Mathematical Modelling and Numerical Analysis M2AN, Vol. 33, N° 2, 1999, p. 439–441
Modélisation Mathématique et Analyse Numérique

Erratum

A KINETIC EQUATION FOR GRANULAR MEDIA

Dario Benedetto[1], Emanuele Caglioti[1] and Mario Pulvirenti[1]

Abstract. In this short note we correct a conceptual error in the heuristic derivation of a kinetic equation used for the description of a one-dimensional granular medium in the so called quasi-elastic limit, presented by the same authors in reference [1]. The equation we derived is however correct so that, the rigorous analysis on this equation, which constituted the main purpose of that paper, remains unchanged.

AMS Subject Classification. 82C21, 82C40.

Received: January 20, 1999.

1. The kinetic equation

We refer to [1] and use the same notation.

Let us start from the basic master (Liouville) equation describing the evolution of a probability density μ^N associated to the dynamics of N inelastic point particles in the line. In equation (1.4) of [1] appears the factor $\delta(x_i - x_j)$ because of the strictly local interaction. In order to avoid mistakes in the correct interpretation of this δ, we mollify the interaction replacing equation (1.3) by its regularized version:

$$\dot{x}_i = v_i, \qquad \dot{v}_i = \alpha \sum_{j=1}^{N} \delta_\eta(x_i - x_j)(v_j - v_i)|v_j - v_i|, \tag{1}$$

where δ_η is an approximation of δ as the parameter $\eta \to 0$. A simple calculation on the two particle scattering problem shows that $\alpha = \alpha(\varepsilon) = -\frac{1}{2}\log(1-\varepsilon)$ so that $\alpha \approx \varepsilon$ only in the limit $\varepsilon \to 0$. As a consequence of Eq. (1) we have the following master equation for the regularized problem:

$$(\partial_t + \sum_{i=1}^{N} v_i \partial_{x_i})\mu^N(x_1, v_1, \dots, x_N, v_N) = -\alpha \sum_{i \neq j} \delta_\eta(x_i - x_j)\partial_{v_i}[\phi(v_j - v_i)\mu(x_1, v_1, \dots, x_N, v_N)] \tag{2}$$

Keywords and phrases. Granular media, inelastic collisions, kinetic equations.

[1] Università degli Studi di Roma “La Sapienza”, Dipartimento di Matematica, Instituo “Guido Castelnuovo”, Piazzale Aldo Moro 2, 00185 Roma, Italy. e-mail: `pulvirenti@axcasp.caspur.it`

and the following hierarchy for the the j-particle distribution functions:

$$(\partial_t + \sum_{i=1}^{j} v_i \partial_{x_i}) f_j^N(x_1, v_1, \dots, x_j, v_j) = -\alpha \sum_{i,k=1; i \neq k}^{j} \delta_\eta(x_i - x_k)\phi(v_k - v_i)\partial_{v_i} f_j^N(x_1, v_1, \dots, x_j, v_j)$$

$$- \alpha(N-j) \sum_{i=1}^{j} \partial_{v_i} \int dv_{j+1} \int dx_{j+1} \delta_\eta(x_i - x_{j+1})\phi(v_{j+1} - v_i) f_{j+1}^N(x_1, v_1, \dots, x_{j+1}, v_{j+1}). \quad (3)$$

Since the regularized dynamics converges to the true dynamics pathwise, we should perform the limit $\eta \to 0$ for fixed N and ε. Here arises our error in [1] which is twofold. From one side $\alpha \neq \varepsilon$. From the other the limit $\eta \to 0$ is not innocent and it does not give equation (1.6) as asserted in [1]: we did confusion between the notion of δ as a distribution in time and as a limit of regularized versions δ_η.

We note also that, interchanging the limit $\eta \to 0$ with the quasi-elastic limit $N \to \infty$, $\varepsilon \to 0$, $N\varepsilon \to \lambda$, we would obtain equation (1.7) of [1] which is actually correct.

In order to derive the evolution equation for the j–particle distribution, instead of studying the limit $\eta \to 0$ (which is possible but involved) it is more natural to look directly at the true dynamics, considering the interaction as a boundary term, in the same spirit of the derivation of the BBKGY hierarchy for hard spheres (see *e.g.* Ref. [2]). We shortly outline the argument.

Let $\mu_0^N = \mu_0^N(x_1, v_1, \dots, x_N, v_N)$ be a probability density for the system at time 0. We assume μ_0^N continuous and symmetric in the exchange of particles. The time evolved probability density is defined as:

$$\mu^N(x_1, v_1, \dots, x_N, v_N, t) = \mu_0^N(T^{-t}(x_1, v_1, \dots, x_N, v_N)) J_\varepsilon^{-n}, \quad (4)$$

where T^t is the flow in the phase space generated by the dynamics, n is the number of collisions delivered by the phase point $(x_1, v_1, \dots, x_N, v_N)$ during the backward dynamics up to the time t and $J_\varepsilon = (1-2\varepsilon)^2$ denotes the Jacobian of the transformation induced in the phase space by a single collision.

Note that T^t is ambiguously defined on the manifold $\{x_i = x_j |$ for some $i \neq j\}$. Indeed in this case we do not know whether the velocities v_i and v_j have to be understood as incoming or outgoing. However such a manifold has zero measure so that this ambiguity is irrelevant in the definition of $\mu^N(\cdot, t)$.

Note also that if t is a collision instant involving the i and j particle, for the phase point $(\bar{X}, \bar{V}) = (\bar{x}_1, \bar{v}_1, \dots, \bar{x}_N, \bar{v}_N)$ in the forward dynamics, then

$$\lim_{\tau \to t^+} \mu^N(T^\tau(\bar{X}, \bar{V}), \tau) = J_\varepsilon^{-1} \lim_{\tau \to t^-} \mu^N(T^\tau(\bar{X}, \bar{V}), \tau),$$

provided that μ_0^N is continuous. This expression can be rewritten in terms of

$$(x_1, v_1, \dots, x_i, v_i, \dots, x_i, v_j, \dots, x_N, v_N) = \lim_{\tau \to t^-} T^\tau(\bar{X}, \bar{V})$$

and

$$(x_1, v_1, \dots, x_i, v_i', \dots, x_i, v_j', \dots, x_N, v_N) = \lim_{\tau \to t^+} T^\tau(\bar{X}, \bar{V}),$$

(where v_i', v_j' are the outgoing velocities and v_i,v_j are the incoming ones) as

$$\lim_{\tau \to t^+} \mu^N(x_1, v_1, \dots, x_i, v_i', \dots, x_i, v_j', \dots, x_N, v_N, \tau) = J_\varepsilon^{-1} \lim_{\tau \to t^-} \mu^N(x_1, v_1, \dots, x_i, v_i, \dots, x_i, v_j, \dots, x_N, v_N, \tau).$$

Now we want to derive an equation for the j–particle distribution functions.

We first note that if $(x_1, v_1, \dots, x_N, v_N)$ is not in the collision manifold, then:

$$\partial_t \mu^N(x_1, v_1, \dots, x_N, v_N, t) + \sum_{k=1}^{N} v_k \partial_{x_k} \mu^N(x_1, v_1, \dots, x_N, v_N, t) = 0. \tag{5}$$

Our next step is to integrate over the last $N - j$ variables and this generates boundary terms which give rise to the collision operator. To compute these terms explicitly we consider the simple case in which $N = 2$ and $j = 1$. An easy calculation shows that:

$$\partial_t f_1(x_1, v_1, t) + v_1 \partial_{x_1} f_1(x_1, v_1, t) = \int dv_2 (v_2 - v_1) \left\{ \mu^2(x_1, v_1, x_1^-, v_2, t) - \mu^2(x_1, v_1, x_1^+, v_2, t) \right\}, \tag{6}$$

where $\mu^2(x_1, v_1, x_1^{\pm}, v_2, t)$ denotes the right and left limit for $x \to x_1$ respectively. We note that for the configuration point $(x_1 - \delta, x_1)$ for a positive small δ, the velocities v_2, v_1 are incoming or outgoing if $v_1 > v_2$ or $v_1 < v_2$ respectively. Taking into account this fact we readily arrive to the following equation:

$$\partial_t f_1(x_1, v_1, t) + v_1 \partial_{x_1} f_1(x_1, v_1, t) = \int dv_2 |v_2 - v_1| \left\{ J_\varepsilon^{-1} \mu^2(x_1, v_1^*, x_1, v_2^*, t) - \mu^2(x_1, v_1, x_1, v_2, t) \right\}, \tag{7}$$

where v_1^* and v_2^* denote the precollisional pair

$$v_1^* = v_1 + \frac{\varepsilon}{1 - 2\varepsilon}(v_1 - v_2) \qquad v_2^* = v_2 - \frac{\varepsilon}{1 - 2\varepsilon}(v_1 - v_2).$$

Note that, as in the case of the Boltzmann equation, we represent μ^2 in terms of the precollisional variables, so that the time t appearing in the right hand side of equation (7) is the left limit.

For the general case we easily deduce the following hierarchy of equations:

$$(\partial_t + \mathcal{L}_j) f_j^N(x_1, v_1, \dots, x_j, v_j) = (N - j) \sum_{k=1}^{j} \int dv_{j+1} |v_k - v_{j+1}| \cdot \tag{8}$$

$$\left\{ J_\varepsilon^{-1} f_{j+1}^N(x_1, v_1, \dots x_k, v_k^*, \dots, x_k, v_{j+1}^*) - f_{j+1}^N(x_1, v_1, \dots x_k, v_k, \dots, x_k, v_{j+1}) \right\},$$

for $i = 1, n$. Here $\mathcal{L}_j$ denotes the generator of the j-particle dynamics. equations (8) are the analogue of the BBGKY hierarchy for Hamiltonian systems.

The integral in the right end side of (8) is $O(\varepsilon)$, so that we are lead to consider the scaling limit $\varepsilon \to 0$, $N \to \infty$ in such a way that $N\varepsilon \to \lambda$, where λ is a positive parameter. Using the Taylor formula and neglecting terms of $o(\varepsilon)$, integrating by parts and performing the limit we arrive to the hierarchy of equations (1.7) of reference [1].

Finally, propagation of chaos implies, as usual, the kinetic equation (1.8) which is the object of investigation in [1].

References

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