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A PRIORI AND A POSTERIORI ERROR BOUNDS FOR A NONCONFORMING LINEAR FINITE ELEMENT APPROXIMATION OF A NON-NEWTONIAN FLOW (*)

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Abstract. — *We consider a nonconforming linear finite element approximation of a non-Newtonian flow, where the viscosity obeys a Carreau type law for a pseudo-plastic. We prove optimal a priori error bounds for both the velocity and pressure. In addition we present a posteriori error estimators, which are based on the local evaluation of the residuals. These yield global upper and local lower bounds for the error. Finally, we perform some numerical experiments, which confirm our a priori error bounds. © Elsevier, Paris*

Key words : finite elements, nonconforming, error bound, a posteriori error estimators, non-Newtonien flow, Carreau law.

Résumé. — *On s'intéresse à une approximation par éléments finis linéaire non-conforme d'un fluide non-Newtonien, dont la viscosité suit une loi de Carreau pseudo-plastique. Un encadrement optimal de l'erreur, à la fois pour la vitesse et la pression, est déterminé a priori. Par ailleurs sont présentés des estimateurs d'erreur a posteriori, basés sur une évaluation locale de l'erreur résiduelle. Ces estimateurs permettent une majoration globale et une minoration locale de l'erreur. Enfin, des applications numériques viennent valider l'encadrement a priori de l'erreur. © Elsevier, Paris*

1. INTRODUCTION

Let Ω be a bounded open connected set in $\mathbb{R}^2$ with a Lipschitz boundary Γ. Given $f \in [L^2(\Omega)]^2$, we consider the following non-Newtonian flow problem:

$(\mathscr{P})$ Find (u, p) such that

$$-\sum_{j=1}^{2} \frac{\partial}{\partial x_j} [\mu(|D(u)|) D_{ij}(u)] + \frac{\partial p}{\partial x_i} = f_i \quad \text{in } \Omega \quad i = 1 \to 2\,, \tag{1.1a}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega\,, \tag{1.1b}$$

$$u = 0 \quad \text{on } \Gamma \quad \text{and} \quad \int_\Omega p\, dx = 0\,; \tag{1.1c}$$

where $u \equiv (u_1, u_2)^t$ is the velocity, p is the pressure, $f \equiv (f_1, f_2)^t$ is the applied body force and $D(u)$ is the rate of deformation tensor with entries

$$D_{ij}(u) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad i, j = 1 \to 2 \tag{1.2a}$$

and

$$|D(u)|^2 := \sum_{i,j=1}^{2} [D_{ij}(u)]^2\,. \tag{1.2b}$$

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We will assume throughout for ease of exposition that the viscosity μ satisfies the following assumption.

(A) $\mu \in C^1[0, \infty)$ and that there exist positive constants C_μ and M_μ such that

$$M_\mu(t-s) \leqslant \mu(t)\,t - \mu(s)\,s \leqslant C_\mu(t-s) \quad \forall t \geqslant s \geqslant 0\,. \tag{1.3}$$

For example the Carreau law

$$\mu(t) := \mu_\infty + (\mu_0 - \mu_\infty)\,(1 + \lambda t^2)^{\frac{r-2}{2}}\,, \tag{1.4}$$

where $0 < \lambda$, $1 < r \leqslant 2$ and $0 < \mu_\infty < \mu_0$ satisfies (1.3) with $C_\mu = \mu_0$ and $M_\mu = \mu_\infty$.

During recent years, many authors have analysed the finite element approximation of non-Newtonian flow problems using conforming elements. For a priori error bounds, see Baranger and Najib [3], Barrett and Liu [4, 5]; Du and Gunzburger [10] and Sandri [16]. For a posteriori error bounds, see Baranger and El Amri [1, 2]. The nonconforming linear element of Crouzeix and Raviart [8] has proved very effective for the Stokes and Navier-Stokes equations; see Verfürth [18] and Dari, Durán and Padra [9] for the corresponding a posteriori error bounds. Unfortunately, Falk and Morley [11] have shown that this simple nonconforming element does not satisfy a discrete Korn inequality, see (2.13) below, and therefore cannot be used with confidence to approximate the non-Newtonian flow problem $(\mathscr{P})$.

Recently a linear nonconforming element has been introduced by Kouhia and Stenberg [14] for nearly incompressible linear elasticity and Stokes flow in two dimensions. This element consists of a conforming linear approximation for one velocity component, a nonconforming linear approximation for the other component with a piecewise constant approximation for the pressure. In [14] it is shown that this element satisfies the discrete Babuška-Brezzi condition, see (2.12b), and the discrete Korn inequality, see (2.13) below. Furthermore optimal a priori error bounds, $O(h)$, are proved for the velocity approximation in a discrete $H^1(\Omega)$ norm and the pressure approximation in the $L^2(\Omega)$ norm, on assuming that $u \in [H^2(\Omega)]^2$ and $p \in H^1(\Omega)$. In this paper we extend these results to the non-Newtonian flow problem $(\mathscr{P})$. In addition we present a posteriori error estimators, which are based on the local evaluation of the residuals. These yield global upper and local lower bounds for the error. We note that more general assumptions than (1.3) on the viscosity, which allow for singular/degenerate power laws as in [5], lead to a number of technical difficulties with this nonconforming element. These will be addressed elsewhere.

The layout of this paper is as follows. In the next section we introduce our finite element approximation and prove an a priori error bound. In Section 3 we establish an a posteriori error estimator. Finally in Section 4 we report on some numerical experiments. Throughout we adopt the standard notation for Sobolev spaces. For any open bounded set G of $\mathbb{R}^2$, with Lipschitz boundary ∂G, we denote the norm and standard semi-norm of $W^{m,\nu}(G)$ for any nonnegative integer m and $\nu \in [1, \infty]$ by $\|\cdot\|_{m,\nu,G}$ and $|\cdot|_{m,\nu,G}$, respectively. For $\nu = 2$ we adopt the standard notation $H^m(G) \equiv W^{m,2}(G)$, $\|\cdot\|_{m,2,G} \equiv \|\cdot\|_{m,G}$ and $|\cdot|_{m,2,G} \equiv |\cdot|_{m,G}$. We adopt similar notation for the product spaces $[W^{m,\nu}(G)]^2 \equiv W^{m,\nu}(G) \times W^{m,\nu}(G)$ and the trace space $W^{m,\nu}(\gamma)$, where $\gamma \subseteq \partial G$. Finally C and C_i denote positive generic constants independent of the mesh size h.

2. A PRIORI ERROR BOUND

We introduce the following spaces

$$X := H_0^1(\Omega) \times H_0^1(\Omega) \quad \text{with norm} \quad \|v\|_X := \sqrt{|v_1|_{1,\Omega}^2 + |v_2|_{1,\Omega}^2}\,,$$

where $v \equiv (v_1, v_2)^t$, on noting the Poincaré inequality; and

$$M := \left\{ q \in L^2(\Omega) : \int_\Omega q\,dx = 0 \right\} \quad \text{with norm} \quad \|q\|_M := \|q\|_{0,\Omega}\,.$$

Then the weak formulation of problem $(\mathscr{P})$ is:

$(\mathscr{L})$ Find $(u,p)\in X\times M$ such that

$$A(u,v)+B(v,p)=f(v)\quad \forall v\in X, \tag{2.1a}$$

$$B(u,q)=0\quad \forall q\in M; \tag{2.1b}$$

where

$$A(w,v):=\int_\Omega \mu(|D(w)|)\,D(w):D(v)\,dx, \tag{2.2a}$$

$$B(v,q):=-\int_\Omega q\,\mathrm{div}\,v\,dx,\quad f(v):=\int_\Omega f\cdot v\,dx \tag{2.2b}$$

and $D(w):D(v):=\sum_{i,j=1}^{2} D_{ij}(w)\,D_{ij}(v)$.

When the viscosity μ satisfies (1.3) it is easy to show, see [5, Lemma 2.1], that there exist two positive constants C_1 and C_2 such that for all 2×2 real symmetric matrices Y and Z

$$|\mu(|Y|)\,Y-\mu(|Z|)\,Z|\leqslant C_1|Y-Z|, \tag{2.3a}$$

$$C_2|Y-Z|^2\leqslant(\mu(|Y|)\,Y-\mu(|Z|)\,Z):(Y-Z). \tag{2.3b}$$

Here and throughout $|\,.\,|$ is the Euclidean norm as in (1.2b) above. Noting the Korn inequality

$$C\|v\|_X^2\leqslant\int_\Omega |D(v)|^2\,dx\leqslant \|v\|_X^2\quad \forall v\in X, \tag{2.4}$$

see for example [15]; it follows from (2.2a) and (2.3a, b) that there exist two positive constants C_3 and C_4 such that

$$|A(w,v)-A(z,v)|\leqslant C_3\|w-z\|_X\|v\|_X\quad \forall w,z,v\in X, \tag{2.5a}$$

$$C_4\|w-v\|_X^2\leqslant A(w,w-v)-A(v,w-v)\quad \forall w,v\in X. \tag{2.5b}$$

Furthermore it is easy to show that the bilinear form $B(\,\cdot\,,\cdot\,)$ is bounded on $X\times M$ and satisfies the Babuška-Brezzi (BB) inf-sup condition, see [13, p. 81], i.e. there exist positive constants C_5 and C_6 such that

$$|B(v,q)|\leqslant C_5\|v\|_X\|q\|_M\quad \forall v\in X\quad \forall q\in M, \tag{2.6a}$$

$$C_6\|q\|_M\leqslant \sup_{v\in X\setminus\{0\}}\frac{B(v,q)}{\|v\|_X}\quad \forall q\in M. \tag{2.6b}$$

It follows immediately from (2.5a, b) and (2.6a, b) that the problem $(\mathscr{L})$ is well-posed; that is, for all $f\in X'$, the dual of X, there exists a unique solution $(u,p)\in X\times M$ solving $(\mathscr{L})$ and

$$\|u\|_X+\|p\|_M\leqslant C\|f\|_{X'}. \tag{2.7}$$

We now apply the nonconforming linear finite element approximation of Kouhia and Stenberg [14] to the problem $(\mathscr{L})$. For ease of exposition, we assume that Ω is polygonal. Let $\mathscr{T}^h$ be a regular triangulation of Ω which also satisfies the very weak assumption:

(B) Every triangle T of $\mathscr{T}^h$ has a least one vertex in the interior of Ω.

Let

$$S_1^h := \{\chi^h \in L^2(\Omega) : \chi^h|_T \text{ is linear } \forall T \in \mathcal{T}^h, \text{ continuous at the midpoints of the interelement boundaries}\},$$

$$S_2^h := \{\chi^h \in C(\bar{\Omega}) : \chi^h|_T \text{ is linear } \forall T \in \mathcal{T}^h\}.$$

We then set $X^h := X_1^h \times X_2^h$, where

$$X_1^h := \{\chi^h \in S_1^h : \chi^h \text{ vanishes at the midpoints of the edges lying on } \Gamma\},$$

$$X_2^h := S_2^h \cap H_0^1(\Omega)$$

and

$$M^h := \{q^h \in M : q^h|_T \text{ is constant } \forall T \in \mathcal{T}^h\}.$$

Let $h := \max h_T$, where $h_T := \operatorname{diam}(T)$ for all $T \in \mathcal{T}^h$. Then standard approximation theory, see e.g. [6, p. 123], and Jensen's inequality, see e.g. [13, p. 103], yields for $\nu \in (1, 2]$ that

$$\inf_{v^h \in X^h} \|u - v^h\|_{X^h} \leq C\left[\sum_{T \in \mathcal{T}^h} h_T^{2\nu - 2} |u|_{2,\nu,T}^{\nu}\right]^{\frac{1}{\nu}} \leq C h^{2 - \frac{2}{\nu}} |u|_{2,\nu,\Omega}, \tag{2.8a}$$

$$\inf_{q^h \in M^h} \|p - q^h\|_{M} \leq C\left[\sum_{T \in \mathcal{T}^h} h_T^{2\nu - 2} |p|_{1,\nu,T}^{\nu}\right]^{\frac{1}{\nu}} \leq C h^{2 - \frac{2}{\nu}} |p|_{1,\nu,\Omega}, \tag{2.8b}$$

where the unique solution (u, p) of $\mathcal{L}$ is assumed to be such that $u \in [W^{2,\nu}(\Omega)]^2$ and $p \in W^{1,\nu}(\Omega)$. We note that such regularity has been proved for $\nu = 2$ if $f \in [L^2(\Omega)]^2$ and $\Gamma \in C^2$, see [12]. See also [12] for some regularity results in the case when Ω is non-convex polygonal. We note that the results of this paper can be easily adapted to the alternative choice $X^h := X_2^h \times X_1^h$.

Then the corresponding finite element approximation of problem $\mathcal{L}$ is:

$(\mathcal{L}^h)$ Find $(u^h, p^h) \in X^h \times M^h$ such that

$$A_h(u^h, v^h) + B_h(v^h, p^h) = f(v^h) \quad \forall v^h \in X^h, \tag{2.9a}$$

$$B_h(u^h, q^h) = 0 \quad \forall q^h \in M^h; \tag{2.9b}$$

where

$$A_h(w, v) := \sum_{T \in \mathcal{T}^h} \int_T \mu(|D(w)|)\, D(w) : D(v)\, dx \quad \forall w, v \in X + X^h, \tag{2.10a}$$

$$B_h(v, q) := -\sum_{T \in \mathcal{T}^h} \int_T q \operatorname{div} v\, dx \quad \forall v \in X + X^h \quad \forall q \in M. \tag{2.10b}$$

We introduce the following mesh-dependent semi-norm

$$\|v\|_{X_h} := \sqrt{[\|v_1\|_h^2 + \|v_2\|_h^2]} \quad \forall v \equiv (v_1, v_2)^t \in X + X^h, \tag{2.11a}$$

where for $i = 1$ and 2

$$\| v_i \|_h := \sqrt{\sum_{T \in \mathcal{T}^h} |v_i|_{1,T}^2} \,. \tag{2.11b}$$

(2.11a, b) is a norm on X since $\| v \|_{X^h} \equiv \| v \|_X$ if $v \in X$. (2.11a,b) is a norm on X^h since if $\| v^h \|_{X^h} = 0$ for $v^h \equiv (v_1^h, v_2^h)^t \in X^h$, then $\| v_2^h \|_h = 0 \Rightarrow v_2^h = 0$ as $X_2^h \subset H_0^1(\Omega)$ and $\| v_1^h \|_h = 0 \Rightarrow v_1^h$ is piecewise constant, and the zero boundary condition together with continuity at the midpoints yields $v_1^h = 0$.

From Lemma 4.3 and Remark 2.3 in [14], we have that if the triangulation $\mathcal{T}^h$ satisfies the assumption (B) then the bilinear form $B_h(\cdot, \cdot)$ is bounded on $(X + X^h) \times M$ and satisfies the discrete (BB) condition, i.e. there exist positive constants C_7 and C_8 such that

$$|B_h(v, q)| \leqslant C_7 \| v \|_{X^h} \| q \|_M \quad \forall v \in X + X^h \quad \forall q \in M \,, \tag{2.12a}$$

$$C_8 \| q^h \|_M \leqslant \sup_{v^h \in X^h \setminus \{0\}} \frac{B_h(v^h, q^h)}{\| v^h \|_{X^h}} \quad \forall q^h \in M^h \,. \tag{2.12b}$$

Furthermore we have from Lemma 4.5 and Remark 2.3 in [14], (2.11a, b) and (2.4) that the discrete Korn inequality,

$$C \| v \|_{X^h}^2 \leqslant \sum_{T \in \mathcal{T}^h} \int_T |D(v)|^2 \, dx \leqslant \| v \|_{X^h}^2 \quad \forall v \in X + X^h \,, \tag{2.13}$$

holds. It follows immediately from (2.10a), (2.3a,b), (2.13) and (2.11a, b) that there exist two positive constants C_9 and C_{10} such that

$$|A_h(w, v) - A_h(z, v)| \leqslant C_9 \| w - z \|_{X^h} \| v \|_{X^h} \quad \forall w, z, v \in X + X^h \,, \tag{2.14a}$$

$$C_{10} \| w - v \|_{X^h}^2 \leqslant A_h(w, w - v) - A_h(v, w - v) \quad \forall w, v \in X + X^h \,. \tag{2.14b}$$

It follows immediately from (2.14a, b) and (2.12a, b) that the problem $(\mathcal{L}^h)$ is well-posed; that is, for all $f \in [L^2(\Omega)]^2$, there exists a unique solution $(u^h, p^h) \in X^h \times M^h$ solving $(\mathcal{L}^h)$ and

$$\| u^h \|_{X^h} + \| p^h \|_M \leqslant C[\, \| f_1 \|_{0,\Omega} + \| f_2 \|_{0,\Omega}] \,. \tag{2.15}$$

LEMMA 2.1: *Let (u, p) be the unique solution of problem $(\mathcal{L})$ and (u^h, p^h) be the unique solution of problem $(\mathcal{L}^h)$. Then we have the following abstract error bound:*

$$\| u - u^h \|_{X^h} + \| p - p^h \|_M \leqslant C\left[\inf_{v^h \in X^h} \| u - v^h \|_{X^h} + \inf_{q^h \in M^h} \| p - q^h \|_M + \Phi^h \right], \tag{2.16a}$$

where

$$\Phi^h := \sup_{w^h \in X^h \setminus \{0\}} \left[\frac{A_h(u, w^h) + B_h(w^h, p) - f(w^h)}{\| w^h \|_{X^h}} \right]. \tag{2.16b}$$

Proof: Let

$$\tilde{X}^h := \{ v^h \in X^h \colon B_h(v^h, q^h) = 0 \quad \forall q^h \in M^h \} \,. \tag{2.17}$$

For any $v^h \in \tilde{X}^h$ and $q^h \in M^h$, let

$$e := u - u^h, \quad e^a := u - v^h, \quad e^h := v^h - u^h,$$
$$\xi := p - p^h, \quad \xi^a := p - q^h, \quad \xi^h := q^h - p^h. \tag{2.18}$$

Then from (2.9a, b), (2.14a,b), (2.17), (2.12a) and (2.16b) we have that

$$\begin{aligned} C_{10}\|e\|^2_{X^h} &\leqslant A_h(u, e^a) - A_h(u^h, e^a) + A_h(u, e^h) - A_h(u^h, e^h) \\ &= A_h(u, e^a) - A_h(u^h, e^a) + A_h(u, e^h) + B_h(e^h, p^h) - f(e^h) \\ &= A_h(u, e^a) - A_h(u^h, e^a) + A_h(u, e^h) + B_h(e^h, p) - f(e^h) - B_h(e^h, \xi^a) \\ &\leqslant C_9\|e\|_{X^h}\|e^a\|_{X^h} + [\Phi^h + C_7\|\xi^a\|_M]\,[\|e\|_{X^h} + \|e^a\|_{X^h}] \\ &\leqslant C[\|e^a\|_{X^h} + \|\xi^a\|_M + \Phi^h]^2. \end{aligned} \tag{2.19}$$

Furthermore following the standard argument, see [13, p. 155], we have from (2.12a, b) and (2.1b) that

$$\inf_{v^h \in \tilde{X}^h} \|u - v^h\|_{X^h} \leqslant C \inf_{v^h \in X^h} \|u - v^h\|_{X^h}. \tag{2.20}$$

Therefore combining (2.19) and (2.20) we obtain the desired result for $\|u - u^h\|_{X^h}$ in (2.16a).

We now estimate $\|p - p^h\|_M$. From (2.9a) we have for any $v^h \in X^h$ and $q^h \in M^h$ on adopting the notation (2.18) that

$$\begin{aligned} B_h(v^h, \xi^h) &= -B_h(v^h, \xi^a) + B_h(v^h, p) + A_h(u^h, v^h) - f(v^h) \\ &= -B_h(v^h, \xi^a) + A_h(u^h, v^h) - A_h(u, v^h) \\ &\quad + [A_h(u, v^h) + B_h(v^h, p) - f(v^h)]. \end{aligned} \tag{2.21}$$

Therefore from (2.12a, b), (2.21), (2.14a) and (2.16b) we have that

$$\begin{aligned} \|\xi\|_M &\leqslant \|\xi^a\|_M + \|\xi^h\|_M \leqslant \|\xi^a\|_M + (C_8)^{-1} \sup_{v^h \in X^h \setminus \{0\}} \left[\frac{B_h(v^h, \xi^h)}{\|v^h\|_{X^h}}\right] \\ &\leqslant C[\|\xi^a\|_M + \|e\|_{X^h} + \Phi^h]. \end{aligned} \tag{2.22}$$

Therefore we obtain the desired bound for $\|p - p^h\|_M$ in (2.16a). □

In order to analyse the term Φ^h in (2.16a), we introduce some additional notation. For any $T \in \mathcal{T}^h$ we denote its boundary by ∂T and its corresponding outward unit normal by n_T. We set

$$\ell^h := \bigcup_{T \in \mathcal{T}^h} \ell(T) \equiv \ell^h_\Omega \cup \ell^h_\Gamma; \tag{2.23a}$$

where $\ell(T)$ for any $T \in \mathcal{T}^h$ is the set of edges of T,

$$\ell^h_\Omega := \{E \in \ell^h \mid E \cap \Omega \neq \phi\}, \quad \ell^h_\Gamma := \{E \in \ell^h \mid E \subset \Gamma\}. \tag{2.23b}$$

For any $E \in \ell^h$, let h_E denote its length. For $E \in \ell^h_\Omega$ we define its unit normal by $n_E = (n_E^1, n_E^2)^t$ with an arbitrary, but fixed, sign. For all $\chi^h \in X_1^h$ and $E \in \ell^h_\Omega$, we define the jump in χ^h across E by

$$[\chi^h]_E(x) := \chi^h|_{T_2}(x) - \chi^h|_{T_1}(x) \quad \forall x \in E\,, \tag{2.24}$$

where $T_1, T_2 \in \mathcal{T}^h$ are such that $E = \bar{T}_1 \cap \bar{T}_2$ and n_E points out of T_1 and into T_2. For $E \in \ell^h_\Gamma$, its unit normal n_E is chosen to be the outward normal to Ω. For any $E \in \ell^h$ we denote the L^2 projector of $L^1(E)$ onto the space of constant functions on E by Π^h_E; that is,

$$\Pi^h_E\, g := \frac{1}{|E|} \int_E g\, ds \quad \forall g \in L^1(E)\,. \tag{2.25}$$

For all $\chi^h \in X_1^h$, it follows from the continuity at the midpoint edges, the zero boundary condition and (2.24) that for all $g \in L^1(E)$

$$\int_E [\chi^h]_E\, \Pi^h_E\, g\, ds = 0 \quad \forall E \in \ell^h_\Omega, \quad \int_E \chi^h\, \Pi^h_E\, g\, ds = 0 \quad \forall E \in \ell^h_\Gamma\,. \tag{2.26}$$

Finally we have for all $E \in \ell(T)$, $T \in \mathcal{T}^h$ and $\nu \in (1, 2]$ that

$$\left| \int_E (g_1 - \Pi^h_E\, g_1)\, g_2\, ds \right| \leq C h_T^{2 - \frac{2}{\nu}} |g_1|_{1,\nu,T} |g_2|_{1,T} \quad \forall g_1 \in W^{1,\nu}(T) \quad \forall g_2 \in H^1(T)\,. \tag{2.27}$$

This result is proved in Lemma 3 in [8] in the case $\nu = 2$. The proof there is easily generalized to the case $\nu \in (1, 2]$.

THEOREM 2.1: *Let $(u, p) \in [W^{2,\nu}(\Omega)]^2 \times W^{1,\nu}(\Omega)$, $\nu \in (1, 2]$, be the unique solution of $(\mathcal{L})$. Let (u^h, p^h) be the unique solution of $(\mathcal{L}^h)$. Then there exists a positive constant C such that*

$$\begin{aligned} \| u - u^h \|_{X^h} + \| p - p^h \|_M &\leq C \left[\sum_{T \in \mathcal{T}^h} h_T^{2\nu - 2} \left(|u|^\nu_{2,\nu,T} + |p|^\nu_{1,\nu,T} \right) \right]^{\frac{1}{\nu}} \\ &\leq C h^{2 - \frac{2}{\nu}} \left[|u|_{2,\nu,\Omega} + |p|_{1,\nu,\Omega} \right]. \end{aligned} \tag{2.28}$$

Proof: We set I to be the 2×2 unit matrix and

$$\begin{aligned} \mathcal{D}_E(u) &:= \mu(|D(u)|)\,(D_{11}(u)\, n_E^1 + D_{12}(u)\, n_E^2)\,, \\ \mathcal{F}_E(u, p) &:= \mathcal{D}_E(u) - p n_E^1 \quad \forall E \in \ell^h\,. \end{aligned} \tag{2.29}$$

The assumption (1.3) yields immediately that

$$M_\mu \leq \mu(t) \leq C_\mu \quad \text{and} \quad M_\mu \leq [\mu(t)\, t]' \leq C_\mu \quad \forall t \geq 0 \tag{2.30a}$$

and hence there exists a positive constant C such that

$$|\mu'(t)\, t| \leq C \quad \forall t \geq 0\,. \tag{2.30b}$$

From (2.29) and (2.30a, b) it follows for $\nu \in (1, 2]$ that

$$|\mathcal{F}_E(u, p)|_{1,\nu,T} \leq C[\,|u|_{2,\nu,T} + |p|_{1,\nu,T}] \quad \forall T \in \mathcal{T}^h. \tag{2.31}$$

Then for any $w^h \in X^h$ we have on recalling (2.10a, b), (1.1a), integrating by parts and noting (2.29), (2.26), (2.27), (2.31) and Jensen's inequality that

$$
\begin{aligned}
&|A_h(u, w^h) + B_h(w^h, p) - f(w^h)| \\
&= \left| \sum_{T \in \mathcal{T}^h} \int_{\partial T} (\mu(|D(u)|)\, D(u) - pI)\, n_T \cdot w^h \, ds \right| \\
&\leqslant \sum_{E \in \ell^h_\Omega} \left| \int_E [w^h_1]_E\, \mathcal{F}_E(u, p)\, ds \right| + \sum_{E \in \ell^h_\Gamma} \left| \int_E w^h_1\, \mathcal{F}_E(u, p)\, ds \right| \\
&\leqslant \sum_{T \in \mathcal{T}^h} \sum_{E \in \ell(T)} \left| \int_E w^h_1 [\mathcal{F}_E(u, p) - \Pi^h_E\, \mathcal{F}_E(u, p)]\, ds \right| \\
&\leqslant C \sum_{T \in \mathcal{T}^h} h_T^{2 - \frac{2}{\nu}} [\,|u|_{2, \nu, T} + |p|_{1, \nu, T}]\ |w^h_1|_{1, T} \\
&\leqslant C \left[\sum_{T \in \mathcal{T}^h} h_T^{2\nu - 2} (\,|u|^\nu_{2, \nu, T} + |p|^\nu_{1, \nu, T}) \right]^{\frac{1}{\nu}} \|w^h\|_{X^h} .
\end{aligned}
\tag{2.32}
$$

Therefore the desired result (2.28) follows from combining (2.16a, b), (2.8a, b) and (2.32). □

3. A POSTERIORI ERROR ESTIMATORS

We set

$$
R_{E, n} := \begin{cases} [\mu(|D(u^h)|)\,(D_{21}(u^h)\, n^1_E + D_{22}(u^h)\, n^2_E) - p^h\, n^2_E]_E & E \in \ell^h_\Omega , \\ 0 & E \in \ell^h_\Gamma , \end{cases}
\tag{3.1}
$$

and

$$
R_{E, s} := \begin{cases} \left[\dfrac{\partial u^h_1}{\partial s_E} \right]_E & E \in \ell^h_\Omega , \\ -\dfrac{\partial u^h_1}{\partial s_E} & E \in \ell^h_\Gamma ; \end{cases}
\tag{3.2}
$$

where $n_E = (n^1_E, n^2_E)^t$ is the chosen unit normal to E, $s_E = (-n^2_E, n^1_E)^t$ the corresponding unit tangent on E and $[\,\cdot\,]_E$ the jump across E, as defined in (2.24). We note that the constants $R_{E,n}$ and $R_{E,s}$ are independent of the chosen sign for n_E for $E \in \ell^h_\Omega$.

Let $\Pi^h_1 : H^1(\Omega) \to S^h_1$ be the approximation operator which satisfies

$$
\int_E \Pi^h_1 \phi \, ds = \int_E \phi \, ds \quad \forall E \in \ell^h, \quad \forall \phi \in H^1(\Omega) ,
\tag{3.3}
$$

see [8, §5]. Let $\Pi_2^h : H^1(\Omega) \to S_2^h$ be the standard genereliazed interpolation operator, see [7]. For all $T \in \mathcal{T}^h$ and $E \in \ell^h$, we set

$$\omega_T := \bigcup_{\ell(T) \cap \ell(T') \neq \emptyset} T' \quad \text{and} \quad \omega_E := \bigcup_{E \in \ell(T')} T' . \tag{3.4}$$

Then for all $g \in H^1(\Omega)$, we have for $i = 1$ and 2 that

$$\| g - \Pi_i^h g \|_{0,T} \leq C h_T \| g \|_{1,\omega_T} \quad \forall T \in \mathcal{T}^h , \tag{3.5a}$$

$$\| g - \Pi_i^h g \|_{0,E} \leq C h_E^{\frac{1}{2}} \| g \|_{1,\omega_E} \quad \forall E \in \ell(T) ; \tag{3.5b}$$

see [8, §5] and [7]. Note that $\Pi_1^h : H_0^1(\Omega) \to X_1^h$, since $\int_{E \cap \Gamma} \Pi_1^h \phi \, ds = 0 \Rightarrow \Pi_1^h \phi$ vanishes at the midpoint of E. For $g \in H_0^1(\Omega)$ we set $\Pi_2^h g = 0$ on Γ, so that $\Pi_2^h : H_0^1(\Omega) \to X_2^h$ and note that (3.5a, b) still hold. Then we set $\Pi^h : X \to X^h$ such that for all $v \in X$

$$\Pi^h v := (\Pi_1^h v_1, \Pi_2^h v_2)^t \in X^h . \tag{3.6}$$

Finally it is easily deduced for all $\chi_1^h \in X_1^h$ and $\chi_2^h \in S_2^h$, as $D_{ij}(\chi^h)$ is piecewise constant, that

$$\begin{aligned} \sum_{T \in \mathcal{T}^h} \int_{\partial T} \frac{\partial \chi_1^h}{\partial s} \chi_2^h \, ds &= \sum_{T \in \mathcal{T}^h} \int_T \left[- \frac{\partial \chi_1^h}{\partial x_1} \frac{\partial \chi_2^h}{\partial x_2} + \frac{\partial \chi_1^h}{\partial x_2} \frac{\partial \chi_2^h}{\partial x_1} \right] dx \\ &= - \sum_{T \in \mathcal{T}^h} \int_{\partial T} \chi_1^h \frac{\partial \chi_2^h}{\partial s} \, ds = 0 , \end{aligned} \tag{3.7}$$

where $\frac{\partial}{\partial s}$ denotes the tangential derivative on ∂T.

THEOREM 3.1: *Let (u, p) and (u^h, p^h) be the unique solutions of $(\mathcal{L})$ and $(\mathcal{L}^h)$, respectively. Then there exists a positive constant C such that*

$$\| u - u^h \|_{X^h} + \| p - p^h \|_M \leq C \left[\sum_{T \in \mathcal{T}^h} (\eta_T^2 + h_T^2 \| f - f_T \|_{0,T}^2) \right]^{\frac{1}{2}} , \tag{3.8a}$$

where

$$\eta_T^2 := h_T^2 \| f_T \|_{0,T}^2 + \sum_{E \in \ell(T)} h_E^2 (|R_{E,n}|^2 + |R_{E,s}|^2) \quad \forall T \in \mathcal{T}^h \tag{3.8b}$$

and

$$f_T := \frac{1}{|T|} \int_T f \, dx \quad \forall T \in \mathcal{T}^h . \tag{3.8c}$$

Proof: Let $(w, r) \in X \times M$ be the unique solution of the following problem:

$$A(w, r) + B(v, r) = A_h(u^h, v) + B_h(v, p^h) \quad \forall v \in X , \tag{3.9a}$$

$$B(w, q) = 0 \quad \forall q \in M . \tag{3.9b}$$

Let $z := u - w \in X$. Then from (2.5b), (3.9a, b), (2.1a), (2.9a), integrating by parts and noting (3.3) and (3.5a, b) yields that

$$\begin{aligned}
C_4 \|z\|_X^2 &\leqslant A(u, z) - A(w, z) + B(z, p - r) \\
&= A(u, z) + B(z, p) - A_h(u^h, z) - B_h(z, p^h) \\
&= f(z - \Pi^h z) - A_h(u^h, z - \Pi^h z) - B_h(z - \Pi^h z, p^h) \\
&= f(z - \Pi^h z) - \sum_{T \in \mathcal{T}^h} \int_{\partial T} [\mu(|D(u^h)|) D(u^h) - p^h I]\, n_T \cdot (z - \Pi^h z)\, ds \\
&= - \sum_{T \in \mathcal{T}^h} \int_{\partial T} [\mu(|D(u^h)|)(D_{21}(u^h)\, n_T^1 + D_{22}(u^h)\, n_T^2) - p^h n_T^2]\, (z_2 - \Pi_z^h z_2)\, ds + f(z - \Pi^h z) \\
&= \frac{1}{2} \sum_{T \in \mathcal{T}^h} \sum_{E \in \ell(T)} \int_E R_{E,n}(z_2 - \Pi_2^h z_2)\, ds + \int_\Omega f \cdot (z - \Pi^h z)\, dx \\
&\leqslant C \|z\|_X \left[\sum_{T \in \mathcal{T}^h} \left(h_T^2 \|f\|_{0,T}^2 + \sum_{E \in \ell(T)} h_E^2 |R_{E,n}|^2 \right) \right]^{\frac{1}{2}}.
\end{aligned} \tag{3.10}$$

Thus we have that

$$\|u - w\|_X \leqslant C \left[\sum_{T \in \mathcal{T}^h} \left(h_T^2 \|f_T\|_{0,T}^2 + h_T^2 \|f - f_T\|_{0,T}^2 + \sum_{E \in \ell(T)} h_E^2 |R_{E,n}|^2 \right)^2 \right]^{\frac{1}{2}}. \tag{3.11}$$

We now estimate $\|p - r\|_M$. From (2.1a), (3.9a), (2.9a), integrating by parts and noting (3.3) we have for all $v \in X$ that

$$\begin{aligned}
&B(v, p - r) \\
&= -A(u, v) + f(v) + A(w, v) - A_h(u^h, v) - B_h(v, p^h) \\
&= A(w, v) - A(u, v) + f(v - \Pi^h v) - A_h(u^h, v - \Pi^h v) - B_h(v - \Pi^h v, p^h) \\
&= A(w, v) - A(u, v) + f(v - \Pi^h v) \\
&\quad - \sum_{T \in \mathcal{T}^h} \int_{\partial T} [\mu(|D(u^h)|) D(u^h) - p^h I]\, n_T \cdot (v - \Pi^h v)\, ds \\
&= A(w, v) - A(u, v) + f(v - \Pi^h v) \\
&\quad + \frac{1}{2} \sum_{T \in \mathcal{T}^h} \sum_{E \in \ell(T)} \int_E R_{E,n}(v_2 - \Pi_2^h v_2)\, ds.
\end{aligned} \tag{3.12}$$

Therefore from (2.6b), (3.12), (2.5a), (3.5a, b) and (3.11) we obtain that

$$\begin{aligned} \| p - r \|_M &\leqslant C_6^{-1} \sup_{v \in X \setminus \{0\}} \frac{B(v, p-r)}{\| v \|_X} \\ &\leqslant C \left\{ \| u - w \|_X + \left[\sum_{T \in \mathcal{T}^h} \left(h_T^2 \| f \|_{0,T}^2 + \sum_{E \in \ell(T)} h_E^2 |R_{E,n}|^2 \right) \right]^{\frac{1}{2}} \right\} \\ &\leqslant C \left[\sum_{T \in \mathcal{T}^h} \left(h_T^2 \| f_T \|_{0,T}^2 + h_T^2 \| f - f_T \|_{0,T}^2 + \sum_{E \in \ell(T)} h_E^2 |R_{E,n}|^2 \right) \right]^{\frac{1}{2}} . \end{aligned} \tag{3.13}$$

We now estimate $\| w - u^h \|_{X^h}$ and $\| r - p^h \|_M$. From (2.6b), (3.9a) and (2.14a) we have that

$$\begin{aligned} \| r - p^h \|_M &\leqslant C_6^{-1} \sup_{v \in X \setminus \{0\}} \frac{B(v, r - p^h)}{\| v \|_X} = C_6^{-1} \sup_{v \in X \setminus \{0\}} \frac{B(v, r) - B_h(v, p^h)}{\| v \|_X} \\ &= C_6^{-1} \sup_{v \in X \setminus \{0\}} \frac{A_h(u^h, v) - A(w, v)}{\| v \|_X} \leqslant C \| w - u^h \|_{X^h} . \end{aligned} \tag{3.14}$$

Let Ψ denote the Airy operator defined by

$$\Psi(\phi) := \begin{pmatrix} \dfrac{\partial^2 \phi}{\partial x_2^2} & -\dfrac{\partial^2 \phi}{\partial x_1 \partial x_2} \\ -\dfrac{\partial^2 \phi}{\partial x_1 \partial x_2} & \dfrac{\partial^2 \phi}{\partial x_1^2} \end{pmatrix} \quad \forall \phi \in H^2(\Omega) . \tag{3.15}$$

We then have the following orthogonal decomposition, see e.g. [11, §2],

$$\mathbb{L}^2(\Omega) = D(X) \oplus \Psi(H^2(\Omega)) , \tag{3.16}$$

where

$$\begin{aligned} \mathbb{L}^2(\Omega) &:= \{ (\tau_{ij})_{i,j=1,2} : \tau_{ij} \in L^2(\Omega) \quad \text{and} \quad \tau_{ij} = \tau_{ji} \} , \\ D(X) &:= \{ D(v) : v \in X \} , \\ \Psi(H^2(\Omega)) &:= \{ \Psi(\phi) : \phi \in H^2(\Omega) \} . \end{aligned}$$

From (3.9a) and (3.16), we know that there exists a $\phi \in H^2(\Omega)$ such that for all $T \in \mathcal{T}^h$

$$\Psi(\phi) = \mu(|D(w)|) D(w) - \mu(|D(u^h)|) D(u^h) - (r - p^h) I \quad \text{on} \quad T . \tag{3.17}$$

We choose $\phi \in H^2(\Omega)$ uniquely such that for all $g \in H^2(\Omega)$

$$\int_\Omega \Psi(\phi) : \Psi(g) \, dx = \sum_{T \in \mathcal{T}^h} \int_T [\mu(|D(w)|) D(w) - \mu(|D(u^h)|) D(u^h) - (r - p^h) I] : \Psi(g) \, dx , \tag{3.18a}$$

$$\int_\Omega \phi \, dx = \int_\Omega \frac{\partial \phi}{\partial x_1} dx = \int_\Omega \frac{\partial \phi}{\partial x_2} dx = 0 . \tag{3.18b}$$

It then follows from (3.15), (3.18a, b), (2.3a) and (3.14) that

$$\| \phi \|_{2,\Omega} \leq C |\phi|_{2,\Omega} \leq C(\| w - u^h \|_{X^h} + \| r - p^h \|_M) \leq C \| w - u^h \|_{X^h} . \tag{3.19}$$

From (2.14b), (3.9a, b), (2.9b), noting that div $u^h = 0$ a.e. in Ω, (3.17), (3.7), (3.5b) and (3.19) we have that

$$\begin{aligned} C_{10} \| w - u^h \|^2_{X^h} &\leq A_h(w, w - u^h) - A_h(u^h, w - u^h) = A_h(u^h, u^h) - A_h(w, u^h) \\ &= A_h(u^h, u^h) - A_h(w, u^h) + B_h(u^h, p^h - r) \\ &= - \sum_{T \in \mathcal{T}^h} \int_T \Psi(\phi) : D(u^h) \, dx \\ &= \sum_{T \in \mathcal{T}^h} \int_{\partial T} \left[\frac{\partial \phi}{\partial x_2} \frac{\partial u_1^h}{\partial s} - \frac{\partial \phi}{\partial x_1} \frac{\partial u_2^h}{\partial s} \right] ds = \sum_{T \in \mathcal{T}^h} \int_{\partial T} \frac{\partial \phi}{\partial x_2} \frac{\partial u_1^h}{\partial s} \, ds \\ &= \sum_{T \in \mathcal{T}^h} \int_{\partial T} \left[\frac{\partial \phi}{\partial x_2} - \Pi_2^h \left(\frac{\partial \phi}{\partial x_2} \right) \right] \frac{\partial u_1^h}{\partial s} \, ds \\ &= - \sum_{E \in \ell^h} \int_E \left[\frac{\partial \phi}{\partial x_2} - \Pi_2^h \left(\frac{\partial \phi}{\partial x_2} \right) \right] R_{E,s} \, ds \\ &\leq C \left[\sum_{E \in \ell^h} h_E^2 |R_{E,s}|^2 \right]^{\frac{1}{2}} \| \phi \|_{2,\Omega} \leq C \left[\sum_{E \in \ell^h} h_E^2 |R_{E,s}|^2 \right] . \end{aligned} \tag{3.20}$$

Therefore combining (3.11), (3.13), (3.20) and (3.14) we obtain the desired result (3.8a). $\square$

Below we will derive upper bounds for the error estimators η_T^2. Let $\mathcal{P}_k$ denote the space of polynomials of degree $\leq k$ (in 2 variables). Following [17, p. 454], let $\hat{T} := \{ \hat{x} \in \mathbb{R}^2 : 0 \leq \hat{x}_1 + \hat{x}_2 \leq 1, 0 \leq \hat{x}_i \, i = 1, 2 \}$ be the reference triangle with $\hat{E} := \hat{T} \cap \{ \hat{x} \in \mathbb{R}^2 : \hat{x}_2 = 0 \}$. Let $\hat{x}_{\hat{T}}$ and $\hat{x}_{\hat{E}}$ be the barycentres of $\hat{T}$ and $\hat{E}$, respectively. Then there exist two unique functions $\psi_{\hat{T}}, \psi_{\hat{E}} \in C^\infty(\hat{T})$ such that

$$\psi_{\hat{T}} \in \mathcal{P}_3, \quad \psi_{\hat{T}}(\hat{x}_{\hat{T}}) = 1, \quad \psi_{\hat{T}} = 0 \quad \text{on } \partial \hat{T} ; \tag{3.21}$$

$$\psi_{\hat{E}} \in \mathcal{P}_2, \quad \psi_{\hat{E}}(\hat{x}_{\hat{E}}) = 1, \quad \psi_{\hat{E}} = 0 \quad \text{on } \partial \hat{T} \backslash \hat{E} . \tag{3.22}$$

From (3.21) and (3.22) it follows that

$$0 \leq \psi_{\hat{T}}(\hat{x}) \leq 1, \quad 0 \leq \psi_{\hat{E}}(\hat{x}) \leq 1 \quad \forall \hat{x} \in \hat{T} . \tag{3.23}$$

For any $T \in \mathcal{T}^h$ with $E \subset \partial T$, let $F_T : \hat{T} \to T$ be the invertible linear map such that $\hat{T}$ is mapped onto T and $\hat{E}$ is mapped onto E. Then we define the polynomial cut-off function $\psi_T(x) := \psi_{\hat{T}}(F_T^{-1}(x))$ if $x \in T$ and zero otherwise. Similarly, we define $\psi_E(x) := \psi_{\hat{E}}(F_T^{-1}(x))$ if $x \in T \subset \omega_E$ and zero if $x \notin \omega_E$. It follows from (3.21)-(3.23) that

$$\begin{gathered} \operatorname{supp} \psi_T \subset T, \quad \max_{x \in T} \psi_T(x) = 1, \quad \psi_T \in \mathcal{P}_3, \\ \psi_T \geq 0 \text{ on } T \quad \text{and} \quad \psi_T = 0 \text{ on } \partial T ; \end{gathered} \tag{3.24}$$

$$\begin{gathered} \operatorname{supp} \psi_E \subseteq \omega_E, \quad \max_{x \in \omega_E} \psi_E(x) = 1, \quad \psi_E|_{T \subset \omega_E} \in \mathcal{P}_2, \\ \psi_E \geq 0 \text{ on } \omega_E \quad \text{and} \quad \psi_E = 0 \text{ on } \partial \omega_E \backslash E . \end{gathered} \tag{3.25}$$

In addition it is easily established that there exist positive constants C_i, independent of h_T and h_E, such that

$$C_1\, h_T \leqslant \| \psi_T^{\frac{1}{2}} \|_{0,T} \quad \forall T \in \mathcal{T}^h, \quad C_2\, h_E^{\frac{1}{2}} \leqslant \| \psi_E^{\frac{1}{2}} \|_{0,E} \quad \forall E \in \ell^h ; \tag{3.26}$$

and the inverse inequalities

$$| \psi_T |_{1,T} \leqslant C_3\, h_T^{-1} \| \psi_T \|_{0,T} \leqslant C_4 \quad \forall T \in \mathcal{T}^h, \tag{2.27a}$$

$$| \psi_E |_{1,\omega_E} \leqslant C_5\, h_E^{-1} \| \psi_E \|_{0,\omega_E} \leqslant C_6 \quad \forall E \in \ell^h \tag{3.27b}$$

hold; see [17, Lemma 5.1].

THEOREM 3.2: *Let the assumptions of Theorem 3.1 hold. Then there exists a positive constant C such that*

$$\eta_T^2 \leqslant C \sum_{T' \subset \omega_T} [\, | u - u^h |_{1,T'}^2 + \| p - p^h \|_{0,T'}^2 + h_{T'}^2 \| f - f_{T'} \|_{0,T'}^2] \,. \tag{3.28}$$

Proof: Let $T \in \mathcal{T}^h$ and $v \in [H_0^1(\omega_T)]^2$. Extending v by 0 outside ω_T to a function in X and using integration by parts on the terms involving u^h and p^h, we obtain

$$\begin{aligned}
\varepsilon_T(v) := & \sum_{T' \subset \omega_T} \int_{T'} f_{T'} \cdot v \, dx - \sum_{E \in \ell(T)} \int_E R_{E,n}\, v_2 \, ds \\
= & \sum_{T' \subset \omega_T} \int_{T'} (f_{T'} - f) \cdot v \, dx + \sum_{E \in \ell(T)} \int_E \tilde{R}_{E,n}\, v_1 \, ds \\
& + \sum_{T' \subset \omega_T} \int_{T'} [\mu(|D(u)|)\, D(u) - \mu(|D(u^h)|)\, D(u^h) - (p - p^h)\, I] : D(v)\, dx \,;
\end{aligned} \tag{3.29}$$

where

$$\tilde{R}_{E,n} := \begin{cases} [\mu(|D(u^h)|)\,(D_{11}(u^h)\, n_E^1 + D_{12}(u^h)\, n_E^2) - p^h\, n_E^1]_E & E \in \ell_\Omega^h, \\ -\,[\mu(|D(u^h)|)\,(D_{11}(u^h)\, n_E^1 + D_{12}(u^h)\, n_E^2) - p^h\, n_E^1] & E \in \ell_\Gamma^h . \end{cases} \tag{3.30}$$

Choosing $v \equiv v_T := \psi_T f_T$ in (3.29), noting (3.26), (3.24), (3.27a) and (2.3a) we obtain for all $T \in \mathcal{T}^h$ that

$$\begin{aligned}
h_T^2 \| f_T \|_{0,T}^2 & \leqslant C h_T^2\, \varepsilon_T(v_T) \\
& \leqslant C [\, | u - u^h |_{1,T}^2 + \| p - p^h \|_{0,T}^2 + h_T^2 \| f - f_T \|_{0,T}^2] \,.
\end{aligned} \tag{3.31}$$

Choosing $v \equiv v_E := \psi_E(0, R_{E,n})^t$ in (3.29), noting (3.26), (3.25), (3.27b) and (3.31) we obtain for all $E \in \ell(T) \cap \ell_\Omega^h$ that

$$\begin{aligned}
h_E^2 | R_{E,n} |^2 & \leqslant C h_E \left[-\, \varepsilon_T(v_E) + \sum_{T' \subset \omega_T} \int_{T'} f_{T'} \cdot v_E \, dx \right] \\
& \leqslant C \sum_{T' \subset \omega_E} [\, | u - u^h |_{1,T'}^2 + \| p - p^h \|_{0,T'}^2 + h_{T'}^2 \| f - f_{T'} \|_{0,T'}^2] \,.
\end{aligned} \tag{3.32}$$

Let $E \in \ell^h$, and $g \in H^1(\omega_E)$ with $g = 0$ on $\partial\omega_E \backslash E$. Extending g by 0 outside ω_E to a function in $H^1(\Omega)$ and performing integration by parts, we obtain

$$\begin{aligned} \varepsilon_E(g) := \int_E R_{E,s}\, g\, ds &= \sum_{T' \subset \omega_E} \int_{\partial T'} \frac{\partial}{\partial s}(u_1 - u_1^h)\, g\, ds \\ &= \sum_{T' \subset \omega_E} \int_{T'} \left[\frac{\partial}{\partial x_2}(u_1 - u_1^h) \frac{\partial g}{\partial x_1} - \frac{\partial}{\partial x_1}(u_1 - u_1^h) \frac{\partial g}{\partial x_2} \right] dx . \end{aligned} \tag{3.33}$$

Choosing $g \equiv g_E := \psi_E R_{E,s}$ in (3.33), noting (3.26) and (3.27b) we obtain for all $E \in \ell^h$ that

$$\begin{aligned} h_E^2 |R_{E,s}|^2 \leqslant C h_E\, \varepsilon_E(g_E) &\leqslant C h_E |R_{E,s}| \sum_{T' \subset \omega_E} |u_1 - u_1^h|_{1,T'} \\ &\leqslant C \sum_{T' \subset \omega_E} |u_1 - u_1^h|_{1,T'}^2 . \end{aligned} \tag{3.34}$$

Combining (3.31), (3.32) and (3.34) yields the desired result (3.28). □

4. NUMERICAL RESULTS

We consider the case of the Carreau law, (1.4) with $\lambda = \mu_0 = 1$, $\mu_\infty = 0.5$ and various choices of $r \in (1, 2]$. We set $\Omega = (0, 1) \times (0, 1)$ an choose f, for each choice of r, such that the unique solution of (1.1a-c) is

$$u_1(x) = x_1^2(1 - x_1)^2 x_2(1 - x_2)(1 - 2x_2) , \tag{4.1a}$$

$$u_2(x) = -x_1(1 - x_1)(1 - 2x_1) x_2^2(1 - x_2)^2 , \tag{4.1b}$$

$$p(x) = 4 x_1 x_2 - 1 . \tag{4.1c}$$

Table 1. — $r = 2.0$

Mesh	A	B	C	D
$\max \lvert u_1 - u_1^h \rvert$	0.833E-1	0.326E-1	0.952E-2	0.251E-2
$\max \lvert u_2 - u_2^h \rvert$	0.0	0.126E-1	0.383E-2	0.114E-2
$\lVert u_1 - u_1^h \rVert_{0,\Omega}$	0.342E-1	0.119E-1	0.356E-2	0.956E-3
$\lVert u_2 - u_2^h \rVert_{0,\Omega}$	0.298E-2	0.386E-2	0.151E-2	0.422E-3
$\lVert u_1 - u_1^h \rVert_h$	0.239	0.162	0.912E-1	0.477E-1
$\lVert u_2 - u_2^h \rVert_h$	0.204E-1	0.419E-1	0.164E-1	0.558E-2
$\lVert p - p^h \rVert_{0,\Omega}$	0.498	0.249	0.125	0.626E-1

Table 2. — $r = 1.5$

Mesh	A	B	C	D
$\max \lvert u_1 - u_1^h \rvert$	0.839E-1	0.327E-1	0.954E-2	0.252E-2
$\max \lvert u_2 - u_2^h \rvert$	0.0	0.126E-1	0.384E-2	0.114E-2
$\lVert u_1 - u_1^h \rVert_{0,\Omega}$	0.344E-1	0.119E-1	0.357E-2	0.956E-3
$\lVert u_2 - u_2^h \rVert_{0,\Omega}$	0.298E-2	0.387E-2	0.152E-2	0.423E-3
$\lVert u_1 - u_1^h \rVert_h$	0.241	0.162	0.913E-1	0.477E-1
$\lVert u_2 - u_2^h \rVert_h$	0.204E-1	0.420E-1	0.164E-1	0.558E-2
$\lVert p - p^h \rVert_{0,\Omega}$	0.498	0.249	0.125	0.626E-1

Table 3. — $r = 1.001$

Mesh	A	B	C	D
$\max \lvert u_1 - u_1^h \rvert$	0.845E-1	0.328E-1	0.955E-2	0.252E-2
$\max \lvert u_2 - u_2^h \rvert$	0.0	0.127E-1	0.385E-2	0.114E-2
$\lVert u_1 - u_1^h \rVert_{0,\Omega}$	0.346E-1	0.120E-1	0.357E-2	0.957E-3
$\lVert u_2 - u_2^h \rVert_{0,\Omega}$	0.298E-2	0.389E-2	0.152E-2	0.424E-3
$\lVert u_1 - u_1^h \rVert_h$	0.242	0.163	0.914E-1	0.477E-1
$\lVert u_2 - u_2^h \rVert_h$	0.204E-1	0.422E-1	0.164E-1	0.559E-2
$\lVert p - p^h \rVert_{0,\Omega}$	0.498	0.249	0.125	0.626E-1

We computed the finite element approximation (u^h, p^h), see (2.9a, b). For computational ease, we employed numerical integration on the right hand side of (2.9a). We chose a quadrature rule over each triangle, which integrated a cubic exactly. It is simple matter to show that the a priori error estimate (2.28), with $\nu = 2$, remains valid; as it would do even for a cruder rule. Four meshes were used in our computations. Mesh A consisted of eight similar triangles obtained by subdividing Ω using the lines $x_1 = 0.5$, $x_2 = 0.5$, $x_2 = x_1$ and $x_2 = 1 - x_1$. Mesh B was generated by subdividing every triangle in mesh A into four similar triangles. Meshes C and D were similarly generated from meshes B and C. We note that these meshes satisfied the weak assumption (B).

Tables 1-3 show, for $i = 1$ and 2, the maximum of the error $\lvert u_i - u_i^h \rvert$ over the mesh points, $\lVert u_i - u_i^h \rVert_{0,\Omega}$, $\lVert u_i - u_i^h \rVert_h$ and $\lVert p - p^h \rVert_{0,\Omega}$ for $r = 2$, 1.5 and 1.001, respectively. The integral norms were approximated using the same quadrature rule as described above. We see that the numerical results confirm our a priori error bound (2.28) with $\nu = 2$ for this simple model problem. It should be noted that mesh A is very crude; especially for X_2^h, which has only one degree of freedom.

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