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GLOBAL BV SOLUTIONS FOR A MODEL OF MULTI-SPECIES MIXTURE IN POROUS MEDIA (*)

Youcef AMIRAT and Yue-Jun PENG

Abstract. — *We consider an initial boundary value problem for a nonlinear differential system consisting of one equation of parabolic type coupled with a $n \times n$ semi-linear hyperbolic system of first order. This system of equations describes a model of compressible miscible displacement of $n+1$ chemical species in porous media, in the absence of diffusion and dispersion. In one-dimensional space, we construct a global weak solution with bounded total variation for the concentration.* © *Elsevier, Paris*

Résumé. — *On considère un système différentiel constitué d'une équation parabolique couplée avec un système $n \times n$ semi-linéaire hyperbolique du 1er ordre. Ce système d'équations intervient dans la modélisation mathématique de l'écoulement d'un mélange à $n+1$ constituants miscibles compressibles en milieu poreux, en absence de diffusion moléculaire et de dispersion. On considère le cas unidimensionnel et on suppose que les données initiales et aux limites des concentrations sont à variations bornées. On démontre l'existence globale en temps d'une solution faible du problème qui est à variation bornée pour les concentrations.* © *Elsevier, Paris*

1. INTRODUCTION

We consider the compressible miscible displacement of a multi-species mixture in porous media. These kinds of models have been described by many authors, see for instance Bear [8], Scheidegger [16], Peaceman [15], Douglas and Roberts [9]-[10]. When the viscosity of the fluid is taken to be constant, the mathematical analysis of these models has been studied by Amirat-Hamdache-Ziani. They proved the existence of global weak solutions in L^∞ for an initial boundary value problem in three space dimensions, in the case of two species mixture which corresponds to a single concentration equation $n=1$ coupled with a parabolic one. This solution is obtained by a compensated compactness argument applied to the approximate solution constructed by the viscosity method [4]-[5]. By the same technique, they also treated some homogenization problems in one space dimension [3]. All these results have been generalized by Amirat-Peng [6], to the case of $n+1$ species mixture for which the concentration variables satisfy a hyperbolic system of n equations. Thus we need some more techniques to achieve the results.

In this paper, we are interested in a simple one-dimensional model of $n+1$ species mixture, which can be viewed as modelling experiments in a core sample. We neglect the molecular diffusion, the dispersion, the gravitational terms, the injection and production source terms. We assume always that the viscosity of the fluid mixture is constant. Then this model can be written as the following parabolic-hyperbolic system:

$$a(u)\,\partial_t p + \partial_x q = 0, \quad q = -\,\partial_x p\,, \tag{1.1}$$

$$\partial_t u_i + q\,\partial_x u_i + b_i(u)\,\partial_t p = 0, \quad \text{in } \Omega_T, \quad 1 \leqslant i \leqslant n\,, \tag{1.2}$$

equipped with the initial and boundary conditions

$$\begin{cases} p(x,0) = p_0(x), & x \in \Omega\,, \\ p(0,t) = 0, \quad q(1,t) = q_1(t), & t \in \,]0,T[\;, \end{cases} \tag{1.3}$$

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and

$$u(0,t)=u^1(t),\quad u(x,0)=u^0(x),\quad x\in\Omega,\quad t\in\,]0,T[\,, \tag{1.4}$$

where $\Omega=\,]0,1[$ and $\Omega_T=\Omega\times\,]0,T[$ with $T>0$, $u_i\ (1\leq i\leq n)$ denote the volumetric concentrations of the fluid mixture, p is the pressure, q is the Darcy velocity of the fluid mixture. For simplicity, we have supposed the porosity, the permeability and the viscosity constant equal to 1. The pressure equation (1.1) expresses the conservation of the total mass and the empirical Darcy law, and the concentration equation (1.2) means the conservation of mass for the ith component of the mixture. Remark that the boundary conditions treated here for the pressure equation are Dirichlet's type at the inflow end and Neumann's type at the outflow end.

As shown in [6], the admissible domain of the concentration variables $u=(u_1,u_2,\dots,u_n)^t$ is

$$K=\left\{u \mid u_i>0,\ \sum_{i=1}^n u_i<1,\ 1\leq i\leq n\right\}. \tag{1.5}$$

The functions a and $b_i\ (1\leq i\leq n)$ are defined on K and have the form

$$a(u)=\sum_{i=1}^n z_i u_i+z_{n+1}\tilde{u},\quad \text{with}\quad \tilde{u}=1-\sum_{i=1}^n u_i, \tag{1.6}$$

$$b_i(u)=u_i(z_i-a(u)),\quad 1\leq i\leq n, \tag{1.7}$$

where $z_i>0\ (1\leq i\leq n+1)$ is the constant compressibility factor of the ith component. For convenience, we extend continuously the functions a and b to $\mathbb{R}^n$. Hence, in view of (1.7), we can take $b_i(u)=0$ as $u_i\leq 0\ (1\leq i\leq n)$. We note that $a(u)>0$ in the compact $\bar{K}$. If $z_i=z_j$ for some i, j, we replace u_i or u_j by u_i+u_j so that the system (1.2) reduces to $n-1$ equations. For this reason, we suppose throughout this paper that $z_i\neq z_j$ for all $i\neq j$.

Our objective is to construct a global weak solution such that the concentration variables are bounded with bounded total variation. In the case $n=1$, this result has been obtained by Amirat-Moussaoui [2]. Here we assume that there exists a function $\bar{p}\in L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;H^1(\Omega))$ such that

$$\bar{p}(0,t)=0,\quad -\partial_x\bar{p}(0,t)=q_1(t),\quad \bar{p}(x,0)=p_0(x),\quad x\in\Omega,\quad t\in\,]0,T[\,, \tag{1.8}$$

$$0\leq -p_0'(x),\quad q_1(t)\leq M,\quad x\in\Omega,\quad t\in\,]0,T[,\quad \text{where}\quad M>0 \text{ is a constant}, \tag{1.9}$$

$$u^0(x),\quad u^1(t)\in\bar{K},\quad \text{a.e. } x\in\Omega \quad\text{and}\quad \text{a.e. } t\in\,]0,T[\,, \tag{1.10}$$

$$u^0\in BV(\Omega)^n,\quad u^1\in BV(0,T)^n, \tag{1.11}$$

where BV denotes the space of functions with bounded total variation. Condition (1.9) means that the flow comes from the left to the right, that is consistent with the boundary condition for u. Of course, the equation (1.2) and the condition (1.4) should be understood in the sense of distributions. That is, for all test functions g in $C_0^1([0,1[\times[0,T[)$, we have

$$-\int_{\Omega_T} u_i(\partial_t g+\partial_x(gq))\,dx\,dt+\int_{\Omega_T} b_i(u)\,\partial_t p g\,dx\,dt$$

$$=\int_\Omega u_i^0(x)\,g(x,0)\,dx+\int_0^T u_i^1(t)\,q(0,t)\,g(0,t)\,dt,\quad 1\leq i\leq n, \tag{1.12}$$

where $C_0^1([0,1[\times[0,T[)$ denotes the space of C^1-differentiable functions with support contained in $[0,1[\times[0,T[$.

The main result of this paper is

THEOREM 1.1: *Under hypotheses (1.8)-(1.11), problem (1.1)-(1.4) has a global weak solution (p,u) satisfying*

$$\partial_x u \in L^\infty(0,T;\mathcal{M}(\Omega)^n), \quad \partial_t u \in L^2(0,T;\mathcal{M}(\Omega)^n),$$

where $\mathcal{M}(\Omega)$ denotes the space of Radon measures on Ω. □

The remainder of the paper deals with the proof of this theorem, the main steps of which is analogous to that of the scalar case $n=1$, see [2]. In the next section, we introduce a change of variables which is essential to treat the BV estimates. Section 3 is devoted to the regularization of the problem. Some similar results to those of [2] will be used without proof. In Section 4, we introduce an intermediary variable v and establish its BV estimate. Finally, in Section 5 we complete the proof of the theorem by the Schauder's fixed point theorem.

2. REDUCTION OF THE SYSTEM TO A TRANSPORT EQUATION

Following an idea used by Kazhikhov-Shelukin [12] and Serre [17] in solving one dimensional equations of viscous gas, we seek a transformation $u\mapsto G(u)$ which allows to reduce the quasi-linear system (1.1)-(1.2) to a transport equation. Let (p,u) be a smooth solution of (1.1)-(1.2). Then the quantity $h=G(u)+p$ verifies the following equation

$$\partial_t h + q\,\partial_x h = -q^2 \quad \text{in } \Omega_T, \tag{2.1}$$

provided that $G(u)$ satisfies

$$\sum_{i=1}^n b_i(u)\frac{\partial G(u)}{\partial u_i} = 1\,. \tag{2.2}$$

Let us now look for solutions of (2.2) in the special form

$$G(u) = \sum_{i=1}^n \alpha_i \log u_i + \beta \log \tilde{u}\,, \tag{2.3}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ and β are constants, $\tilde{u}$ is defined by (1.6). We have

$$\begin{aligned}\sum_{i=1}^n b_i(u)\frac{\partial G(u)}{\partial u_i} &= \sum_{i=1}^n b_i(u)\left(\frac{\alpha_i}{u_i}-\frac{\beta}{\tilde{u}}\right)\\ &= \sum_{i=1}^n \alpha_i(z_i - a(u)) - \frac{\beta}{\tilde{u}}b_s(u)\,,\end{aligned}$$

with

$$\begin{aligned}b_s(u) := \sum_{i=1}^n b_i(u) = \sum_{i=1}^n (z_i - a(u))\,u_i &= \sum_{i=1}^n z_i u_i - a(u)\sum_{i=1}^n u_i\\ &= a(u) - z_{n+1}\tilde{u} + a(u)\,\tilde{u} - a(u)\\ &= (a(u) - z_{n+1})\,\tilde{u}\,.\end{aligned}$$

So that (2.2) leads to the relation

$$\sum_{i=1}^{n} b_i(u)\frac{\partial G(u)}{\partial u_i} = \sum_{i=1}^{n} \alpha_i z_i - a(u)\left(\sum_{i=1}^{n} \alpha_i + \beta\right) + \beta z_{n+1} .$$

As compared with (2.2), we get

$$\sum_{i=1}^{n} \alpha_i z_i + \beta z_{n+1} = 1, \quad \sum_{i=1}^{n} \alpha_i + \beta = 0 . \tag{2.4}$$

By a suitable choice of the coefficients (α_i, β), we obtain

$$\begin{cases} G_i(u) = \dfrac{1}{z_i - z_{i+1}} \log\left(\dfrac{u_i}{u_{i+1}}\right), & 1 \leqslant i \leqslant n-1 , \\ G_n(u) = \dfrac{1}{z_1 - z_{n+1}} \log\left(\dfrac{u_1}{\tilde{u}}\right), \end{cases} \tag{2.5}$$

which are n particular solutions of (2.2). We show now that $G_1, G_2, ..., G_n$ are independent. Let $J = J(u)$ be the Jacobian matrix of the transformation $G = (G_1, G_2, ..., G_n)^t$. By a straightforward calculation, we have

$$J(u) = \frac{\partial G}{\partial u} = \begin{pmatrix} c_1 w_1 & -c_1 w_2 & 0 & \dots & 0 \\ 0 & c_2 w_2 & -c_2 w_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & -c_{n-1} w_n \\ c_n(w_1 + y) & c_n y & c_n y & \dots & c_n y \end{pmatrix},$$

where we denote by $c_i = \dfrac{1}{z_i - z_{i+1}}\ (1 \leqslant i \leqslant n-1)$, $c_n = \dfrac{1}{z_1 - z_{n+1}}$, $w_i = 1/u_i (1 \leqslant i \leqslant n)$ and $y = 1/\tilde{u}$.

LEMMA 2.1: *The transformation $G = (G_1, G_2, ..., G_n)^t$ defined by (2.5) is a C^1 diffeomorphism from K to $\mathbb{R}^n$.* □

Proof: It is easily seen that the matrix $J(u)$ may be written in the form $J = CW(u)$, where C is the diagonal matrix $C = \text{diag}\,(c_1, c_2, ..., c_n)$ and

$$W(u) = \begin{pmatrix} w_1 & -w_2 & 0 & \dots & \dots & 0 \\ 0 & w_2 & -w_3 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & w_{n-1} & -w_n \\ w_1 + y & y & \dots & \dots & \dots & y \end{pmatrix}.$$

Therefore

$$\det J(u) = \left(\prod_{i=1}^{n} c_i\right) \det W(u) . \tag{2.6}$$

In view of the special form of the matrix $W(u)$, it is not difficult to get

$$\det W(u) = \prod_{i=1}^{n} w_i + y \sum_{i=1}^{n} \prod_{j \neq i} w_j . \tag{2.7}$$

Hence $\det W(u) \neq 0$ for any $u \in K$, and G is locally invertible. To show that G is a C^1 diffeomorphism it remains to prove that G is globally injective in K. To this end, let $u = (u_1, u_2, ..., u_n)^t$ and $u' = (u'_1, u'_2, ..., u'_n)^t$ be two points in K such that $G_i(u) = G_i(u')$ $(1 \leqslant i \leqslant n)$. Then from

$$\log u_i - \log u_{i+1} = \log u'_i - \log u'_{i+1}, \quad 1 \leqslant i \leqslant n-1 ,$$

and

$$\log u_1 - \log \tilde{u} = \log u'_1 - \log \tilde{u}' ,$$

we get

$$\frac{u'_{i+1}}{u_{i+1}} = \frac{u'_i}{u_i} = \frac{\tilde{u}'}{\tilde{u}} = \text{constant}, \quad \forall 1 \leqslant i \leqslant n-1 ,$$

which implies $u'_i = u_i (1 \leqslant i \leqslant n)$. This finishes the proof of Lemma 2.1. □

LEMMA 2.2: *Let $J^{-1}(u) = (\gamma_{ij}(u))_{1 \leqslant i,j \leqslant n}$ be the inverse matrix of $J(u)$ in K. Then*

$$|\gamma_{ij}(u)| \leqslant |c_j^{-1}|, \quad 1 \leqslant i, j \leqslant n, \quad \forall u \in K . \quad \square$$

Proof: Let $u \in K$ fixed. By the Cramer's formula, we have

$$W^{-1}(u) = \frac{1}{\det W(u)} \cdot W^*(u), \quad \text{with} \quad W^*(u) = (w^*_{ij})_{1 \leqslant i,j \leqslant n} .$$

By a straightforward calculation, we can write

$$w^*_{ij} = (-1)^{\alpha(i,j)} \prod_{l \neq i} w_l + y(1 - \delta_{jn}) \prod_{l \neq i, i_0} (-1)^{\beta(i, i_0)} w_l .$$

Therefore

$$\frac{|w^*_{ij}|}{\det W(u)} \leqslant \frac{\prod_{l \neq i} w_l + y \prod_{l \neq i, i_0} w_l}{\prod_1^n w_l + y \sum_1^n \prod_{l \neq i} w_l} \leqslant 1 ,$$

since $w_l \geqslant 1$ $(1 \leqslant l \leqslant n)$. This means that each element of $W^{-1}(u)$ is uniformly bounded by 1 in $\bar{K}$. Thus the result follows from the identity $J^{-1}(u) = W^{-1}(u)\, C^{-1}$. □

3. REGULARIZATION OF THE PROBLEM

Let us first introduce a perturbation of G to avoid its singularities on the boundary ∂K of K. Let $\delta > 0$ be a small parameter. We define the approximate function of $b_i(u)$ by

$$b_i^\delta(u) = (z_i - a^\delta(u))(u_i + \delta), \quad 1 \leqslant i \leqslant n ,$$

with

$$a^{\delta}(u)=\frac{1}{1+2n\delta}\left(\sum_{i=1}^{n} z_i(u_i+\delta)+z_{n+1}\tilde{u}^{\delta}\right), \quad \tilde{u}^{\delta}=\tilde{u}+n\delta .$$

Then we define the approximate transformation $G^{\delta}=(G_1^{\delta}, G_2^{\delta}, \dots, G_n^{\delta})^t$ by

$$G_i^{\delta}(u)=c_i \log\left(\frac{u_i+\delta}{u_{i+1}+\delta}\right), \quad 1 \leqslant i \leqslant n-1 ,$$

and

$$G_n^{\delta}(u)=c_n \log\left(\frac{u_1+\delta}{\tilde{u}^{\delta}}\right).$$

We check easily that

$$\sum_{i=1}^{n} b_i^{\delta}(u)\frac{\partial G_j^{\delta}(u)}{\partial u_i}=1, \quad 1 \leqslant j \leqslant n .$$

Therefore, the transport equation (2.1) is still valid for the approximate transformation G^{δ}, which is defined in $\bar{K}$.

Following [2], we introduce a closed convex set of functions in $L^1(\Omega_T)^n$:

$$\mathscr{B}=\{v \mid v \in L^1(\Omega_T)^n, v(x,t) \in \bar{K} \text{ a.e. in } \Omega_T\} .$$

Since (1.1) is a linear equation for any given function u, the following result is valid in our case, see [2] for the proof.

LEMMA 3.1: *Let $u \in \mathscr{B}$ and assume that conditions (1.8) and (1.9) are fulfilled. Then, there exists a unique solution (p,q) of problem (1.1) and (1.3), such that $p \in W$, where*

$$W=L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)) \cap L^{\infty}(0,T;H^1(\Omega)) .$$

Moreover, we have the estimates:

$$\|p\|_{L^2(0,T;H^2(\Omega))}+\|\partial_t p\|_{L^2(\Omega_T)} \leqslant C, \tag{3.1}$$

and

$$0 \leqslant q=-\partial_x p \leqslant M, \quad \text{a.e. in } \Omega_T, \tag{3.2}$$

where M is the constant in (1.9), and $C>0$ is a constant depending only on the function $\bar{p}$ in (1.8) and $z_1, z_2, \dots, z_n$. □

Next we regularize the solution (p,q) of problem (1.1) and (1.3), denoted by (p^{η}, q^{η}), with $\eta>0$ a small parameter. In the next discussion, we denote by $C>0$ various constants independent of η. The function q^{η} is chosen such that,

(i) $\eta \leqslant q^{\eta}(x,t) \leqslant \eta+M, \quad \forall (x,t) \in \Omega_T,$

(ii) q^{η} converges, as η tends to 0, to q strongly in $L^2(\Omega_T)$ and weakly in $L^2(0,T;H^1(\Omega))$,

(iii) $|\partial_x q^{\eta}| \leqslant C/\eta$.

Similarly, the function p^η is chosen such that,

(i) $p^\eta \in C^2(\bar{\Omega}_T)$,

(ii) $p^\eta(0,t) = 0, \quad \forall t \in \,]0,T[$,

(iii) $\|p^\eta\|_W \leqslant C \|p\|_W$,

(iv) $\|\partial_x p^\eta\|_{L^\infty(\Omega_T)} \leqslant M_1$, where $M_1 > 0$ is a constant independent of η,

(v) $\|p^\eta\|_{C^2(\bar{\Omega}_T)} \leqslant C_0(\eta)\, \|p\|_W$, where $C_0 > 0$ is a constant depending on η,

(vi) p^η converges, as η tends to 0, to p in W.

In the following the constant M will be used in the place of Max (M, M_1). We regularize also the data u^0 and u^1 by $u^{0\eta} \in C^1(\bar{\Omega})^n$ and $u^{1\eta} \in C^1[0,T]^n$ respectively, satisfying for any $\eta > 0$ small enough,

(i) $u^{0\eta}(x) \in \bar{K}$, $u^{1\eta}(t) \in \bar{K}$, $\forall x \in \Omega$, $\forall t \in [0,T]$,

(ii) $u^{0\eta}(0) = u^{1\eta}(0), \qquad (u^{0\eta})'(0) = (u^{1\eta})'(0) = 0$,

(iii) $\|u^{0\eta}\|_{W^{1,1}(\Omega)^n} + \|u^{1\eta}\|_{W^{1,1}(]0,T[)^n} \leqslant C(\|u^0\|_{BV(\Omega)^n} + \|u^1\|_{BV(0,T)^n})$,

(iv) $(u^{0\eta}, u^{1\eta})$ converges to (u^0, u^1) in $L^1(\Omega)^n \times L^1(0,T)^n$ as η tends to 0.

We refer to [2] for these regularizations.

Now we consider the initial-boundary value problem

$$\begin{cases} \partial_t w_i + q^\eta\, \partial_x w_i + b_i^\delta(w)\, \partial_t p^\eta = 0, & \text{in } \Omega_T, \quad 1 \leqslant i \leqslant n\,, \\ w(x,0) = u^{0\eta}(x), \quad x \in \Omega, & w(0,t) = u^{1\eta}(t), \quad t \in \,]0,T[\,. \end{cases} \tag{3.3}$$

Of course, the solution w of (3.3) depends on two parameters η and δ, which will not be explicitly written for the moment. Setting

$$h_i = G_i^\delta(w) + p^\eta, \quad 1 \leqslant i \leqslant n\,.$$

Then it is easy to see that h_i is solution of

$$\begin{cases} \partial_t h_i + q^\eta\, \partial_x h_i = -\,\tilde{q}^\eta\, q^\eta, \quad \text{in } \Omega_T, \quad 1 \leqslant i \leqslant n\,, \\ h_i(x,0) = h_i^0(x) := G_i^\delta(u^{0\eta}(x)) + p^\eta(x,0), \quad x \in \Omega\,, \\ h_i(0,t) = h_i^1(t) := G_i^\delta(u^{1\eta}(t)), \quad t \in \,]0,T[\,, \end{cases} \tag{3.4}$$

where $\tilde{q}^\eta = -\,\partial_x p^\eta$. Moreover we have

$$h^0 \in C^1(\bar{\Omega})^n, \quad h^1 \in C^1[0,T]^n\,,$$

$$h^0(0) = h^1(0), \quad (h^1)'(0) + q^\eta(0,0)\,(h^0)'(0) = -\,\tilde{q}^\eta(0,0)\, q^\eta(0,0)\,,$$

$$\tilde{q}^\eta \text{ and } q^\eta \text{ belong to } C^1(\bar{\Omega}_T) \text{ and } q^\eta \geqslant 0\,.$$

As a consequence of a result in [2], problem (3.4) admits a unique global solution $h = (h_1, h_2, \ldots, h_n)^t \in C^1(\bar{\Omega}_T)^n$. It follows that the vector-valued function $w = (G^\delta)^{-1}(h - p^\eta I)$ is a classical solution in $C^1(\bar{\Omega}_T)^n$ of (3.3). Thus we have defined a sequence $(w^\delta)_{\delta>0}$. The next step consists in showing that the limit of $(w^\delta)_{\delta>0}$ exists and is a solution of

$$\begin{cases} \partial_t w_i + q^\eta\, \partial_x w_i + b_i(w)\, \partial_t p^\eta = 0, & \text{in } \Omega_T, \quad 1 \leqslant i \leqslant n\,, \\ w(x,0) = u^{0\eta}(x), \quad x \in \Omega, & w(0,t) = u^{1\eta}(t), \quad t \in \,]0,T[\,. \end{cases} \tag{3.5}$$

4. BV ESTIMATES FOR AN INTERMEDIARY VARIABLE

In the absence of the regularity for p^η, equation (3.3) is not enough to establish a BV estimate for the concentration. We introduce then a new sequence of vector-valued functions $v^\delta = (v_1^\delta, v_2^\delta, \ldots, v_n^\delta)^t$ defined by

$$v_n^\delta = \frac{e^{h_n^\delta / c_n}}{S^\delta}\,, \tag{4.1}$$

and

$$v_i^\delta = v_{i+1}^\delta \, e^{h_i^\delta / c_i}, \quad 1 \leq i \leq n-1\,, \tag{4.2}$$

where

$$S^\delta = \sum_{i=0}^{n-1} \prod_{j=0}^{i} e^{h_{n-j}^\delta / c_{n-j}} + \prod_{j=1}^{n-1} e^{h_{n-j}^\delta / c_{n-j}}\,. \tag{4.3}$$

The aim of this section is to show a BV estimate for the limit of the sequence (v^δ). We begin by

LEMMA 4.1: *The function v^δ defined by (4.1)-(4.3) satisfies $v^\delta \in K$ and*

$$v_1^\delta = \tilde{v}^{\,\delta} \, v_n^\delta \, S^\delta \tag{4.4}$$

where

$$\tilde{v}^{\,\delta} = 1 - \sum_{i=1}^{n} v_i^\delta\,. \quad \square \tag{4.5}$$

Proof: By definition, we have immediately $v_i^\delta > 0$ $(1 \leq i \leq n)$. Then by a straightforward calculation, we get

$$\sum_{i=1}^{n} v_i^\delta = \frac{1}{S^\delta}\left(S^\delta - \prod_{j=1}^{n-1} e^{h_{n-j}^\delta / c_{n-j}}\right) < 1\,.$$

Therefore

$$\tilde{v}^{\,\delta} = \frac{1}{S^\delta} \prod_{j=1}^{n-1} e^{h_{n-j}^\delta / c_{n-j}} > 0\,, \tag{4.6}$$

which yields $v^\delta \in K$. Now by induction, we can write v_1^δ in the form

$$v_1^\delta = v_n^\delta \prod_{j=1}^{n-1} e^{h_{n-j}^\delta / c_{n-j}}\,.$$

Hence (4.4) follows from (4.5) and (4.6). $\square$

LEMMA 4.2: *For any $\delta > 0$, the vector-valued function v^δ solves the system*

$$\partial_t v_i^\delta + q^\eta \partial_x v_i^\delta = -\tilde{q}^\eta \, q^\eta \left(\sum_{j=i}^{n} \frac{1}{c_j} - L(v^\delta)\right) v_i^\delta, \quad in\ \Omega_T, \quad 1 \leq i \leq n\,, \tag{4.7}$$

$$v_i^\delta(x,0) = (u_i^{0\eta}(x) + \delta) \exp\left(p^\eta(x,0) \sum_{j=i}^{n} \frac{1}{c_j}\right) \frac{1}{R^\delta(x)}, \quad x \in \Omega, \quad 1 \leq i \leq n\,, \tag{4.8}$$

$$v^\delta(0,t) = u^{1\eta}(t) + \delta, \quad t \in \,]0, T[\,. \tag{4.9}$$

Here the operator L and the function R^δ are defined respectively by

$$L(v) = \sum_{i=0}^{n-1} \left(\sum_{j=0}^{i} \frac{1}{c_{n-j}}\right) v_{n-i} + \left(\sum_{j=1}^{n-1} \frac{1}{c_{n-j}}\right) \tilde{v}\,,$$

and

$$R^{\delta}(x)=\sum_{i=0}^{n-1}(u_{n-i}^{0\eta}(x)+\delta)\exp\Big(p^{\eta}(x,0)\sum_{j=0}^{i}\frac{1}{c_{n-j}}\Big)+\tilde{u}_{\delta}^{0\eta}(x)\exp\Big(p^{\eta}(x,0)\sum_{j=1}^{n-1}\frac{1}{c_j}\Big),\quad x\in\Omega,$$

with

$$\tilde{u}_{\delta}^{0\eta}=\tilde{u}^{0\eta}-n\delta. \quad \square$$

Proof: For convenience, we drop the index δ. Let us first derive the equation satisfied by v_n. Differentiating the relation $v_n S=e^{h_n/c_n}$ with respect to t and x and using (3.4), we get

$$\text{(4.10)}\qquad S(\partial_t v_n+q^{\eta}\,\partial_x v_n)=\frac{1}{c_n}e^{h_n/c_n}(\partial_t h_n+q^{\eta}\,\partial_x h_n)-v_n(\partial_t S+q^{\eta}\,\partial_x S)$$

$$=-\frac{1}{c_n}\tilde{q}^{\eta}\,q^{\eta}\,v_n S-v_n(\partial_t S+q^{\eta}\,\partial_x S).$$

By (3.4) and (4.3), we have

$$\partial_t S+q^{\eta}\,\partial_x S=-\tilde{q}^{\eta}\,q^{\eta}\Big[\sum_{i=0}^{n-1}\Big(\sum_{j=0}^{i}\frac{1}{c_{n-j}}\Big)e^{\sum_{j=0}^{i}h_{n-j}/c_{n-j}}+\Big(\sum_{j=1}^{n-1}\frac{1}{c_{n-j}}\Big)e^{\sum_{j=1}^{n-1}h_{n-j}/c_{n-j}}\Big].$$

We deduce from the definition (4.1) and (4.2) that,

$$e^{\sum_{j=1}^{n-1}h_{n-j}/c_{n-j}}=\prod_{j=1}^{n-1}e^{h_{n-j}/c_{n-j}}=\prod_{j=1}^{n-1}\frac{v_{n-j}}{v_{n-j+1}}=\frac{v_1}{v_n},$$

and

$$e^{\sum_{j=0}^{i}h_{n-j}/c_{n-j}}=v_n\,S\prod_{j=1}^{i}\frac{v_{n-j}}{v_{n-j+1}}=Sv_{n-i},\quad 1\leqslant i\leqslant n-1.$$

So that

$$\partial_t S+q^{\eta}\,\partial_x S=-\tilde{q}^{\eta}\,q^{\eta}\Big[S\sum_{i=0}^{n-1}\Big(\sum_{j=0}^{i}\frac{1}{c_{n-j}}\Big)v_{n-i}+\frac{v_1}{v_n}\Big(\sum_{j=1}^{n-1}\frac{1}{c_{n-j}}\Big)\Big],$$

and by (4.4) we obtain,

$$\partial_t S+q^{\eta}\,\partial_x S=-\tilde{q}^{\eta}\,q^{\eta}\,S\Big[\sum_{i=0}^{n-1}\Big(\sum_{j=0}^{i}\frac{1}{c_{n-j}}\Big)v_{n-i}+\tilde{v}\,\Big(\sum_{j=1}^{n-1}\frac{1}{c_{n-j}}\Big)\Big]=-\tilde{q}^{\eta}\,q^{\eta}\,SL(v).$$

Substituting this relation into (4.10), we get (4.7) for $i = n$. By induction we now suppose that (4.6) is true for the index $i+1$ and shall show it for the index i. In fact, by induction and using the relation $v_i = v_{i+1}\, e^{h_i/c_i}$ we have

$$\partial_t v_i + q^\eta\, \partial_x v_i = e^{h_i/c_i}[\partial_t v_{i+1} + q^\eta\, \partial_x v_{i+1}] + \frac{1}{c_i} e^{h_i/c_i}[\partial_t\, h_i + q^\eta\, \partial_x h_i]\, v_{i+1}$$

$$= -\tilde{q}^\eta\, q^\eta\, v_i \left(\sum_{j=i+1}^{n} \frac{1}{c_j} - L(v) \right) - \frac{1}{c_i} \tilde{q}^\eta\, q^\eta\, v_i$$

$$= -\tilde{q}^\eta\, q^\eta\, v_i \left(\sum_{j=i}^{n} \frac{1}{c_j} - L(v) \right),$$

which is (4.7) for the index i.

Let us now prove (4.8) and (4.9). From the definition of G_i and h_i, we have

$$e^{h_i/c_i} = \begin{cases} e^{p^\eta/c_n} \dfrac{w_1}{\tilde{w}} & \text{if } i = n\,, \\ e^{p^\eta/c_i} \dfrac{w_i}{w_{i+1}} & \text{if } 1 \leqslant i \leqslant n-1\,. \end{cases}$$

It follows from (4.3) that

$$S = \sum_{i=0}^{n-1} \left(\prod_{j=1}^{i} e^{p^\eta/c_{n-j}} \frac{w_{n-j}}{w_{n-j+1}} \right) e^{p^\eta/c_n} \frac{w_1}{\tilde{w}} + \prod_{j=1}^{n-1} e^{p^\eta/c_{n-j}} \frac{w_{n-j}}{w_{n-j+1}}$$

$$= \frac{w_1}{w_n \tilde{w}} \sum_{i=0}^{n-1} w_{n-i}\, e^{p^\eta \sum_{j=0}^{i} \frac{1}{c_{n-j}}} + \frac{w_1}{w_n} e^{p^\eta \sum_{j=1}^{n-1} \frac{1}{c_j}}.$$

Hence

$$S(x, 0) = \frac{w_1(x, 0)}{w_n(x, 0)\, \tilde{w}(x, 0)} R(x)\,,$$

which leads to (4.8) for $i = n$. Since $p(0, x) = 0$, we have also

$$S(0, t) = \frac{w_1(0, t)}{w_n(0, t)\, \tilde{w}(0, t)}.$$

Therefore

$$v_n(0, t) = \frac{e^{h_n(0, t)/c_n}}{S(0, t)} = \frac{w_1(0, t)}{\tilde{w}(0, t)\, S(0, t)} = w_n(0, t)\,,$$

which proves (4.9) for $i = n$. Similarly, conditions (4.8) and (4.9) for $i < n$ may be obtained by induction. □

LEMMA 4.3: *As $\delta \to 0$, the sequence (v^δ) converges to v in $H^1(\Omega_T)^n$ weakly and in $L^2(\Omega_T)$ strongly. Moreover, v is a solution of*

$$\partial_t v_i + q^\eta\, \partial_x v_i = -\tilde{q}^\eta\, q^\eta \left(\sum_{j=i}^{n} \frac{1}{c_j} - L(v) \right) v_i \quad \text{in } \Omega_T, \quad 1 \leqslant i \leqslant n, \tag{4.11}$$

$$v_i(x,0) = u_i^{0\eta}(x) \exp\left(p^\eta(x,0) \sum_{j=i}^{n} \frac{1}{c_j} \right) \frac{1}{R(x)}, \quad x \in \Omega, \quad 1 \leqslant i \leqslant n, \tag{4.12}$$

$$v(0,t) = u^{1\eta}(t), \quad 0 < t < T, \tag{4.13}$$

where L is defined in Lemma 4.2 and

$$R(x) = \sum_{i=0}^{n-1} u_{n-i}^{0\eta}(x) \exp\left(p^\eta(x,0) \sum_{j=0}^{i} \frac{1}{c_{n-j}} \right) + \tilde{u}^{0\eta}(x) \exp\left(p^\eta(x,0) \sum_{j=1}^{n-1} \frac{1}{c_j} \right).$$

Furthermore, $v(x,t) \in \bar{K}$ for every (x,t) in Ω_T, and v belongs to $C^1(\bar{\Omega}_T)^n$. □

The proof of this lemma is analogous to that of [2], we refer the reader to that paper.

LEMMA 4.4: *Let v be the solution of problem (4.11)-(4.13). Then*

$$\| v \|_{L^\infty(0,T;\, W^{1,1}(\Omega)^n)} + \| v \|_{W^{1,\infty}(0,T;\, L^1(\Omega)^n)} \leqslant C_1, \tag{4.14}$$

$$\| v \|_{L^\infty(0,T;\, H^1(\Omega)^n)} + \| \partial_t v \|_{L^2(\Omega_T)^n} \leqslant C_2(\eta), \tag{4.15}$$

where $C_1 > 0$ is a constant depending only on the data, $C_2 > 0$ is a constant depending on the data and η, whereas C_1 and C_2 are independent of u. □

Proof: For $1 \leqslant i \leqslant n$, we set

$$L_i(v) = \sum_{j=i}^{n} \frac{1}{c_j} - L(v).$$

From (4.11) and (4.13), we have for $\eta > 0$ small enough,

$$\begin{aligned} |q^\eta(0,t)\, \partial_x v_i(0,t)| &= |-\tilde{q}^\eta(0,t)\, q^\eta(0,t)\, L_i(u^{1\eta}(t))\, u_i^{1\eta}(t) - \partial_t u_i^{1\eta}(t)| \\ &\leqslant CM(M+\eta) + |\partial_t u_i^{1\eta}(t)| \\ &\leqslant 2\,CM^2 + |\partial_t u_i^{1\eta}(t)|, \quad \forall t \in \,]0,T[\,. \end{aligned}$$

Thus

$$\int_0^T |q^\eta(0,t)\, \partial_x v_i(0,t)|\, dt \leqslant 2\,CM^2\, T + \int_0^T |\partial_t u_i^{1\eta}(t)|\, dt. \tag{4.16}$$

Now differentiating (4.11) with respect to x, we get

$$\begin{aligned} \partial_t(\partial_x v_i) + \partial_x(q^\eta \partial_x v_i) = &- v_i\, L_i(v)\, (\tilde{q}^\eta\, \partial_x q^\eta + q^\eta\, \partial_x \tilde{q}^\eta) \\ &- \tilde{q}^\eta\, q^\eta (\partial_x v_i)\, L_i(v) - \tilde{q}^\eta\, q^\eta\, v_i\, \partial_x L(v). \end{aligned} \tag{4.17}$$

Then multiplying (4.17) by sign $(\partial_x v_i)$ and integrating over Ω yields

$$\frac{d}{dt}\int_0^1 |\partial_x v_i|\, dx + \int_0^1 \partial_x (q^\eta |\partial_x v_i|)\, dx \tag{4.18}$$

$$= -\int_0^1 (\tilde{q}^\eta\, \partial_x q^\eta + q^\eta\, \partial_x \tilde{q}^\eta)\, L_i(v)\, v_i \,\text{sign}\,(\partial_x v_i)\, dx - \int_0^1 \tilde{q}^\eta\, q^\eta (L_i(v)\, \partial_x v_i + v_i\, \partial_x L(v))\,\text{sign}\,(\partial_x v_i)\, dx\,.$$

But

$$\partial_x L(v) = \sum_{i=0}^{n-1} \left(\sum_{j=0}^{i} \frac{1}{c_{n-j}}\right) \partial_x v_{n-i} - \left(\sum_{j=1}^{n-1} \frac{1}{c_{n-j}}\right) \sum_{j=1}^{n} \partial_x v_j\,. \tag{4.19}$$

Therefore, integrating (4.18) from 0 to t, and adding the equations for $i = 1$ to n, we obtain

$$\int_0^1 \sum_{i=1}^n |\partial_x v_i|\, dx + \int_0^t q^\eta(1, s) \sum_{i=1}^n |\partial_x v_i(1, s)|\, ds$$

$$\leqslant \int_0^1 \sum_{i=1}^n |\partial_x v_i(x, 0)|\, dx + \int_0^t q^\eta(0, s) \sum_{i=1}^n |\partial_x v_i(0, s)|\, ds$$

$$+ CM \int_0^t\int_0^1 (|\partial_x \tilde{q}^\eta| + |\partial_x q^\eta|)\, dx\, ds + CM^2 \int_0^t\int_0^1 \sum_{i=1}^n |\partial_x v_i|\, dx\, ds\,.$$

Using (4.16), Lemmas 4.2-4.3 and the definition of q^η, we get

$$\int_0^1 \sum_{i=1}^n |\partial_x v_i|\, dx \leqslant C + CM^2 \int_0^t\int_0^1 \sum_{i=1}^n |\partial_x v_i|\, dx\, ds, \quad \forall t \in\,]0, T]\,.$$

Thus, by the Gronwall's lemma, $\partial_x v$ is bounded in $L^\infty(0, T; L^1(\Omega)^n)$ by a constant independent of η. It follows that $\partial_t v$ is also bounded in $L^\infty(0, T; L^1(\Omega)^n)$. This proves (4.14).

To prove (4.15), we multiply (4.17) by $\partial_x v_i$ to obtain

$$\frac{1}{2}\frac{d}{dt} |\partial_x v_i|^2 + \partial_x (q^\eta \partial_x v_i)\, \partial_x v_i = - v_i\, L_i(v)\, \partial_x v_i (\tilde{q}^\eta\, \partial_x q^\eta + q^\eta \partial_x \tilde{q}^\eta)$$

$$- \tilde{q}^\eta\, q^\eta\, |\partial_x v_i|^2 L_i(v) - \tilde{q}^\eta\, q^\eta\, v_i (\partial_x L(v))\, \partial_x v_i\,.$$

Integrating it over $\Omega_t = \Omega \times\,]0, t[$, with $0 < t < T$, we get

$$\frac{1}{2}\int_0^1 |\partial_x v_i|^2\, dx + \frac{1}{2}\int_0^t q^\eta(1, s)\, |\partial_x v_i(1, s)|^2\, ds$$

$$= \frac{1}{2}\int_0^1 |\partial_x v_i(x, 0)|^2\, dx + \frac{1}{2}\int_0^t q^\eta(0, s)\, |\partial_x v_i(0, s)|^2\, ds$$

$$- \frac{1}{2}\int_0^t\int_0^1 \partial_x q^\eta(x, s)\, |\partial_x v_i(x, s)|^2\, dx\, ds$$

$$- \int_0^t\int_0^1 [v_i\, L_i(v)\, \partial_x v_i (\tilde{q}^\eta \partial_x q^\eta + q^\eta \partial_x \tilde{q}^\eta)]\, (x, s)\, dx\, ds$$

$$- \int_0^t\int_0^1 (\tilde{q}^\eta\, q^\eta\, |\partial_x v_i|^2 L_i(v))\, (x, s)\, dx\, ds - \int_0^t\int_0^1 (\tilde{q}^\eta\, q^\eta\, v_i (\partial_x L(v))\, \partial_x v_i)\, (x, s)\, dx\, ds\,.$$

The first term on the right hand side is majorized by a constant depending only on the $H^1(\Omega)$ norm of the data u^0 and p_0. For the second term, we use (4.11) and (4.13) to get

$$\begin{aligned}
& q^\eta(0,s)\,|\partial_x v_i(0,s)|^2 \\
&= |-\tilde{q}^\eta(0,s)\,q^\eta(0,s)\,L_i(u^{1\eta}(s))\,u_i^{1\eta}(s)\,\partial_x v_i(0,s) - \partial_t u_i^{1\eta}(s)\,\partial_x v_i(0,s)| \\
&\leq CM\sqrt{M+\eta}\,\sqrt{q^\eta(0,s)}\,|\partial_x v_i(0,s)| + |\partial_t u_i^{1\eta}(s)|\,\frac{1}{\sqrt{\eta}}\cdot\sqrt{\eta}\,|\partial_x v_i(0,s)| \\
&\leq 2\,C^2 M^3 + \frac{1}{4}\,q^\eta(0,s)\,|\partial_x v_i(0,s)|^2 + \frac{1}{\eta}\,|\partial_t u_i^{1\eta}(s)|^2 + \frac{\eta}{4}\,|\partial_x v_i(0,s)|^2 \\
&\leq 2\,C^2 M^3 + \frac{1}{2}\,q^\eta(0,s)\,|\partial_x v_i(0,s)|^2 + \frac{1}{\eta}\,|\partial_t u_i^{1\eta}(s)|^2,
\end{aligned}$$

since $q^\eta(0,s) \geq \eta$. Hence

$$\frac{1}{2}\int_0^t q^\eta(0,s)\,|\partial_x v_i(0,s)|^2\,ds \leq 2\,C^2 M^3\,t + \frac{1}{\eta}\int_0^t |\partial_t u_i^{1\eta}(s)|^2\,ds\,.$$

For the third term, by the construction of q^η, we have

$$\left|\int_0^t\int_0^1 \partial_x q^\eta(x,s)\,|\partial_x v_i(x,s)|^2\,dx\,ds\right| \leq \frac{C}{\eta}\int_0^t\int_0^1 |\partial_x v_i(x,s)|^2\,dx\,ds\,.$$

For the fourth term, we have

$$\begin{aligned}
&\int_0^t\int_0^1 |(v_i\,L_i(v)\,\partial_x v_i(\tilde{q}^\eta\,\partial_x q^\eta + q^\eta\,\partial_x\tilde{q}^\eta))\,(x,s)|\,dx\,ds \\
&\leq C\int_0^t\int_0^1 |\partial_x v_i(\tilde{q}^\eta\,\partial_x q^\eta + q^\eta\,\partial_x\tilde{q}^\eta)\,(x,s)|\,dx\,ds \\
&\leq C(M+1)\int_0^t\int_0^1 |\partial_x v_i|^2\,(x,s)\,dx\,ds \\
&\qquad + \frac{C(M+1)}{2}\left(\int_0^t\int_0^1 |\partial_x\tilde{q}^\eta|^2\,(x,s)\,dx\,ds + \int_0^t\int_0^1 |\partial_x q^\eta|^2\,(x,s)\,dx\,ds\right).
\end{aligned}$$

For the fifth term, we have

$$\int_0^t\int_0^1 \tilde{q}^\eta q^\eta\,|\partial_x v_i|^2\,(x,s)\,dx\,ds \leq 2\,M^2\int_0^t\int_0^1 |\partial_x v_i|^2\,(x,s)\,dx\,ds\,.$$

For the last term, we have from (4.19)

$$\left|\int_0^t\int_0^1 (\tilde{q}^\eta\,q^\eta\,\partial_x(L(v)))\,(x,s)\,dx\,ds\right| \leq CM^2\int_0^t\int_0^1 \sum_{i=1}^n |\partial_x v_i|^2\,(x,s)\,dx\,ds\,.$$

Finally,

$$\frac{1}{2}\int_0^1 \sum_{i=1}^n |\partial_x v_i|^2 + \frac{1}{2}\int_0^t q^\eta(1,s) \sum_{i=1}^n |\partial_x v_i(1,s)|^2 \, ds$$

$$\leqslant C\left(1+\frac{1}{\eta}\right)\int_0^t \int_0^1 \sum_{i=1}^n |\partial_x v_i|^2 \, dx\, ds, \quad \forall t \in \left]0, T\right] .$$

We conclude again from the Gronwall's lemma that $\partial_x v$ is bounded in $L^\infty(0, T; L^2(\Omega)^n)$, and then from (4.11) $\partial_t v$ is bounded in $L^2(\Omega_T)^n$. This completes the proof of Lemma 4.4. □

5. END OF THE PROOF OF THEOREM

Let us now return to problem (3.3). Recall that its solution w^δ is in $C^1(\bar{\Omega}_T)^n$ and is given by $w^\delta = (G^\delta)^{-1}(h^\delta - p^\eta I)$. Thus, by definition

$$\frac{w_i^\delta}{w_{i+1}^\delta} = e^{(h_i^\delta - p^\eta)/c_i}, \quad 1 \leqslant i \leqslant n-1 ,$$

and

$$\frac{w_1^\delta}{\tilde{w}^\delta} = e^{(h_n^\delta - p^\eta)/c_n} .$$

Using (4.1) and (4.2), we get

$$\frac{w_i^\delta}{w_{i+1}^\delta} = \frac{v_i^\delta}{v_{i+1}^\delta} e^{-p^\eta/c_i}, \quad 1 \leqslant i \leqslant n-1 ,$$

and

$$\frac{w_1^\delta}{\tilde{w}^\delta} = \frac{v_1^\delta}{\tilde{v}^\delta} e^{-p^\eta/c_n} .$$

It follows that

$$\frac{w_i^\delta}{\tilde{w}^\delta} = \frac{v_i^\delta}{\tilde{v}^\delta} e^{-p^\eta\left(\frac{1}{c_n} - \sum_{j=1}^{i-1}\frac{1}{c_j}\right)}, \quad 1 \leqslant i \leqslant n , \tag{5.1}$$

with the convention $\sum_1^0 \frac{1}{c_j} = 0$. Thus

$$\frac{1}{\tilde{w}^\delta}\sum_{i=1}^n w_i^\delta = \frac{1}{\tilde{v}}\sum_{i=1}^n v_i^\delta \exp\left(-p^\eta\left(\frac{1}{c_n} - \sum_{j=1}^{i-1}\frac{1}{c_j}\right)\right).$$

Since $\sum_{i=1}^{n} w_i^{\delta} = 1 - \tilde{w}^{\delta}$, we deduce

$$\tilde{w}^{\delta} = \frac{\tilde{v}^{\,\delta}}{\tilde{v}^{\,\delta} + \sum_{i=1}^{n} v_i^{\delta} \exp\left(-p^{\eta}\left(\frac{1}{c_n} - \sum_{j=1}^{i-1} \frac{1}{c_j}\right)\right)},$$

and therefore

$$\tilde{w}_l^{\delta} = \frac{v_l^{\delta} \exp\left(-p^{\eta}\left(\frac{1}{c_n} - \sum_{j=1}^{l-1} \frac{1}{c_j}\right)\right)}{\tilde{v}^{\,\delta} + \sum_{i=1}^{n} v_i^{\delta} \exp\left(-p^{\eta}\left(\frac{1}{c_n} - \sum_{j=1}^{i-1} \frac{1}{c_j}\right)\right)}, \quad 1 \leqslant l \leqslant n\,.$$

We easily see that, as δ tends to 0, w^{δ} converges in $H^1(\Omega_T)^n$ weakly to w given by

$$w_l = \frac{v_l \exp\left(-p^{\eta}\left(\frac{1}{c_n} - \sum_{j=1}^{l-1} \frac{1}{c_j}\right)\right)}{\tilde{v} + \sum_{i=1}^{n} v_i \exp\left(-p^{\eta}\left(\frac{1}{c_n} - \sum_{j=1}^{i-1} \frac{1}{c_j}\right)\right)}. \tag{5.2}$$

Clearly w is a solution of problem (3.5) and satisfies, $w(x, t) \in \bar{K}$ for every $(x, t) \in \Omega_T$. Furthermore, from the inequality $-M \leqslant p^{\eta} \leqslant 0$ we get

$$\tilde{v} + \sum_{i=1}^{n} v_i \exp\left(-M\left|\frac{1}{c_n} - \sum_{j=1}^{i-1} \frac{1}{c_j}\right|\right) \leqslant \tilde{v} + \sum_{i=1}^{n} v_i \exp\left(-p^{\eta}\left(\frac{1}{c_n} - \sum_{j=1}^{i-1} \frac{1}{c_j}\right)\right)$$
$$\leqslant \tilde{v} + \sum_{i=1}^{n} v_i \exp\left(M\left|\frac{1}{c_n} - \sum_{j=1}^{i-1} \frac{1}{c_j}\right|\right).$$

Now we introduce a mapping $\mathcal{T}_{\eta}$ defined on $\mathcal{B}$ by

$$\mathcal{T}_{\eta}(u) = w^{\eta}, \quad \forall u \in \mathcal{B}\,.$$

We will prove that $\mathcal{T}_{\eta}$ has a fixed point in $\mathcal{B}$. Firstly, using (5.2) and Lemma 4.4, we have the following result.

LEMMA 5.1: *For any u in $\mathcal{B}$, the mapping $\mathcal{T}_{\eta}$ is well-defined and satisfies:*

$$w^{\eta}(x, t) \in \bar{K} \text{ for every } (x, t) \text{ in } \Omega_T, \tag{5.3}$$

$$\| w^{\eta} \|_{L^{\infty}(0, T; W^{1,1}(\Omega)^n)} + \| \partial_t w^{\eta} \|_{L^2(0, T; L^1(\Omega)^n)} \leqslant C, \tag{5.4}$$

$$\| w^{\eta} \|_{L^{\infty}(0, T; H^1(\Omega)^n)} + \| \partial_t w^{\eta} \|_{L^2(\Omega_T)^n} \leqslant C_1(\eta), \tag{5.5}$$

where $C_1 > 0$ is a constant depending on the data and η, but independent of u. □

LEMMA 5.2: *For any $\eta > 0$, the mapping $\mathcal{T}_{\eta}$ has a fixed point in $\mathcal{B}$.* □

Proof: Clearly by (5.3), $\mathcal{T}_{\eta}(\mathcal{B})$ is contained in $\mathcal{B}$, and by (5.4), $\mathcal{T}_{\eta}(\mathcal{B})$ is a bounded set of $W^{1,1}(\Omega_T)$. Therefore it is relatively compact in $\mathcal{B}$. To apply the Schauder's fixed point theorem, it remains to prove that the mapping $\mathcal{T}_{\eta}$ is continuous.

For this purpose, let (u^m) be a sequence in $\mathcal{B}$ which converges strongly, as m tends to infinity, to a function u in $L^1(\Omega_T)^n$, and let (p_m, q_m) be the corresponding solution of problem (1.1) and (1.3). According to Lemma 3.1, there is a subsequence, still denoted by (p_m), and a function $p \in W$ such that $p_m \to p$, a.e. in Ω_T, in $L^\infty(0, T; H^1(\Omega))$ weak-$*$, and in $H^1(0, T; L^2(\Omega))$ weak. It is easily seen that p is a solution of

$$a(u)\,\partial_t p - \partial_x^2 p = 0 \quad \text{in} \quad \Omega_T,$$

$$p(0, t) = 0, \quad -\partial_x p(1, t) = q_1(t), \quad t \in\,]0, T[\,,$$

$$p(x, 0) = p_0(x), \quad x \in \Omega .$$

Since the solution of this problem is unique, the whole sequence (p_m) converges to p. In addition, the sequence $q_m = -\partial_x p_m$ converges, as $m \to \infty$, to $q = -\partial_x p$ strongly in $L^2(\Omega_T)$. Indeed, (q_m) is bounded in $L^2(0, T; H^1(\Omega))$ and $\partial_t q_m$ is bounded in $L^2(0, T; H^{-1}(\Omega))$. Thus the result follows from the compactness argument of Aubin's type, see [7] or [14]. The sequence q_m^η converges, as $m \to \infty$, to q^η strongly in $L^2(\Omega_T)$.

Let us now consider the solutions w and w^m of problem (3.3) with respective coefficients (p^η, q^η) and (p_m^η, q_m^η). We introduce two auxiliary functions v and v^m given by

$$v_l = \frac{w_l \exp\left(p^\eta\left(\frac{1}{c_n} - \sum_{j=1}^{l-1} \frac{1}{c_j}\right)\right)}{\tilde{w} + \sum_{i=1}^{n} w_i \exp\left(p^\eta\left(\frac{1}{c_n} - \sum_{j=1}^{i-1} \frac{1}{c_j}\right)\right)},$$

and

$$v_l^m = \frac{w_l^m \exp\left(p_m^\eta\left(\frac{1}{c_n} - \sum_{j=1}^{l-1} \frac{1}{c_j}\right)\right)}{\tilde{w}^m + \sum_{i=1}^{n} w_i^m \exp\left(p_m^\eta\left(\frac{1}{c_n} - \sum_{j=1}^{i-1} \frac{1}{c_j}\right)\right)}.$$

It is clear that v and (v^m) are bounded in $H^1(\Omega_T)^n$ and are solutions of the corresponding equations (4.11), with respective coefficients $(\tilde{q}^\eta, q^\eta)$ and $(\tilde{q}_m^\eta, q_m^\eta)$, and with the same boundary and initial conditions (4.12)-(4.13). Then, the difference $z^m = v - v^m$ satisfies

$$\begin{aligned} \partial_t z_i^m + q_m^\eta\, \partial_x z_i^m = {} & (q_m^\eta - q^\eta)\, \partial_x v_i - (\tilde{q}^\eta q^\eta - \tilde{q}_m^\eta q_m^\eta)\, L_i(v)\, v_i \\ & + \tilde{q}_m^\eta q_m^\eta L_i(v)\, z_i^m + \tilde{q}_m^\eta q_m^\eta (L_i(v) - L_i(v^m))\, v_i^m . \end{aligned} \tag{5.6}$$

We observe that

$$L_i(v) - L_i(v^m) = L(v) - L(v^m) = \sum_{l=0}^{n-1} \left(\sum_{j=0}^{l} \frac{1}{c_{n-j}}\right) z_{n-l}^m - \left(\sum_{j=1}^{n-1} \frac{1}{c_{n-j}}\right) \sum_{l=1}^{n-1} z_l^m .$$

Therefore, multiplying (5.6) by sign (z_i^m) and integrating it over Ω, it follows

$$\begin{aligned} \frac{d}{dt}\int_0^1 |z_i^m|\, dx + q_m^\eta(1, t)\, |z_i^m(1, t)| \le {} & \int_0^1 |\partial_x(q_m^\eta)|\, |z_i^m|\, dx + \int_0^1 |q_m^\eta - q^\eta|\, |\partial_x v_i|\, dx \\ & + C_1 \int_0^1 |\tilde{q}^\eta q^\eta - \tilde{q}_m^\eta q_m^\eta|\, dx + C_2 M^2 \int_0^1 |z_i^m|\, dx , \end{aligned}$$

where C_1 and C_2 are constants independent of η and m. Now integrating it again over $]0, t[$, we obtain since $q_m^\eta \geqslant 0$,

$$\int_0^1 |z_i^m(x,t)|\,dx \leqslant \left(C_2 M^2 + \frac{C_3}{\eta}\right)\int_0^t\int_0^1 |z_i^m(x,s)|\,dx\,ds + C_4 M \int_0^t\int_0^1 (|\tilde{q}^\eta - \tilde{q}_m^\eta| + |q^\eta - q_m^\eta|)\,dx\,ds + \frac{C_5}{\sqrt{\eta}}\left(\int_0^t\int_0^1 |q_m^\eta - q^\eta|^2\,dx\,ds\right)^{\frac{1}{2}}.$$

Then for any $\varepsilon > 0$ we can find an integer N such that $m > N$ implies

$$\int_0^1 |z_i^m(x,t)|\,dx \leqslant C_6(\eta)\int_0^t\int_0^1 |z_i^m(x,s)|\,dx\,ds + C_7(\eta)\,\varepsilon\,,$$

where $C_6(\eta)$ and $C_7(\eta)$ are constants independent of m. So that, by the Gronwall's Lemma, we have

$$\int_0^1 |z_i^m(x,t)|\,dx \leqslant C_8(\eta)\,\varepsilon\,,$$

where $C_8(\eta)$ is a constant independent of m. This means that v^m converges, as $m \to \infty$, to v in $L^\infty(0, T; L^1(\Omega))$. Using the convergence of (p_m) to p, together with the relations linking v and w, we deduce that w^m converges, as $m \to \infty$, to w in $L^1(\Omega_T)$. Hence the mapping $\mathcal{T}_\eta$ is continuous, and the proof is finished. □

Proof of Theorem 1.1: Take a sequence (η_m) of nonnegative real numbers which tends to zero as m tends to ∞. With every integer $m \geqslant 1$ we associate the fixed point $u^m \in \mathcal{B}$ of the mapping $\mathcal{T}_{\eta_m}$. Let (p_m, q_m) be the corresponding solution to (1.1) and (1.3) associated with u^m. Then (p_m, u^m) satisfies the system

$$a(u^m)\,\partial_t p_m - \partial_x^2 p_m = 0 \quad \text{in } \Omega_T\,, \tag{5.7}$$

$$\partial_t u_i^m + q_m^\eta\,\partial_x u_i^m + b_i(u^m)\,\partial_t p_m = 0 \quad \text{in } \Omega_T, \quad 1 \leqslant i \leqslant n\,. \tag{5.8}$$

Since (u^m) is bounded in the space $L^\infty(0, T; W^{1,1}(\Omega)^n) \cap H^1(0, T; L^1(\Omega)^n)$, there is a subsequence, still denoted (u^m), which converges to a function u in $L^1(\Omega_T)^n$ strongly and therefore in $L^r(\Omega_T)^n$ strongly for any $r \in [1, +\infty[$. The sequence (p_m) is bounded in W. Then, there is a subsequence, still denoted (p_m), which converges, as $m \to \infty$, weakly in $L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega))$ to a function p. By a compactness argument, $q_m = -\partial_x p_m$ converges, as $m \to \infty$, to $q = -\partial_x p$ in $L^r(\Omega_T)$ strongly for any $r \in [1, +\infty[$. Therefore, $q_m^{\eta_m}$ converges, as $m \to \infty$, to q in $L^r(\Omega_T)$ strongly for any $r \in [1, +\infty[$, and also in $L^2(0, T; H^1(\Omega))$ weakly.

Of course, $a(u^m)$ (resp. $(b_i(u^m))$ converges, as $m \to \infty$, to $a(u)$ (resp. $(b_i(u))$ in $L^r(\Omega_T)$ strongly for any $r \in [1, +\infty[$. Then we can pass to the limit in (5.7). On the other hand, for any $g \in C_0^1(]0, 1[\times [0, T[)$, we have

$$\int_0^T\int_0^1 (u_i^m\,\partial_t g + u_i^m\,\partial_x(q_m^{\eta_m} g) - b_i(u^m)\,\partial_t p_m\, g)\,dx\,ds = -\int_0^1 u_0(x)\,g(x,0)\,dx - \int_0^T u_1(t)\,q_m^{\eta_m}(0,t)\,g(0,t)\,dt, \quad 1 \leqslant i \leqslant n\,.$$

Then we can pass to the limit in each term. Note in particular that the weak convergence of $q_m^{\eta_m}$ to q in $L^2(0, T; H^1(\Omega))$ implies that of $q_m^{\eta_m}(0, t)$ to $q(0, t)$ in $L^2(0, T)$ weakly. Finally, the BV estimates of u come from (5.4). Thus the proof of Theorem 1.1 is completed.

6. REMARK

Consider the system (1.1) and (1.2) with the following particular initial and boundary conditions

$$\begin{cases} p(x,0) = p_0(x), & x \in \Omega, \\ p(0,t) = p_1, \quad p(1,t) = p_2, & t \in]0,T[, \end{cases} \tag{6.1}$$

and

$$u(0,t) = u^1(t), \quad u(x,0) = u^0(x), \quad x \in \Omega, \quad t \in]0,T[, \tag{6.2}$$

where p_1 and p_2 are constants. Comparing this problem to (1.1)-(1.4), we see that only (1.3) is replaced by (6.1). Assume that condition (1.10) holds and in addition there exist two constants $m > 0$, $\bar{\alpha} > 0$ such that,

$$\begin{cases} p_0 \in C^{2,\bar{\alpha}}(\bar{\Omega}), \quad p_0(0) = p_1, \quad p_0(1) = p_2, \\ p_0''(0) = p_0''(1) = 0, \quad -p_0'' \geq 0, \quad -p_0' \geq m, \end{cases} \tag{6.3}$$

and

$$\begin{cases} u^0 \in C^{1,\bar{\alpha}}(\bar{\Omega})^n, \quad u^1 \in C^{1,\bar{\alpha}/2}([0,T])^n, \quad \text{with } u^0(0) = u^1(0), \\ (u_i^1)'(0) - p_0'(0)\,(u_i^0)'(0) = 0, \quad 1 \leq i \leq n. \end{cases} \tag{6.4}$$

By using a result in [1], we can prove that for any u given in K, the solution p of problem (1.1) and (6.1) satisfies the estimate:

$$\|p\|_{W_r^{2,1}(\Omega_T)} \leq C,$$

where $C > 0$ is a constant depending only on the initial and boundary conditions of p, and

$$W_r^{2,1}(\Omega_T) = \{v \in L^2(\Omega_T), \partial_t v, \partial_x v, \partial_{xx} v \in L^2(\Omega_T)\},$$

with $2 < r < 3$. Therefore, on repeating the same arguments as [2] for $n = 1$, we are able to prove the following theorem. The proof will be omitted.

THEOREM 6.1: *There is a constant β, $0 < \beta \leq \bar{\alpha}$, such that problem (1.1)-(1.2) and (6.1)-(6.2) is uniquely solvable in the space $C^{2+\beta, 1+\beta/2}(\bar{\Omega}_T) \times C^{\beta}(\Omega_T)^n$. If in addition, $u^0(x) \in K$, $u^1(t) \in K$ for every (x,t) in Ω_T, then u lies in $C^{1+\beta/2}(\bar{\Omega}_T)^n$.* □

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