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PREDICTION-CORRECTION LEGENDRE SPECTRAL SCHEME FOR INCOMPRESSIBLE FLUID FLOW

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Abstract. The initial-boundary value problem of two-dimensional incompressible fluid flow in stream function form is considered. A prediction-correction Legendre spectral scheme is proposed, which is easy to be performed. The numerical solution possesses the accuracy of second-order in time and higher order in space. The numerical experiments show the high accuracy of this approach.

Résumé. Le problème de fluide incompressible à deux dimensions est considéré sous forme de fonction-courant. Un schéma de type prédiction-correction spectrale de Legendre est proposé, ce dernier étant facile à mettre en œuvre. La solution numérique possède une précision de second ordre en temps et d'ordre supérieur en espace. Les résultats numériques montrent la grande préscison de cette approche.

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1. Introduction

Since Navier-Stokes equation plays an important role in studying incompressible fluid flow, there have been a lot of literature concerning the existence, uniqueness, regularity of its solution. Usually the primitive equation is considered, *e.g.*, see [1,2]. Many methods are used for the numerical simulation of this problem, such as finite difference method, finite element method and spectral method, *e.g.*, see [2–8]. But we meet several difficulties in calculation. For instance, if we use finite difference method, then we have to evaluate the pressure at each time step. Some authors developed artificial compressibility method or small parameter method (see [9–12]). But the accuracy is lowered usually. On the other hand, it is not easy to deal with the boundary value of the pressure (see [13]). If we use finite element method or spectral method, then we need to construct a trial function space with the incompressibility and some conditions ensuring the convergence. It is also a difficult job. Therefore we look for alternative formulations of Navier-Stokes equation. One of them is the vorticity-stream function form (see [1, 3, 4]). Since the incompressibility is included automatically and the pressure no longer appears in this case, the above difficulties are removed in theoretical analysis and numerical experiments. However, for the initial-boundary value problem, another difficulty comes out. It is how to deal with the value of the vorticity near the boundary. For simplicity of analysis, it is assumed sometimes that the vorticity is given on the boundary, *e.g.*, see [1, 4]. However it is not physical, and so brings in the errors. The third is the stream function form in which only the stream function appears in the equation with the physical boundary conditions. Let $T > 0$ and $\Omega = \{(x, y) \mid \ |x| < 1, |y| < 1\}$ with the boundary $\partial\Omega$. $u(x, y, t)$ and μ are the stream function and the kinetic viscosity, respectively. $f(x, y)$ and $u_0(x, y)$ describe the source term and the initial state. Δ is

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the Laplacian. For simplicity, let $u_x = \dfrac{\partial u}{\partial x}$, $u_y = \dfrac{\partial u}{\partial y}$ and define

$$G(u,v) = u_y(\Delta v)_x - u_x(\Delta v)_y.$$

Then the stream function form of Navier-Stokes equation is as follows (see [6])

$$\begin{cases} \dfrac{\partial}{\partial t}\Delta u + G(u,u) - \mu\Delta^2 u = f, & \text{in} \quad \Omega\times(0,T], \\ u = \dfrac{\partial u}{\partial n} = 0, & \text{on} \quad \partial\Omega\times(0,T], \\ u(x,y,0) = u_0(x,y), & \text{in} \quad \Omega\cup\partial\Omega. \end{cases} \tag{1.1}$$

The main merits of this expression are remedying the troubles mentioned above and keeping the physical boundary value conditions naturally. But in opposite, the appearance of the biharmonic operator and the nonlinear terms involving the third order derivatives with respects to the spatial variables bring in some other difficulties in analysis. To our knowledge, there have been very few results for this problem. For instance, in the field of spectral method, it still remains on the level of biharmonic equation. Two outstanding results were due to Bernardi, Coppoletta, Maday [14] and Jie Shen [15], which increase the possibility of precise analysis and efficient algorithm for the stream function form. Recently the authors considered this problem. The existence, uniqueness and regularity of the weak solution are proved under some conditions in [16]. While a fully discrete Legendre spectral scheme is proposed in [17]. Numerical results show the advantages of this approach. However, since the nonlinear term is approximated by partially implicit technique in [17], the numerical solution possesses the accuracy of first-order in time only.

In this paper, we propose a prediction-correction Legendre spectral scheme for (1.1). In Section 2, we introduce the weak formulation of (1.1) and some results in [16, 17]. In Section 3, we construct the prediction-correction scheme. At each step, the nonlinear term is approximated explicitly. Thus we can solve it explicitly and save the work. In Section 4, we list some numerical results which show the high accuracy of this method. The main idea and techniques in this paper are also useful for other nonlinear time-dependent problems with high order derivatives in space.

2. The weak formulation and related results

In this section, we introduce some notations and the weak formulation of (1.1). We also list some properties of the trilinear form used for dealing with the nonlinear term $G(u,u)$.

Throughout this paper, we use Sobolev spaces $W^{r,p}(\Omega)$ and $W_0^{r,p}(\Omega)$. For $p=2$, we denote these spaces by $H^r(\Omega)$ and $H_0^r(\Omega)$. Their definitions and properties can be found in [18, 19]. We denote by $\|\cdot\|_{r,p}$ the norm of $W^{r,p}(\Omega)$. If $p=2$, then the index p is omitted. In addition, let $\|\cdot\| = \|\cdot\|_0$. We recall that the usual semi-norm $|\cdot|_r$ is equivalent to the norm $\|\cdot\|_r$ in $H_0^r(\Omega)$. Moreover for $u\in H_0^2(\Omega)$,

$$| u |_2 = \| \Delta u \| . \tag{2.1}$$

These notations are also applied to the vector function spaces $(W^{r,p}(\Omega))^2$ and $(H^r(\Omega))^2$. Let $(\cdot,\cdot)$ be the usual inner product of either $(L^2(\Omega))^2$ or $L^2(\Omega)$. Denote the usual inner product of $H^s(\Omega)$ by $(\cdot,\cdot)_s$. The space $H^\sigma(0, T; W^{r,p}(\Omega))$ is defined as in [19].

For any $u,w\in W^{1,4}(\Omega)$ and $v\in H^2(\Omega)$, we define the trilinear form

$$J(u,v,w) = (\Delta v, u_y w_x - u_x w_y). \tag{2.2}$$

It is shown in [16] that

$$J(u,v,u) = 0, \ \forall u\in W^{1,4}(\Omega), \ v\in H^2(\Omega), \tag{2.3}$$

$$J(u,v,w) = -J(w,v,u),\ \forall u,\ w \in W^{1,4}(\Omega),\ v \in H^2(\Omega), \tag{2.4}$$

$$J(u,v,w) = -(G(u,v),w),\ \forall u \in H^2(\Omega),\ v \in H^3(\Omega),\ w \in H_0^1(\Omega), \tag{2.5}$$

and for any $u,\ v,\ w \in H^2(\Omega)$,

$$\mid J(u,v,w) \mid \leq\ C(\Omega) \parallel u \parallel_2 \parallel v \parallel_2 \parallel w \parallel_2 . \tag{2.6}$$

Hereafter $C(\Omega)$ is a certain positive constant depending only on Ω, which could be different in different cases. Some more precise estimations for $\mid J(u,v,w) \mid$ were established in [16].

We shall also use the following lemma.

Lemma 2.1. *Let $u,\ v,\ w \in H_0^2(\Omega)$. If, in addition, $v \in W^{2,p}(\Omega)$ with $p > 2$, then*

$$\mid J(u,v,w) \mid \leq C(\Omega) \parallel \Delta u \parallel \parallel v \parallel_{W^{2,p}(\Omega)} \parallel \nabla w \parallel . \tag{2.7}$$

If $u \in W^{2,p}(\Omega)$ with $p > 2$, then

$$\mid J(u,v,w) \mid \leq C(\Omega) \parallel u \parallel_{W^{2,p}(\Omega)} \parallel \Delta v \parallel \parallel \nabla w \parallel \tag{2.8}$$

and

$$\mid J(u,v,w) \mid \leq C(\Omega) \parallel u \parallel_{W^{2,p}(\Omega)} \parallel \nabla v \parallel \parallel \Delta w \parallel . \tag{2.9}$$

Proof. Clearly

$$\frac{1}{2} + \frac{1}{p} + \frac{p-2}{2p} = 1.$$

By the Hölder inequality and $H^1(\Omega) \hookrightarrow L^{\frac{2p}{p-2}}(\Omega)$,

$$\begin{aligned}
\mid J(u,v,w) \mid\ &\leq \mid \int_\Omega \Delta v u_x w_y \mathrm{d}\Omega \mid + \mid \int_\Omega \Delta v u_y w_x \mathrm{d}\Omega \mid \\
&\leq \parallel \Delta v \parallel_{L^p(\Omega)} (\parallel u_x \parallel_{L^{\frac{2p}{p-2}}(\Omega)} \parallel w_y \parallel + \parallel u_y \parallel_{L^{\frac{2p}{p-2}}(\Omega)} \parallel w_x \parallel) \\
&\leq\ C(\Omega) \parallel \Delta u \parallel \parallel v \parallel_{W^{2,p}(\Omega)} \parallel \nabla w \parallel .
\end{aligned}$$

Thus (2.7) follows. Next, since $W^{2,p}(\Omega) \hookrightarrow C^1(\Omega)$, we obtain

$$\begin{aligned}
\mid J(u,v,w) \mid\ &\leq \mid \int_\Omega \Delta v u_x w_y \mathrm{d}\Omega \mid + \mid \int_\Omega \Delta v u_y w_x \mathrm{d}\Omega \mid \\
&\leq \parallel u_x \parallel_{C(\Omega)} \parallel \Delta v \parallel \parallel w_y \parallel + \parallel u_y \parallel_{C(\Omega)} \parallel \Delta v \parallel \parallel w_x \parallel \\
&\leq\ C(\Omega) \parallel u \parallel_{W^{2,p}(\Omega)} \parallel \Delta v \parallel \parallel \nabla w \parallel .
\end{aligned}$$

Finally we prove (2.9). By integrating by parts, we have

$$\begin{aligned}
J(u,v,w) = &\int_\Omega (v_x w_y u_{xx} - v_x w_x u_{xy} + v_y w_y u_{xy} - v_y w_x u_{yy}) \mathrm{d}\Omega \\
&+ \int_\Omega (u_x v_x w_{xy} - u_y v_x w_{xx} + u_x v_y w_{yy} - u_y v_y w_{xy}) \mathrm{d}\Omega.
\end{aligned}$$

We have

$$\mid \int_\Omega u_y v_x w_{xx} \mathrm{d}\Omega \mid \leq \parallel u_y \parallel_{C(\Omega)} \parallel \nabla v \parallel \parallel \Delta w \parallel$$

and

$$\mid \int_\Omega u_{xy} v_x w_x \mathrm{d}\Omega \mid \leq \parallel u_{xy} \parallel_{L^p(\Omega)} \parallel v_x \parallel_{L^2(\Omega)} \parallel w_x \parallel_{L^{\frac{2p}{p-2}}(\Omega)} .$$

The other terms can be estimated similarly. The proof is completed.

Now, let $H^{-s}(\Omega)$ be the dual space of $H_0^s(\Omega)$, and $\langle u, v\rangle$ be the duality paring between $H^{-s}(\Omega)$ and $H_0^s(\Omega)$. For any $u, v \in H_0^2(\Omega)$, we define $J(u,v) \in H^{-2}(\Omega)$ by

$$\langle J(u,v), w\rangle_{L(H^{-2},H_0^2)} = J(u,v,w),\ \forall w \in H_0^2(\Omega). \tag{2.10}$$

Let $J(u) = J(u,u)$. Clearly (2.6) implies

$$\| J(u) \|_{-2} \leq C(\Omega) \| u \|_2^2,\ \forall u \in H_0^2(\Omega). \tag{2.11}$$

We now turn to the weak formulation of (1.1). For given functions

$$f \in L^2(0,T; H^{-2}(\Omega)) \tag{2.12}$$

and

$$u_0 \in H_0^1(\Omega), \tag{2.13}$$

the weak solution of (1.1) is a function $u \in L^2(0,T; H_0^2(\Omega))$ such that

$$\begin{cases} (\dfrac{\partial}{\partial t}\nabla u, \nabla v) + \mu(\Delta u, \Delta v) + J(u,u,v) + \langle f, v\rangle_{L(H^{-2},H_0^2)} = 0,\ \forall v \in H_0^2(\Omega), \\ u(x,y,0) = u_0(x,y). \end{cases} \tag{2.14}$$

It is shown in [2, 16] that if $f \in L^2(0,T;H^{-2}(\Omega))$ and $u_0 \in H_0^1(\Omega)$, then (2.14) has unique solution $u \in L^2(0,T;H_0^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega))$.

3. Prediction-correction spectral scheme

In this section, we present the prediction-correction Legendre spectral scheme for (1.1), and a theorem concerning the convergence.

We denote by $P_N(\Omega)$ the set of all algebraic polynomials of degree at most N in each variable. The subspace $P_N(\Omega) \cap H_0^2(\Omega)$ is denoted by V_N. We shall use the following lemmas.

Lemma 3.1. *If $p \geq 2$ and $u \in P_N(\Omega)$, then*

$$\| u \|_{L^p(\Omega)} \leq 2^{1-\frac{2}{p}} N^{2-\frac{4}{p}} \| u \| .$$

Proof. Let $L_k(x)$ be the Legendre polynomials of degree k, and

$$u(x,y) = \sum_{k,l=0}^{N} a_{k,l} L_k(x) L_l(y) \tag{3.1}$$

where

$$a_{k,l} = (k+\frac{1}{2})(l+\frac{1}{2}) \int_{-1}^{1}\int_{-1}^{1} u(x,y) L_k(x) L_l(y) \mathrm{d}x\mathrm{d}y.$$

By the orthogonality of Legendre polynomials,

$$\| u \| = \left(\sum_{k,l=0}^{N} | a_{k,l} |^2 \left(k+\frac{1}{2}\right)^{-1} \left(l+\frac{1}{2}\right)^{-1} \right)^{\frac{1}{2}}.$$

Therefore

$$\sum_{k,l=0}^{N} \mid a_{k,l} \mid \leq (\sum_{k,l=0}^{N} \mid a_{k,l} \mid^2 (k+\frac{1}{2})^{-1}(l+\frac{1}{2})^{-1})^{\frac{1}{2}} (\sum_{k,l=0}^{N} (k+\frac{1}{2})(l+\frac{1}{2}))^{\frac{1}{2}} \leq \frac{1}{2}(N+1)^2 \parallel u \parallel .$$

Since $\mid L_k(x) \mid \leq 1$ and $\mid L_l(y) \mid \leq 1$, we obtain from (3.1) that

$$\parallel u \parallel_{L^\infty(\Omega)} \leq \sum_{k,l=0}^{N} \mid a_{k,l} \mid \leq \frac{1}{2}(N+1)^2 \parallel u \parallel .$$

Hence

$$\int_{-1}^{1}\int_{-1}^{1} \mid u(x,y) \mid^p \mathrm{d}x\mathrm{d}y \leq \parallel u \parallel^{p-2}_{L^\infty(\Omega)} \parallel u \parallel^2 \leq 2^{p-2}N^{2p-4} \parallel u \parallel^p .$$

Lemma 3.2 *(see [20]). For any function $u \in V_N$,*

$$\parallel \Delta u \parallel \leq C_p N^2 \parallel \nabla u \parallel,$$

C_p being a certain positive constant independent of any function u and N.

We define the orthogonal projection operator $\Pi_{2,N}: H_0^2(\Omega) \to V_N$, such that

$$(u - \Pi_{2,N}u, \psi)_2 = 0, \ \forall \psi \in V_N.$$

Lemma 3.3 *(see [14]). If $0 \leq r \leq 2 \leq \sigma$, then*

$$\parallel u - \Pi_{2,N}u \parallel_r \leq C(r,\sigma)N^{r-\sigma} \parallel u \parallel_\sigma, \ \forall u \in H^\sigma(\Omega) \cap H_0^2(\Omega)$$

where $C(r,\sigma)$ is a certain positive constant depending only on r and σ.

Now we consider the basis functions of V_N. The Legendre polynomial of degree k is of the form

$$L_k(x) = \frac{1}{2^k k!}\frac{d^k}{dx^k}(x^2-1)^k, -1 \leq x \leq 1.$$

A good choice of basis functions was due to Jie Shen [15]. In this case, we take

$$\begin{aligned} \psi_k(x) &= d_k(L_k(x) - \frac{2(2k+5)}{2k+7}L_{k+2}(x) + \frac{2k+3}{2k+7}L_{k+4}(x)), \\ d_k &= \frac{1}{\sqrt{2(2k+3)^2(2k+5)}}, \ 0 \leq k \leq N-4. \end{aligned}$$

Then $V_N = \{\Psi_k / k \in \aleph_N\}$, $\Psi_k(x,y) = \psi_{k_1}(x)\psi_{k_2}(y)$ and $\aleph_N = \{(i,j)/0 \leq i,j \leq N-4\}$. The matrices with elements $(\psi_k, \psi_{k'})$ and $(\frac{d\psi_k(x)}{dx}, \frac{d\psi_{k'}(x)}{dx})$, $0 \leq k, \ k' \leq N-4$ are pentadiagonal. This feature provides an efficient algorithm. In particular, $(\frac{d^2\psi_k(x)}{dx^2}, \frac{d^2\psi_{k'}(x)}{dx^2}) = \delta_{k,k'}$ which simplifies the calculation and leads to a good condition number of the corresponding matrix.

We now describe the scheme. We divide the interval $[0,T]$ into M uniform subintervals. Let $\tau = \frac{T}{M}$ be the step size and $S_\tau = \{t = l\tau \mid 0 \leq l \leq M-1\}$. We denote by $\tilde{u}(x,y,t+\tau)$ the predicted value of $u(x,y,t+\tau)$, and

$$u_t(x,y,t) = \frac{u(x,y,t+\tau) - u(x,y,t)}{\tau}, \ \tilde{u}_t(x,y,t) = \frac{\tilde{u}(x,y,t+\tau) - u(x,y,t)}{\tau}.$$

Let $\eta(x,y,t)$ be the approximation to $u(x,y,t)$. For each $t\in S_\tau,\ \eta(x,y,t)\in V_N$. Set $\eta(x,y,0)=\Pi_{2,N}u_0$. The prediction-correction scheme is composed of two steps as follows,

$$(\nabla\tilde{\eta}_t(t),\nabla v)+\frac{\mu}{2}(\Delta(\eta(t)+\tilde{\eta}(t+\tau)),\Delta v)+J(\eta(t),\eta(t),v)+(f(t),v)=0,\ \forall v\in V_N, \tag{3.2}$$

$$\begin{aligned}&(\nabla\eta_t(t),\nabla v)+\frac{\mu}{2}(\Delta(\eta(t)+\eta(t+\tau)),\Delta v)+\frac{1}{2}J(\eta(t),\eta(t),v)\\&+\frac{1}{2}J(\tilde{\eta}(t+\tau),\tilde{\eta}(t+\tau),v)+\frac{1}{2}(f(t+\tau)+f(t),v)=0,\ \forall v\in V_N.\end{aligned} \tag{3.3}$$

Obviously the approximate solution at the initial time is well defined. To evaluate it, we only need to solve a linear system. Now assume that the numerical solution at the time $t=n\tau$ is known. Let

$$a(u,v)=(\nabla u,\nabla v)+\frac{1}{2}\mu\tau(\Delta u,\Delta v),\ \forall u,\ v\ \in V_N.$$

Clearly $a(u,v)$ is a bilinear continuous and coercive form on $V_N\times V_N$. Hence by the Lax-Milgram Theorem, the numerical solution at the time $t=(n+1)\tau$ is determined uniquely. So this scheme has a unique solution as long as $f\in C(0,T;L^2(\Omega))$ and $u_0\in H_0^2(\Omega)$.

Now, let $r_0\geq 0$ and $m\geq 1$ be some constants. Define

$$E(z,t)=\|\nabla z(t)\|^2+\frac{\mu\tau}{2}(1+m)\|\Delta z(t)\|^2+\frac{\mu\tau}{2}\sum_{\substack{t'\leq t-\tau\\ t'\in S_\tau}}(\|\Delta z(t')\|^2+r_0\tau\|\nabla z_t(t')\|^2).$$

By using Lemma 2.1 and Lemmas 3.1–3.3, we obtain the following result.

Theorem 3.1. *Assume that $f\in H^2(0,T;L^2(\Omega))$, $u_0\in H_0^2(\Omega)$ and for $l_0>3,\ l_1>2$, $u\in L^2(0,T;H^{l_0}(\Omega))\cap H^1(0,T;H^{l_1}(\Omega))\cap H^2(0,T;H^2(\Omega))\cap H^3(0,T;H^1(\Omega))$. If $\tau=O(N^{-4})$, then there exists a positive constant M^* depending only on μ, Ω, and the norms of u and f in the spaces mentioned above, such that for all $t\in S_\tau$,*

$$E(u-\eta,t)\leq M^*(\tau^4+N^{2-2l_0}+N^{-2l_1}+\tau N^{2-2l_1}).$$

The proof of this theorem is given in [21].

4. Numerical results

This section is devoted to numerical experiments. We compare the results of Scheme (3.2)-(3.3) to the results of the following scheme (see [17])

$$\begin{cases}(\nabla\eta_t(t),\nabla v)+\mu(\Delta(\eta(t)+\sigma\tau\eta_t(t)),\Delta v)\\ +J(\eta(t)+\delta\tau\eta_t(t),\eta(t),v)+(f(t),v)=0,\ \forall v\in V_N,\ t\in S_\tau,\\ \eta(x,y,0)=\Pi_{2,N}u_0(x,y)\end{cases} \tag{4.1}$$

where σ,δ are parameters, $0\leq\sigma,\ \delta\leq 1$. In all calculations, we take $\sigma=0.5,\delta=0$. For describing the errors, we denote by (x_i,y_j) and w_{ij} the nodes and the weights of the two-dimensional Legendre-Gauss quadrature respectively. Define the relative error and the absolute error as

$$E(u(t))=\left(\frac{\sum_{i,j=1}^{20}|\eta(x_i,y_j,t)-u(x_i,y_j,t)|^2 w_{ij}}{\sum_{i,j=1}^{20}|u(x_i,y_j,t)|^2 w_{ij}}\right)^{\frac{1}{2}}$$

and

$$E^*(u(t))=(\sum_{i,j=1}^{20}|\eta(x_i,y_j,t)-u(x_i,y_j,t)|^2 w_{ij})^{\frac{1}{2}}.$$

Example 1. Consider (1.1) with the exact solution

$$u(x, y, t) = Ae^{Bt}(1 + \cos \pi x)(1 + \cos \pi y).$$

We take $A = B = 0.1$. The numerical results are presented in Tables 1 and 3. Table 1 shows the high accuracy of Scheme (3.2-3.3). In particular, for suitably large N, *e.g.*, $N = 14$, the numerical solution has the accuracy of second order in time. Table 2 indicates that the numerical solution of Scheme (3.2-3.3) converges to the exact solution as $N \to \infty$ and $\tau \to 0$. It also shows that this scheme is very stable for long time calculation even for small N and big τ. In paricular, for $N = 10$, the accuracy of Scheme (3.2-3.3) with $\tau = 0.1$ is as good as Scheme 4.1 with $\tau = 0.001$. So we can save the work. Table 3 shows that the calculations are still stable for very small μ, even if $\mu = 0$. It shows again that Scheme (3.2-3.3) provides more accurate numerical results than (4.1).

Table 1. The errors $E(u(t))$ of Scheme (3.2-3.3), $\mu = 0.5, N = 14$.

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
$\tau = 0.1$	3.919E-6	4.331E-6	4.782E-6	5.279E-6	5.825E-6
$\tau = 0.01$	5.905E-8	6.524E-8	7.208E-8	7.964E-8	8.798E-8
$\tau = 0.001$	6.576E-10	7.189E-10	7.873E-10	8.635E-10	9.482E-10

Table 2. The errors $E(u(30))$, $\mu = 0.5$.

	Scheme (3.2-3.3)			Scheme (4.1)	
	$N = 8$	$N = 10$	$N = 12$	$N = 8$	$N = 10$
$\tau = 0.5$	3.612E-4	3.933E-4	4.070E-4	2.444E-2	2.438E-2
$\tau = 0.1$	9.155E-5	3.675E-5	3.712E-5	4.920E-3	4.880E-3
$\tau = 0.01$	9.672E-5	1.749E-6	8.469E-7	5.367E-4	4.879E-4
$\tau = 0.001$	–	–	–	1.301E-4	4.876E-5

Table 3. The errors $E(u(20))$, $N = 10$.

	Scheme (3.2-3.3)		Scheme (4.1)	
	$\tau = 0.04$	$\tau = 0.025$	$\tau = 0.01$	$\tau = 0.001$
$\mu = 10^{-3}$	3.487E-6	2.861E-6	4.325E-4	4.214E-5
$\mu = 10^{-4}$	2.103E-5	1.658E-6	4.418E-4	4.248E-5
$\mu = 0$	2.114E-5	1.469E-6	4.452E-4	4.272E-5

Example 2. Consider (1.1) with the exact solution

$$u(x, y, t) = \frac{(1 - x^2)^2(1 - y^2)^2}{H + Gt^2 + x^2 + y^2}.$$

The numerical results are listed in Table 4. It also shows that Scheme (3.2-3.3) provides more accurate numerical results than Scheme (4.1). In particular, it is stable even for long time calculation.

Table 4. The errors $E(u(t))$ and $E^*(u(t))$, $\mu = 0.05, H = 1, G = 0.01, N = 12$.

	Scheme (3.2-3.3)				Scheme (4.1)	
	$\tau = 0.1$		$\tau = 0.4$		$\tau = 0.01$	
	$E^*(u(t))$	$E(u(t))$	$E^*(u(t))$	$E(u(t))$	$E^*(u(t))$	$E(u(t))$
$t = 20$	4.232E-7	2.694E-6	5.189E-6	3.309E-5	6.051E-5	3.853E-4
$t = 40$	2.462E-8	5.205E-7	3.854E-7	8.148E-6	1.101E-5	2.099E-4
$t = 60$	3.553E-9	1.625E-7	5.657E-8	2.588E-6	3.527E-6	1.613E-4
$t = 80$	8.648E-10	6.936E-8	1.384E-8	1.110E-6	1.530E-6	1.227E-4
$t = 100$	2.858E-10	3.559E-8	4.588E-9	5.712E-7	7.938E-7	9.883E-5

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