphism $F_Q : \hat{Q} \to Q$ which maps $\hat{S}$ onto S. Denote by β_Q the Gram determinant of the transformation $\hat{S} \to S$ induced by F_Q. In most applications, the components of F_Q are n-linear functions. The edges of Q then are straight lines. Isoparametric elements of higher order, however, also fall into the category considered here. Let $A_Q : \hat{T} \to \mathbb{R}^n$ be the invertible affine transformation which is uniquely determined by the condition that $A_Q^{-1} \circ F_Q$ leaves invariant the vertices of the reference simplex $\hat{T}$. Denote by α_Q the Gram determinant of the transformation of $\hat{E}$ induced by A_Q. Replacing $\hat{T}$, $\hat{E}$, T, E, and F_T by $\hat{Q}$, $\hat{S}$, Q, S, and F_Q respectively, the results of Proposition 3.3 remain valid provided

$$\| \,|||I - DF_Q^{-1} DA_Q||| \,\|_{0,\infty;\hat{T}}, \quad \| \,|||I - DA_Q^{-1} DF_Q||| \,\|_{0,\infty;\hat{Q}},$$

$$\| 1 - |\det DF_Q|^{-1} |\det DA_Q| \,\|_{0,\infty;\hat{Q}}, \quad \| 1 - \beta_Q^{-1} \alpha_Q \|_{0,\infty;\hat{Q}}$$

are smaller than a positive threshold which only depends on the corresponding estimates on $\hat{Q}$. Geometrically this means that each $Q \in \mathcal{T}_h$ is close to a parallelogram, if $n = 2$, or to a parallelepiped, if $n = 3$. □

4. QUASI-LINEAR ELLIPTIC EQUATIONS OF 2nd ORDER

We consider boundary value problems of the form

$$\begin{aligned} -\nabla \cdot \underline{a}(x, u, \nabla u) &= b(x, u, \nabla u) && \text{in } \Omega \\ u &= 0 && \text{on } \Gamma \end{aligned}$$

where $b \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and $\underline{a} \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ are such that the matrix $A(x, y, z) := \left(\frac{1}{2}(\partial_{z_i} a_j(x, y, z) + \partial_{z_j} a_i(x, y, z))\right)_{1 \le i, j \le n}$ is positive definite for all $x \in \Omega$, $y \in \mathbb{R}$, $z \in \mathbb{R}^n$. In addition we assume that $\underline{a}$ and b are affine with respect to ∇u, i.e.

$$\partial_{z_i} \partial_{z_j} a_k(x, y, z) = \partial_{z_i} \partial_{z_j} b(x, y, z) = 0 \quad \forall 1 \le i, j, k \le n, \quad x \in \Omega, y \in \mathbb{R}, z \in \mathbb{R}^n .$$

This condition was not needed in [18] where we considered Problem (4.1) in the framework of $W^{1,r}$-spaces. Here, it is needed in order to ensure Condition (2.1).

Under suitable growth conditions on $\underline{a}$, b, and their derivatives there are numbers $1 < r$, $p < \infty$ and $r \le s \le \infty$ such that the weak formulation of Problem (4.1) fits into the abstract framework of Section 2 with

$$\begin{aligned} X &:= W_0^{1,r}(\Omega), & \|\cdot\|_X &:= \|\cdot\|_{1,r}, \\ X_- &:= L^r(\Omega), & \|\cdot\|_{X_-} &:= \|\cdot\|_{0,r}, \\ X_+ &:= W_0^{1,s}(\Omega), & \|\cdot\|_{X_+} &:= \|\cdot\|_{1,s}, \\ Y &:= W_0^{1,p'}(\Omega), & \|\cdot\|_Y &:= \|\cdot\|_{1,p'}, \\ Y_+ &:= W^{2,p'}(\Omega) \cap Y, & \|\cdot\|_Y &:= \|\cdot\|_{2,p'}, \\ \langle F(u), \varphi \rangle_Y &:= \int_\Omega \underline{a}(x, u, \nabla u) \cdot \nabla \varphi - \int_\Omega b(x, u, \nabla u)\, \varphi . \end{aligned} \tag{4.2}$$

The condition $DF(u)^* \in \operatorname{Isom}(Y_+, X_-^*)$ is satisfied if and only if the linearized adjoint boundary value problem

$$\begin{aligned} -\nabla \cdot (A(x, u, \nabla u)\nabla w) + \partial_y \underline{a}(x, u, \nabla u) \cdot \nabla w & \\ + \nabla \cdot (\nabla_z b(x, u, \nabla u)\, w) + \partial_y b(x, u, \nabla u) w &= g \quad \text{in } \Omega \\ w &= 0 \quad \text{on } \Gamma \end{aligned}$$

admits for each right-hand side $g \in L^{r'}(\Omega)$ a unique weak solution $w \in W^{2,p'}(\Omega)$ such that $\|w\|_{2,p'} \leq c \|g\|_{0,r'}$. This is an additional condition about elliptic regularity which, in general, is only satisfied if the interior angles of the piecewise straight boundary Γ satisfy additional conditions depending on p and r (*cf.*, e.g., Chapters 4 and 5 in [14] and [8]). If, e.g., $n = 2$ and $p = r > 2$, the claimed elliptic regularity holds for convex polygonal domains Ω (*cf.* Theorem 4.4.7 in [14]).

As a specific example we may consider a nonlinear convection-diffusion equation:

$$\underline{a}(x, u, \nabla u) = k(u)\nabla u$$

$$b(x, u, \nabla u) = f - \underline{c}(x, u) \cdot \nabla u$$

$$f \in L^\infty(\Omega), \underline{c} \in C^1(\Omega \times \mathbb{R}, \mathbb{R}^n), k \in C^2(\mathbb{R})$$

$$k(s) \geq \alpha > 0, |k^{(l)}(s)| \leq \gamma \quad \forall s \in \mathbb{R}, l = 0, 1, 2,$$

$$|\nabla^l \underline{c}(x, s)| \leq \gamma \quad \forall x \in \Omega, s \in \mathbb{R}, l = 0, 1,$$

$$r = p \in (n, 4),$$

$$r < s < \infty .$$

Compare also [4] for a more detailed estimate of the constants $\|DF(u)^*\|_{\mathscr{L}(Y_+, X_-^*)}$ and $\|DF(u)^{*-1}\|_{\mathscr{L}(X_-^*, Y_+)}$ in the case of specific semilinear elliptic pdes within a Hilbert space setting, i.e. $r = p = 2$.

We de not specify the discretization of Problem (4.1) in detail. We only assume that $X_h \subset X \cap W^{1,\infty}(\Omega)$ and $Y_h \subset Y \cap W^{1,\infty}(\Omega)$ are finite element spaces corresponding to $\mathscr{T}_h$ consisting of affinely equivalent elements in the sense of [6] and that $S_{h,0}^{1,0} \subset Y_h$.

In order to construct R_h, $\tilde{F}_h$, and $\tilde{Y}_h$, we fix an integer $\lambda > 0$ and define approximations $\underline{a}_h$ and b_h of $\underline{a}$ and b by

$$\underline{a}_h(x, v_h, \nabla v_h) = \sum_{T \in \mathscr{T}_h} \pi_{\lambda, T}\, \underline{a}(x, v_h, \nabla v_h),$$

$$b_h(x, v_h, \nabla v_h) = \sum_{T \in \mathscr{T}_h} \pi_{\lambda - 1, T}\, b(x, v_h, \nabla v_h).$$

Here, $v_h \in X_h$ is arbitrary. Now, $\tilde{F}_h$ is defined in the same way as F with $\underline{a}$ and b replaced by $\underline{a}_h$ and b_h, respectively, $R_h := I_h$, and

$$\tilde{Y}_h := \operatorname{span}\{\psi_T v, \psi_E P\sigma : v \in \Pi_{\lambda-1|T}, \sigma \in \Pi_{\lambda|E}, T \in \mathscr{T}_h, E \in \mathscr{E}_{h,\Omega}\}.$$

For abbreviation we define for every $T \in \mathscr{T}_h$

$$\varepsilon_T := \Big\{ h_T^{2p} \|\nabla \cdot (\underline{a}(\cdot, u_h, \nabla u_h) - \underline{a}_h(\cdot, u_h, \nabla u_h)) + b(\cdot, u_h \nabla u_h) - b_h(\cdot, u_h, \nabla u_h)\|_{0,p;T}^p$$
$$\Big\{ + \sum_{E \subset \partial T \setminus \Gamma} h_E^{p+1} \|[\underline{n}_E \cdot (\underline{a}(\cdot, u_h, \nabla u_h) - \underline{a}_h(\cdot, u_h, \nabla u_h))]_E\|_{p;E}^p \Big\}^{1/p} \tag{4.3}$$

and

$$\eta_T := \left\{ h_T^{2p} \| \nabla \cdot \underline{a}_h(\,.\,, u_h, \nabla u_h) + b_h(\,.\,, u_h, \nabla u_h) \|_{0,p;T}^p + \sum_{E \subset \partial T \setminus \Gamma} h_E^{p+1} \| [\underline{n}_E \cdot \underline{a}_h(\,.\,, u_h, \nabla u_h)]_E \|_{p;E}^p \right\}^{1/p}. \tag{4.4}$$

Here, $u_h \in X_h$ is an approximate solution of Problem (1.2). The quantity ε_T obviously measures the quality of the approximation of $\underline{a}$ and b by $\underline{a}_h$ and b_h, respectively, and can be estimated explicitly. Below we will show that $\| (Id_{Y_+} - R_h)^* [F(u_h) - \tilde{F}_h(u_h)] \|_{Y_+^*}$ and $\| F(u_h) - \tilde{F}_h(u_h) \|_{Y_h^*}$ are bounded from above by $\left\{ \sum_{T \in \mathscr{E}_h} \varepsilon_T^p \right\}^{1/p}$. When using piecewise linear finite elements, i.e. $X_h \subset S_h^{1,0}$, for the nonlinear convection-diffusion equation given above, the quantity ε_T may roughly be estimated by

$$\varepsilon_T \leqslant c h_T^{\left(2 + \lambda - \frac{n}{p}\right)} \| \nabla u \|_{0,p;T}^2 .$$

The quantity η_T will be used as error estimator.

Using integration by parts elementwise, we obtain for all $\varphi \in Y$

$$\langle F(u_h), \varphi \rangle_Y = \sum_{T \in \mathcal{T}_h} \int_T \{ -\nabla \cdot \underline{a}(x, u_h, \nabla u_h) - b(x, u, \nabla u_h) \}_\varphi + \sum_{E \in \mathscr{E}_{h,\Omega}} \int_E [\underline{n}_E \cdot \underline{a}(x, u_h, \nabla u_h)]_E \, \varphi \tag{4.5}$$

and

$$\langle \tilde{F}(u_h), \varphi \rangle_Y = \sum_{T \in \mathcal{T}_h} \int_T \{ -\nabla \cdot \underline{a}_h(x, u_h, \nabla u_h) - b_h(x, u, \nabla u_h) \}_\varphi + \sum_{E \in \mathscr{E}_{h,\Omega}} \int_E [\underline{n}_E \cdot \underline{a}_h(x, u_h, \nabla u_h)]_E \, \varphi . \tag{4.6}$$

Inequalities (3.2), (3.3), and Hölder's inequality yield for every $\varphi \in Y_+$ with $\| \varphi \|_{Y_+} = 1$

$$\begin{aligned}
&\sum_{T \in \mathcal{T}_h} \int_T \{ -\nabla \cdot (\underline{a}(x, u_h, \nabla u_h) - \underline{a}_h(x, u_h, \nabla u_h)) - b(x, u_h, \nabla u_h) + b_h(x, u_h, \nabla u_h) \} \{ \varphi - I_h \varphi \} \\
&\quad + \sum_{E \in \mathscr{E}_{h,\Omega}} \int_E [\underline{n}_E \cdot (\underline{a}(x, u_h, \nabla u_h) - \underline{a}_h(x, u_h, \nabla u_h))]_E \{ \varphi - I_h \varphi \} \\
&\leqslant \sum_{T \in \mathcal{T}_h} c_{I1} h_T^2 \| \nabla \cdot (\underline{a}(x, u_h, \nabla u_h) - \underline{a}_h(\,.\,, u_h, \nabla u_h)) + b(\,.\,, u_h, \nabla u_h) - b_h(\,.\,, u_h, \nabla u_h) \|_{0,p,T} \| \varphi \|_{2,p';\tilde{\omega}_T} \\
&\quad + \sum_{E \in \mathscr{E}_h, \Omega} c_{I2} h_E^{2 - 1/p'} \| [\underline{n}_E \cdot (\underline{a}(\,.\,, u_h, \nabla u_h) - \underline{a}_h(\,.\,, u_h, \nabla u_h))]_E \|_{p;E} \| \varphi \|_{2,p';\tilde{\omega}_E} \\
&\leqslant c \left\{ \sum_{T \in \mathcal{T}_h} \varepsilon_T^p \right\}^{1/p},
\end{aligned}$$

and

$$\sum_{T \in \mathcal{T}_h} \int_T \{- \nabla \cdot \underline{a}_h(x, u_h, \nabla u_h) - \underline{b}_h(x, u_h, \nabla u_h)\} \{\varphi - I_h \varphi\}$$
$$+ \sum_{E \in \mathcal{E}_{h,\Omega}} \int_E [\underline{n}_E \cdot \underline{a}_h(x, u_h, \nabla u_h)]_E \{\varphi - I_h \varphi\}$$
$$\leq \sum_{T \in \mathcal{T}_h} c_{I1}\, h_T^2 \| \nabla \cdot \underline{a}_h(\,.\,, u_h, \nabla u_h) + b_h(\,.\,, u_h, \nabla u_h) \|_{0,p;T} \| \varphi \|_{2,p';\tilde{\omega}_T}$$
$$+ \sum_{E \in \mathcal{E}_{h,\Omega}} c_{I2}\, h_E^{2-1/p'} \| [\underline{n}_E \cdot \underline{a}_h(\,.\,, u_h, \nabla u_h)]_E \|_{p;E} \| \varphi \|_{2,p';\tilde{\omega}_E}$$
$$\leq c \left\{ \sum_{T \in \mathcal{T}_h} \eta_T^p \right\}^{1/p} .$$

Together with Equations (4.5) and (4.6) this proves that

$$\| (Id_{Y_+} - R_h)^* [F(u_h) - \tilde{F}_h(u_h)] \|_{Y_+^*} \leq c \left\{ \sum_{T \in \mathcal{T}_h} \varepsilon_T^p \right\}^{1/p} \tag{4.7}$$

and

$$\| (Id_{Y_+} - R_h)^* \tilde{F}_h(u_h) \|_{Y_+^*} \leq c \left\{ \sum_{T \in \mathcal{T}_h} \eta_T^p \right\}^{1/p} . \tag{4.8}$$

Consider arbitrary $T \in \mathcal{T}_h$, $E \in \mathcal{E}_{h,\Omega}$, $v \in \Pi_{\lambda - 1|T}$, and $\sigma \in \Pi_{\lambda|E}$. Since $\psi_T v \in C^\kappa(\mathbb{R}^n)$, $\operatorname{supp}(\psi_T v) \subset T$, $\psi_E P\sigma \in C^\kappa(\mathbb{R}^n)$, $\operatorname{supp}(\psi_E P\sigma) \subset \omega_E$, and since $P\sigma$ is constant along lines perpendicular to E, we conclude that

$$\| \psi_T v \|_{0,p';T} \leq c_1\, h_T^2 \| \psi_T v \|_{2,p';T} \tag{4.9}$$

and

$$\| \psi_E P\sigma \|_{0,p';\omega_E} + h_E^{1/p'} \| \psi_E P\sigma \|_{p';E} \leq c_2\, h_E^2 \| \psi_E P\sigma \|_{2,p';\omega_E} \tag{4.10}$$

with constants c_1 and c_2 which only depend on $\sup_{h>0} \sup_{T \in \mathcal{T}_h} h_T / \rho_T$. From Equations (4.5), (4.6), Estimates (4.9), (4.10), and Hölder's inequality we conclude that the following inequalities hold for every $\varphi_h \in \tilde{Y}_h$ with $\| \varphi_h \|_{Y_+} = 1$:

$$\sum_{T \in \mathcal{T}_h} \int_T \{- \nabla \cdot (\underline{a}(x, u_h, \nabla u_h) - \underline{a}_h(x, u_h, \nabla u_h)) - b(x, u_h, \nabla u_h) + b_h(x, u_h, \nabla u_h)\} \varphi_h$$
$$+ \sum_{E \in \mathcal{E}_{h,\Omega}} \int_E [\underline{n}_E \cdot (\underline{a}(x, u_h, \nabla u_h) - \underline{a}_h(x, u_h, \nabla u_h))]_E \varphi_h$$
$$\leq c_1 \sum_{T \in \mathcal{T}_h} h_T^2 \| \nabla \cdot (\underline{a}(\,.\,, u_h, \nabla u_h) - \underline{a}_h(\,.\,, u_h, \nabla u_h)) + b(\,.\,, u_h, \nabla u_h) - b_h(\,.\,, u_h, \nabla u_h) \|_{0,p,T} \| \varphi_h \|_{2,p';T}$$
$$+ c_2 \sum_{E \in \mathcal{E}_{h}, \Omega} h_E^{2-1/p'} \| [\underline{n}_E \cdot (\underline{a}(\,.\,, u_h, \nabla u_h) - \underline{a}_h(\,.\,, u_h, \nabla u_h))]_E \|_{p;E} \| \varphi_h \|_{2,p';\omega_E}$$
$$\leq c' \left\{ \sum_{T \in \mathcal{T}_h} \varepsilon_T^p \right\}^{1/p}$$

and

$$\sum_{T \in \mathcal{T}_h} \int_T \{-\nabla \cdot (\underline{a}(x, u_h, \nabla u_h) - \underline{b}_h(x, u_h, \nabla u_h))\} \varphi_h$$

$$+ \sum_{E \in \mathcal{E}_{h,\Omega}} \int_E [\underline{n}_E \cdot \underline{a}_h(x, u_h, \nabla u_h)]_E \varphi_h$$

$$\leqslant c_1 \sum_{T \in \mathcal{T}_h} h_T^2 \| \nabla \cdot \underline{a}_h(\,.\,, u_h, \nabla u_h) + b_h(\,.\,, u_h, \nabla u_h) \|_{0,p;T} \| \varphi \|_{2,p';T}$$

$$+ c_2 \sum_{E \in \mathcal{E}_{h,\Omega}} h_E^{2-1/p'} \| [\underline{n}_E \cdot \underline{a}_h(\,.\,, u_h, \nabla u_h)]_E \|_{p;E} \| \varphi_h \|_{2,p';\omega_E}$$

$$\leqslant c' \left\{ \sum_{T \in \mathcal{T}_h} \eta_T^p \right\}^{1/p}.$$

Since $\tilde{Y}_h$ is equipped with the norm of Y_+, this yields

$$\| F(u_h) - \tilde{F}_h(u_h) \|_{\tilde{Y}_h^*} \leqslant c \left\{ \sum_{T \in \mathcal{T}_h} \varepsilon_T^p \right\}^{1/p} \tag{4.11}$$

and

$$\| \tilde{F}(u_h) \|_{\tilde{Y}_h^*} \leqslant c \left\{ \sum_{T \in \mathcal{T}_h} \eta_T^p \right\}^{1/p}. \tag{4.12}$$

In order to establish inequality (2.3), consider an arbitrary simplex $T \in \mathcal{T}_h$ and an arbitrary face $E \in \mathcal{E}_{h,\Omega}$ of T and denote by $\tilde{Y}_{h|\omega}$, $\omega \in \{T, \omega_E, \omega_T\}$, the set of all functions $\varphi \in \tilde{Y}_h$ with $\operatorname{supp} \varphi \subset \omega$. Proposition 3.3, Equation (4.6), and the definition of $\tilde{Y}_h$ then yield

$$c_1 c_3^{-1} h_T^2 \| \nabla \cdot \underline{a}_h(\,.\,, u_h, \nabla u_h) + b_h(\,.\,, u_h, \nabla u_h) \|_{0,p;T}$$

$$\leqslant \sup_{v \in \Pi_{\lambda-1|T} \setminus \{0\}} \| \psi_T v \|_{2,p';T}^{-1} \int_T \{-\nabla \cdot \underline{a}_h(x, u_h, \nabla u_h) - b_h(x, u_h, \nabla u_h)\} \psi_T v$$

$$= \sup_{v \in \Pi_{\lambda-1|T} \setminus \{0\}} \| \psi_T v \|_{2,p';T}^{-1} \langle \tilde{F}_h(u_h), \psi_T v \rangle_{Y_+} \tag{4.13}$$

$$\leqslant \sup_{\substack{\varphi_h \in \tilde{Y}_{h|T} \\ \| \varphi_h \|_{Y_+} = 1}} \langle \tilde{F}_h(u_h), \varphi \rangle_{Y_+}$$

and, using Inequality (4.13),

$$c_2 c_4^{-1} h_E^{2-1/p'} \| [\underline{n}_E \cdot \underline{a}_h(. , u_h)]_E \|_{p;E}$$

$$\leqslant \sup_{\sigma \in \Pi_{\lambda|E} \setminus \{0\}} c_4^{-1} h_E^{2-1/p'} \| \sigma \|_{p';E}^{-1} \int_E [\underline{n}_E \cdot \underline{a}_h(. , u_h, \nabla u_h)]_E \psi_E P\sigma$$

$$= \sup_{\sigma \in \Pi_{\lambda|E} \setminus \{0\}} c_4^{-1} h_E^{2-1/p'} \| \sigma \|_{p';E}^{-1} \left\{ \langle \tilde{F}_h(u_h), \psi_E P\sigma \rangle_{Y_+} + \int_{\omega_E} \{ \nabla \cdot \underline{a}_h(x, u_h, \nabla u_h) + b_h(x, u_h, \nabla u_h) \} \psi_E P\sigma \right\} \tag{4.14}$$

$$\leqslant \sup_{\substack{\varphi \in \tilde{Y}_{h|\omega_E} \\ \| \varphi \|_{Y_+} = 1}} \langle \tilde{F}_h(u_h), \varphi \rangle_{Y_+} + h_E^2 \| \nabla \cdot \underline{a}_h(. , u_h, \nabla u_h) + b_h(. , u_h, \nabla u_h) \|_{0,p;\omega_E}$$

$$\leqslant c \sup_{\substack{\varphi \in \tilde{Y}_{h|\omega_E} \\ \| \varphi \|_{Y_+} = 1}} \langle \tilde{F}_h(u_h), \varphi \rangle_{Y_+} .$$

Inequalities (4.13) and (4.14) imply that

$$\eta_T \leqslant c \sup_{\substack{\varphi \in \tilde{Y}_{h|\omega_T} \\ \| \varphi \|_{Y_+} = 1}} \langle \tilde{F}_h(u_h), \varphi \rangle_{Y_+} \tag{4.15}$$

and

$$\left\{ \sum_{T \in \mathcal{T}_h} \eta_T^p \right\}^{1/p} \leqslant c \| \tilde{F}_h(u_h) \|_{\tilde{Y}_h^*} . \tag{4.16}$$

Inequalities (4.8) and (4.16), in particular, prove Inequality (2.3).

The previous estimates together with the abstract results of Section 2 yield the following a posteriori error estimates for Problem (4.1).

PROPOSITION 4.1: *Let $u \in X$ a weak solution of Problem (4.1) which satisfies the conditions of Proposition 2.1 and let $u_h \in X_h$ be an approximate solution of the corresponding discrete problem which is sufficiently close to u in the sense of Proposition 2.1. Then the following a posteriori error estimates hold:*

$$\| u - u_h \|_{0,r} \leqslant c_1 \left\{ \sum_{T \in \mathcal{T}h} \eta_T^p \right\}^{1/p} + c_2 \left\{ \sum_{T \in \mathcal{T}h} \varepsilon_T^p \right\}^{1/p} + c_3 \| R_h^* [F(u_h) - F_h(u_h)] \|_{Y_+^*} + c_4 \| R_h^* F_h(u_h) \|_{Y_+^*}$$

and

$$\eta_T \leqslant c_5 \| u - u_h \|_{0,r;\omega_T} + c_6 \left\{ \sum_{T' \subset \omega_T} \varepsilon_{T'}^p \right\}^{1/p} .$$

Here, ε_T and η_T are given by Equations (4.3) and (4.4) and $\| R_h^* [F(u_h) - F_h(u_h)] \|_{Y_+^*}$ and $\| R_h^* F_h(u_h) \|_{Y_+^*}$ are the consistency error of the discretization and the residual of the discrete problem, respectively. The constants $c_1, \dots c_6$ only depend on $\sup_{h>0} \sup_{T \in \mathcal{T}h} h_T / \rho_T$.

Remark 4.2: When comparing Proposition 4.1 with Proposition 6.1 in [18] we see that we now obtain error estimates which are better by a factor of h. Apart from the different scaling, the structure of the estimator and of the perturbation terms, however, remains unchanged. □

Remark 4.3: Proposition 4.1 also holds for problems with Neumann boundary conditions. One only has to replace Γ by that part of the boundary on which Dirichlet conditions are imposed.

The first estimate of Proposition 4.1 also holds if η_T is defined using the original coefficients $\underline{a}$ and b. The ε_T-term then of course vanishes. □

We conclude this section with a simple example of an a posteriori error estimator which is based on the solution of auxiliary local problems and which generalizes the estimator introduced in [2, 3]. For simplicity we assume that $p = r = 2$. We choose an arbitrary vertex $x_0 \in \mathcal{N}_{h,\Omega}$ and keep it fixed in what follows. Set $\mathcal{T}_0 := \{T \in \mathcal{T}_h : x_0 \in \mathcal{N}(T)\}$ and $\omega_0 := \omega_{x_0}$. Let

$$\hat{X}_h := \hat{Y}_h := \tilde{Y}_{h|\omega_0}$$

and define the operator $B \in \mathcal{L}(\hat{X}_h, \hat{Y}_h^*)$ by

$$\langle Bu, \varphi \rangle_Y := \int_{\omega_0} \nabla\varphi^t A_0 \nabla u \quad \forall u \in \hat{X}_h, \varphi \in \hat{Y}_h ,$$

whee

$$A_0 := A(x_0, u_h(x_0), \pi_{0,\omega_0}(\nabla u_h)) .$$

Note that the operator B is obtained by first linearizing around u_h the differential operator associated with Problem (4.1), then freezing at x_0 the coefficients of the resulting linear operator, and then retaining only the principal part of the linear constant-coefficient operator. Since ∇u_h may be discontinuous, its value at x_0 is approximated by the L^2-projection $\pi_{0,\omega_0}(\nabla u_h)$. Other constructions are of course also possible.

Denote by $0 < \lambda_- \leqslant \lambda_+$ the minimal and maximal eigenvalue of A_0, respectively. We then have

$$\lambda_- \| \nabla u \|^2_{0,2;\omega_0} \leqslant \langle Bu, u \rangle_Y ,$$

$$\langle Bu, \varphi \rangle_Y \leqslant \lambda_+ \| \nabla u \|_{0,2;\omega_0} \| \nabla \varphi \|_{0,2;\omega_0} \quad \forall u \in \hat{X}_h, \varphi \in \hat{Y}_h .$$

This, together with the definition of $\hat{X}_h$, $\hat{Y}_h$, Proposition 3.3, and Inequalities (4.9), (4.10) implies that

$$0 < \underline{c} := \inf_{u \in \hat{X}_h \setminus \{0\}} \sup_{\varphi \in \hat{Y}_h \setminus \{0\}} \frac{\langle Bu, \varphi \rangle_Y}{\| u \|_{X_-} \| \varphi \|_{Y_+}}$$

$$\leqslant \bar{c} := \sup_{u \in \hat{X}_h \setminus \{0\}} \sup_{\varphi \in \hat{Y}_h \setminus \{0\}} \frac{\langle Bu, \varphi \rangle_Y}{\| u \|_{X_-} \| \varphi \|_{Y_+}} < \infty .$$

The constants $\underline{c}$ and $\bar{c}$ only depend on λ_-, λ_+, and $\sup_{h>0} \sup_{T \in \mathcal{T}_h} h_T/\rho_T$. Hence, $B \in \operatorname{Isom}(\hat{X}_h, \hat{Y}_h^*)$ when $\hat{X}_h$ is equipped with the norm of X_- and $\hat{Y}_h$ is endowed with the norm of Y_+. Due to the construction of $\hat{X}_h$ and $\hat{Y}_h$, condition (2.6) is obviously satisfied.

Let $u_0 \in \hat{X}_h$ be the unique solution of

$$\langle Bu_0, \varphi \rangle_Y = \langle \tilde{F}_h(u_h), \varphi \rangle_Y \quad \forall \varphi \in \hat{Y}_h \tag{4.17}$$

and set

$$\eta_{x_0} := \| u_0 \|_{0,2;\omega_0} . \tag{4.18}$$

Note that Problem (4.17) is equivalent to

$$\int_{\omega_0} \nabla\varphi^t A_0 \nabla u_0 = \int_{\omega_0} \underline{a}_h(x, u_h, \nabla u_h) \cdot \nabla\varphi - \int_{\omega_0} b_h(x, u_h, \nabla u_h)\, \varphi \quad \forall \varphi \in \hat{Y}_h .$$

This shows that η_{x_0} falls into the class of error estimators originally introduced in [2, 3].

Proposition 3.3, Equation (4.6), and Inequality (4.15) imply that

$$\underline{c} \| \tilde{F}_h(u_h) \|_{\hat{Y}_h^*} \leq \left\{ \sum_{T \in \mathcal{T}_0} \eta_T^2 \right\}^{1/2} \leq \bar{c} \| \tilde{F}_h(u_h) \|_{\hat{Y}_h^*} .$$

Here, $\hat{Y}_h$ is equipped with the norm of Y_+; the constants $\bar{c}$ und $\underline{c}$ only depend on $\sup_{h>0} \sup_{T \in \mathcal{T}_h} h_T / \rho_T$. Together with Proposition 2.7 this yields the following result.

PROPOSITION 4.4: *Let $x_0 \in \mathcal{N}_{h,\Omega}$ be an arbitrary vertex in the triangulation. Then there are two constants c_1, c_2, which only depend on the polynomial degree of the space X_h and on the quantity $\sup_{h>0} \sup_{T \in \mathcal{T}_h} h_T / \rho_T$, such that*

$$c_1 \left\{ \sum_{T \in \mathcal{T}_0} \eta_T^2 \right\}^{1/2} \leq \eta_{x_0} \leq c_2 \left\{ \sum_{T \in \mathcal{T}_0} \eta_T^2 \right\}^{1/2} .$$

Here η_T and η_{x_0} are given by Equations (4.4) and (4.18), respectively.

Remark 4.5: When comparing η_{x_0} with the corresponding estimator in [18] (*cf.* Equ. (6.15) in [18]), we observe that we use the same auxiliary local problem. But, now, the estimator is the L^2-norm of the solution of the auxiliary local problem instead of its $W^{1,2}$-norm in [18]. □

5. STATIONARY, INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

As an example for the treatment of elliptic systems we consider the stationary, incompressible Navier-Stokes equations

$$\begin{aligned} -\nu \Delta \underline{u} + (\underline{u} \cdot \nabla)\, \underline{u} + \nabla p &= \underline{f} \quad \text{in } \Omega , \\ \nabla \cdot \underline{u} &= 0 \quad \text{in } \Omega , \\ \underline{u} &= 0 \quad \text{on } \Gamma , \end{aligned} \tag{5.1}$$

where $\nu > 0$ is the constant viscosity of the fluid.

In order to cast Problem (5.1) into the framework of Section 2, set

$$M := W_0^{1,2}(\Omega)^n, \quad Q := \left\{ p \in L^2(\Omega) : \int_\Omega p = 0 \right\} ,$$

and define

$$X := M \times Q\,, \quad \|\cdot\|_X := \{\|\cdot\|_{1,2}^2 + \|\cdot\|_{0,2}^2\}^{1/2}\,,$$

$$X_- := L^2(\Omega)^n \times (W^{1,2}(\Omega) \cap Q)^*\,, \quad \|\cdot\|_{X_-} := \{\|\cdot\|_{0,2}^2 + \|\cdot\|_{-1,2}^2\}^{1/2}\,,$$

$$X_+ := X\,, \quad \|\cdot\|_{X_+} := \|\cdot\|_X\,,$$

$$Y := X\,, \quad \|\cdot\|_Y := \|\cdot\|_Y\,,$$

$$Y_+ := (W^{2,2}(\Omega)^n \cap M) \times (W^{1,2}(\Omega) \cap Q)\,, \quad \|\cdot\|_{Y_+} := \{\|\cdot\|_{2,2}^2 + \|\cdot\|_{1,2}^2\}^{1/2}\,,$$

$$\langle F([\underline{u}, p]), [\underline{v}, q]\rangle_Y := \int_\Omega \{\nu\, \nabla \underline{u} \nabla \underline{v} + (\underline{u}\underline{v} - p \nabla \cdot \underline{v} + q \nabla \cdot \underline{u} - \underline{f}\underline{v})\}\,.$$

Since $W^{2,2}(\Omega)$ is continuously imbedded in $L^\infty(\Omega)$, we have for all $\underline{u}$, $\underline{v} \in X$, $\underline{w} \in Y_+$

$$\int_\Omega (\underline{u} \cdot \nabla)\, \underline{v}\underline{w} \leq c\, \|\underline{u}\|_{X_-} \|\underline{v}\|_{X_+} \|\underline{w}\|_{Y_+}\,.$$

Hence, $F \in C^1(X, Y^*)$ and DF satisfies condition (2.1). The condition $DF([\underline{u}, p])^* \in \operatorname{Isom}(Y_+, X_-^*)$ is satisfied if and only if the adjoint, linearized Navier-Stokes problem

$$-\nu\, \Delta \underline{v} - (\underline{u} \cdot \nabla)\, \underline{v} + \underline{v} \cdot (\nabla \underline{u}) - \nabla q = \underline{w} \quad \text{in } \Omega\,,$$

$$-\nabla \cdot \underline{v} = r \quad \text{in } \Omega\,,$$

$$\underline{v} = 0 \quad \text{on } \Gamma\,,$$

admits for each right-hand side $[\underline{w}, r] \in L^2(\Omega)^n \times (W^{1,2}(\Omega) \cap Q)$ a unique weak solution $[\underline{v}, q] \in Y_+$ such that $\|[\underline{v}, q]\|_{Y_+} \leq c\{\|\underline{w}\|_{0,2}^2 + \|r\|_{1,2}^2\}^{1/2}$. Once more, this is an additional condition about elliptic regularity. It is certainly satisfied if Ω is convex and $\nu^{-2}\|f\|_{0,2}$ is sufficiently small.

Let $M_h \subset M$ and $Q_h \subset Q$ be two finite element spaces corresponding to $\mathcal{T}_h$ consisting of affinely equivalent elements in the sense of [6]. We assume that there are two integers k, $l \geq 1$ such that

$$[S_{h,0}^{1,0}]^n \subset M_h \subset [S_h^{k,0}]^n$$

and

$$S_h^{1,0} \cap Q \subset Q_h \subset S_h^{l,0} \quad \text{or} \quad S_h^{0,-1} \cap Q \subset Q_h \subset S_h^{l,-1}\,.$$

Moreover, the spaces M_h, Q_h must satisfy the Babuška-Brezzi condition

$$\inf_{p_h \in Q_h \setminus \{0\}} \sup_{\underline{u}_h \in M_h \setminus \{0\}} \frac{\int_\Omega p_h \nabla \cdot \underline{u}_h}{\|p_h\|_{0,2} \|\underline{u}_h\|_{1,2}} \geq \beta > 0 \tag{5.2}$$

with a constant β independent of h. Examples of spaces M_h, Q_h satisfying the Babuška-Brezzi condition may found, e.g. in [13].

Within the framework of Section 2 we set

$$X_h := Y_h := M_h \times Q_h ,$$

$$\langle F_h([\underline{u}_h, p_h]), [\underline{v}_h, q_h]\rangle_Y := \langle F([\underline{u}_h, p_h], [\underline{v}_h, q_h])\rangle_Y \quad \forall [\underline{u}_h, p_h], [\underline{v}_h, q_h] \in X_h . \tag{5.3}$$

Obviously, the consistency error of this discretization vanishes, i.e.

$$\| R_h^* [F([\underline{u}_h, p_h]) - F_h([\underline{u}_h, p_h])] \|_{Y_+^*} = 0$$

whatever restriction operator we choose. $\tilde{F}_h$ is defined in the same way as F with $\underline{f}$ replaced by

$$\underline{f}_h := \sum_{T \in \mathcal{T}_h} \pi_{0,T} \underline{f} .$$

If the discrete pressures are discontinuous, we denote by J_h the L^2-projection onto $S_h^{0,-1} \cap Q$. Otherwise, we set $J_h := I_h$ with the obvious modifications for the nodes on the boundary Γ. Note, that the error estimate (3.2) also holds for the operator J_h. Using this convention we define

$$R_h[\underline{u}, p] := [I_h u_1, \ldots, I_h u_n, J_h p] ,$$

$$\tilde{Y}_h := \operatorname{span} \{ [\psi_T \underline{v}, 0], [\psi_E P\sigma, 0], [0, \psi_T p] : \underline{v} \in [\Pi_{m|T}]^n, \sigma \in [\Pi_{m'|E}]^n, p \in \Pi_{k-1|T}, T \in \mathcal{T}_h, E \in \mathcal{E}_{h,\Omega} \} ,$$

where $m := \max\{2k-1, l-1\}$ and $m' := \max\{k-1, l\}$. Recalling that $\tilde{Y}_h$ is equipped with the norm of Y_+, we condlude from Inequalities (3.2) and (4.9) that

$$\begin{aligned} &\| (Id_{Y_+} - R_h)^* [\tilde{F}_h(\underline{u}_h, p_h)] - F([\underline{u}_h, p_h]) \|_{Y_+^*} \\ &= \sup_{\substack{[\underline{v}, q] \in Y_+ \\ \|[\underline{v}, q]\|_{Y_+} = 1}} \sum_{T \in \mathcal{T}_h} \sum_{i=1}^n \int_T (f_i - \pi_{0,T} f_i)(v_i - I_h v_i) \\ &\leqslant \sup_{\substack{[\underline{v}, q] \in Y_+ \\ \|[\underline{v}, q]\|_{Y_+} = 1}} \sum_{T \in \mathcal{T}_h} \sum_{i=1}^n c_{I1} h_T^2 \| f_i - \pi_{0,T} f_i \|_{0,2;T} \| v_i \|_{2,2;\tilde{\omega}_T} \\ &\leqslant c \left\{ \sum_{T \in \mathcal{T}_h} h_T^4 \| \underline{f} - \pi_{0,T} \underline{f} \|_{0,2;T}^2 \right\}^{1/2} \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} &\| \tilde{F}_h([\underline{u}_h, p_h]) - F([\underline{u}_h, p_h]) \|_{\tilde{Y}_h^*} \\ &= \sup_{\substack{[\underline{v}_h, q_h] \in \tilde{Y}_h \\ \|[\underline{v}_h, q_h]\|_{Y_+} = 1}} \sum_{T \in \mathcal{T}_h} \int_T (\underline{f} - \pi_{0,T} \underline{f}) \underline{v}_h \\ &\leqslant \sup_{\substack{[\underline{v}_h, q_h] \in \tilde{Y}_h \\ \|[\underline{v}_h, q_h]\|_{Y_+} = 1}} \sum_{T \in \mathcal{T}_h} c_1 h_T^2 \| \underline{f} - \pi_{0,T} \underline{f} \|_{0,2;T} \| v_h \|_{2,2;T} \\ &\leqslant c \left\{ \sum_{T \in \mathcal{T}_h} h_T^4 \| \underline{f} - \pi_{0,T} \underline{f} \|_{0,2;T}^2 \right\}^{1/2} . \end{aligned} \tag{5.5}$$

For abbreviation, we define for $T \in \mathcal{T}_h$

$$\eta_T := \{h_T^4 \| -\nu\,\Delta \underline{u}_h + (\underline{u}_h \cdot \nabla)\,\underline{u}_h + \nabla p_h - \underline{f}_h \|_{0,2;T}^2$$

$$\left\{ + \sum_{E \subset \partial T \setminus \Gamma} h_E^3 \| [\nu \underline{n}_E \nabla \underline{u}_h - p_h\,\underline{n}_E]_E \|_{2;E}^2 + h_T^2 \| \nabla \cdot \underline{u}_h \|_{0,2;T}^2 \right\}^{1/2}. \tag{5.6}$$

This will be our error estimator.

Integration by parts elementwise yields the following representation of the residual

$$\langle \tilde{F}_h([\underline{u}_h, p_h]), [\underline{v}, q] \rangle_Y = \sum_{T \in \mathcal{T}_h} \int_T \{-\nu\,\Delta \underline{u}_h + (\underline{u}_h \cdot \nabla)\,\underline{u}_h + \nabla p_h - \underline{f}_h\}\,\underline{v}$$

$$+ \sum_{E \in \mathcal{E}_{h,\Omega}} \int_E [\underline{n}_E \nabla \underline{u}_h - p_h\,\underline{n}_E]_E\,\underline{v} \tag{5.7}$$

$$+ \sum_{T \in \mathcal{T}_h} \int_T q \nabla \cdot \underline{u}_h \quad \forall [\underline{v}, q] \in Y.$$

Estimates (3.2), (3.3), and Hölder's inequality yield for every $[\underline{v}, q] \in Y_+$ with $\| [\underline{v}, q] \|_{Y_+} = 1$

$$\sum_{T \in \mathcal{T}_h} \int_T \{-\nu\,\Delta \underline{u}_h + (\underline{u}_h \cdot \nabla)\,\underline{u}_h + \nabla p_h - \underline{f}_h\}\,\{\underline{v} - I_h\,\underline{v}\}$$

$$+ \sum_{E \in \mathcal{E}_{h,\Omega}} \int_E [\underline{n}_E \nabla \underline{u}_h - p_h\,n_E]_E \{\underline{v} - I_h\,\underline{v}\} + \sum_{T \in \mathcal{T}_h} \int_T \nabla \cdot \underline{u}_h \{q - J_h\,q\}$$

$$\leqslant \sum_{T \in \mathcal{T}_h} c_{I1}\,h_T^2 \| -\nu\,\Delta \underline{u}_h + (\underline{u}_h \cdot \nabla)\,\underline{u}_h + \nabla p_h - \underline{f}_h \|_{0,2;T} \| \underline{v} \|_{2,2;\tilde{\omega}_T}$$

$$+ \sum_{E \in \mathcal{E}_{h,\Omega}} c_{I2}\,h_E^{3/2} \| [\underline{n}_E \nabla \underline{u}_h - \underline{n}_E\,p_h]_E \|_{2;E} \| \underline{v} \|_{2,2;\tilde{\omega}_E}$$

$$+ \sum_{T \in \mathcal{T}_h} c_{I1}\,h_T \| \nabla \cdot \underline{u}_h \|_{0,2;T} \| q \|_{1,2;\tilde{\omega}_T}$$

$$\leqslant c \left\{ \sum_{T \in \mathcal{T}_h} \eta_T^2 \right\}^{1/2}$$

and, using Equation (5.7),

$$\| [Id_{Y_+} - R_h]^* \tilde{F}_h([\underline{u}_h, p_h]) \|_{Y_+^*} \leqslant \left\{ \sum_{T \in \mathcal{T}_h} \right\}^{1/2}. \tag{5.8}$$

Inequalities (4.9), (4.10), and Hölder's inequality on the other hand imply that for every $[\underline{v}_h, q_h] \in \tilde{Y}_h$ with $\| [\underline{v}_h, q_h] \|_{Y_+} = 1$

$$\begin{aligned}
&\sum_{T \in \mathcal{T}_h} \int_T \{ \nu \Delta \underline{u}_h + (\underline{u}_h \cdot \nabla) \underline{u}_h + \nabla p_h - \underline{f}_h \} \underline{v}_h \\
&\quad + \sum_{E \in \mathcal{E}_{h,\Omega}} \int_E [\underline{n}_E \nabla \underline{u}_h - p_h n_E]_E \underline{v}_h + \sum_{T \in \mathcal{T}_h} \int_T \nabla \cdot \underline{u}_h q_h \\
&\leq \sum_{T \in \mathcal{T}_h} c_1 h_T^2 \| - \nu \Delta \underline{u}_h + (\underline{u}_h \cdot \nabla) \underline{u}_h + \nabla p_h - \underline{f}_h \|_{0,2;T} \| \underline{v}_h \|_{2,2;T} \\
&\quad + \sum_{E \in \mathcal{E}_{h,\Omega}} c_2 h_E^{3/2} \| [\underline{n}_E \nabla \underline{u}_h - p_h \underline{n}_E]_E \|_{2;E} \| \underline{v}_h \|_{2,2;\omega_E} \\
&\quad + \sum_{T \in \mathcal{T}_h} c_1 h_E \| \nabla \cdot \underline{u}_h \|_{0,2;T} \| q_h \|_{1,2;T} \\
&\leq c \left\{ \sum_{T \in \mathcal{T}_h} \eta_T^2 \right\}^{1/2}
\end{aligned}$$

and, using once more Equation (5.7),

$$\| \tilde{F}_h([\underline{u}_h, p_h]) \|_{\tilde{Y}_h^*} \leq c \left\{ \sum_{T \in \mathcal{T}_h} \eta_T^2 \right\}^{1/2}. \tag{5.9}$$

In order to establish Inequality (2.3), we proceed as in Section 4 and use the same notations, too. Equation (5.7), the definition of $\tilde{Y}_h$, and Proposition 3.3 imply that

$$\begin{aligned}
&c_1 c_3^{-1} h_T \| \nabla \cdot \underline{u}_h \|_{0,2;T} \\
&\leq \sup_{q \in \Pi_{k-1|T} \setminus \{0\}} c_3^{-1} h_T \| \psi_T q \|_{0,2;T}^{-1} \int_T \nabla \cdot \underline{u}_h q \psi_T \\
&= \sup_{q \in \Pi_{k-1|T} \setminus \{0\}} c_3^{-1} h_T \| \psi_T q \|_{0,2;T}^{-1} \langle \tilde{F}_h([\underline{u}_h, p_h]), [0, \psi_T q] \rangle_Y \qquad (5.10) \\
&\leq \sup_{\substack{[\underline{v}, q] \in \tilde{Y}_{h|T} \\ \| [\underline{v}, q] \|_{Y_+} = 1}} \langle \tilde{F}_h([\underline{u}_h, p_h]), [\underline{v}, q] \rangle_Y .
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&c_1 c_3^{-1} h_T^2 \| \nu \Delta \underline{u}_h + (\underline{u}_h \cdot \nabla) \underline{u}_h + \nabla p_h - \underline{f}_h \|_{0,2;T} \\
&\leq \sup_{q \in \Pi_{m|T} \setminus \{0\}} \| \psi_T \underline{v} \|_{2,2;T}^{-1} \int_T \{ - \nu \Delta \underline{u}_h + (\underline{u}_h \cdot \nabla) \underline{u}_h + \nabla p_h - f_h \} \psi_T \underline{v} \\
&= \sup_{q \in \Pi_{m|T} \setminus \{0\}} \| \psi_T \underline{v} \|_{0,2;T}^{-1} \langle \tilde{F}_h([\underline{u}_h, p_h]), [\psi_T \underline{v}, 0] \rangle_Y \qquad (5.11) \\
&\leq \sup_{\substack{[\underline{v}, q] \in \tilde{Y}_{h|T} \\ \| [\underline{v}, q] \|_{Y_+} = 1}} \langle \tilde{F}_h([\underline{u}_h, p_h]), [\underline{v}, q] \rangle_Y
\end{aligned}$$

and, using Estimate (5.11),

$$
\begin{aligned}
& c_2 c_4^{-1} h_E^{3/2} \| [\underline{n}_E \nabla \underline{u}_h - p_h \underline{n}_E] \|_{2;E} \\
& \leqslant \sup_{\sigma \in \Pi_{m|E} \setminus \{0\}} c_4^{-1} h_E^{3/2} \| \sigma \|_{2;E}^{-1} \int_E [\underline{n}_E \nabla \underline{u}_h - p_h \underline{n}_E]_E \psi_E P\sigma \\
& = \sup_{\sigma \in \Pi_{m|E} \setminus \{0\}} c_4^{-1} h_E^{3/2} \| \sigma \|_{2;E}^{-1} \Big\{ \langle \tilde{F}_h([\underline{u}_h, p_h]), [\psi_E P\sigma, 0] \rangle_Y \\
& \quad + \int_{\omega_E} \{ -\nu \Delta \underline{u}_h + (\underline{u}_h \cdot \nabla) \underline{u}_h + \nabla p_h - \underline{f}_h \} \psi_E P\sigma \Big\} \\
& \leqslant \sup_{\substack{[\underline{v}, q] \in \tilde{Y}_{h|\omega_E} \\ \| [\underline{v}, q] \|_{Y_+} = 1}} \langle \tilde{F}_h([\underline{u}_h, p_h]), [\underline{v}, q] \rangle_Y \\
& \quad + h_E^2 \| -\nu \Delta \underline{u}_h + (\underline{u}_h \cdot \nabla) \underline{u}_h + \nabla p_h - \underline{f}_h \|_{0,2;\omega_E} \\
& \leqslant c \sup_{\substack{[\underline{v}, q] \in \tilde{Y}_{h|\omega_E} \\ \| [\underline{v}, q] \|_{Y_+} = 1}} \langle \tilde{F}_h([\underline{u}_h, p_h]), [\underline{v}, q] \rangle_Y .
\end{aligned} \tag{5.12}
$$

Inequalities (5.10)-(5.12) imply

$$
\eta_T \leqslant c \sup_{\substack{[\underline{v}, q] \in \tilde{Y}_{h|\omega_T} \\ \| [\underline{v}, q] \|_{Y_+} = 1}} \langle \tilde{F}_h([\underline{u}_h, p_h]), [\underline{v}, q] \rangle_Y \tag{5.13}
$$

and

$$
\left\{ \sum_{T \in \mathcal{T}_h} \eta_T^2 \right\}^{1/2} \leqslant c \| \tilde{F}_h([\underline{u}_h, p_h]) \|_{\tilde{Y}_h^*} , \tag{5.14}
$$

where $\tilde{Y}_h$ is equipped with the norm of Y_+. Inequalities (5.8) and (5.14) in particular establish Condition (2.3).

The previous estimates combined with the results of Section 2 yield the following a posteriori error estimates.

PROPOSITION 5.1: *Let $[\underline{u}, p] \in X$ be a weak solution of Problem (5.1) which satisfies the conditions of Proposition 2.1 and let $[\underline{u}_h, p_h] \in X_h$ be an approximate solution of the corresponding discrete problem which is sufficiently close to $[\underline{u}, p]$ in the sense of Proposition 2.1. Then the following a posteriori error estimates hold:*

$$
\begin{aligned}
\| \underline{u} - \underline{u}_h \|_{0,2} + \| p - p_h \|_{-1,2} \leqslant{} & c_1 \left\{ \sum_{T \in \mathcal{T}_h} \eta_T^2 \right\}^{1/2} + c_2 \left\{ \sum_{T \in \mathcal{T}_h} h_T^4 \| \underline{f} - \pi_{0,T} \underline{f} \|_{0,2;T}^2 \right\}^{1/2} \\
& + c_3 \| R_h^* F_h([\underline{u}_h, p_h]) \|_{Y_+^*}
\end{aligned}
$$

and

$$
\eta_T \leqslant c_4 \{ \| \underline{u} - \underline{u}_h \|_{0,2;\omega_T} + \| p - p_h \|_{-1,2;\omega_T} \} + c_5 \left\{ \sum_{T' \subset \omega_T} h_{T'}^4 \| \underline{f} - \pi_{0,T'} \underline{f} \|_{0,2;T'}^2 \right\}^{1/2} .
$$

Here, η_T is given by Equation (5.6) and $\| R_h^* F_h([\underline{u}_h, p_h]) \|_{Y_+^*}$ is the residual of the discrete problem. The constants $c_1, \ldots, c_5$ only depend on $\sup_{h>0} \sup_{T \in \mathcal{T}_h} h_T / \rho_T$.

Remark 5.2: When comparing Proposition 5.1 with Proposition 8.1 in [18] we see that we now obtain error estimates which are better by a factor of h. Apart from the different scaling the structure of the estimator and of the perturbation terms, however, is the same. □

Remark 5.3: In [18] we considered discretizations of Problem (5.1) which contained additional stabilization terms of the form

$$\delta \sum_{T \in \mathcal{T}_h} h_T^2 \int_T \{ - \nu \, \Delta \underline{u}_h + (\underline{u}_h \cdot \nabla) \, \underline{u}_h + \nabla p_h - \underline{f} \} \{ (\underline{u}_h \cdot \nabla) \, \underline{v}_h + \nabla q_h \}$$

$$+ \delta \sum_{E \in \mathcal{E}_{h,\Omega}} h_E \int_E [p_h]_E [q_h]_E + \alpha\delta \int_\Omega \nabla \cdot \underline{u}_h \, \nabla \cdot \underline{v}_h \, .$$

A proper choice of the parameters $\alpha > 0, \delta > 0$ then yields a stable discretization without any condition on the spaces M_h, Q_h and on the Peclet number $h_T \nu^{-1}$. The corresponding consistency error $\| F([\underline{u}_h, p_h]) - F_h([\underline{u}_h, p_h]) \|_{Y_h^*}$ could be absorbed by the error estimator. In the present context, however, the corresponding consistency error $\| R_h^* [F([\underline{u}_h, p_h]) - F_h([\underline{u}_h, p_h])] \|_{Y_+^*}$ cannot be balanced by the estimator. This is due to a lack of powers of h in the second and third term of the stabilization. □

Remark 5.4: Proposition 5.1 can be extended to the slip boundary condition

$$\underline{u} \cdot \underline{n} = \underline{\underline{T}}(\nu \underline{u}, p) - [\underline{n} \cdot \underline{\underline{T}}(\nu \underline{u}, p) \cdot \underline{n}] \, \underline{n} = 0 \, ,$$

where

$$\underline{\underline{T}}(\underline{u}, p) := \left(\frac{1}{2} (\partial_i u_j + \partial_j u_i) - p \delta_{ij} \right)_{1 \leq i,j \leq n}$$

denotes the stress tensor. One only has to replace $\nu \nabla \underline{u} - p \underline{\underline{I}}$ in Equation (5.6) by $\underline{\underline{T}}(\nu \underline{u}, p)$, and Γ by the part of the boundary on which the no-slip condition $\underline{u} = 0$ is imposed. Here, $\underline{\underline{I}} := (\delta_{ij})_{1 \leq i,j \leq n}$ denotes the unit tensor. □

We conclude with an error estimator which is based on the solution of auxiliary local Stokes problems and which fits into the framework of Proposition 2.7. To this end we asume that ψ_T is constructed as in Example 3.1. We choose an arbitrary vertex $x_0 \in \mathcal{N}_{h,\Omega}$ and keep it fixed in what follows. Let ω_0 and $\mathcal{T}_0$ be as in Section 4.

Put

$$M_0 := \operatorname{span} \{ \psi_T \underline{v}, \psi_E P \sigma : [\underline{v} \in \Pi_{m''|T}]^n, \sigma \in [\Pi_{m'|E}]^n, T \in \mathcal{T}_0, E \in \mathcal{E}(T) \backslash \Gamma \}$$

$$Q_0 := \operatorname{span} \{ \psi_T p : p \in \Pi_{k-1|T}, T \in \mathcal{T}_0 \} \, ,$$

where

$$m := \max \{ 2k - 1, l - 1 \} \, ,$$

$$m' := \max \{ k - 1, l \} \, ,$$

$$m'' := \max \{ m, k - 2 + (k+1)(n+1) \} \, ,$$

and define

$$\hat{X}_h := \hat{Y}_h := M_0 \times Q_0 ,$$

$$\langle B([\underline{v}, q]), [\underline{w}, r] \rangle_Y := \int_{\omega_0} \{ \nu \nabla \underline{v} \nabla \underline{w} - q \nabla \cdot \underline{w} + r \nabla \cdot \underline{v} \} \quad \forall [\underline{v}, q], [\underline{w}, r] \in \hat{X}_h .$$

The definition of m'' and the particular choice of ψ_T imply that $\psi_T \nabla q \in M_0$ for all $q \in Q_0$. Together with Proposition 3.3 this shows that the spaces M_0, Q_0 satisfy an analogue of the Babuška-Brezzi condition (5.2). Hence, $B \in \text{Isom}(\hat{X}_h, \hat{Y}_h^*)$ when $\hat{X}_h$ and $\hat{Y}_h$ are equipped with the norms of X and Y, respectively. This together with Proposition 3.3 and Estimates (4.9), (4.10) implies that we also have $B \in \text{Isom}(\hat{X}_h, \hat{Y}_h^*)$ when $\hat{X}_h$ is equipped with the norm of X_- and $\hat{Y}_h$ is endowed with the norm of Y_+.

Let $[\underline{u}_0, p_0]$ be the unique solution of

$$\langle B([\underline{u}_0, p_0]), [\underline{w}, r] \rangle_Y = \langle \tilde{F}_h([\underline{u}_h, p_h]), [\underline{w}, r] \rangle_Y \quad \forall [\underline{w}, r] \in \hat{Y}_h \tag{5.15}$$

and set

$$\eta_{x_0} := \{ \nu \| \underline{u}_0 \|^2_{0,2;\omega_0} + h^2_{\omega_0} \| p_0 \|^2_{0,2;\omega_0} \}^{1/2} . \tag{5.16}$$

Note, that Problem (5.15) is equivalent to

$$\nu \int_{\omega_0} \nabla \underline{u}_0 \nabla w - \int_{\omega_0} p_0 \nabla \cdot \underline{w} = \int_{\omega_0} \{ \nu \nabla \underline{u}_h \nabla \underline{w} + (\underline{u}_h \cdot \nabla) \underline{u}_h \underline{w} - p_h \nabla \cdot \underline{w} - \underline{f}_h \underline{w} \} \quad \forall \underline{w} \in M_0$$

$$\int_{\omega_0} r \nabla \cdot \underline{u}_0 = \int_{\omega_0} r \nabla \cdot \underline{u}_h \quad \forall r \in Q_0 .$$

Hence, it is a local discrete Stokes problem. We also note, that, on Q_0, we have replaced $\| \cdot \|_{-1,2;\omega_0}$ by the equivalent and more tractable norm $h_{\omega_0} \| \cdot \|_{0,2;\omega_0}$.

Obviously, we have $\tilde{Y}_{h|\omega_0} \subset \hat{Y}_h$. Inequalities (4.9), (4.10) and Equation (5.7) on the other hand imply that

$$\begin{aligned} &\| \tilde{F}_h([\underline{u}_h, p_h]) \|_{\tilde{Y}_h^*} \\ &= \sup_{\substack{[\underline{v}, q] \in \tilde{Y}_h \\ \| [\underline{v}, q] \|_{Y_+} = 1}} \left\{ \sum_{T \in \mathcal{T}_h} \int_T \{ -\nu \Delta \underline{u}_h + (\underline{u}_h \cdot \nabla) \underline{u}_h + \nabla p_h - \underline{f}_h \} \underline{v} \right. \\ &\quad \left. + \sum_{E \in \mathcal{E}_{h,\Omega}} \int_E [\underline{n}_E \nabla \underline{u}_h - p_h \underline{n}_E]_E \underline{v} + \sum_{T \in \mathcal{T}_h} \int_T \nabla \cdot \underline{u}_h q \right\} \\ &\leqslant c \left\{ \sum_{T \in \mathcal{T}_h} \eta_T^2 \right\} . \end{aligned}$$

Together with Inequality (5.13) this proves

$$\| \tilde{F}_h([\underline{u}_h, p_h]) \|_{\tilde{Y}_h^*} \leqslant c \sup_{\substack{[\underline{v}, q] \in \tilde{Y}_{h|\omega_0} \\ \| [\underline{v}, q] \|_{Y_+} = 1}} \langle \tilde{F}_h([\underline{u}_h, p_h]), [\underline{v}, q] \rangle$$

and thus establishes Condition (2.6). Proposition 2.7 therefore yields the following analogue of Proposition 4.4.

PROPOSITION 5.5: *Let $x_0 \in \mathcal{N}_{h,\Omega}$ be an arbitrary vertex in the triangulation. Then there are two constants c_1, c_2, which only depend on the polynomial degree of the space X_h and on the quantity $\sup_{h>0} \sup_{T \in \mathcal{T}_h} h_T/\rho_T$, such that*

$$c_1 \left\{ \sum_{T \in \mathcal{T}_0} \right\}^{1/2} \leq \eta_{x_0} \leq c_2 \left\{ \sum_{T \in \mathcal{T}_0} \eta_T^2 \right\}^{1/2} .$$

Here, η_T and η_{x_0} are given by Equations (5.6) and (5.16), respectively.

Remark 5.6: When comparing η_{x_0} with the corresponding estimator in [18] (*cf.* Equ. (8.17) in [18]) we observe that we use the same auxiliary problem but evaluate different norms of the corresponding solution. □

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