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MULTI-PARAMETER ASYMPTOTIC ERROR RESOLUTION OF THE MIXED FINITE ELEMENT METHOD FOR THE STOKES PROBLEM

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Abstract. In this paper, a multi-parameter error resolution technique is applied into a mixed finite element method for the Stokes problem. By using this technique and establishing a multi-parameter asymptotic error expansion for the mixed finite element method, an approximation of higher accuracy is obtained by multi-processor computers in parallel.

Résumé. Dans cet article une méthode de résolution d'erreurs multi-paramétrique est appliquée à la méthode des éléments finis mixte pour le problème de Stokes. À l'aide de cette technique, et en utilisant un développement asymptotique multi-paramétrique de l'erreur, on obtient une plus grande précision par calcul parallèle sur des machines multi-processeurs.

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1. Introduction

The Stokes equations have become an important model problem in computational fluid dynamics for designing and analyzing finite element algorithms. In the context of the Galerkin variational formulation, the computational accuracy for the velocity components can be compared to that known for the Poisson equation. For the pressure, however, one frequently observes a drastic reduction of the accuracy along the boundary. Therefore, one has anyway to employ some techniques or arguments to recover and increase the accuracy.

The extrapolation approach established by Richardson in 1926, is an important technique to obtain approximations of higher accuracy. The application of this approach in the finite difference method can be found in [13]. In 1983, Lin, Lü and Shen [10] have applied the technique to the finite element method. Development in this direction can be found in [3,12,14]. However, this approach has a limitation since it requires a global refinement and hence wastes computing time and memory. Recently, a so-called *multi-parameter error resolution* technique has been introduced to obtain approximations of higher accuracy (see [19,20]). This new approach is based on a multi-parameter asymptotic error expansion and only uses partial refinements. It is shown that a combination of discrete solutions related to such partial refinements can produce an approximation of higher accuracy and the procedure can be done by multi-processor computers in parallel. In this paper, we apply this technique to a mixed finite element method for the Stokes problem.

Consider the homogenous Stokes problem

$$\begin{cases} -\Delta \mathbf{u} + \nabla p &= \mathbf{f}, \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0, \quad \text{in } \Omega, \\ \mathbf{u} &= 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega = [0,1]^2$, $\mathbf{u} = (u_1, u_2)$ is the velocity, p is the pressure and $\mathbf{f}$ indicates the external force. We know that (1.1) is equivalent to the variational formulation([8]): Find $(\mathbf{u}, p) \in (H_0^1(\Omega))^2 \times L_0^2(\Omega)$ such that

$$B(\mathbf{u}, p; \mathbf{v}, q) = (\mathbf{f}, \mathbf{v}), \quad \forall (\mathbf{v}, q) \in (H_0^1(\Omega))^2 \times L_0^2(\Omega), \tag{1.2}$$

where

$$L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q = 0\},$$

$$B(\mathbf{u}, p; \mathbf{v}, q) = (\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \operatorname{div} \mathbf{v}) + (q, \operatorname{div} \mathbf{u})$$

and other notations are identical to these of references [5, 8].

Divide Ω into subrectangles $\Omega_j (j = 1, \cdots, m)$ whose edges are parallel to the x-axis and the y-axis respectively. On each $\Omega_j (j = 1, \cdots, m)$ a uniform mesh is imposed, and globally a quasi-uniform partition on Ω is formed. Denote the mesh sizes on Ω_j by $h_{j,1}$ in x-direction and $h_{j,2}$ in y-direction $(j = 1, \cdots, m)$. Among these mesh parameters, some are independent, say $h_1, \cdots, h_s$. It can be proved that $2 \le s \le m + 1$ and one can choose $s > 2$. Let $h = \max\{h_j, \cdots, h_s\}, T_h$ be the partition on Ω and $(\mathbf{V}_h, L_h)$ be a pair of finite element spaces on T_h, *e.g.* the Bernardi-Raugel finite element space or the filtered bilinear-constant finite element space, satisfying the Babuska-Brezzi conditon (see [2, 8]). The interpolated finite element approximations corresponding to $h_1, \cdots, h_s$ is denoted by $(\mathbf{u}(h_1, \cdots, h_s), p(h_1, \cdots, h_s))$. Then there exist *multi-parameter error resolution* expressions on Ω (see Sect. 4)

$$\left\{ \begin{array}{rcl} \mathbf{u}(h_1, \cdots, h_s) & = & \mathbf{u} + \sum_{j=1}^s \mathbf{a}_j h_j^2 + O(h^3), \text{ in } H^1 - \text{norm}, \\ \mathbf{u}(h_1, \cdots, h_s) & = & \mathbf{u} + \sum_{j=1}^s \mathbf{a}_j h_j^2 + O(h^4), \text{ in } L^2 - \text{norm}, \\ p(h_1, \cdots, h_s) & = & p + \sum_{j=1}^s b_j h_j^2 + O(h^3), \text{ in } L^2 - \text{norm} \end{array} \right. \tag{1.3}$$

where $\mathbf{a}_j, b_j (j = 1, \cdots, s)$ are independent of $(h_1, \cdots, h_s)$. Based on (1.3), a multi-parameter extrapolation yields approximations of higher accuracy

$$\left\{ \begin{array}{l} (4 \sum_{j=1}^s \mathbf{u}_j - (4s - 3)\mathbf{u}_0)/3 = \mathbf{u} + O(h^3), \text{ in } H^1 - \text{norm}, \\ (4 \sum_{j=1}^s \mathbf{u}_j - (4s - 3)\mathbf{u}_0)/3 = \mathbf{u} + O(h^4), \text{ in } L^2 - \text{norm}, \\ (4 \sum_{j=1}^s p_j - (4s - 3)p_0)/3 = p + O(h^3), \text{ in } L^2 - \text{norm}, \end{array} \right. \tag{1.4}$$

where $(\mathbf{u}_0, p_0) = (\mathbf{u}(h_1, \cdots, h_s), p(h_1, \cdots, h_s))$ and $(\mathbf{u}_j, p_j) = (\mathbf{u}(h_1, \cdots, h_{j-1}, h_j/2, h_{j+1}, \cdots, h_s),$ $p(h_1, \cdots, h_{j-1}, h_j/2, h_{j+1}, \cdots, h_s))$.

Therefore, a parallel algorithm for approximations of higher accuracy can be designed as follows.

Algorithm.

Step 1. Compute $(\mathbf{u}_j, p_j)(0 \le j \le s)$ in parallel.

Step 2. Set

$$\left\{ \begin{array}{l} \mathbf{u}^c = (4 \sum_{j=1}^s \mathbf{u}_j - (4s - 3)\mathbf{u}_0)/3, \\ p^c = (4 \sum_{j=1}^s p_j - (4s - 3)p_0)/3. \end{array} \right. \tag{1.5}$$

2. Preliminaries

For any element $e = [x_e - h_e, y_e - k_e]^2$, we define the interpolation operators $\mathbf{i}_h = (i_{h,1}, i_{h,2}) : (C(e))^2 \longrightarrow Q_{1,2} \times Q_{2,1}$ and $j_h : L^2(e) \longrightarrow Q_0$ by (*cf.* [2, 8])

$$\begin{cases} (\mathbf{v} - \mathbf{i}_h \mathbf{v})(x_e \pm h_e, y_e \pm k_e) = 0, \ \forall \mathbf{v} \in (C(e))^2, \\ \displaystyle\int_{(x_e \pm h_e, y_e - k_e)}^{(x_e \pm h_e, y_e + k_e)} (\varphi - i_{h,1}\varphi) dy = 0, \ \forall \varphi \in C(e), \\ \displaystyle\int_{(x_e - h_e, y_e \pm k_e)}^{(x_e + h_e, y_e \pm k_e)} (\varphi - i_{h,2}\varphi) dx = 0, \ \forall \varphi \in C(e), \end{cases} \tag{2.1}$$

$$j_h \varphi = \frac{1}{|\, e \,|} \int_e \varphi, \ \forall \varphi \in L^2(e), \tag{2.2}$$

where $Q_{r,n} = span\{x^i y^j : 0 \le i \le r, \ 0 \le j \le n\}$ and $Q_r \equiv Q_{r,r}$. Introduce two other postprocessing interpolation operators. Let T_h be a rectangular partition on Ω with size h. We assume that T_h has been obtained from T_{3h} by dividing each element into nine congruent rectangles. Let $\tau = \cup_{i=1}^{9} e_i \in T_{3h}$ with $e_i \in T_h$, and $\mathbf{I}_{3h} = (I_{3h,1}, I_{3h,2})$ and J_{3h} are defined as

$$\begin{cases} I_{3h,j} w|_\tau \in Q_3(\tau), \ j = 1, 2, \\ I_{3h,j} w(p_i) = w(p_i), \quad i = 1, \cdots, 16, \ j = 1, 2, \end{cases} \tag{2.3}$$

$$\begin{cases} J_{3h} p|_\tau \in Q_2(\tau), \\ \int_{e_i} (J_{3h} p - p) = 0, \quad i = 1, \cdots, 9, \end{cases} \tag{2.4}$$

where $p_i \, (i = 1, \cdots, 16)$ are all vertices of $e_1, \cdots, e_9$. It can be proved that (*cf.* [11, 17, 18])

$$\begin{cases} \mathbf{I}_{3h} \mathbf{i}_h = \mathbf{I}_{3h}, \\ \|\mathbf{I}_{3h} \mathbf{v}\|_1 \le c \|\mathbf{v}\|_1, \quad \forall \mathbf{v} \in \mathbf{V}_h, \\ h^{-1} \|\mathbf{I}_{3h} \mathbf{w} - \mathbf{w}\|_0 + \|\mathbf{I}_{3h} \mathbf{w} - \mathbf{w}\|_1 \le c h^r \|\mathbf{w}\|_{1+r}, \quad \forall \mathbf{w} \in H^{1+r}(\Omega), 1 \le r \le 3, \end{cases} \tag{2.5}$$

$$\begin{cases} J_{3h} j_h = J_{3h}, \\ \|J_{3h} q\|_0 \le c \|q\|_0, \quad \forall q \in L_h, \\ \|J_{3h} w - w\|_0 \le c h^r \|w\|_r, \quad \forall w \in H^r(\Omega), 1 \le r \le 3. \end{cases} \tag{2.6}$$

In order to solve the problem (1.1), many pairs of finite element spaces were introduced (see [2, 8]). One of them is the so-called Bernardi-Raugel element, which is defined by

$$\begin{cases} \mathbf{V}_h = \{\mathbf{v} \in (C(\overline{\Omega}))^2, \quad \mathbf{v}\,|_e \in Q_{1,2}(e) \times Q_{2,1}(e), \ \mathbf{v}\,|_{\partial\Omega} = 0, \, e \in T_h\}, \\ L_h = \{q \in L_0^2(\Omega), \quad q|_e \in Q_0(e), \quad \int_\Omega q = 0, \, e \in T_h\}. \end{cases}$$

We mention here that the Bernardi-Raugel element was presented but not enalysed in [7].

3. Error resolution for interpolants

We have the following *multi-parameter error resolution* for interpolants.

Theorem 3.1. *For $(\mathbf{u},p)\in(H^4(\Omega))^2\times H^3(\Omega)$, there exist $\mathbf{f}_j\in(L^2(\Omega))^2, \mathbf{g}_j\in(H^{1/2}(\Gamma))^2 (j=1,\cdots,s)$ such that*

$$\begin{aligned}&B(\mathbf{u}-i_h\mathbf{u},p-j_hp;\mathbf{v},q)\\=&\sum_{j=1}^s h_j^2(\int_\Omega \mathbf{f}_j\mathbf{v}+\int_\Gamma \mathbf{g}_j\mathbf{v})+O(h^3)(\|\mathbf{u}\|_4+\|p\|_3)\|\mathbf{v}\|_1,\ \forall(\mathbf{v},q)\in\mathbf{V}_h\times L_h,\end{aligned}$$

where $\Gamma=\cup_{1\le i,j\le s}\partial\Omega_i\cap\partial\Omega_j$.

Proof. Since

$$(q,\operatorname{div}(\mathbf{u}-\mathbf{i}_h\mathbf{u}))=0,\quad \forall q\in L_h,$$

it is only necessary to prove two expansions

$$(\nabla(\mathbf{u}-i_h\mathbf{u}),\nabla\mathbf{v})=\frac{1}{3}\sum_{e\in T_h}\int_e(h_e^2\partial_{xxyy}u_1,k_e^2\partial_{xxyy}u_2)\mathbf{v}+O(h^3)\|\mathbf{u}\|_4\|\mathbf{v}\|_1 \tag{3.1}$$

for $\mathbf{v}=(v_1,v_2)\in\mathbf{V}_h$, and

$$\begin{aligned}(p-j_hp,\operatorname{div}\mathbf{v}) =& \frac{1}{3}\sum_{e\in T_h}\int_e(k_e^2\partial_{xyyy}p,h_e^2\partial_{xxxy}p)\mathbf{v}-\frac{1}{3}\sum_{s_{e,4}\cup s_{e,2}\subset\Gamma_x}k_e^2(\int_{s_{e,4}}-\int_{s_{e,2}})\partial_{xy}pv_1\\&-\frac{1}{3}\sum_{s_{e,3}\cup s_{e,1}\subset\Gamma_y}h_e^2(\int_{s_{e,3}}-\int_{s_{e,1}})\partial_{xy}pv_2+O(h^3)\|p\|_3\|\mathbf{v}\|_1,\end{aligned} \tag{3.2}$$

where Γ_x,Γ_y is the part of Γ parallel to the x-direction and the y-direction respectively, and

$$s_{e,1}=\{(x,y):x=x_e-h_e,y_e-k_e\le y\le y_e+k_e\},$$
$$s_{e,2}=\{(x,y):x_e-h_e\le x\le x_e+h_e,y=y_e-k_e\},$$
$$s_{e,3}=\{(x,y):x=x_e+h_e,y_e-k_e\le y\le y_e+k_e\},$$
$$s_{e,4}=\{(x,y):x_e-h_e\le x\le x_e+h_e,y=y_e+k_e\}.$$

We employ the technique in [11, 17, 18] and introduce the two error functions

$$F\equiv F_e=\frac{1}{2}((y-y_e)^2-k_e^2),\quad E\equiv E_e=\frac{1}{2}((x-x_e)^2-h_e^2),$$

then, it is obvious that $F(y)=0$ on $s_{e,2}$ and $s_{e,4}$, $E(x)=0$ on $s_{e,1}$ and $s_{e,3}$.

$$\left\{\begin{array}{rcl}E&=&-\frac{1}{3}h_e^2+\frac{1}{6}(E^2)_{xx},\\ x-x_e&=&\frac{1}{6}(E^2)_{xxx},\\ (x-x_e)^2&=&\frac{1}{45}(E^3)_{xxxx}+\frac{1}{5}h_e^2,\end{array}\right.$$

$$\begin{cases} F &= -\frac{1}{3}k_e^2 + \frac{1}{6}(F^2)_{yy}, \\ y - y_e &= \frac{1}{6}(F^2)_{yyy}, \\ (y-y_e)^2 &= \frac{1}{45}(F^3)_{yyyy} + \frac{1}{5}k_e^2. \end{cases}$$

Let $\mathbf{u} = (u_1, u_2)$, we only need to go to prove

$$(\nabla(u_1 - i_{h,1}u_1), \nabla v_1) = \frac{1}{3}\sum_{e\in T_h} h_e^2 \int_e \partial_{xxyy}u_1 v_1 + O(h^3)\|\mathbf{u}\|_4\|\mathbf{v}\|_1. \tag{3.3}$$

A Taylor expansion yields

$$\partial_x v_1 = \partial_x v_1(x, y_e) + \frac{1}{6}(F^2)_{yyy}\partial_{xy}v_1(x,y_e) + (\frac{1}{90}(F^3)_{yyyy} + \frac{1}{10}k_e^2)\partial_{xyy}v_1.$$

The definition (2.1), integration by parts and inverse estimates lead to

$$\int_e \partial_x(u_1 - i_{h,1}u_1)\partial_x v_1(x,y_e) = 0,$$

$$\begin{aligned} \frac{1}{6}\int_e (F^2)_{yyy}\partial_x(u_1 - i_{h,1}u_1)\partial_{xy}v_1(x,y_e) &= -\frac{1}{6}\int_e F^2\partial_{xyyy}u_1\partial_{xy}v_1(x,y_e) \\ &= O(h^3)\|\mathbf{u}\|_4\|\mathbf{v}\|_1, \end{aligned}$$

$$\begin{aligned} \int_e \partial_x(u_1 - i_{h,1}u_1)(\frac{1}{90}(F^3)_{yyyy} + \frac{1}{10}k_e^2)\partial_{xyy}v_1 &= -\frac{1}{90}\int_e (F^3)_y\partial_{xyyy}u_1\partial_{xyy}v_1 \\ &= O(h^3)\|\mathbf{u}\|_4\|\mathbf{v}\|_1. \end{aligned}$$

Thus, we obtain

$$(\partial_x(u_1 - i_{h,1}u_1), \partial_x v_1) = O(h^3)\|\mathbf{u}\|_4\|\mathbf{v}\|_1 \tag{3.4}$$

For $(\partial_y(u_1 - i_{h,1}u_1), \partial_y v_1)$, using the definition (2.1),

$$\partial_y v_1 = E_{xx}\partial_y v_1(x_e, y) + \frac{1}{6}(E^2)_{xxx}\partial_{xy}v_1,$$

and integration by parts,

$$\begin{aligned} &\int_e \partial_y(u_1 - i_{h,1}u_1)(E)_{xx}\partial_y v_1(x_e,y) \\ = \ & (\int_{s_{e,3}} - \int_{s_{e,1}})E_x\partial_y(u_1 - i_{h,1}u_1)\partial_y v_1(x_e,y)dy - \int_e E_x\partial_{xy}(u_1 - i_{h,1}u_1)\partial_y v_1(x_e,y) \\ = \ & -(\int_{s_{e,3}} - \int_{s_{e,1}})E_x(u_1 - i_{h,1}u_1)\partial_{yy}v_1(x_e,y)dy + \int_e E\partial_{xxy}u_1\partial_y v_1(x_e,y) \\ = \ & -\frac{1}{3}h_e^2\int_e \partial_{xxy}u_1(\partial_y v_1 - E_x\partial_{xy}v_1) + \frac{1}{6}\int_e (E^2)_{xx}\partial_{xxy}u_1\partial_y v(x_e,y) \\ = \ & -\frac{1}{3}h_e^2\int_e \partial_{xxy}u_1\partial_y v_1 + \int_e \partial_{xxxy}u_1(-\frac{1}{3}h_e^2 E\partial_{xy}v_1 - \frac{1}{6}(E^2)_x\partial_y v_1 + \frac{1}{6}(E^2)_x E_x\partial_{xy}v_1) \\ = \ & -\frac{1}{3}h_e^2(\int_{s_{e,4}} - \int_{s_{e,2}})\partial_{xxy}u_1 v_1 dx + \frac{1}{3}h_e^2\int_e \partial_{xxyy}u_1 v_1 + O(h^3)\|\mathbf{u}\|_4\|\mathbf{v}\|_1, \end{aligned}$$

$$\begin{aligned}
&\frac{1}{6}\int_e (E^2)_{xxx}\partial_y(u_1 - i_h u_1)\partial_{xy}v_1 \\
&= \frac{1}{6}(\int_{s_{e,3}} - \int_{s_{e,1}})(E^2)_{xx}\partial_y(u_1 - i_h u_1)\partial_{xy}v_1 - \frac{1}{6}\int_e (E^2)_{xx}\partial_{xy}(u_1 - i_{h,1}u_1)\partial_{xy}v_1 \\
&= -\frac{1}{6}(\int_{s_{e,3}} - \int_{s_{e,1}})(E^2)_{xx}(u_1 - i_h u_1)\partial_{xyy}v_1 + \frac{1}{6}\int_e (E^2)_x \partial_{xxy}u_1\partial_{xy}v_1 \\
&= \frac{1}{6}\int_e (E^2)_x\partial_{xxy}u_1\partial_{xy}v_1 \\
&= O(h^3)\|\mathbf{u}\|_4\|\mathbf{v}\|_1.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
(\partial_y(u_1 - i_h u_1), \partial_y v_1) \quad = \quad & \frac{1}{3}\sum_{e\in T_h} h_e^2((\int_{s_{e,4}} - \int_{s_{e,2}})\partial_{xxy}u_1 v_1 dx + \int_e \partial_{xxyy}u_1 v_1) \\
& + O(h^3)\|\mathbf{u}\|_4\|\mathbf{v}\|_1. \qquad (3.5)
\end{aligned}$$

By the quasi-uniformity of T_h, one sees that line integrals in (3.5) disappear during the summation, thus (3.3) follows from (3.4, 3.5).

We turn to prove (3.2). Similar to the proof of (3.1), we only need to prove

$$\begin{aligned}
(p - j_h p, \partial_x v_1) \quad = \quad & \frac{1}{3}\sum_{e\in T_h} k_e^2 \int_e \partial_{xyy}p v_1 - \frac{1}{3}\sum_{s_{e,4}\cup s_{e,2}\subset\Gamma_x} k_e^2(\int_{s_{e,4}} - \int_{s_{e,2}})\partial_{xy}p v_1 \\
& + O(h^3)\|p\|\|_3\|\mathbf{v}\|_1. \qquad (3.6)
\end{aligned}$$

By the Taylor expansion, we have

$$\begin{aligned}
\int_e (p - j_h p)\partial_x v_1 \quad &= \quad \int_e (p - j_h p)(\partial_x v_1(x, y_e) + F_y\partial_{xy}v_1(x, y_e) + \frac{1}{6}((F^2)_{yy} + k_e^2)\partial_{xyy}v_1) \\
&\equiv \quad I + II + III. \qquad (3.7)
\end{aligned}$$

The definition (2.2) shows that $I = 0$ and

$$\begin{aligned}
II \quad = \quad & -\int_e F\partial_y p\partial_{xy}v_1(x, y_e) \\
= \quad & \frac{1}{3}k_e^2\int_e \partial_y p\partial_{xy}v_1(x, y_e) - \frac{1}{6}\int_e (F^2)_{yy}\partial_y p\partial_{xy}v_1(x, y_e) \\
= \quad & \frac{1}{3}k_e^2(\int_{s_{e,3}} - \int_{s_{e,1}})\partial_y p\partial_y v_1(x, y_e)dy - \frac{1}{3}k_e^2\int_e \partial_{xy}p(\partial_y v_1 - F_y\partial_{yy}v_1) + O(h^3)\|p\|_3\|\mathbf{v}\|_1 \\
= \quad & \frac{1}{3}k_e^2(\int_{s_{e,3}} - \int_{s_{e,1}})\partial_y p\partial_y v_1(x, y_e)dy - \frac{1}{3}k_e^2\int_e \partial_{xy}p\partial_y v_1 + O(h^3)\|p\|_3\|\mathbf{v}\|_1 \\
= \quad & \frac{1}{3}k_e^2(\int_{s_{e,3}} - \int_{s_{e,1}})\partial_y p\partial_y v_1(x, y_e)dy - \frac{1}{3}k_e^2(\int_{s_{e,4}} - \int_{s_{e,2}})\partial_{xy}p v_1 dx \\
& + \frac{1}{3}k_e^2\int_e \partial_{xyy}p v_1 + O(h^3)\|p\|_3\|\mathbf{v}\|_1,
\end{aligned}$$

$$
\begin{aligned}
III &= \frac{1}{6}\int_e F^2\partial_{yy}p\partial_{xyy}v_1\\
&= \frac{1}{6}(\int_{s_{e,3}} - \int_{s_{e,1}})F^2\partial_{yy}p\partial_{yy}v_1 dy - \frac{1}{6}\int_e F^2\partial_{xyy}p\partial_{yy}v_1\\
&= \frac{1}{6}(\int_{s_{e,3}} - \int_{s_{e,1}})F^2\partial_{yy}p\partial_{yy}v_1 dy + O(h^3)\|p\|_3\|\mathbf{v}\|_1.
\end{aligned}
$$

Similarly, line integrals along y-direction in the above terms will disappear during the summation of $e \in T_h$. Thus we complete the proof.

From the proof of Theorem 3.1, we can also obtain the following

Theorem 3.2. *For* $(\mathbf{u},p) \in (H^4(\Omega))^2 \times H^3(\Omega)$, *there exist* $\mathbf{f}_j \in (L^2(\Omega))^2, \mathbf{g}_j \in (H^{1/2}(\Gamma))^2 (j = 1, \cdots, s)$ *such that*

$$
\begin{aligned}
B(\mathbf{u} - i_h\mathbf{u}, p - j_hp; \mathbf{v}, q) &= \sum_{j=1}^{s} h_j^2(\int_\Omega \mathbf{f}_j\mathbf{v} + \int_\Gamma \mathbf{g}_j\mathbf{v})\\
&\quad + O(h^4)(\|\mathbf{u}\|_4 + \|p\|_3)(h^{-1}\|\mathbf{v} - \mathbf{w}\|_1 + \|\mathbf{w}\|_2),\ \forall(\mathbf{v}, q) \in \mathbf{V}_h \times L_h, \quad (3.8)
\end{aligned}
$$

where $\mathbf{w} \in (H_0^1(\Omega) \cap H^4(\Omega))^2$.

4. Error resolution for finite element solutions

A mixed finite element method based on (1.1) reads as follows: Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times L_h$ such that

$$B(\mathbf{u}_h, p_h; \mathbf{v}, q) = (\mathbf{f}, \mathbf{v}), \quad \forall(\mathbf{v}, q) \in \mathbf{V}_h \times L_h. \tag{4.1}$$

or

$$B(\mathbf{u}_h - u, p_h - p; \mathbf{v}, q) = 0, \quad \forall(\mathbf{v}, q) \in \mathbf{V}_h \times L_h. \tag{4.2}$$

One has the following error estimate (see [2,8])

$$h^{-1}\|\mathbf{u} - \mathbf{u}_h\|_0 + \|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \le ch(\|\mathbf{u}\|_2 + \|p\|_1). \tag{4.3}$$

Now, we shall present the *multi-parameter error resolution* for the numerical solution.

Theorem 4.1. *If* $(\mathbf{u}, p) \in (H_0^1(\Omega) \cap H^4(\Omega))^2 \times (L_0^2(\Omega) \cap H^3(\Omega))$ *is the exact solution of (1.1), then there exist* $(\mathbf{a}_j, b_j) \in (H_0^1(\Omega) \cap H^2(\Omega \setminus \Gamma))^2 \times (L_0^2(\Omega) \cap H^1(\Omega \setminus \Gamma))(j = 1, \cdots, s)$ *such that*

$$
\begin{cases}
\mathbf{I}_{3h}\mathbf{u}_h - \mathbf{u} = \sum_{j=1}^{s} \mathbf{a}_j h_j^2 + O(h^3),\ \textit{in } H^1(\Omega \setminus \Gamma)\\
J_{3h}p_h - p = \sum_{j=1}^{s} b_j h_j^2 + O(h^3),\ \textit{in } L^2(\Omega \setminus \Gamma).
\end{cases}
$$

Proof. Let $(\mathbf{a}_j, b_j) \in (H_0^1(\Omega))^2 \times L_0(\Omega)$ $(j = 1, \cdots, s)$ satisfy

$$B(\mathbf{a}_j, b_j; \mathbf{v}, q) = (f_j, \mathbf{v}) + \int_\Gamma \mathbf{g}_j\mathbf{v},\ \forall(\mathbf{v}, q) \in (H_0^1(\Omega))^2 \times L_0^2(\Omega) \tag{4.4}$$

and $(\mathbf{a}_{j,h}, b_{j,h}) \in \mathbf{V}_h \times L_h$ $(j = 1, \cdots, s)$

$$B(\mathbf{a}_{j,h}, b_{j,h}; \mathbf{v}, q) = (f_j, \mathbf{v}) + \int_\Gamma \mathbf{g}_j\mathbf{v},\ \forall(\mathbf{v}, q) \in \mathbf{V}_h \times L_h, \tag{4.5}$$

then we have

$$B(\mathbf{u}-i_h\mathbf{u}-\sum_{j=1}^{s}\mathbf{a}_{j,h}h_j^2,p-j_hp-\sum_{j=1}^{s}b_{j,h}h_j^2;\mathbf{v},q)=O(h^3)\|\mathbf{v}\|_1.$$

Taking $\mathbf{v}=\mathbf{u}_h-\mathbf{i}_h\mathbf{u}-\sum_{j=1}^{s}\mathbf{a}_{j,h}h_j^2$ and $q=p_h-j_hp-\sum_{j=1}^{s}b_{j,h}h_j^2$ in the above equation, we obtain by (4.2) and the Babuska-Brezzi condition (see *e.g.* [1,2,4])

$$\left\{\begin{array}{l}\mathbf{u}_h-\mathbf{i}_h\mathbf{u}=\sum_{j=1}^{s}\mathbf{a}_{j,h}h_j^2+O(h^3),\ \text{in } H^1(\Omega),\\ p_h-j_hp=\sum_{j=1}^{s}b_{j,h}h_j^2+O(h^3),\ \text{in } L^2(\Omega).\end{array}\right.\tag{4.6}$$

Regularity results imply that $(\mathbf{a}_j,b_j)\in(H_0^1(\Omega)\cap H^2(\Omega\setminus\Gamma))^2\times(L_0^2(\Omega)\times H^1(\Omega\setminus\Gamma))(j=1,\cdots,s)$ (see [9]) and a similar estimate to (4.3) yields

$$\|\mathbf{a}_j-\mathbf{a}_{j,h}\|_1+\|b_j-b_{j,h}\|_0\le ch(\|\mathbf{a}_j\|_2'+\|b_j\|_1'),\ j=1,\cdots,s,\tag{4.7}$$

where $\|\cdot\|_i'$ means piecewise H^i-norm(i=1,2). Thus, (4.7) together with (2.5, 2.6, and 4.6) complete the proof.

Theorem 4.2. *If $(\mathbf{u},p)\in(H_0^1(\Omega)\cap H^4(\Omega))^2\times(L_0^2(\Omega)\cap H^3(\Omega))$ is the exact solution of (1.1), then there exist $\mathbf{a}_j\in(H_0^1(\Omega)\cap H^2(\Omega\setminus\Gamma))^2(j=1,\cdots,s)$ such that*

$$\mathbf{I}_{3h}\mathbf{u}_h-\mathbf{u}=\sum_{j=1}^{s}\mathbf{a}_jh_j^2+O(h^4),\ \textit{in } L^2(\Omega\setminus\Gamma).\tag{4.8}$$

Proof. Let $\mathbf{a}_j,b_j,\mathbf{a}_{j,h}$ and $b_{j,h}$ satisfy (4.4, 4.5) and set $\eta=\mathbf{u}_h-\mathbf{i}_h\mathbf{u}-\sum_{j=1}^{s}\mathbf{a}_{j,h}h_j^2$ as well as $\xi=p_h-j_hp-\sum_{j=1}^{s}b_{j,h}h_j^2$, then there exists $(\mathbf{w},\varphi)\in(H_0^1(\Omega)\cap H^2(\Omega))^2\times(L_0^2(\Omega)\cap H^1(\Omega))$ satisfying

$$\left\{\begin{array}{l}B(\mathbf{w},\varphi;\mathbf{v},q)=(\eta,\mathbf{v}),\quad\forall(\mathbf{v},q)\in(H_0^1(\Omega))^2\times L_0^2(\Omega),\\ \|\mathbf{w}\|_2+\|\varphi\|_1\le c\|\eta\|_0.\end{array}\right.\tag{4.9}$$

Thus

$$\begin{aligned}\|\eta\|_0^2&=B(\mathbf{w},\varphi;\eta,\xi)\\&=B(\mathbf{w}-\mathbf{i}_h\mathbf{w},\varphi-j_h\varphi;\eta,\xi)+B(\eta,\xi;\mathbf{i}_h\mathbf{w},-j_h\varphi).\end{aligned}\tag{4.10}$$

From Theorem 3.1, Theorem 3.2, (4.6, 4.9), we have

$$\begin{aligned}|\ B(\mathbf{w}-\mathbf{i}_h\mathbf{w},\varphi-j_h\varphi;\eta,\xi)\ |&\le ch(\|\mathbf{w}\|_2+\|\varphi\|_1)(\|\eta\|_1+\|\xi\|_0)\\&\le ch^4\|\eta\|_0\end{aligned}\tag{4.11}$$

and

$$\begin{aligned}|\ B(\eta,\xi;\mathbf{i}_h\mathbf{w},-j_h\varphi)\ |&\le ch^4(h^{-1}\|\mathbf{w}-\mathbf{i}_h\mathbf{w}\|_1+\|\mathbf{w}\|_2)\\&\le ch^4\|\mathbf{w}\|_2\\&\le ch^4\|\eta\|_0.\end{aligned}\tag{4.12}$$

Combining (2.5, 4.10, 4.11) and (4.12), we achieve the proof.

5. Remarks

Only the *multi-paramemter error resolution* for the Bernardi-Raugel finite element spaces has been discussed above. In fact, the error resolution arguments presented in this paper can be applied to other pairs of finite element spaces with similar results, for instance, the filtered bilinear-constant finite element spaces. Moreover, if the Green-function technique as in [6] is adapted, then maximum norm estimates can also be obtained.

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