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FINITE VOLUME SCHEMES FOR A NONLINEAR HYPERBOLIC EQUATION. CONVERGENCE TOWARDS THE ENTROPY SOLUTION AND ERROR ESTIMATE

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Abstract. In this paper, we study some finite volume schemes for the nonlinear hyperbolic equation $u_t(x,t) + \mathrm{div}F(x,t,u(x,t)) = 0$ with the initial condition $u_0 \in L^\infty(\mathbb{R}^N)$. Passing to the limit in these schemes, we prove the existence of an entropy solution $u \in L^\infty(\mathbb{R}^N \times \mathbb{R}_+)$. Proving also uniqueness, we obtain the convergence of the finite volume approximation to the entropy solution in $L^p_{loc}(\mathbb{R}^N \times \mathbb{R}_+)$, $1 \le p \le +\infty$. Furthermore, if $u_0 \in L^\infty \cap \mathrm{BV}_{loc}(\mathbb{R}^N)$, we show that $u \in \mathrm{BV}_{loc}(\mathbb{R}^N \times \mathbb{R}_+)$, which leads to an "$h^{\frac{1}{4}}$" error estimate between the approximate and the entropy solutions (where h defines the size of the mesh).

Résumé. Dans cet article, on étudie des schémas volumes finis pour l'équation hyperbolique non linéaire $u_t(x,t) + \mathrm{div}F(x,t,u(x,t)) = 0$, avec comme condition initiale $u_0 \in L^\infty(\mathbb{R}^N)$. En passant à la limite dans ces schémas numériques, on obtient l'existence d'une solution entropique $u \in L^\infty(\mathbb{R}^N \times \mathbb{R}_+)$, puis son unicité. On montre aussi la convergence dans $L^p_{loc}(\mathbb{R}^N \times \mathbb{R}_+)$, $(1 \le p \le +\infty)$ de la solution approchée donnée par le schéma vers la solution entropique. De plus, si $u_0 \in L^\infty \cap \mathrm{BV}_{loc}(\mathbb{R}^N)$, on prouve que $u \in \mathrm{BV}_{loc}(\mathbb{R}^N \times \mathbb{R}_+)$, ce qui implique une estimation d'erreur de l'ordre de $h^{\frac{1}{4}}$ entre solution approchée et solution entropique (h étant le pas du maillage).

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1. Introduction

1.1. Presentation of the problem

The aim of this paper is to define and study some finite volume schemes which approach the following nonlinear hyperbolic equation with some initial condition:

$$\begin{cases} u_t(x,t) + \mathrm{div}(F(x,t,u(x,t))) &= 0, \qquad \forall x \in \mathbb{R}^N, \forall t \in \mathbb{R}_+, \\ u(x,0) &= u_0(x), \quad \forall x \in \mathbb{R}^N \end{cases} \tag{1}$$

where

$$\begin{array}{rccl} F : & \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R} & \to & \mathbb{R}^N, \ N \ge 1 \\ & (x,t,s) & \mapsto & F(x,t,s). \end{array}$$

The problems of physical interest often have fluxes of the form $F(x,t,s) = v(x,t)f(s)$ with $v : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}^N$ and $f : \mathbb{R} \to \mathbb{R}$. In [2,3], Eymard, Gallouët and Herbin consider finite volume schemes in this case. Other authors like Cockburn, Coquel and Lefloch in [1] and Vila in [8] study finite volume schemes in the case where $F(x,t,s) = F(s)$ with $F : \mathbb{R} \to \mathbb{R}^N$, which might be the first step in the study of hyperbolic systems. The

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equation (1) is a generalization of these two cases and the study of this general case allows us to emphasize the methods developped in [2, 3].

Our interest lies in the study of convergence – it means: convergence and rate of convergence – of finite volume schemes towards the entropy solution to (1) (see below for the definition of entropy solution). However, the study of these schemes permits us to prove also existence and uniqueness of the entropy solution.

The following hypotheses are made on the data:

- $u_0 \in L^\infty(\mathbb{R}^N)$: $\exists A,\ B \in \mathbb{R}$ such that $A \le u_0 \le B$ almost everywhere,
- $F \in C^1(\mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R})$ and $\dfrac{\partial F}{\partial s}$ is locally Lipschitz continuous,
- $\mathrm{div}_x F(x,t,s) = \displaystyle\sum_{i=1}^N \frac{\partial F_i}{\partial x_i}(x,t,s) = 0\ \forall\ (x,t,s) \in \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}, \qquad (2)$
- for all compact set $\mathcal{K} \subset \mathbb{R}$, there exists $V_\mathcal{K} < +\infty$ such that $|\dfrac{\partial F}{\partial s}(x,t,s)| \le V_\mathcal{K}$ for almost every $(x,t,s) \in \mathbb{R}^N \times \mathbb{R}_+ \times \mathcal{K}$.

Definition 1. *We say that $u \in L^\infty(\mathbb{R}^N \times]0,+\infty[)$ is an entropy solution to (1) if:*

$$\int_{\mathbb{R}^N\times\mathbb{R}_+} \Big[|u(x,t)-\kappa|\varphi_t(x,t) + (F(x,t,u(x,t)\top\kappa) - F(x,t,u(x,t)\perp\kappa)).\nabla\varphi(x,t)\Big]dxdt$$
$$+ \int_{\mathbb{R}^N} |u_0(x)-\kappa|\varphi(x,0)dx \ge 0,\ \forall \kappa\in\mathbb{R},\ \forall\varphi\in C_c^1(\mathbb{R}^N\times\mathbb{R}_+,\mathbb{R}_+), \qquad (3)$$

where $a\top b$ denotes $\max\{a,b\}$ and $a\perp b$, $\min\{a,b\}$.

In [5], Kruskov proved existence and uniqueness of entropy solution to (1), if $F \in C^3(\mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R}^N)$, by using a parabolic regularization of (1). In [3, 4], Eymard, Gallouët and Herbin showed existence and uniqueness of the entropy solution in the case $F(x,t,s) = v(x,t)f(s)$ with $v \in L^\infty \cap C^1(\mathbb{R}^N \times \mathbb{R}_+, \mathbb{R}^N)$ and $f \in C^1(\mathbb{R},\mathbb{R})$ by passing to the limit in some finite volume schemes. That is the way we use to obtain this result in our case. This proof needs the notions of entropy process solution and of nonlinear weak $*$ convergence that we introduce hereafter.

Definition 2. *A function $v \in L^\infty(\mathbb{R}^N \times \mathbb{R}_+ \times]0,1[,\mathbb{R})$ is an entropy process solution to (1) if it satisfies:*

$$\int_{\mathbb{R}^N\times\mathbb{R}_+}\int_0^1 [|v(x,t,\alpha)-\kappa|\varphi_t(x,t) + (F(x,t,v(x,t,\alpha)\top\kappa) - F(x,t,v(x,t,\alpha)\perp\kappa)).$$
$$\nabla\varphi(x,t)]dxdtd\alpha + \int_{\mathbb{R}^N}|u_0(x)-\kappa|\varphi(x,0)dx \ge 0,\ \forall\kappa\in\mathbb{R},\ \forall\varphi\in C_c^1(\mathbb{R}^N\times\mathbb{R}_+,\mathbb{R}_+). \qquad (4)$$

This concept has been introduced by Eymard, Gallouët and Herbin in [3, 4]. It is closely related to the concept of measure valued solution due to Di Perna [6].

Definition 3. *If Ω is an open set of $\mathbb{R}^N$ and $(u_n)_{n\in\mathbb{N}}$ is a sequence of $L^\infty(\Omega)$. We say that $(u_n)_{n\in\mathbb{N}}$ converges in a nonlinear weak $*$ sense if there exists $u \in L^\infty(\Omega\times]0,1[)$ s.t.:*

$$\int_\Omega h(u_n(x))\varphi(x)dx \overset{n\to\infty}{\longrightarrow} \int_\Omega\int_0^1 h(u(x,\alpha))\varphi(x)dxd\alpha \quad \forall\varphi\in L^1(\Omega) \quad \forall h\in C(\mathbb{R},\mathbb{R}).$$

This kind of convergence permits us to pass to the limit in the numerical scheme and thus to show the existence of an entropy process solution.

We can remark that the nonlinear weak $*$ convergence corresponds to the convergence towards a Young measure (*cf.* [6]) for any sequence of $L^\infty(\Omega)$. With the notations of Definition 3, the sequence $(u_n)_{n\in\mathbb{N}}$ converges towards the Young measure $\nu = (\nu_x)_{x\in\Omega}$ defined by:

$$\langle \nu_x, h\rangle = \int_0^1 h(u(x,\alpha))d\alpha \quad \forall h \in C(\mathbb{R},\mathbb{R}), \text{ for a.e. } x \in \Omega.$$

We can prove the uniqueness of the entropy process solution, which is moreover the unique entropy solution, by a technique of regularization due to Kruskov. At the same time, we show the convergence of the schemes towards the entropy solution.

Furthermore, we obtain an error estimate between the approximate solution given by the scheme and the entropy solution, provided that u_0 is in a "good" functional space. Such a result was proved in [2] in the case $F = vf$ with $u_0 \in L^\infty \cap \mathrm{BV}(\mathbb{R}^N)$.

Definition 4. *For $\Omega \subset \mathbb{R}^p$, the functional space $BV(\Omega)$ is defined as follows:*

$$BV(\Omega) = \Big\{ g : \sup\{\int_{\mathbb{R}^p} g(x)\, div\varphi(x)dx,\ \varphi \in C_c^\infty(\Omega,\mathbb{R}^p),\ \|\varphi\|_\infty = \sup_{x\in\Omega}|\varphi(x)| \le 1\} < +\infty \Big\}.$$

On $BV(\Omega)$, we define a seminorm:

$$|g|_{BV(\Omega)} = \sup\{\int_{\mathbb{R}^p} g(x)\, div\varphi(x)dx\ ;\ \varphi \in C_c^\infty(\Omega,\mathbb{R}^p),\ \|\varphi\|_\infty \le 1\}.$$

We also consider $BV_{loc}(\Omega)$:

$$BV_{loc}(\Omega) = \{g; g \in BV(K) \text{ for all compact set } K \subset \Omega\}.$$

1.2. Main results

In Section 2, we present the schemes that we consider: these schemes are Euler explicit in time and finite volume in space. They are first order in space and time. Then, we prove some stability properties which are verified by the approximate solution given by these schemes.

The aim of Section 3 is to prove in which way the approximate solution is close to the entropy solution. We show in Theorem 1, page 137, that the approximate solution satisfies the inequality (27), page 137, similar to (3), page 130. The difference between (3) and (27) provides from the error terms, which are well-controlled. It is the key of all the following results.

In Section 4, we pass to the limit in the numerical scheme and therefore we prove the existence of an entropy process solution (Lemma 4, page 143). Then, by a technique of regularization due to Kruskov, we show that this solution is an entropy solution and is the unique one (Lemma 5, page 144). Moreover, we obtain the convergence of the approximate solution towards the entropy solution. All these results are expressed in Theorem 2, page 141.

Until this point, we just have to assume the initial condition u_0 to belong to $L^\infty(\mathbb{R}^N)$. However, in order to obtain an error estimate in $h^{\frac14}$, where h is the size of the mesh, between the approximate and the entropy solution, we need $u_0 \in L^\infty \cap BV_{loc}(\mathbb{R}^N)$ and $u \in L^\infty \cap BV_{loc}(\mathbb{R}^N \times \mathbb{R}_+)$. But, in Section 4.4, we prove, using a particular scheme on a structured mesh, that $u_0 \in L^\infty \cap BV_{loc}(\mathbb{R}^N)$ implies $u \in BV_{loc}(\mathbb{R}^N \times [0,T[)$ for all $T > 0$.

Therefore, in Section 5, we can show an error estimate of order $h^{\frac14}$ under assumption $u_0 \in L^\infty \cap BV_{loc}(\mathbb{R}^N)$.

2. Properties of the schemes

2.1. Presentation of the schemes

Let $\mathcal{T}$ be a mesh of $\mathbb{R}^N$ such that the common interface between two cells of $\mathcal{T}$ is included in a hyperplane of $\mathbb{R}^N$. We assume that there exist $h > 0$ and $\alpha > 0$ such that, for any $p \in \mathcal{T}$:

$$\begin{array}{lcl} \alpha h^N & \leq & m(p), \\ m(\partial p) & \leq & \dfrac{1}{\alpha} h^{N-1}, \\ \delta(p) & \leq & h. \end{array} \tag{5}$$

where $m(p)$ denotes the N-dimensional Lebesgue measure of the cell p, $m(\partial p)$ denotes the $(N-1)$-dimensional Lebesgue measure of its boundary and $\delta(p)$ denotes its diameter.

With these notations, the parameter h defines the size of the mesh and α its regularity. Under the hypotheses (5), it is quite easy to verify that each cell has a finite number, bounded by a quantity depending only on N and α, of neighbours.

For any control volume p we denote by $N(p)$ the set of the neighbours of p. If $q \in N(p)$, σ_{pq} is the common interface between p and q and $n_{p,q}$ is the unit normal vector to σ_{pq} oriented from p to q.

Let $k > 0$ be the time step and $t^n = nk$ for all $n \in \mathbb{N}$.

For all $(p,q) \in \mathcal{T}^2$, $q \in N(p)$, for all $n \in \mathbb{N}$, we consider some numerical fluxes $F^n_{p,q} \in C(\mathbb{R}^2, \mathbb{R}) : (u,v) \to F^n_{p,q}(u,v)$ that satisfy:

$$\left\{ \begin{array}{rl} (i) & F^n_{p,q}(u,v) \text{ is nondecreasing w.r.t. } u \text{ and nonincreasing w.r.t. } v, \text{ for } (u,v) \in [A,B]^2 \text{ (}A \\ & \text{and } B \text{ are defined in (2)),} \\ (ii) & F^n_{p,q}(u,v) = -F^n_{q,p}(v,u) \text{ for all } (u,v) \in [A,B]^2, \\ (iii) & F^n_{p,q}(u,v) \text{ is Lipschitz continuous over } [A,B]^2 \text{ with the same Lipschitz constant w.r.t.} \\ & u \text{ and } v\text{: } m(\sigma_{pq})M, \text{ where } M \text{ only depends on } F \text{ and } u_0, \\ (iv) & F^n_{p,q}(s,s) = \dfrac{1}{k}\displaystyle\int_{t^n}^{t^{n+1}} \int_{\sigma_{pq}} F(\gamma,t,s).n_{p,q} d\gamma dt \text{ for all } s \in [A,B]^2. \end{array} \right. \tag{6}$$

The hypothesis ($6i$) ensures the monotony of the scheme, ($6ii$) its conservativity, ($6iii$) its regularity and ($6iv$) its consistency. We can note that, under the assumption $\mathrm{div}_x(F) = 0$, ($6iv$) implies:

$$\sum_{q \in N(p)} F^n_{p,q}(s,s) = 0 \quad \forall p \in \mathcal{T}, \quad \forall n \in \mathbb{N}, \quad \forall s \in [A,B]. \tag{7}$$

But, the hypothesis ($6iv$) may also be replaced by:

$$\left\{ \begin{array}{l} \left| F^n_{p,q}(s,s) - \dfrac{1}{k}\displaystyle\int_{t^n}^{t^{n+1}} \int_{\sigma_{pq}} F(\gamma,t,s).n_{p,q} d\gamma dt \right| \leq C_{F,\alpha}(k+h)h^{N-1} \quad \forall s \in [A,B] \\ \displaystyle\sum_{q \in N(p)} F^n_{p,q}(s,s) = 0 \quad \forall p \in \mathcal{T}, \quad \forall n \in \mathbb{N}, \quad \forall s \in [A,B] \end{array} \right. \tag{8}$$

where $C_{F,\alpha}$ only depends on F and α. It means that we can, for instance, replace ($6iv$) by:

$$F^n_{p,q}(s,s) = \int_{\sigma_{pq}} F(\gamma, t^n, s).n_{p,q} d\gamma.$$

We give here a very classical example of functions $F^n_{p,q}$ that satisfy (6). This definition of the fluxes leads to the 1-dimensional by interface Godunov scheme.

$$F^n_{p,q}(u,v) = \begin{cases} \displaystyle\min_{u\le s\le v} \frac{1}{k}\int_{t^n}^{t^{n+1}} \int_{\sigma_{pq}} F(\gamma,t,s).n_{p,q}d\gamma dt & \text{if } u \le v \\ \displaystyle\max_{u\ge s\ge v} \frac{1}{k}\int_{t^n}^{t^{n+1}} \int_{\sigma_{pq}} F(\gamma,t,s).n_{p,q}d\gamma dt & \text{if } u > v. \end{cases}$$

In Section 4.4, we propose another example of fluxes in a particular case.

The discrete unknowns are the u^n_p, $p \in \mathcal{T}$, $n \in \mathbb{N}$. Let us consider the following numerical scheme:

$$\begin{cases} m(p)\dfrac{u_p^{n+1}-u_p^n}{k} + \displaystyle\sum_{q\in N(p)} F^n_{p,q}(u^n_p, u^n_q) = 0, \quad p \in \mathcal{T},\ n \in \mathbb{N}, \\ u^0_p = \dfrac{1}{m(p)}\displaystyle\int_p u_0(x)dx, \quad p \in \mathcal{T}. \end{cases} \tag{9}$$

The time step must verify:

$$k \le \frac{(1-\xi)\alpha^2 h}{2M},\ \xi \in]0,1[. \tag{10}$$

The approximate solution $u_{\mathcal{T},k}$ is defined by:

$$u_{\mathcal{T},k}(x,t) = u^n_p \text{ for } x \in p \text{ and } t \in [t^n, t^{n+1}[. \tag{11}$$

2.2. $\mathbf{L^\infty}$-stability

Lemma 1. *Assume (2), (5), (6) and (10) hold. Then, the approximate solution $u_{\mathcal{T},k}$ defined by (9) and (11) verifies:*

$$A \le u^n_p \le B,\ \forall n \in \mathbb{N},\ \forall p \in \mathcal{T}, \tag{12}$$

and

$$\|u_{\mathcal{T},k}\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}_+)} \le \|u_0\|_{L^\infty(\mathbb{R}^N)}. \tag{13}$$

Proof. We prove (12) by induction; (13) is a consequence of (12).

The inequality (12) holds for $n = 0$ because $A \le u_0 \le B$ *a.e.* We assume that it holds for n. Introducing (7) in (9), we get:

$$u^{n+1}_p = u^n_p - \frac{k}{m(p)} \sum_{\substack{q\in N(p),\\ u^n_p \ne u^n_q}} \frac{(F^n_{p,q}(u^n_p,u^n_q) - F^n_{p,q}(u^n_p,u^n_p))}{u^n_p - u^n_q}(u^n_p - u^n_q).$$

Thanks to (6i), (6iii), (5) and (10), we obtain that u^{n+1}_p is a convex combination of u^n_p and u^n_q, $q \in N(p)$. Therefore, $A \le u^{n+1}_p \le B$ $\forall p \in \mathcal{T}$ and it concludes the proof.

2.3. **BV weak stability**

We give here some notations that will be used in all the sequel. Let $T > 0$ and $R > 0$.

$$\begin{aligned} N_T &= \max\{n \in \mathbb{N},\ n \leq \frac{T}{k} + 1\}, \\ \mathcal{T}_R &= \{p \in \mathcal{T},\ p \subset B(0,R)\}, \\ \mathcal{E}^n &= \{(p,q) \in \mathcal{T}^2,\ q \in N(p),\ u_p^n > u_q^n\}, \\ \mathcal{E}_R^n &= \{(p,q) \in \mathcal{T}^2,\ p \text{ or } q \in \mathcal{T}_R,\ q \in N(p),\ \sigma_{pq} \subset B(0,R) \text{ and } u_p^n > u_q^n\}. \end{aligned}$$

The assumptions made on the mesh (5) ensure that there exists $C_{R,N,\alpha}$ which only depends on R, N and α such that Card $\mathcal{T}_R \leq C_{R,N,\alpha} h^{-N}$, Card $\mathcal{E}_R^n \leq C_{R,N,\alpha} h^{-N}$ and Card $\{\sigma_{pq} \in B(0,R) \setminus B(0,R-h),\ (p,q) \in \mathcal{T}^2\} \leq C_{R,N,\alpha} h^{1-N}$.

The following lemma gives some estimates on the time and space derivatives of the approximate solution $u_{\mathcal{T},k}$. We call them BV-weak estimates.

Lemma 2. *Assume (2), (5), (6), (10). Let $u_{\mathcal{T},k}$ be defined by (9), (11), let $T > 0$ and $R > 0$. Then, there exists $C_{bv} \in \mathbb{R}$ depending only on F, u_0, M, α, ξ, R and T such that:*

$$\sum_{n=0}^{N_T} k \sum_{(p,q)\in\mathcal{E}_R^n} \left[\max_{u_q^n \leq c \leq d \leq u_p^n} \left(F_{p,q}^n(d,c) - F_{p,q}^n(d,d) \right) + \max_{u_q^n \leq c \leq d \leq u_p^n} \left(F_{p,q}^n(d,c) - F_{p,q}^n(c,c) \right) \right] \leq \frac{C_{bv}}{\sqrt{h}}, \quad \forall h < R, \tag{14}$$

and

$$\sum_{n=0}^{N_T} \sum_{p\in\mathcal{T}_R} m(p)|u_p^{n+1} - u_p^n| \leq \frac{C_{bv}}{\sqrt{h}}, \quad \forall h < R. \tag{15}$$

Proof. In this proof, we denote by $(C_i)_{i\in\mathbb{N}}$ some quantities that only depend on F, u_0, M, α, ξ, R and T. The size of the mesh is chosen small enough ($h < R$) so that $\mathcal{T}_R$ is not empty.

We first prove (14). We multiply the scheme (9) by $k u_p^n$ and we sum the result over $n \in \{0,..,N_T\}$ and $p \in \mathcal{T}_R$. We obtain:

$$B_1 + B_2 = 0 \tag{16}$$

where

$$\begin{aligned} B_1 &= \sum_{n=0}^{N_T} \sum_{p\in\mathcal{T}_R} m(p) u_p^n (u_p^{n+1} - u_p^n), \\ B_2 &= \sum_{n=0}^{N_T} \sum_{p\in\mathcal{T}_R} k \sum_{q\in N(p)} u_p^n (F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n)). \end{aligned}$$

The term B_2 can be turned into a sum on the edges of the mesh instead of a sum on the cells. That is the reason why we introduce B_3:

$$B_3 = \sum_{n=0}^{N_T} \sum_{(p,q)\in\mathcal{E}_R^n} k \left[u_p^n (F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_p^n, u_p^n)) - u_q^n (F_{p,q}^n(u_p^n, u_q^n) - F_{p,q}^n(u_q^n, u_q^n)) \right].$$

The quantity $|B_3 - B_2|$ only contains a sum of terms concentrated on the boundary of $B(0,R)$. Each term is bounded by $C_1 h^{N-1}$ and the number of such terms is lower than the number of edges σ_{pq} included in $B(0,R) \setminus B(0,R-h)$. Therefore,

$$|B_3 - B_2| \leq C_2. \tag{17}$$

For all $p \in \mathcal{T}$, $q \in N(p)$, $n \in \mathbb{N}$ we denote by $\Psi^n_{p,q}$ the following function:

$$\Psi^n_{p,q}(x) = \int_0^x s\left(\frac{\partial F^n_{p,q}}{\partial u}(s,s) + \frac{\partial F^n_{p,q}}{\partial v}(s,s)\right) ds = \int_0^x s\frac{d}{ds}(F^n_{p,q}(s,s))ds.$$

Integrating by parts, we get, for all $(a,b) \in \mathbb{R}^2$:

$$\Psi^n_{p,q}(b) - \Psi^n_{p,q}(a) = b(F^n_{p,q}(b,b) - F^n_{p,q}(a,b)) - a(F^n_{p,q}(a,a) - F^n_{p,q}(a,b)) - \int_a^b \left(F^n_{p,q}(s,s) - F^n_{p,q}(a,b)\right) ds.$$

This equality permits us to rewrite B_3 as:

$$B_3 = B_4 + B_5 \tag{18}$$

with

$$\begin{aligned} B_4 &= \sum_{n=0}^{N_T} \sum_{(p,q)\in\mathcal{E}^n_R} k(\Psi^n_{p,q}(u^n_p) - \Psi^n_{p,q}(u^n_q)), \\ B_5 &= \sum_{n=0}^{N_T} \sum_{(p,q)\in\mathcal{E}^n_R} k\int_{u^n_q}^{u^n_p} \left(F^n_{p,q}(u^n_p, u^n_q) - F^n_{p,q}(s,s)\right) ds. \end{aligned}$$

Because of (7), $\sum_{q\in N(p)} \Psi^n_{p,q}(x) = 0 \ \forall p \in \mathcal{T} \ \forall x \in [A,B]$ and B_4 is again reduced to a sum of terms included in $B(0,R) \setminus B(0,R-h)$ and bounded by $C_3 h^{N-1}$. Therefore:

$$|B_4| \leq C_4. \tag{19}$$

We now have to estimate B_5. Using the monotony properties of the numerical fluxes ($6i$) and a technical lemma given in [2], we get:

$$\int_{u^n_q}^{u^n_p} \left(F^n_{p,q}(u^n_p, u^n_q) - F^n_{p,q}(s,s)\right) ds \geq \frac{1}{4m(\sigma_{pq})M}\Bigg(\max_{u^n_q\leq c\leq d\leq u^n_p}(F^n_{p,q}(d,c) - F^n_{p,q}(d,d))^2 + \max_{u^n_q\leq c\leq d\leq u^n_p}(F^n_{p,q}(d,c) - F^n_{p,q}(c,c))^2\Bigg). \tag{20}$$

Then:

$$B_5 \geq \frac{1}{4M}\sum_{n=0}^{N_T} k \sum_{(p,q)\in\mathcal{E}^n_R}\Bigg(\frac{1}{m(\sigma_{pq})}\max_{u^n_q\leq c\leq d\leq u^n_p}(F^n_{p,q}(d,c) - F^n_{p,q}(d,d))^2 + \frac{1}{m(\sigma_{pq})}\max_{u^n_q\leq c\leq d\leq u^n_p}(F^n_{p,q}(d,c) - F^n_{p,q}(c,c))^2\Bigg). \tag{21}$$

Let us now turn to an estimate of B_1.

$$B_1 = -\frac{1}{2}\sum_{n=0}^{N_T}\sum_{p\in\mathcal{T}_R} m(p)(u_p^{n+1}-u_p^n)^2 + \underbrace{\frac{1}{2}\sum_{p\in\mathcal{T}_R} m(p)(u_p^{N_T+1})^2}_{\geq 0} - \underbrace{\frac{1}{2}\sum_{p\in\mathcal{T}_R} m(p)(u_p^0)^2}_{\leq C_5}.$$

We apply Cauchy-Schwarz inequality to the scheme (9). Hence,

$$(u_p^{n+1}-u_p^n)^2 \leq \frac{k^2}{m(p)^2}\sum_{q\in N(p)} m(\sigma_{pq})\sum_{q\in N(p)}\frac{1}{m(\sigma_{pq})}(F_{p,q}^n(u_p^n,u_q^n)-F_{p,q}^n(u_p^n,u_p^n))^2, \tag{22}$$

and

$$B_1 \geq -C_5 - \frac{(1-\xi)}{4M}\sum_{n=0}^{N_T}\sum_{(p,q)\in\mathcal{E}_R^n} k\left(\frac{1}{m(\sigma_{pq})}\max_{u_q^n\leq c\leq d\leq u_p^n}(F_{p,q}^n(d,c)-F_{p,q}^n(d,d))^2 \right.$$
$$\left. +\frac{1}{m(\sigma_{pq})}\max_{u_q^n\leq c\leq d\leq u_p^n}(F_{p,q}^n(d,c)-F_{p,q}^n(c,c))^2\right). \tag{23}$$

Then, we can deduce, from (16), (17), (18), (19), (21) and (23), that:

$$\sum_{n=0}^{N_T} k\sum_{(p,q)\in\mathcal{E}_R^n}\frac{1}{m(\sigma_{pq})}\left(\max_{u_q^n\leq c\leq d\leq u_p^n}(F_{p,q}^n(d,c)-F_{p,q}^n(d,d))^2 + \max_{u_q^n\leq c\leq d\leq u_p^n}(F_{p,q}^n(d,c)-F_{p,q}^n(c,c))^2\right)\leq C_6$$

We now just have to apply the Cauchy-Schwarz inequality to get the BV-weak estimate on the space derivatives (14). The estimate on the time derivatives (15) is a straightforward consequence of (9) and (14).

3. Entropy inequalities for the approximate solution

In this section, we show how the approximate solution $u_{\mathcal{T},k}$ is close to the entropy solution. First, we derive a discrete entropy inequality which is a consequence of the monotony of the scheme. Then, we prove that $u_{\mathcal{T},k}$ verifies an inequality, similar to (3), but with the add of some error terms.

3.1. Discrete entropy inequality

Lemma 3. *Assume (2), (5), (6) and (10). Let $u_{\mathcal{T},k}$ be given by (9), (11). Then, for all $\kappa\in\mathbb{R}$, $p\in\mathcal{T}$ and $n\in\mathbb{N}$, the following inequality holds:*

$$\frac{|u_p^{n+1}-\kappa|-|u_p^n-\kappa|}{k} + \frac{1}{m(p)}\sum_{q\in N(p)}\Big(F_{p,q}^n(u_p^n\top\kappa,u_q^n\top\kappa)-F_{p,q}^n(u_p^n\bot\kappa,u_q^n\bot\kappa)\Big)\leq 0. \tag{24}$$

Proof. The scheme (9) writes:

$$u_p^{n+1} = u_p^n - \frac{k}{m(p)}\sum_{q\in N(p)} F_{p,q}^n(u_p^n,u_q^n) = G(u_p^n, u_q^n\ q\in N(p))$$

where G is a nondecreasing function whith respect to the u_q^n, $q \in N(p)$, and to u_p^n when k satisfies (10). Furthermore, $G(\kappa,\kappa) = \kappa$, $\forall \kappa \in \mathbb{R}$. Therefore:

$$u_p^{n+1} \top \kappa \quad \leq \quad u_p^n \top \kappa - \frac{k}{m(p)} \sum_{q \in N(p)} F_{p,q}^n(u_p^n \top \kappa, u_q^n \top \kappa), \tag{25}$$

$$u_p^{n+1} \bot \kappa \quad \geq \quad u_p^n \bot \kappa - \frac{k}{m(p)} \sum_{q \in N(p)} F_{p,q}^n(u_p^n \bot \kappa, u_q^n \bot \kappa). \tag{26}$$

The difference between (25) and (26) leads directly to (24).

3.2. Continuous entropy estimate for the approximate solution

In Theorem 1, we prove that the approximate solution verifies an entropy inequality with some error terms. These error terms are expressed with the help of some measures. For $\Omega = \mathbb{R}^N$ or $\Omega = \mathbb{R}^N \times \mathbb{R}_+$, we denote by $\mathcal{M}(\Omega)$ the set of measures on Ω, *i.e.* the set of positive continuous linear forms on $C_c(\Omega)$. If $\mu \in \mathcal{M}(\Omega)$, we set: $\langle \mu, g \rangle = \int_\Omega g d\mu$, for all $g \in C_c(\Omega)$. The estimates (28), (29) and (30) give a control of the error terms.

Theorem 1. *Assume (2), (5),(6) and (10). Let $u_{\mathcal{T},k}$ be defined by (9), (11). Then, there exist $\mu_{\mathcal{T}} \in \mathcal{M}(\mathbb{R}^N)$ and $\mu_{\mathcal{T},k} \in \mathcal{M}(\mathbb{R}^N \times \mathbb{R}_+)$ such that, $\forall \kappa \in \mathbb{R}$, $\forall \varphi \in C_c^\infty(\mathbb{R}^N \times \mathbb{R}_+, \mathbb{R}_+)$:*

$$\int_{\mathbb{R}^N \times \mathbb{R}_+} [|u_{\mathcal{T},k}(x,t) - \kappa| \varphi_t(x,t) + (F(x,t,u_{\mathcal{T},k}(x,t) \top \kappa) - F(x,t,u_{\mathcal{T},k}(x,t) \bot \kappa)).\nabla \varphi(x,t)] dx dt$$
$$+ \int_{\mathbb{R}^N} |u_0(x) - \kappa| \varphi(x,0) dx \geq - \int_{\mathbb{R}^N \times \mathbb{R}_+} (|\varphi_t(x,t)| + |\nabla \varphi(x,t)|) d\mu_{\mathcal{T},k}(x,t) - \int_{\mathbb{R}^N} \varphi(x,0) d\mu_{\mathcal{T}}(x). \tag{27}$$

Furthermore the measures $\mu_{\mathcal{T},k}$ and $\mu_{\mathcal{T}}$ verify the following properties:

1. *For all $R > 0$ and $T > 0$, there exists C_m depending only on F, u_0, M, α, ξ, R and T such that*

$$\mu_{\mathcal{T},k}(B(0,R) \times [0,T]) \leq C_m(h + \sqrt{h}), \ \forall h < R. \tag{28}$$

2. *The measure $\mu_{\mathcal{T}}$ is the measure of density $|u_0 - u_{\mathcal{T},0}|$, where $u_{\mathcal{T},0}(x) = u_p^0$ $\forall x \in p$, with respect to the Lebesgue measure. For all $R > 0$, we have:*

$$\lim_{h \to 0} (\mu_{\mathcal{T}}(B(0,R))) = 0, \tag{29}$$

and if $u_0 \in L^\infty \cap BV_{loc}(\mathbb{R}^N)$, there exists D_m only depending on u_0, α and R such that:

$$\mu_{\mathcal{T}}(B(0,R)) \leq D_m h, \ \forall h < R. \tag{30}$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^N \times \mathbb{R}_+, \mathbb{R}_+)$ and $\kappa \in \mathbb{R}$. Let $T > 0$ and $R > 0$ such that $\varphi(x,t) \neq 0$ implies $|x| \leq R - h$ and $t \in [0,T]$. Let us multiply (24) by $\int_{t^n}^{t^{n+1}} \int_p \varphi(x,t) dx dt$ and sum the result for all p and n. It yields:

$$T_1 + T_2 \leq 0 \tag{31}$$

with

$$T_1 = \sum_{n=0}^{N_T} \sum_{p \in \mathcal{T}_R} \frac{|u_p^{n+1} - \kappa| - |u_p^n - \kappa|}{k} \int_{t^n}^{t^{n+1}} \int_p \varphi(x,t) dx dt \tag{32}$$

and

$$T_2 = \sum_{n=0}^{N_T} \sum_{p\in\mathcal{T}_R} \frac{1}{m(p)} \int_{t^n}^{t^{n+1}} \int_p \varphi(x,t)dxdt \sum_{q\in N(p)} (F^n_{p,q}(u^n_p\top\kappa, u^n_q\top\kappa) - F^n_{p,q}(u^n_p\top\kappa, u^n_p\top\kappa) - F^n_{p,q}(u^n_p\bot\kappa, u^n_q\bot\kappa) + F^n_{p,q}(u^n_p\bot\kappa, u^n_p\bot\kappa)). \quad (33)$$

The term T_1 contains the discrete time derivatives of $|u_{\mathcal{T},k} - \kappa|$ and T_2 the discrete space derivatives of $F(.,.,u_{\mathcal{T},k}\top\kappa) - F(.,.,u_{\mathcal{T},k}\bot\kappa)$. The proof lies in the comparison between T_1 and T_1^* and between T_2 and T_2^*, where T_1^* and T_2^* are respectively the temporal and the spatial term in (27):

$$\begin{aligned} T_1^* &= -\int_{\mathbb{R}^N\times\mathbb{R}_+} |u_{\mathcal{T},k}(x,t) - \kappa|\varphi_t(x,t)dxdt - \int_{\mathbb{R}^N} |u_0(x) - \kappa|\varphi(x,0)dx \\ T_2^* &= -\int_{\mathbb{R}^N\times\mathbb{R}_+} (F(x,t,u_{\mathcal{T},k}(x,t)\top\kappa) - F(x,t,u_{\mathcal{T},k}(x,t)\bot\kappa)).\nabla\varphi(x,t)dxdt. \end{aligned}$$

Comparison between $\mathbf{T_1}$ and $\mathbf{T_1^*}$

Using the definition of $u_{\mathcal{T},k}$, (11), and introducing $u_{\mathcal{T},0}(x) = u^0_p,\ \forall x \in p$, we get:

$$T_1^* = \sum_{n=0}^{N_T} \sum_{p\in\mathcal{T}_R} \frac{|u^{n+1}_p - \kappa| - |u^n_p - \kappa|}{k} \int_{t^n}^{t^{n+1}} \int_p \varphi(x,t^{n+1})dxdt + \int_{\mathbb{R}^N} (|u_{\mathcal{T},0}(x) - \kappa| - |u_0(x) - \kappa|)\varphi(x,0)dx.$$

Hence,

$$|T_1 - T_1^*| \le \sum_{n=0}^{N_T} \sum_{p\in\mathcal{T}_R} |u^{n+1}_p - u^n_p| \int_{t^n}^{t^{n+1}} \int_p |\varphi_t(x,t)|dxdt + \int_{\mathbb{R}^N} |u_0(x) - u_{\mathcal{T},0}(x)|\varphi(x,0)dx. \quad (34)$$

We define two measures $\mu_\mathcal{T} \in \mathcal{M}(\mathbb{R}^N)$ and $\lambda_{\mathcal{T},k} \in \mathcal{M}(\mathbb{R}^N\times\mathbb{R}_+)$ by their action on $C_c(\mathbb{R}^N)$ and $C_c(\mathbb{R}^N\times\mathbb{R}_+)$:

$$\begin{aligned} \langle\mu_\mathcal{T}, g\rangle &= \int_{\mathbb{R}^N} |u_0(x) - u_{\mathcal{T},0}(x)|g(x)dx,\ \forall g \in C_c(\mathbb{R}^N), \\ \langle\lambda_{\mathcal{T},k}, g\rangle &= \sum_{n\in\mathbb{N}} \sum_{p\in\mathcal{T}} |u^{n+1}_p - u^n_p| \int_{t^n}^{t^{n+1}} \int_p g(x,t)dxdt,\ \forall g \in C_c(\mathbb{R}^N\times\mathbb{R}_+). \end{aligned}$$

Inequality (34) gives:

$$|T_1 - T_1^*| \le \int_{\mathbb{R}^N\times\mathbb{R}_+} |\varphi_t(x,t)|d\lambda_{\mathcal{T},k}(x,t) + \int_{\mathbb{R}^N} \varphi(x,0)d\mu_\mathcal{T}(x). \quad (35)$$

The properties (29) and (30) can be proved, first for $u_0 \in C^\infty_c(\mathbb{R}^N)$ and, then, by density for $u_0 \in L^\infty(\mathbb{R}^N)$ or $u_0 \in L^\infty \cap \mathrm{BV}_{loc}(\mathbb{R}^N)$. Furthermore, $\lambda_{\mathcal{T},k}$ will contribute to the measure $\mu_{\mathcal{T},k}$ and, thanks to (15), we have:

$$\lambda_{\mathcal{T},k}(B(0,R)\times[0,T]) \le TC_{bv}\sqrt{h} \quad \forall R > 0,\ \forall T > 0.$$

Comparison between $\mathbf{T_2}$ and $\mathbf{T_2^*}$

In T_2 we gather the terms by edges. Thus, $T_2 = T_{21} - T_{22}$, with:

$$T_{21} = \sum_{n=0}^{N_T} \sum_{(p,q)\in\mathcal{E}_R^n} \frac{1}{m(p)} \int_{t^n}^{t^{n+1}} \int_p \varphi(x,t)dxdt \left[F_{p,q}^n(u_p^n\top\kappa, u_q^n\top\kappa) - F_{p,q}^n(u_p^n\top\kappa, u_p^n\top\kappa)\right.$$
$$\left. - F_{p,q}^n(u_p^n\bot\kappa, u_q^n\bot\kappa) + F_{p,q}^n(u_p^n\bot\kappa, u_p^n\bot\kappa)\right]$$

and

$$T_{22} = \sum_{n=0}^{N_T} \sum_{(p,q)\in\mathcal{E}_R^n} \frac{1}{m(q)} \int_{t^n}^{t^{n+1}} \int_q \varphi(x,t)dxdt \left[F_{p,q}^n(u_p^n\top\kappa, u_q^n\top\kappa) - F_{p,q}^n(u_q^n\top\kappa, u_q^n\top\kappa)\right.$$
$$\left. - F_{p,q}^n(u_p^n\bot\kappa, u_q^n\bot\kappa) + F_{p,q}^n(u_q^n\bot\kappa, u_q^n\bot\kappa)\right].$$

Using the fact that $\mathrm{div}_x F = 0$, we can also gather the terms of T_2^* by edges and afterwards decompose T_2^* as $T_{21}^* - T_{22}^*$, with:

$$T_{21}^* = \sum_{n=0}^{N_T} \sum_{(p,q)\in\mathcal{E}_R^n} \int_{\sigma_{pq}} \int_{t^n}^{t^{n+1}} \left(\frac{1}{m(\sigma_{pq})} F_{p,q}^n(u_p^n\top\kappa, u_q^n\top\kappa) - F(\gamma,t,u_p^n\top\kappa).n_{p,q}\right.$$
$$\left. - \frac{1}{m(\sigma_{pq})} F_{p,q}^n(u_p^n\bot\kappa, u_q^n\bot\kappa) + F(\gamma,t,u_p^n\bot\kappa).n_{p,q}\right)\varphi(\gamma,t)dtd\gamma,$$

$$T_{22}^* = \sum_{n=0}^{N_T} \sum_{(p,q)\in\mathcal{E}_R^n} \int_{\sigma_{pq}} \int_{t^n}^{t^{n+1}} \left(\frac{1}{m(\sigma_{pq})} F_{p,q}^n(u_p^n\top\kappa, u_q^n\top\kappa) - F(\gamma,t,u_q^n\top\kappa).n_{p,q}\right.$$
$$\left. - \frac{1}{m(\sigma_{pq})} F_{p,q}^n(u_p^n\bot\kappa, u_q^n\bot\kappa) + F(\gamma,t,u_q^n\bot\kappa).n_{p,q}\right)\varphi(\gamma,t)dtd\gamma.$$

In order to compare T_{21} with T_{21}^*, we add and substract the following quantity in T_{21}^*:

$$\int_{\sigma_{pq}} \int_{t^n}^{t^{n+1}} \frac{1}{m(\sigma_{pq})} (F_{p,q}^n(u_p^n\top\kappa, u_p^n\top\kappa) - F_{p,q}^n(u_p^n\bot\kappa, u_p^n\bot\kappa))\varphi(\gamma,t)d\gamma dt.$$

Then, we can see that some terms of T_{21}^* and T_{21} are similar (you just have to replace the mean value of φ over a cell by the mean value over the edge of the cell). The other terms are due to the dependence of F upon x and t. In the following comparison between T_{21} and T_{21}^* the term containing $\varphi(\xi,\tau)$ has no real influence because of the consistency of the scheme. Therefore,

$$
\begin{aligned}
T_{21} - T_{21}^* = &\sum_{n=0}^{N_T} \sum_{(p,q)\in\mathcal{E}_R^n} k \left[\begin{array}{c} F_{p,q}^n(u_p^n \top \kappa, u_q^n \top \kappa) - F_{p,q}^n(u_p^n \top \kappa, u_p^n \top \kappa) \\ -F_{p,q}^n(u_p^n \bot \kappa, u_q^n \bot \kappa) + F_{p,q}^n(u_p^n \bot \kappa, u_p^n \bot \kappa) \end{array} \right] \\
&\times \left(\frac{1}{k^2 m(p) m(\sigma_{pq})} \int_{t^n}^{t^{n+1}} \int_p \int_{t^n}^{t^{n+1}} \int_{\sigma_{pq}} (\varphi(x,t) - \varphi(\gamma,s))\, dx dt d\gamma ds \right) \\
&+ \sum_{n=0}^{N_T} \sum_{(p,q)\in\mathcal{E}_R^n} \int_{t^n}^{t^{n+1}} \int_{\sigma_{pq}} \left[\begin{array}{c} F(\gamma,s,u_p^n \top \kappa).n_{p,q} - \dfrac{1}{m(\sigma_{pq})} F_{p,q}^n(u_p^n \top \kappa, u_p^n \top \kappa) \\ -F(\gamma,s,u_p^n \bot \kappa).n_{p,q} + \dfrac{1}{m(\sigma_{pq})} F_{p,q}^n(u_p^n \bot \kappa, u_p^n \bot \kappa) \end{array} \right] \\
&\times \left(\frac{1}{k m(\sigma_{pq})} \int_{t^n}^{t^{n+1}} \int_{\sigma_{pq}} (\varphi(\gamma,s) - \varphi(\xi,\tau)) d\xi d\tau \right) d\gamma ds.
\end{aligned}
\tag{36}
$$

As F is C^1, there exists $C_{F,u_0,T,R}$ depending only on F, u_0, T and R such that, $\forall (p,q) \in \mathcal{E}_R^n$, $\forall \gamma \in \sigma_{pq}$, $\forall s \in [t^n, t^{n+1}[$, $\forall v \in [A,B]$:

$$
|F(\gamma,s,v).n_{p,q} - \frac{1}{m(\sigma_{pq})} F_{p,q}^n(v,v)| \le C_{F,u_0,T,R}(h+k). \tag{37}
$$

Moreover, for all $(\gamma,\xi,s,\tau) \in \sigma_{pq}^2 \times [t^n, t^{n+1}[^2$, we have:

$$
|\varphi(\gamma,s) - \varphi(\xi,\tau)| \le \int_0^1 (h+k)\,(|\nabla\varphi| + |\varphi_t|)\,(\xi + \theta(\gamma-\xi), \tau + \theta(s-\tau)) d\theta \tag{38}
$$

and, $\forall (x,\gamma,t,s) \in p \times \sigma_{pq} \times [t^n, t^{n+1}[^2$:

$$
|\varphi(x,t) - \varphi(\gamma,s)| \le \int_0^1 (h+k)\,(|\nabla\varphi| + |\varphi_t|)\,(\gamma + \theta(x-\gamma), s + \theta(t-s)) d\theta. \tag{39}
$$

For all $p \in \mathcal{T}$, $q \in N(p)$, $n \in \mathbb{N}$, we define some measures $\mu_{p,q}^n \in \mathcal{M}(\mathbb{R}^N \times \mathbb{R}_+)$ and $\nu_{p,q}^n \in \mathcal{M}(\mathbb{R}^N \times \mathbb{R}_+)$ by their action on $C_c(\mathbb{R}^N \times \mathbb{R}_+)$:

$$
\langle \mu_{p,q}^n, g \rangle = \frac{1}{k^2 m(p) m(\sigma_{pq})} \int_{t^n}^{t^{n+1}} \int_p \int_{t^n}^{t^{n+1}} \int_{\sigma_{pq}} \int_0^1 (h+k) g(\gamma + \theta(x-\gamma), s + \theta(t-s)) d\theta dx dt d\gamma ds,
$$

and

$$
\langle \nu_{p,q}^n, g \rangle = \frac{1}{k m(\sigma_{pq})} \int_{t^n}^{t^{n+1}} \int_{\sigma_{pq}} \int_{t^n}^{t^{n+1}} \int_{\sigma_{pq}} \int_0^1 (h+k)^2 g(\xi + \theta(\gamma-\xi), \tau + \theta(s-\tau)) d\theta d\xi d\tau d\gamma ds.
$$

Then, the expressions (36), (37), (38) and (39) lead to:

$$|T_{21}-T_{21}^*| \leq \sum_{n=0}^{N_T}\sum_{(p,q)\in\mathcal{E}_R^n} k\left[F_{p,q}^n(u_p^n\top\kappa,u_q^n\top\kappa)-F_{p,q}^n(u_p^n\top\kappa,u_p^n\top\kappa)+F_{p,q}^n(u_p^n\bot\kappa,u_q^n\bot\kappa)\right.$$
$$\left.-F_{p,q}^n(u_p^n\bot\kappa,u_p^n\bot\kappa)\right]\langle\mu_{p,q}^n,|\nabla\varphi|+|\varphi_t|\rangle+C_{F,u_0,T,R}\sum_{n=0}^{N_T}\sum_{(p,q)\in\mathcal{E}_R^n}\langle\nu_{p,q}^n,|\varphi|+|\varphi_t|\rangle. \quad (40)$$

We obtain as well:

$$|T_{22}-T_{22}^*| \leq \sum_{n=0}^{N_T}\sum_{(p,q)\in\mathcal{E}_R^n} k\left[F_{p,q}^n(u_p^n\top\kappa,u_q^n\top\kappa)-F_{p,q}^n(u_q^n\top\kappa,u_q^n\top\kappa)+F_{p,q}^n(u_p^n\bot\kappa,u_q^n\bot\kappa)\right.$$
$$\left.-F_{p,q}^n(u_q^n\bot\kappa,u_q^n\bot\kappa)\right]\langle\mu_{q,p}^n,|\nabla\varphi|+|\varphi_t|\rangle+C_{F,u_0,T,R}\sum_{n=0}^{N_T}\sum_{(p,q)\in\mathcal{E}_R^n}\langle\nu_{p,q}^n,|\varphi|+|\varphi_t|\rangle. \quad (41)$$

The monotony of $F_{p,q}^n$ implies that $\forall (p,q)\in\mathcal{E}_R^n$, $\forall\kappa\in\mathbb{R}$,

$$\begin{aligned} 0\leq F_{p,q}^n(u_p^n\top\kappa,u_q^n\top\kappa)-F_{p,q}^n(u_p^n\top\kappa,u_p^n\top\kappa) &\leq \max_{u_q^n\leq c\leq d\leq u_p^n}(F_{p,q}^n(d,c)-F_{p,q}^n(d,d)),\\ 0\leq F_{p,q}^n(u_p^n\top\kappa,u_q^n\top\kappa)-F_{p,q}^n(u_q^n\top\kappa,u_q^n\top\kappa) &\leq \max_{u_q^n\leq c\leq d\leq u_p^n}(F_{p,q}^n(d,c)-F_{p,q}^n(c,c)) \end{aligned} \quad (42)$$

and these properties are always true if we replace $\top$ by $\bot$. Finally, we can define $\mu_{\mathcal{T},k}\in\mathcal{M}(\mathbb{R}^N\times\mathbb{R}_+)$ by its action on $C_c(\mathbb{R}^N\times\mathbb{R}_+)$:

$$\langle\mu_{\mathcal{T},k},g\rangle=\langle\lambda_{\mathcal{T},k},g\rangle+2\sum_{n\in\mathbb{N}}\sum_{(p,q)\in\mathcal{E}^n}k\big[\max_{u_q^n\leq c\leq d\leq u_p^n}(F_{p,q}^n(d,c)-F_{p,q}^n(d,d))\langle\mu_{p,q}^n,g\rangle$$
$$+\max_{u_q^n\leq c\leq d\leq u_p^n}(F_{p,q}^n(d,c)-F_{p,q}^n(c,c))\langle\mu_{q,p}^n,g\rangle\big]+2C_{F,u_0,T,R}\sum_{n\in\mathbb{N}}\sum_{(p,q)\in\mathcal{E}^n}\langle\nu_{p,q}^n,g\rangle.$$

Lemma 2 gives the bound (28). Thanks to (31), (35), (40), (41) and (42), we get:

$$-T_1^*-T_2^*\geq-\int_{\mathbb{R}^N\times\mathbb{R}_+}(|\varphi_t(x,t)|+|\nabla\varphi(x,t)|)d\mu_{\mathcal{T},k}(x,t)-\int_{\mathbb{R}^N}\varphi(x,0)d\mu_{\mathcal{T}}(x).$$

This puts an end to the proof of Theorem 1.

4. Existence and uniqueness of the entropy solution

The aim of this section is to prove Theorem 2; it gives simultaneously the existence and the uniqueness of the entropy solution to (1) and the convergence of the scheme towards this solution. Furthermore, in Section 4.4, we study the case where the initial condition u_0 belongs to $L^\infty\cap BV_{loc}(\mathbb{R}^N)$.

Theorem 2. *Under assumptions (2), the nonlinear hyperbolic problem (1) has a unique entropy solution u and the approximate solution defined by (9)-(11) with the hypotheses (5), (6), (10) converges towards u in $L^p_{loc}(\mathbb{R}^N\times\mathbb{R}_+)$ for $1\leq p<+\infty$.*

The proof splits up into 3 steps. First, we give a property of the bounded sequences in $L^\infty(\mathbb{R}^N \times \mathbb{R}_+)$. Then, it permits us to pass to the limit in (27) and therefore to prove the existence of an entropy process solution. Finally, we use a technique of regularization to prove that the entropy process solution is unique and is the unique entropy solution.

4.1. A property of bounded sequences in $\mathbf{L^\infty(\mathbb{R}^N \times \mathbb{R}_+)}$

Proposition 1. *Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^\infty(\mathbb{R}^N \times \mathbb{R}_+)$. Then, there exists a subsequence of $(u_n)_{n\in\mathbb{N}}$ (still denoted by $(u_n)_{n\in\mathbb{N}}$) and $u \in L^\infty(\mathbb{R}^N \times \mathbb{R}_+ \times]0,1[)$ such that $(u_n)_{n\in\mathbb{N}}$ converges towards u in a nonlinear weak $*$ sense. Furthermore, if u does not depend on α (i.e $u(x,t,\alpha) = v(x,t)$ for a.e. (x,t), for a.e. α), then $(u_n)_{n\in\mathbb{N}}$ converges towards v in $L^p_{loc}(\mathbb{R}^N \times \mathbb{R}_+)$ for all $1 \le p < \infty$.*

The first part of the proof (convergence in a nonlinear weak $*$ sense) is given in [4]. We get the convergence in L^p_{loc} by choosing some particular functions h (power functions) in the definition of the nonlinear weak $*$ convergence (*cf.* [3]).

In order to pass to the limit in (27), we also need the following corollary:

Corollary 1. *Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of $L^\infty(\mathbb{R}^N \times \mathbb{R}_+)$ and $u \in L^\infty(\mathbb{R}^N \times \mathbb{R}_+ \times]0,1[)$ such that $(u_n)_{n\in\mathbb{N}}$ converges towards u in a nonlinear weak $*$ sense.*

Then, $\forall g \in C(\mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, $\forall K \subset \mathbb{R}^N \times \mathbb{R}_+$, $\forall \varphi \in L^1(K)$,

$$\int_K g(x,t,u_n(x,t))\varphi(x,t)dxdt \overset{n\to\infty}{\longrightarrow} \int_K \int_0^1 g(x,t,u(x,t,\alpha))\varphi(x,t)dxdtd\alpha.$$

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of $L^\infty(\mathbb{R}^N \times \mathbb{R}_+)$. Proposition 1 gives the existence of a subsequence $(u_n)_{n\in\mathbb{N}}$ and of $u \in L^\infty(\mathbb{R}^N \times \mathbb{R}_+ \times]0,1[)$ such that $(u_n)_{n\in\mathbb{N}}$ converges towards u in a nonlinear weak $*$ sense. Therefore, there exists $U \ge 0$ such that $\|u\|_\infty \le U$ and $\|u_n\|_\infty \le U$ $\forall n \in \mathbb{N}$.

Let $g \in C(\mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ and K a compact set of $\mathbb{R}^N \times \mathbb{R}_+$, equipped with the usual distance d. As the set $K \times [-U,U]$ is a compact set of $\mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}$, g is uniformly continuous on $K \times [-U,U]$.

Let $\varepsilon > 0$. There exists $\eta > 0$ such that for all $(x,t),(x',t') \in K$, for all $s \in [-U,U]$:

$$d((x,t),(x',t')) < \eta \Longrightarrow |g(x,t,s) - g(x',t',s)| < \varepsilon. \tag{43}$$

Moreover, $K \subset \bigcup_{(x,t)\in K} B((x,t),\eta)$ and K is a compact set. Then, we can extract a finite covering from this open covering of K:

$$K \subset \bigcup_{l=1}^{L} B((x_l,t_l),\eta).$$

On K, we can define $\pi_l(x,t) = \big(\eta - d((x,t),(x_l,t_l))\big)\top 0$. Each function π_l is continuous and nonnegative on K and $\pi_l(x,t) > 0$ if and only if $(x,t) \in B((x_l,t_l),\eta)$. Therefore, $\sum_{l=1}^L \pi_l(x,t) \neq 0$ for all $(x,t) \in K$ and we can set: $\psi_l(x,t) = \pi_l(x,t)/\sum_{l=1}^{L}\pi_l(x,t)$, which verifies $\sum_{l=1}^{L}\psi_l(x,t) = 1$, $\forall (x,t) \in K$.

Let $\varphi \in L^1(K)$, we have:

$$\left|\int_K g(x,t,u_n(x,t))\varphi(x,t)dxdt - \int_K \int_0^1 g(x,t,u(x,t,\alpha))\varphi(x,t)dxdtd\alpha\right|$$
$$\leq \int_K \sum_{l=1}^{L} |g(x,t,u_n(x,t)) - g(x_l,t_l,u_n(x,t))|\, \psi_l(x,t)\varphi(x,t)dxdt$$
$$+ \sum_{l=1}^{L} \left| \int_K \left(g(x_l,t_l,u_n(x,t)) - \int_0^1 g(x_l,t_l,u(x,t,\alpha))d\alpha \right) \psi_l(x,t)\varphi(x,t)dxdt \right|$$
$$+ \int_K \int_0^1 \sum_{l=1}^{L} |g(x_l,t_l,u(x,t,\alpha)) - g(x,t,u(x,t,\alpha))|\, \psi_l(x,t)\varphi(x,t)dxdtd\alpha.$$

We use the uniform continuity of g and the result of Proposition 1. There exists $P \in \mathbb{N}$ such that for all $n \geq P$, for all $l \in \{1,..,L\}$,

$$\left| \int_K \left(g(x_l,t_l,u_n(x,t)) - \int_0^1 g(x_l,t_l,u(x,t,\alpha))d\alpha \right) \psi_l(x,t)\varphi(x,t)dxdtd\alpha \right| \leq \frac{\varepsilon}{L}.$$

Thus, for all $\varepsilon > 0$, there exists $P \in \mathbb{N}$ such that for all $n \geq P$

$$\left|\int_K g(x,t,u_n(x,t))\varphi(x,t)dxdt - \int_K \int_0^1 g(x,t,u(x,t,\alpha))\varphi(x,t)dxdtd\alpha\right| \leq \varepsilon(2\|\varphi\|_{L^1(K)} + 1).$$

This concludes the proof of Corollary 1.

4.2. **Existence of an entropy process solution**

Lemma 4. *Assume (2), then there exists an entropy process solution to the problem (1).*

Proof. We consider a sequence of meshes $(\mathcal{T}_n)_{n\in\mathbb{N}}$ and a sequence of time steps $(k_n)_{n\in\mathbb{N}}$ that satisfy the hypotheses (5) and the C.F.L. condition with $h = h_n$, $k = k_n$ and α and ξ not depending on n. We assume that $(h_n)_{n\in\mathbb{N}}$ goes to 0 when n goes to infinity. Then, we consider the sequence of approximate solutions $(u_{\mathcal{T}_n,k_n})_{n\in\mathbb{N}}$ given by (9)-(11) and the hypotheses (6) and (10).

Lemma 1 proves that $(u_{\mathcal{T}_n,k_n})_{n\in\mathbb{N}}$ is a bounded sequence of $L^\infty(\mathbb{R}^N \times \mathbb{R}_+)$. Therefore, Proposition 1 gives the existence of $v \in L^\infty(\mathbb{R}^N \times \mathbb{R}_+ \times]0,1[)$ and of a subsequence, still denoted by $(u_{\mathcal{T}_n,k_n})_{n\in\mathbb{N}}$, such that $(u_{\mathcal{T}_n,k_n})_{n\in\mathbb{N}}$ converges towards v in a nonlinear weak $*$ sense. Let us prove that v is an entropy process solution to (1).

Let $\varphi \in C_c^\infty(\mathbb{R}^N \times \mathbb{R}_+, \mathbb{R}_+)$ and $\kappa \in \mathbb{R}$. Let $R > 0$ and $T > 0$ be such that Supp$\varphi \subset B(0,R) \times [0,T]$. We pass to the limit in the expression (27). Applying the definition of the nonlinear weak $*$ convergence and Corollary 1, we get:

$$\lim_{n\to\infty} \int_{\mathbb{R}^N \times \mathbb{R}_+} |u_{\mathcal{T}_n,k_n}(x,t) - \kappa|\varphi_t(x,t)dxdt = \int_{\mathbb{R}^N \times \mathbb{R}_+} \int_0^1 |v(x,t,\alpha) - \kappa|\varphi_t(x,t)dxdtd\alpha,$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N \times \mathbb{R}_+} (F(x,t,u_{\mathcal{T}_n,k_n}(x,t)\top\kappa) - F(x,t,u_{\mathcal{T}_n,k_n}(x,t)\bot\kappa)).\nabla\varphi(x,t)dxdt$$
$$= \int_{\mathbb{R}^N \times \mathbb{R}_+} \int_0^1 (F(x,t,v(x,t,\alpha)\top\kappa) - F(x,t,v(x,t,\alpha)\bot\kappa)).\nabla\varphi(x,t)dxdtd\alpha.$$

The error terms are bounded as follows:

$$\int_{\mathbb{R}^N\times\mathbb{R}_+}(|\varphi_t(x,t)|+|\nabla\varphi(x,t)|)d\mu_{\mathcal{T}_n,k_n}(x,t) \;\le\; (\|\varphi_t\|_\infty+\|\nabla\varphi\|_\infty)\mu_{\mathcal{T}_n,k_n}(B(0,R)\times[0,T]),$$
$$\int_{\mathbb{R}^N}\varphi(x,0)d\mu_{\mathcal{T}_n}(x) \;\le\; \|\varphi(.,0)\|_\infty\mu_{\mathcal{T}_n}(x)(B(0,R)),$$

and the properties (28) and (29) ensure that these two expressions go to 0 when n goes to infinity. Therefore, v is an entropy process solution.

4.3. Uniqueness

Lemma 5. *The entropy process solution to (1) is unique and it is the unique entropy solution to (1).*

Proof. Assume that there exist two entropy process solutions $v\in L^\infty(\mathbb{R}^N\times\mathbb{R}_+\times]0,1[)$ and $w\in L^\infty(\mathbb{R}^N\times\mathbb{R}_+\times]0,1[)$. It means that, $\forall\kappa\in\mathbb{R},\ \forall\varphi\in C_c^1(\mathbb{R}^N\times\mathbb{R}_+,\mathbb{R}_+)$,

$$\int_{\mathbb{R}^N\times\mathbb{R}_+}\int_0^1[|v(x,t,\alpha)-\kappa|\varphi_t(x,t)+(F(x,t,v(x,t,\alpha)\top\kappa)-F(x,t,v(x,t,\alpha)\bot\kappa)).\nabla\varphi(x,t)]dxdtd\alpha$$
$$+\int_{\mathbb{R}^N}|u_0(x)-\kappa|\varphi(x,0)dx\ge 0, \quad (44)$$

and

$$\int_{\mathbb{R}^N\times\mathbb{R}_+}\int_0^1[|w(y,s,\beta)-\kappa|\varphi_s(y,s)+(F(y,s,w(y,s,\beta)\top\kappa)-F(y,s,w(y,s,\beta)\bot\kappa)).\nabla\varphi(y,s)]dydsd\beta$$
$$+\int_{\mathbb{R}^N}|u_0(y)-\kappa|\varphi(y,0)dy\ge 0. \quad (45)$$

The proof of Lemma 5 splits up into 2 steps. In Step 1, we prove that:
For all $\psi\in C_c^\infty(\mathbb{R}^N\times\mathbb{R}_+,\mathbb{R}_+)$,

$$\int_{\mathbb{R}^N\times\mathbb{R}_+\times]0,1[^2}|v(x,t,\alpha)-w(x,t,\beta)|\psi_t(x,t)+(F(x,t,v(x,t,\alpha)\top w(x,t,\beta))$$
$$-F(x,t,v(x,t,\alpha)\bot w(x,t,\beta)).\nabla\psi(x,t))dxdtd\alpha d\beta\ge 0. \quad (46)$$

Then, choosing a good function ψ, we obtain in Step 2 that, for all compact set $K\subset\mathbb{R}^N\times\mathbb{R}_+$,

$$\int_{K\times]0,1[^2}|v(x,t,\alpha)-w(x,t,\beta)|dxdtd\alpha d\beta=0. \quad (47)$$

It implies that $v(x,t,\alpha)=w(x,t,\beta)$ for almost all $(x,t,\alpha,\beta)\in\mathbb{R}^N\times\mathbb{R}_+\times]0,1[^2$. Then, $v(x,t,\alpha)$ and $w(x,t,\beta)$ do not respectively depend on α and β. But, if we know an entropy solution u of the problem (1), we can build an entropy process solution v defined by $v(x,t,\alpha)=u(x,t)$ for a.e. $(x,t,\alpha)\in\mathbb{R}^N\times\mathbb{R}_+\times]0,1[$; reciprocally, if v is an entropy process solution of (1) and if there exists u such that $v(x,t,\alpha)=u(x,t)$ for almost every $(x,t,\alpha)\in\mathbb{R}^N\times\mathbb{R}_+\times]0,1[$, then u is an entropy solution of (1). Thus, the entropy process solution is unique and it is the unique entropy solution u.

Furthermore, Proposition 1 implies that the approximate solution given by the scheme converges towards u in $L^p_{loc}(\mathbb{R}^N\times\mathbb{R}_+)$ for $1\le p<+\infty$ and it concludes the proof of Theorem 2.

It remains to prove (46) and (47).

4.3.1. *Step 1*

The idea is to take $\kappa = w(y,s,\beta)$ in (44) and to choose a function φ that makes y close to x and s close to t. We introduce two functions $\rho_N \in C_c^\infty(\mathbb{R}^N,\mathbb{R})$ and $\bar\rho_1 \in C_c^\infty(\mathbb{R},\mathbb{R})$ that satisfy:

$$\left\{\begin{array}{ll} \text{Supp }(\rho_N) \subset \{x \in \mathbb{R}^N;\ |x| \le 1\}, & \text{Supp }(\bar\rho_1) \subset [-1,0], \\ \rho_N(x) \ge 0,\ \forall x \in \mathbb{R}^N, & \bar\rho_1(x) \ge 0,\ \forall x \in \mathbb{R}, \\ \displaystyle\int_{\mathbb{R}^N} \rho_N(x)dx = 1, & \displaystyle\int_{\mathbb{R}} \bar\rho_1(x)dx = 1. \end{array}\right\}. \tag{48}$$

For all $r \ge 1$ we define $\rho_{N,r} : x \to r^N \rho_N(rx)$ and $\bar\rho_{1,r} : x \to r\bar\rho_1(rx)$. We have:

$$\int_{\mathbb{R}^N\times\mathbb{R}_+} \rho_{N,r}(x-y)\bar\rho_{1,r}(t-s)dyds = 1 \quad \forall x \in \mathbb{R}^N,\ \forall t \in \mathbb{R}_+. \tag{49}$$

Let $\psi \in C_c^\infty(\mathbb{R}^N \times \mathbb{R}_+, \mathbb{R}_+)$, we set $\varphi(x,t,y,s) = \psi(x,t)\rho_{N,r}(x-y)\bar\rho_{1,r}(t-s)$. We rewrite (44) with $\varphi(.,.,y,s)$ and $\kappa = w(y,s,\beta)$ and we integrate with respect to y, s and β. It yields:

$$A_1 + A_2 + A_3 + A_4 + A_5 \ge 0 \tag{50}$$

with

$$\begin{aligned} A_1 &= \int_{(\mathbb{R}^N\times\mathbb{R}_+\times]0,1[)^2} |v(x,t,\alpha) - w(y,s,\beta)|\psi_t(x,t)\rho_{N,r}(x-y)\bar\rho_{1,r}(t-s)dxdtd\alpha dydsd\beta, \\ A_2 &= \int_{(\mathbb{R}^N\times\mathbb{R}_+\times]0,1[)^2} |v(x,t,\alpha) - w(y,s,\beta)|\psi(x,t)\rho_{N,r}(x-y)\bar\rho'_{1,r}(t-s)dxdtd\alpha dydsd\beta, \\ A_3 &= \int_{(\mathbb{R}^N\times\mathbb{R}_+\times]0,1[)^2} (F(x,t,v(x,t,\alpha)\top w(y,s,\beta)) - F(x,t,v(x,t,\alpha)\bot w(y,s,\beta)). \\ &\qquad\qquad \nabla\psi(x,t)\rho_{N,r}(x-y)\bar\rho_{1,r}(t-s)dxdtd\alpha dydsd\beta, \\ A_4 &= \int_{(\mathbb{R}^N\times\mathbb{R}_+\times]0,1[)^2} (F(x,t,v(x,t,\alpha)\top w(y,s,\beta)) - F(x,t,v(x,t,\alpha)\bot w(y,s,\beta)). \\ &\qquad\qquad \nabla\rho_{N,r}(x-y)\psi(x,t)\bar\rho_{1,r}(t-s)dxdtd\alpha dydsd\beta, \\ A_5 &= \int_{(\mathbb{R}^N)^2\times\mathbb{R}_+\times]0,1[} |u_0(x) - w(y,s,\beta)|\psi(x,0)\rho_{N,r}(x-y)\bar\rho_{1,r}(-s)dxdydsd\beta. \end{aligned}$$

We will obtain (46) by passing to the limit on r in (50). Indeed, if we set:

$$A_{10} = \int_{\mathbb{R}^N\times\mathbb{R}_+\times]0,1[^2} |v(x,t,\alpha) - w(x,t,\beta)|\psi_t(x,t)dxdtd\alpha d\beta,$$

and

$$A_{30} = \int_{\mathbb{R}^N\times\mathbb{R}_+\times]0,1[^2} (F(x,t,v(x,t,\alpha)\top w(x,t,\beta)) - F(x,t,v(x,t,\alpha)\bot w(x,t,\beta)).\nabla\psi(x,t)dxdtd\alpha d\beta,$$

we prove that

$$A_1 \overset{r\to\infty}{\longrightarrow} A_{10}, \tag{51}$$

and

$$A_3 \overset{r\to\infty}{\longrightarrow} A_{30}. \tag{52}$$

Furthermore, we show that $\liminf(-A_2 - A_4 - A_5) \geq 0$. In the sequel, we will use the following result:

Lemma 6. *Let $g \in L^\infty(\mathbb{R}^q)$. For all compact set $K \subset \mathbb{R}^q$:*

$$\int_{\mathbb{R}^q} |g(x+h) - g(x)| 1_K(x) dx \overset{h\to 0}{\longrightarrow} 0.$$

The term A_1

Thanks to (49) and the triangular inequality we get:

$$|A_1 - A_{10}| \leq \int_{(\mathbb{R}^N\times\mathbb{R}_+\times]0,1[)^2} |w(x,t,\beta) - w(y,s,\beta)| \psi_t(x,t) \rho_{N,r}(x-y) \bar{\rho}_{1,r}(t-s) dxdtd\alpha dydsd\beta.$$

We set:

$$\varepsilon(r,K,w) = \sup\{\int_K \int_0^1 |w(x,t,\beta) - w(x+\eta, t+\tau, \beta)| dxdtd\beta,\ |\eta| \leq \frac{1}{r},\ 0 \leq \tau \leq \frac{1}{r}\}.$$

Therefore, $|A_1 - A_{10}| \leq \|\psi_t\|_\infty \varepsilon(r,K,w)$ and, as $\varepsilon(r,K,w) \overset{r\to\infty}{\longrightarrow} 0$ (Lemma 6), we get (51).

The term A_3

Similarly, there exists $C_{F,\psi,v,w}$ that only depends on F, ψ, $\|v\|_\infty$ and $\|w\|_\infty$ such that:

$$|A_3 - A_{30}| \leq C_{F,\psi,v,w} \int_{(\mathbb{R}^N\times\mathbb{R}_+\times]0,1[)^2} |w(x,t,\beta) - w(y,s,\beta)| |\nabla\psi(x,t)| \\ \rho_{N,r}(x-y) \bar{\rho}_{1,r}(t-s) dxdtd\alpha dydsd\beta \leq C_{F,\psi,v,w} \|\nabla\psi\|_\infty \varepsilon(r,K,w),$$

and it implies (52).

The term $A_2 + A_4$

We rewrite (45) with the test function $\varphi(x,t,.,.)$ and $\kappa = v(x,t,\alpha)$, and we integrate with respect to x and t. It yields:

$$-A_2 - A_{40} \geq 0$$

where

$$A_{40} = \int_{(\mathbb{R}^N\times\mathbb{R}_+\times]0,1[)^2} (F(y,s,v(x,t,\alpha)\top w(y,s,\beta)) - F(y,s,v(x,t,\alpha)\bot w(y,s,\beta))). \\ \nabla\rho_{N,r}(x-y)\psi(x,t)\bar{\rho}_{1,r}(t-s) dxdtd\alpha dydsd\beta.$$

Therefore,

$$-A_2 - A_4 \geq A_{40} - A_4 = \mathcal{A}_4 \tag{53}$$

with

$$\mathcal{A}_4 = \int_{(\mathbb{R}^N\times\mathbb{R}_+\times]0,1[)^2} \Big(F(y,s,v(x,t,\alpha)\top w(y,s,\beta)) - F(y,s,v(x,t,\alpha)\bot w(y,s,\beta)) \\ - F(x,t,v(x,t,\alpha)\top w(y,s,\beta)) + F(x,t,v(x,t,\alpha)\bot w(y,s,\beta)) \Big) . \nabla\rho_{N,r}(x-y)\psi(x,t)\bar{\rho}_{1,r}(t-s) dxdtd\alpha dydsd\beta.$$

If we replace $w(y,s,\beta)$ by $w(x,t,\beta)$ in $\mathcal{A}_4$, we get:

$$\mathcal{A}_{4b} = \int_{(\mathbb{R}^N\times\mathbb{R}_+\times]0,1[)^2} \Big(F(y,s,v(x,t,\alpha)\top w(x,t,\beta)) - F(y,s,v(x,t,\alpha)\bot w(x,t,\beta)) \\ - F(x,t,v(x,t,\alpha)\top w(x,t,\beta)) + F(x,t,v(x,t,\alpha)\bot w(x,t,\beta))\Big).\nabla\rho_{N,r}(x-y)\psi(x,t)\bar\rho_{1,r}(t-s)dxdtd\alpha dydsd\beta.$$

But, for all (x,t,α,β), the function:

$$y \mapsto F(y,s,v(x,t,\alpha)\top w(x,t,\beta)) - F(y,s,v(x,t,\alpha)\bot w(x,t,\beta)) \\ - F(x,t,v(x,t,\alpha)\top w(x,t,\beta)) + F(x,t,v(x,t,\alpha)\bot w(x,t,\beta))$$

is a divergence-free function. Hence, $\mathcal{A}_{4b} = 0$. Moreover,

$$|\mathcal{A}_4 - \mathcal{A}_{4b}| \le \int_{(\mathbb{R}^N\times\mathbb{R}_+\times]0,1[)^2} |w(x,t,\beta) - w(y,s,\beta)| \\ \Bigg(\Bigg|\int_0^1 \Big(\frac{\partial F}{\partial s}(x,t,w(x,t,\beta)\top v(x,t,\alpha) + \theta(w(x,t,\beta)\top v(x,t,\alpha) - w(y,s,\beta)\top v(x,t,\alpha))) \\ -\frac{\partial F}{\partial s}(y,s,w(x,t,\beta)\top v(x,t,\alpha) + \theta(w(x,t,\beta)\top v(x,t,\alpha) - w(y,s,\beta)\top v(x,t,\alpha)))\Big) d\theta\Bigg| \\ + \Bigg|\int_0^1 \Big(\frac{\partial F}{\partial s}(x,t,w(x,t,\beta)\bot v(x,t,\alpha) + \theta(w(x,t,\beta)\bot v(x,t,\alpha) - w(y,s,\beta)\bot v(x,t,\alpha))) \\ -\frac{\partial F}{\partial s}(y,s,w(x,t,\beta)\bot v(x,t,\alpha) + \theta(w(x,t,\beta)\bot v(x,t,\alpha) - w(y,s,\beta)\bot v(x,t,\alpha)))\Big) d\theta\Bigg|\Bigg) \\ |\nabla\rho_{N,r}(x-y)|\psi(x,t)\bar\rho_{1,r}(t-s)dxdtd\alpha dydsd\beta.$$

As $\dfrac{\partial F}{\partial s}$ is locally Lipschitz continuous, there exists $C_{F,\psi,v,w}$ depending only on F, ψ, $\|v\|_\infty$ and $\|w\|_\infty$ such that for all $(x,t)\in K$, for all (y,s) s.t. $|x-y| \le \dfrac{1}{r}$ and $|t-s| \le \dfrac{1}{r}$:

$$|\frac{\partial F}{\partial s}(x,t,p) - \frac{\partial F}{\partial s}(y,s,p)| \le \frac{C'_{F,\psi,v,w}}{r} \quad \forall p \in [-(\|v\|_\infty\top\|w\|_\infty),(\|v\|_\infty\top\|w\|_\infty)].$$

Finally,

$$|\mathcal{A}_4| = |\mathcal{A}_4 - \mathcal{A}_{4b}| \le C'_{F,\psi,v,w}\|\psi\|_\infty\varepsilon(r,K,w).$$

and

$$\mathcal{A}_4 \overset{r\to\infty}{\longrightarrow} 0. \tag{54}$$

<u>The termA_5</u>

We rewrite (45) with $\varphi(y,s) = \psi(x,0)\rho_{N,r}(x-y)\int_s^\infty \bar\rho_{1,r}(-\tau)d\tau$ and $\kappa = u_0(x)$, and we integrate with respect to x. We get:

$$-A_5 \ge -A_6 - A_7, \tag{55}$$

where:

$$A_6 = -\int_{(\mathbb{R}^N)^2\times\mathbb{R}^+\times]0,1[} (F(y,s,w(y,s,\beta)\top u_0(x)) - F(y,s,w(y,s,\beta)\bot u_0(x))).\nabla\rho_{N,r}(x-y) \\ \psi(x,0)\int_s^\infty \bar{\rho}_{1,r}(-\tau)d\tau dxdydsd\beta,$$

and

$$A_7 = \int_{(\mathbb{R}^N)^2} |u_0(y)-u_0(x)|\psi(x,0)\rho_{N,r}(x-y)\int_0^\infty \bar{\rho}_{1,r}(-\tau)d\tau dxdy.$$

We set:

$$\varepsilon(r,K_0,u_0) = \sup\{\int_{K_0} |u_0(x)-u_0(x+\eta)|dx;\ |\eta|\le \frac{1}{r}\}. \tag{56}$$

Then, $|A_7| \le \|\psi(.,0)\|_\infty \varepsilon(r,K_0,u_0)$ and

$$A_7 \overset{r\to\infty}{\longrightarrow} 0. \tag{57}$$

Replacing $u_0(x)$ by $u_0(y)$ in A_6, we get:

$$A_{60} = -\int_{(\mathbb{R}^N)^2\times\mathbb{R}^+\times]0,1[} (F(y,s,w(y,s,\beta)\top u_0(y)) - F(y,s,w(y,s,\beta)\bot u_0(y))).\nabla\rho_{N,r}(x-y) \\ \psi(x,0)\int_s^\infty \bar{\rho}_{1,r}(-\tau)d\tau dxdydsd\beta.$$

Let us integrate by parts with respect to x:

$$A_{60} = \int_{(\mathbb{R}^N)^2\times\mathbb{R}^+\times]0,1[} (F(y,s,w(y,s,\beta)\top u_0(y)) - F(y,s,w(y,s,\beta)\bot u_0(y))).\nabla\psi(x,0) \\ \rho_{N,r}(x-y)\int_s^\infty \bar{\rho}_{1,r}(-\tau)d\tau dxdydsd\beta.$$

Then, there exists C_{F,ψ,w,u_0} depending only on F, ψ, $\|w\|_\infty$ and $\|u_0\|_\infty$ such that:

$$|A_{60}| \le \frac{C_{F,\psi,w,u_0}}{r}.$$

Furthermore, there exists C'_{F,ψ,w,u_0} depending only on F, ψ, $\|w\|_\infty$ and $\|u_0\|_\infty$ such that:

$$|A_6 - A_{60}| \le C'_{F,\psi,v,u_0}\|\psi(.,0)\|\varepsilon(r,K_0,u_0).$$

Therefore,

$$A_6 \overset{r\to\infty}{\longrightarrow} 0. \tag{58}$$

Finally, (50), (51), (52), (53), (54), (55), (57) and (58) lead to (46).

4.3.2. *Step 2*

We now prove (47). Let K be a compact set of $\mathbb{R}^N \times \mathbb{R}_+$. Let $\omega \in \mathbb{R}$, $R > 0$ and $T \in]0, \frac{R}{\omega}[$ such that

$$K \subset \bigcup_{0 \le t \le T} (B(0, R - \omega t) \times \{t\}).$$

Let $\rho \in C_c^\infty(\mathbb{R}^+, [0,1])$ verify:

$$\begin{aligned} &\rho(r) = 1 \text{ si } r \in [0, R],\\ &\rho(r) = 0 \text{ si } r \in [R+1, +\infty[,\\ &\rho'(r) \le 0 \text{ for all } r \in \mathbb{R}^+. \end{aligned}$$

Let ψ be defined by:

$$\psi(x,t) = \begin{cases} \rho(|x| + \omega t)\dfrac{T-t}{T} \text{ for } x \in \mathbb{R} \text{ and } t \in [0,T], \\ \\ 0 \text{ for } x \in \mathbb{R} \text{ and } t \ge T. \end{cases} \tag{59}$$

Applying (46) with ψ, we get:

$$\int_{\mathbb{R}^N \times \mathbb{R}_+ \times]0,1[^2} \Big[|v(x,t,\alpha) - w(x,t,\beta)| (\frac{T-t}{T}\omega\rho'(|x| + \omega t) - \frac{1}{T}\rho(|x| + \omega t))$$
$$+ (F(x,t,v(x,t,\alpha) \top w(x,t,\beta)) - F(x,t,v(x,t,\alpha) \bot w(x,t,\beta))).\frac{x}{|x|}\frac{T-t}{T}\rho'(|x| + \omega t)\Big] dx dt d\alpha d\beta \ge 0.$$

We take $\omega = V_{\mathcal{K}}$ where $\mathcal{K} = [-(\|v\|_\infty \top \|w\|_\infty), (\|v\|_\infty \top \|w\|_\infty)]$. As $\rho' \le 0$, we have:

$$\int_{K \times]0,1[^2} |v(x,t,\alpha) - w(x,t,\beta)| dx dt d\alpha d\beta \le 0,$$

and it yields (47). It concludes the proof of Theorem 2.

4.4. Study of the case $\mathbf{u_0} \in \mathbf{BV_{loc}}(\mathbb{R}^{\mathbf{N}})$

In order to prove the existence and the uniqueness of the entropy solution and the convergence of the numerical scheme, we just needed $u_0 \in L^\infty(\mathbb{R}^N)$. But if we want to prove an error estimate between the approximate solution given by the scheme and the entropy solution in $L^1_{loc}(\mathbb{R}^N \times \mathbb{R}_+)$, which is the aim of Section 5, we have to assume some regularity on u_0 and u. Indeed, if $u_0 \in \mathrm{BV}_{loc}(\mathbb{R}^N)$, we get an error estimate of order $h^{\frac{1}{4}}$ because of the following result:

Theorem 3. *Assume (2) and $u_0 \in BV_{loc}(\mathbb{R}^N)$. Then the entropy solution to (1) u belongs to $BV_{loc}(\mathbb{R}^N \times [0,T[)$ for all $T > 0$ and, for all compact set $K \subset \mathbb{R}^N \times \mathbb{R}_+$, there exists $C_{K,u_0,F}$ depending only on K, u_0 and F such that*

$$\forall \eta \in B_{\mathbb{R}^N}(0,1),\ \forall \tau \in [0,1],\ \int_K |u(x+\eta, t+\tau) - u(x,t)| dx dt \le C_{K,u_0,F}(|\eta| + \tau). \tag{60}$$

We note that (60) is a consequence of the fact that $u \in \mathrm{BV}_{loc}(\mathbb{R}^N \times [0,T[)$. Indeed:

Lemma 7. *Let $\Omega \subset \mathbb{R}^p$ and $\xi \in \mathbb{R}_+$. For all $g \in BV(\Omega \cup (\Omega+\xi))$, we have:*

$$\int_\Omega |g(x+\xi) - g(x)|dx \le |g|_{BV(\Omega\cup(\Omega+\xi))}|\xi|.$$

In order to prove Theorem 3, we consider the entropy solution as the limit of the approximate solution given by a particular scheme. We know that, in the monodimensional case, the scheme (9) has a nonincreasing total variation (see [7]) ; it means that we have a strong BV estimate on the approximate solution. But, in the multidimensional case, we actually just have a weak BV estimate (Lemma 2). However, for some particular schemes, we can obtain some strong BV estimate, even in the multidimensional case.

We consider a scheme on a structured mesh. We give the proof for $N = 2$ for the sake of simplicity. Then, we denote by $z = (x, y)$ a point of $\mathbb{R}^2$ and by (F^x, F^y) the function F and the problem (1) rewrites:

$$\begin{array}{rcll} u_t(z,t) + (F^x(z,t,u(z,t)))_x + (F^y(z,t,u(z,t)))_y, &=& 0, & \forall z \in \mathbb{R}^2, \forall t \in \mathbb{R}_+, \\ u(z,0) &=& u_0(z), & \forall z \in \mathbb{R}^2. \end{array} \tag{61}$$

The mesh is made up squares of side h numbered in a Cartesian way. Let us denote by $p_{i,j}$ the cell of centre $(x_i = ih, y_j = jh)$. The vertices of this cell are the points of coordinates $(x_{i-\frac12}, y_{j-\frac12})$, $(x_{i-\frac12}, y_{j+\frac12})$, $(x_{i+\frac12}, y_{j+\frac12})$, $(x_{i+\frac12}, y_{j-\frac12})$.

The discrete unknowns are the $u^n_{i,j}$ where $(i,j) \in \mathbb{Z}^2$, $n \in \mathbb{N}$. The approximate solution $u_{\mathcal{T},k}$ is defined by:

$$u_{\mathcal{T},k}(x,t) = u^n_{i,j} \quad \forall x \in p_{i,j}, \quad \forall t \in [t^n, t^{n+1}[. \tag{62}$$

The BV-norm of $u_{\mathcal{T},k}$ can be written as follows:

$$|u_{\mathcal{T},k}|_{BV(\mathbb{R}^2\times[0,T[)} = \sum_{n=0}^{N_T}\sum_{i,j\in\mathbb{Z}} h^2|u^{n+1}_{i,j} - u^n_{i,j}| + \sum_{n=0}^{N_T} k \sum_{i,j\in\mathbb{Z}} \left(h|u^n_{i+1,j} - u^n_{i,j}| + h|u^n_{i,j+1} - u^n_{i,j}|\right). \tag{63}$$

We split each component of F into two parts: the first part must be nondecreasing w.r.t. s and the second part nonincreasing w.r.t. s. We set:

$$\begin{array}{rcl} F^x(z,t,s) &=& a(z,t,s) + b(z,t,s), \\ F^y(z,t,s) &=& c(z,t,s) + d(z,t,s). \end{array}$$

For instance, we can take:

$$\begin{array}{rclrcl} a(z,t,s) &=& \frac12(F^x(z,t,s) + Ms), & b(z,t,s) &=& \frac12(F^x(z,t,s) - Ms), \\ c(z,t,s) &=& \frac12(F^y(z,t,s) + Ms), & d(z,t,s) &=& \frac12(F^y(z,t,s) - Ms), \end{array}$$

where the parameter M is well-chosen. If $M = V_{[A,B]}$, we note that a, b, c and d are Lipschitz continuous w.r.t. s with M as Lipschitz constant. We consider the following scheme:

$$\begin{array}{rcll} u^{n+1}_{i,j} = u^n_{i,j} &-& \frac{k}{h} & \left(a^n_{i+\frac12,j}(u^n_{i,j}) - a^n_{i-\frac12,j}(u^n_{i-1,j}) + b^n_{i+\frac12,j}(u^n_{i+1,j}) - b^n_{i-\frac12,j}(u^n_{i,j})\right) \\ &-& \frac{k}{h} & \left(c^n_{i,j+\frac12}(u^n_{i,j}) - c^n_{i,j-\frac12}(u^n_{i,j-1}) + d^n_{i,j+\frac12}(u^n_{i,j+1}) - d^n_{i,j-\frac12}(u^n_{i,j})\right) \end{array} \tag{64}$$

with the initial condition:

$$u^0_{i,j} = \frac{1}{h^2}\int_{p_{ij}} u_0(z)dx, \tag{65}$$

and the following definition of the fluxes:

$$\left\{\begin{array}{lcl} a^n_{i+\frac{1}{2},j}(s) & = & \dfrac{1}{h}\displaystyle\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} a(x_{i+\frac{1}{2}},y,t^n,s)dy, \\ b^n_{i+\frac{1}{2},j}(s) & = & \dfrac{1}{h}\displaystyle\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} b(x_{i+\frac{1}{2}},y,t^n,s)dy, \\ c^n_{i,j+\frac{1}{2}}(s) & = & \dfrac{1}{h}\displaystyle\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} c(x,y_{j+\frac{1}{2}},t^n,s)dx, \\ d^n_{i,j+\frac{1}{2}}(s) & = & \dfrac{1}{h}\displaystyle\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} d(x,y_{j+\frac{1}{2}},t^n,s)dx. \end{array}\right. \tag{66}$$

All these fluxes are Lipschitz continuous w.r.t. s with M as Lipschitz constant. The fluxes $a^n_{i+\frac{1}{2},j}$, $c^n_{i,j+\frac{1}{2}}$ (*resp.* $b^n_{i+\frac{1}{2},j}$, $d^n_{i,j+\frac{1}{2}}$) are nondecreasing (*resp.* nonincreasing) w.r.t. s. The hypothesis $\mathrm{div}_x F = 0$ implies:

$$(a^n_{i+\frac{1}{2},j}+b^n_{i+\frac{1}{2},j}-a^n_{i-\frac{1}{2},j}-b^n_{i-\frac{1}{2},j}+c^n_{i,j+\frac{1}{2}}+d^n_{i,j+\frac{1}{2}}-c^n_{i,j-\frac{1}{2}}-d^n_{i,j-\frac{1}{2}})(s)=0 \forall s\in[A,B],\ \forall (i,j)\in\mathbb{Z}^2,\ \forall n\in\mathbb{N}.$$

Thus, the scheme (64) rewrites:

$$\begin{array}{rcl} u^{n+1}_{i,j}=u^n_{i,j} & - & \dfrac{k}{h}\left(a^n_{i-\frac{1}{2},j}(u^n_{i,j})-a^n_{i-\frac{1}{2},j}(u^n_{i-1,j})+b^n_{i+\frac{1}{2},j}(u^n_{i+1,j})-b^n_{i+\frac{1}{2},j}(u^n_{i,j})\right) \\ & - & \dfrac{k}{h}\left(c^n_{i,j-\frac{1}{2}}(u^n_{i,j})-c^n_{i,j-\frac{1}{2}}(u^n_{i,j-1})+d^n_{i,j+\frac{1}{2}}(u^n_{i,j+1})-d^n_{i,j+\frac{1}{2}}(u^n_{i,j})\right). \end{array} \tag{67}$$

We study it under the following C.F.L. condition:

$$k \le \frac{h}{4M}\cdot \tag{68}$$

This condition ensures the L^∞-stability of the scheme. The BV-stability is given by the following lemma:

Lemma 8. *Assume (2), $u_0\in BV(\mathbb{R}^2)$, and $\frac{\partial F}{\partial s}$ globally Lipschitz continuous. Then there exists $C_{F,u_0}>0$ depending only on F and u_0 such that $u_{\mathcal{T},k}$ defined by (62),(66), (67), (65) and (68) verifies:*

$$\sum_{i,j\in\mathbb{Z}}\left(h|u^n_{i+1,j}-u^n_{i,j}|+h|u^n_{i,j+1}-u^n_{i,j}|\right) \le (1+C_{F,u_0}k)^n|u_0|_{BV(\mathbb{R}^2)},\ \forall n\in\mathbb{N}, \tag{69}$$

$$\sum_{i,j\in\mathbb{Z}} h^2|u^{n+1}_{i,j}-u^n_{i,j}| \le k(1+C_{F,u_0}k)^n|u_0|_{BV(\mathbb{R}^2)}\ \forall n\in\mathbb{N}, \tag{70}$$

and

$$|u_{\mathcal{T},k}|_{BV(\mathbb{R}^2\times[0,T[)} \le 2Te^{C_{F,u_0}T}|u_0|_{BV(\mathbb{R}^2)}. \tag{71}$$

Proof. First, we note that (71) is a straightforward consequence of (69), (70) and (63). Then, we prove (69) by induction. Lemma 7 implies: $BV(u,0)\le |u_0|_{BV(\mathbb{R}^2)}$. We assume that the property (69) holds for n. We set $BV(u,n)=BV(u,n)_x+BV(u,n)_y$ with:

$$BV(u,n)_x=\sum_{i,j\in\mathbb{Z}} h|u^n_{i+1,j}-u^n_{i,j}| \text{ and } BV(u,n)_y=\sum_{i,j\in\mathbb{Z}} h|u^n_{i,j+1}-u^n_{i,j}|.$$

We show here how to bound $BV(u,n+1)_x$.

The definition of a and b and the condition (68) ensure that, for all $(i,j) \in \mathbb{Z}^2$, there exist $\alpha^n_{i-\frac{1}{2},j} \in [0,\frac{1}{4}]$ and $\beta^n_{i+\frac{1}{2},j} \in [0,\frac{1}{4}]$ such that the scheme (67) may rewrite:

$$u^{n+1}_{i,j} = u^n_{i,j} - \alpha^n_{i+\frac{1}{2},j}(u^n_{i,j} - u^n_{i-1,j}) - \beta^n_{i+\frac{1}{2},j}(u^n_{i,j} - u^n_{i+1,j}) \\ - \frac{k}{h}\left(c^n_{i,j-\frac{1}{2}}(u^n_{i,j}) - c^n_{i,j-\frac{1}{2}}(u^n_{i,j-1}) + d^n_{i,j+\frac{1}{2}}(u^n_{i,j+1}) - d^n_{i,j+\frac{1}{2}}(u^n_{i,j})\right).$$

Therefore, we have:

$$u^{n+1}_{i+1,j} - u^{n+1}_{i,j} = (u^n_{i+1,j} - u^n_{i,j})(1 - \alpha^n_{i+\frac{1}{2},j} - \beta^n_{i+\frac{1}{2},j}) + \alpha^n_{i-\frac{1}{2},j}(u^n_{i,j} - u^n_{i-1,j}) \\ + \beta^n_{i+\frac{3}{2},j}(u^n_{i+2,j} - u^n_{i+1,j}) + N_c + N_d, \quad (72)$$

with:

$$\begin{aligned} N_c &= \frac{k}{h}\left((c^n_{i,j-\frac{1}{2}}(u^n_{i,j}) - c^n_{i,j-\frac{1}{2}}(u^n_{i,j-1})) - (c^n_{i+1,j-\frac{1}{2}}(u^n_{i+1,j}) - c^n_{i+1,j-\frac{1}{2}}(u^n_{i+1,j-1}))\right), \\ N_d &= \frac{k}{h}\left((d^n_{i,j+\frac{1}{2}}(u^n_{i,j+1}) - d^n_{i,j+\frac{1}{2}}(u^n_{i,j})) - (d^n_{i+1,j+\frac{1}{2}}(u^n_{i+1,j+1}) - d^n_{i+1,j+\frac{1}{2}}(u^n_{i+1,j}))\right). \end{aligned}$$

We introduce some "decaled" fluxes defined by:

$$\begin{aligned} c^n_{i+\frac{1}{2},j-\frac{1}{2}}(s) &= c(x_{i+\frac{1}{2}}, y_{j-\frac{1}{2}}, t^n, s) \quad \forall s \in [A,B], \\ d^n_{i+\frac{1}{2},j+\frac{1}{2}}(s) &= d(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t^n, s) \quad \forall s \in [A,B], \end{aligned}$$

and we set:

$$\begin{aligned} N^*_c &= \frac{k}{h}\left((c^n_{i+\frac{1}{2},j-\frac{1}{2}}(u^n_{i,j}) - c^n_{i+\frac{1}{2},j-\frac{1}{2}}(u^n_{i+1,j})) - (c^n_{i+\frac{1}{2},j-\frac{3}{2}}(u^n_{i,j-1}) - c^n_{i+\frac{1}{2},j-\frac{3}{2}}(u^n_{i+1,j-1}))\right), \\ N^*_d &= \frac{k}{h}\left((d^n_{i+\frac{1}{2},j+\frac{1}{2}}(u^n_{i,j+1}) - d^n_{i+\frac{1}{2},j+\frac{1}{2}}(u^n_{i+1,j+1})) - (d^n_{i+\frac{1}{2},j-\frac{1}{2}}(u^n_{i,j}) - d^n_{i+\frac{1}{2},j-\frac{1}{2}}(u^n_{i+1,j}))\right). \end{aligned}$$

There also exist $\gamma^n_{i+\frac{1}{2},j-\frac{1}{2}} \in [0,\frac{1}{4}]$ and $\delta^n_{i+\frac{1}{2},j-\frac{1}{2}} \in [0,\frac{1}{4}]$ such that:

$$\begin{aligned} N^*_c &= \gamma^n_{i+\frac{1}{2},j-\frac{1}{2}}(u^n_{i,j} - u^n_{i+1,j}) - \gamma^n_{i+\frac{1}{2},j-\frac{3}{2}}(u^n_{i,j-1} - u^n_{i+1,j-1}), \\ N^*_d &= \delta^n_{i+\frac{1}{2},j+\frac{1}{2}}(u^n_{i+1,j+1} - u^n_{i,j+1}) - \delta^n_{i+\frac{1}{2},j-\frac{1}{2}}(u^n_{i+1,j} - u^n_{i,j}). \end{aligned} \quad (73)$$

Now, we have to bound $|N_c - N^*_c|$ and $|N_d - N^*_d|$. Denoting by $L^{x,s}_c$ and $L^{x,s}_d$ (*resp.* $L^{y,s}_c$ and $L^{y,s}_d$) the Lipschitz constants of $\partial c/\partial s$ and $\partial d/\partial s$ w.r.t. x (*resp.* y), we get:

$$\begin{aligned} |N_c - N^*_c| &\leq kL^{x,s}_c(|u^n_{i,j} - u^n_{i,j-1}| + |u^n_{i+1,j} - u^n_{i+1,j-1}|) + kL^{y,s}_c|u^n_{i,j-1} - u^n_{i+1,j-1}|, \\ |N_d - N^*_d| &\leq kL^{x,s}_d(|u^n_{i,j+1} - u^n_{i,j}| + |u^n_{i+1,j+1} - u^n_{i+1,j}|) + kL^{y,s}_d|u^n_{i,j} - u^n_{i+1,j}|. \end{aligned} \quad (74)$$

But, thanks to (72) and (73), we have:

$$u^{n+1}_{i+1,j} - u^{n+1}_{i,j} = (u^n_{i+1,j} - u^n_{i,j})(1 - \alpha^n_{i+\frac{1}{2},j} - \beta^n_{i+\frac{1}{2},j} - \gamma^n_{i+\frac{1}{2},j-\frac{1}{2}} - \delta^n_{i+\frac{1}{2},j-\frac{1}{2}}) \\ + \alpha^n_{i-\frac{1}{2},j}(u^n_{i,j} - u^n_{i-1,j}) + \beta^n_{i+\frac{3}{2},j}(u^n_{i+2,j} - u^n_{i+1,j}) + \gamma^n_{i+\frac{1}{2},j-\frac{3}{2}}(u^n_{i+1,j-1} - u^n_{i,j-1}) \\ + \delta^n_{i+\frac{1}{2},j+\frac{1}{2}}(u^n_{i+1,j+1} - u^n_{i,j+1}) + (N_c - N^*_c) + (N_d - N^*_d).$$

Therefore, taking the absolute value of this last expression, summing over i and j and setting $\mathcal{L} = \max_{w\in\{a,b,c,d\}} (L_w^{x,s}, L_w^{y,s})$, we get:

$$BV(u,n+1)_x \leq BV(u,n)_x + k\mathcal{L}(2BV(u,n)_x + 4BV(u,n)_y).$$

We obtain a similar bound for $BV(u,n+1)_y$ and finally,

$$BV(u,n+1) \leq (1+6k\mathcal{L})^n |u_0|_{BV(\mathbb{R}^2)}.$$

It proves (69). In order to obtain (70), we use the scheme (67) as we did for (15).

Lemma 9. *Assume (2) and $u_0 \in BV_{loc}(\mathbb{R}^2)$. Then, for all compact set $\Omega \in \mathbb{R}^2$, for all $T > 0$, there exist $C_{F,u_0,\Omega,T,\frac{k}{h}}$ and $C'_{F,u_0,\Omega,T,\frac{k}{h}}$ depending only u_0, F, Ω, T and k/h such that:*

$$|u_{\mathcal{T},k}|_{BV(\Omega\times[0,T[)} \leq Te^{C_{F,u_0,\Omega,T,\frac{k}{h}}T} C'_{F,u_0,\Omega,T,\frac{k}{h}}. \tag{75}$$

Lemma 9 is a consequence of Lemma 8. If we want to compute $|u_{\mathcal{T},k}|_{BV(\Omega\times[0,T[)}$ we just need the values of $u_{i,j}^n$ on a compact set K and therefore, we just need the knowledge of u_0 on a compact set K_0, which depends on K and k/h. Then, we consider the truncature of u_0 to K_0 and the associated approximate solution and we apply Lemma 8.

Proof of Theorem 3. Lemma 9 generalized to $\mathbb{R}^N$ shows that $u_{\mathcal{T},k}$ belongs to BV$(\Omega\times[0,T[)$ for all compact set $\Omega\subset\mathbb{R}^N$, for all $T>0$. Therefore, $\forall\psi \in C_c^\infty(\Omega\times[0,T[,\mathbb{R})$ such that $\|\psi\|_\infty \leq 1$,

$$\int_{\mathbb{R}^N\times[0,T[} u_{\mathcal{T},k}(x,t)\frac{\partial\psi}{\partial x_j}(x,t) \leq Te^{C_{F,u_0,\Omega,T,\frac{k}{h}}T} C'_{F,u_0,\Omega,T,\frac{k}{h}} \quad \forall j\in\{1,..,N\}, \tag{76}$$

$$\int_{\mathbb{R}^N\times[0,T[} u_{\mathcal{T},k}(x,t)\frac{\partial\psi}{\partial t}(x,t) \leq Te^{C_{F,u_0,\Omega,T,\frac{k}{h}}T} C'_{F,u_0,\Omega,T,\frac{k}{h}}. \tag{77}$$

But $u_{\mathcal{T},k}$ converges towards u in weak $*$-$L^\infty(\mathbb{R}^N\times[0,T[)$. Therefore, if k/h stays constant, we can pass to the limit in (76) and (77) and then we get the same inequalities with u instead of $u_{\mathcal{T},k}$. It proves that u belongs to BV$_{loc}(\mathbb{R}^N\times[0,T[)$, for all $T>0$.

5. Error estimate

In this last section, we prove an error estimate between the approximate solution given by the scheme and the entropy solution to (1):

Theorem 4. *Assume (2), (5), (6) and (10). Let u be the entropy solution of (1) defined by (3) and $u_{\mathcal{T},k}$ the approximate solution given by (9),(11). If $u_0 \in BV_{loc}(\mathbb{R}^N)$, we have the following error estimate: for any compact set $E\subset\mathbb{R}^N\times\mathbb{R}_+$, there exists K depending only on E, F, u_0, M, α and ξ such that*

$$\int_E |u_{\mathcal{T},k}(x,t) - u(x,t)|dxdt \leq Kh^{\frac{1}{4}}. \tag{78}$$

Theorem 4 is a straightforward consequence of the following lemma:

Lemma 10. *Assume (2) and* $u_0 \in L^\infty \cap BV_{loc}(\mathbb{R}^N)$*. Let* $\tilde{u} \in L^\infty(\mathbb{R}^N \times \mathbb{R}_+)$ *such that* $A \leq \tilde{u} \leq B$ *a.e. Assume that there exist* $\mu \in \mathcal{M}(\mathbb{R}^N \times \mathbb{R}_+)$ *and* $\mu_0 \in \mathcal{M}(\mathbb{R}^N)$ *such that:*

$$\int_{\mathbb{R}^N\times\mathbb{R}_+} [|\tilde{u}(x,t)-\kappa|\varphi_t(x,t) + (F(x,t,\tilde{u}(x,t)\top\kappa) - F(x,t,\tilde{u}(x,t)\bot\kappa)).\nabla\varphi(x,t)]dxdt$$
$$+\int_{\mathbb{R}^N} |u_0(x)-\kappa|\varphi(x,0)dx \geq -\int_{\mathbb{R}^N\times\mathbb{R}_+} (|\varphi_t(x,t)| + |\nabla\varphi(x,t)|)d\mu(x,t) - \int_{\mathbb{R}^N} \varphi(x,0)d\mu_0(x),$$
$$\forall \kappa\in\mathbb{R},\ \forall\varphi\in C_c^\infty(\mathbb{R}^N\times\mathbb{R}_+,\mathbb{R}_+). \quad (79)$$

Let u *be the unique entropy solution to the problem (1):*

$$\int_{\mathbb{R}^N\times\mathbb{R}_+} [|u(y,s)-\kappa|\varphi_s(y,s) + (F(y,s,u(y,s)\top\kappa) - F(y,s,u(y,s)\bot\kappa)).\nabla\varphi(y,s)]dyds$$
$$+\int_{\mathbb{R}^N} |u_0(y)-\kappa|\varphi(y,0)dy \geq 0, \quad \forall\kappa\in\mathbb{R},\ \forall\varphi\in C_c^\infty(\mathbb{R}^N\times\mathbb{R}_+,\mathbb{R}_+). \quad (80)$$

Then, for all compact set $E\subset\mathbb{R}^N\times\mathbb{R}_+$*, there exists* C_{E,F,u_0}*,* R *and* T *which only depend on* E*,* F*,* u_0 *such that:*

$$\int_E |\tilde{u}(x,t)-u(x,t)|dxdt \leq C_{E,F,u_0}(\mu_0(B(0,R)) + \mu(B(0,R) + (\mu(B(0,R)\times[0,T]))^{\frac{1}{2}}). \quad (81)$$

Indeed, Theorem 1 proves the existence of $\mu = \mu_{\mathcal{T},k}$ and $\mu_0 = \mu_{\mathcal{T}}$ such that $u_{\mathcal{T},k}$ verifies (79). Moreover, as $\mu_{\mathcal{T},k}$ and $\mu_{\mathcal{T}}$ satisfy respectively (28) and (30), we get (78).

It remains to prove Lemma 10. The proof is close to the proof of Theorem 5. In a first step, we show that, for all $\psi\in C_c^\infty(\mathbb{R}^N\times\mathbb{R}_+,\mathbb{R}_+)$, there exists C_{ψ,F,u_0} depending only on ψ, F and u_0 such that:

$$\int_{\mathbb{R}^N\times\mathbb{R}_+} [|\tilde{u}(x,t)-u(x,t)|\psi_t(x,t) + (F(x,t,\tilde{u}(x,t)\top u(x,t)) - F(x,t,\tilde{u}(x,t)\bot u(x,t))).\nabla\psi(x,t)]dxdt \geq$$
$$-C_{\psi,F,u_0}\left\{\mu_0(\{\psi(.,0)\neq 0\}) + (\mu(\{\psi\neq 0\}))^{\frac{1}{2}} + \mu(\{\psi\neq 0\})\right\}. \quad (82)$$

Then, let E be a compact set of $\mathbb{R}^N\times\mathbb{R}_+$. Let $\omega\in\mathbb{R}$, $R>0$ and $T\in]0,\dfrac{R}{\omega}[$ such that

$$E\subset\bigcup_{0\leq t\leq T}(B(0,R-\omega t)\times\{t\}).$$

Applying (82) with the function ψ defined by (59), we get:

$$\int_{\mathbb{R}^N\times[0,T]}\Bigg[|\tilde{u}(x,t)-u(x,t)|(\frac{T-t}{T}\omega\rho'(|x|+\omega t) - \frac{1}{T}\rho(|x|+\omega t))$$
$$+ (F(x,t,\tilde{u}(x,t)\top u(x,t)) - F(x,t,\tilde{u}(x,t))\bot u(x,t))).\frac{x}{|x|}\frac{T-t}{T}\rho'(|x|+\omega t)\Bigg]dxdt \geq$$
$$-C_{\psi,F,u_0}\Big(\mu_0(B(0,R+1)) + \mu(B(0,R+1)\times[0,T]) + (\mu(B(0,R+1)\times[0,T]))^{\frac{1}{2}}\Big).$$

We take $\omega = V_{\mathcal{K}}$, with $\mathcal{K} = [-(\|\tilde{u}\|_\infty \top \|u\|_\infty), (\|\tilde{u}\|_\infty \top \|u\|_\infty)]$ and, finally,

$$\int_E |\tilde{u}(x,t) - u(x,t)| dxdt \le C_{\psi,F,u_0} T\Big(\mu_0(B(0,R+1)) + \mu(B(0,R+1)\times[0,T]) + (\mu(B(0,R+1)\times[0,T]))^{\frac{1}{2}}\Big).$$

It remains to prove (82).

Let $\psi \in C_c^\infty(\mathbb{R}^N \times \mathbb{R}_+, \mathbb{R}_+)$, we set: $\varphi(x,t,y,s) = \psi(x,t)\rho_{N,r}(x-y)\bar{\rho}_{1,r}(t-s)$. Applying (79) with $\varphi(.,.,y,s)$ and $\kappa = u(y,s)$, and integrating with respect to y and s, we get:

$$D_1 + D_2 + D_3 + D_4 + D_5 \ge -E. \tag{83}$$

Each term D_i can be obtained by replacing in A_i, encountered in the proof of Lemma 5, $v(x,t,\alpha)$ by $\tilde{u}(x,t)$ and $w(y,s,\beta)$ par $u(y,s)$. The term E is due to the measure terms in (79):

$$\begin{aligned} E = \int_{(\mathbb{R}^N\times\mathbb{R}_+)^2} (|\rho_{N,r}(x-y)(\psi_t(x,t)\bar{\rho}_{1,r}(t-s) + \psi(x,t)\bar{\rho}'_{1,r}(t-s)| \\ + |\bar{\rho}_{1,r}(t-s)(\nabla\psi(x,t)\rho_{N,r}(x-y) + \psi(x,t)\nabla\rho_{N,r}(x-y)|)d\mu(x,t)dyds \\ + \int_{\mathbb{R}^N\times\mathbb{R}_+\times\mathbb{R}^N} |\psi(x,0)\rho_{N,r}(x-y)\bar{\rho}_{1,r}(-s)|d\mu_0(x)dyds. \end{aligned} \tag{84}$$

Let us introduce $K = \{(x,t) \in \mathbb{R}^N \times \mathbb{R}_+;\ \psi(x,t) \ne 0\}$ and $K_0 = \{x \in \mathbb{R}^N;\ \psi(x,0) \ne 0\}$. The properties (48) imply:

$$E \le C_{F,\psi}((r+1)\mu(K) + \mu_0(K_0)). \tag{85}$$

For the D_i, we do the same as in the proof of Lemma 5. It yields:

$$|D_1 - \int_{\mathbb{R}^N\times\mathbb{R}_+} |u(x,t) - \tilde{u}(x,t)|\psi_t(x,t)dxdt| \le \|\psi_t\|_\infty \varepsilon(r,K,u), \tag{86}$$

$$|D_3 - \int_{\mathbb{R}^N\times\mathbb{R}_+} (F(x,t,\tilde{u}(x,t)\top u(x,t)) - F(x,t,\tilde{u}(x,t)\bot u(x,t))).\nabla\psi(x,t)]dxdt| \le C_{F,\psi,A,B}\|\nabla\psi\|_\infty \varepsilon(r,K,u), \tag{87}$$

$$-D_2 - D_4 \ge -C_{F,\psi,A,B}\|\psi\|_\infty \varepsilon(r,K,u), \tag{88}$$

$$-D_5 \ge -C_{F,\psi,A,B}(\frac{1}{r} + \varepsilon(r,K_0,u_0)). \tag{89}$$

Then, the inequalities (83), (85), (86), (87), (88) and (89) lead to:

$$\begin{aligned} \int_{\mathbb{R}^N\times\mathbb{R}_+} [|\tilde{u}(x,t) - u(x,t)|\psi_t(x,t) + (F(x,t,\tilde{u}(x,t)\top u(x,t)) - F(x,t,\tilde{u}(x,t)\bot u(x,t))).\nabla\psi(x,t)]dxdt \ge \\ - C_{F,\psi,A,B}((r+1)\mu(K) + \mu_0(K_0) + \frac{1}{r} + \varepsilon(r,K,u) + \varepsilon(r,K_0,u_0)). \end{aligned}$$

If $u_0 \in \mathrm{BV}_{loc}(\mathbb{R}^N)$, Lemma 7 and Theorem 3 ensure that there exist C_{K_0,u_0} and $C_{K,u_0,F}$ such that:

$$\varepsilon(r, K_0, u_0) \leq \frac{C_{K_0,u_0}}{r}, \quad \varepsilon(r, K, u) \leq \frac{C_{K,u_0,F}}{r}\cdot$$

We conclude the proof of (82) by taking $r = \dfrac{1}{\sqrt{\mu(K)}}$ (or $r \to \infty$ if $\mu(K) = 0$). This puts an end to the proof of Theorem 4.

Remark 1. *The estimate (78) is probably not optimal. Indeed, when the mesh is rectangular (in the case $N = 2$), Lemma 8 gives some strong BV estimates on the approximate solution. They lead to an "$h^{\frac{1}{2}}$" error estimate between the approximate and the entropy solutions.*

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