Mathematical Modelling and Numerical Analysis M2AN, Vol. 33, N° 4, 1999, p. 807–836
Modélisation Mathématique et Analyse Numérique

ON THE COMBINED EFFECT OF BOUNDARY APPROXIMATION AND NUMERICAL INTEGRATION ON MIXED FINITE ELEMENT SOLUTION OF 4TH ORDER ELLIPTIC PROBLEMS WITH VARIABLE COEFFICIENTS

Pulin K. Bhattacharyya[1] and Neela Nataraj[2]

Abstract. Error estimates for the mixed finite element solution of 4th order elliptic problems with variable coefficients, which, in the particular case of aniso-/ortho-/isotropic plate bending problems, gives a direct, simultaneous approximation to bending moment tensor field $\Psi = (\psi_{ij})_{1\leq i,j\leq 2}$ and displacement field 'u', have been developed considering the combined effect of boundary approximation and numerical integration.

AMS Subject Classification. 35J40, 65N30.

Received: January 6, 1998. Revised: October 23, 1998.

1. Introduction

In [5] a new mixed finite element method for 4th order elliptic partial differential equations with variable/constant coefficients defined in convex polygonal domain, from which the mixed method scheme of Hellan-Hermann-Miyoshi [15, 22, 23, 28] for the biharmonic problem in convex polygonal domain can be retrieved as a particular case with a proper choice of coefficients a_{ijkl} of the equation [see (2.2)], was developed with all details of mathematical analysis of convergence. This mixed finite element method found its application in the mixed method analysis of shell problems in [31] and also specific mention in [33]. But for the same isotropic plate bending problem, the mixed method scheme of [5] and that of Hellan-Hermann-Miyoshi are *different*. Error estimates of order $O(h^{m-1})$ have been obtained in [5] under the assumption that an exact integration of the integrals of the bilinear forms is possible, the domain being a convex polygonal one (*i.e.* **no** approximation of the boundary is necessary), the convexity of the polygonal domain (in *all* papers) being a requirement for the regularity [21, 24] of the solution on which the proof of the existence of solution of the continuous *mixed* variational problem and error estimates are based. But in many practical situations both approximation of the curved boundary of the convex domain by a polygonal one or some other suitable curved boundary and numerical integration for the evaluation of bilinear forms are to be performed. In such situations an estimate for the combined effect of the numerical integration and approximation of the curved boundary of the convex domain on the mixed finite element solution of the problem is essential. Such estimates for *classical* finite element methods of solution of second order problems have been obtained in [17, 19, 32, 35, 36, 38–40], and of fourth

Keywords and phrases. Mixed finite element method, fourth order equations, convergence analysis.

[1] Professor, Department of Mathematics, Indian Institute of Technology, New Delhi 110016, India.
Present Address: Honorary Visiting Professor, School of Computer and Systems Sciences, Jawaharlal Nehru University, New Delhi 110067, India. e-mail: `pulinkum@hotmail.com`

[2] Research Scholar, Department of Mathematics, Indian Institute of Technology, New Delhi 110016, India.
e-mail: `neela@maths.iitd.ernet.in`

order problems in [8, 27], but to our knowledge such results for *mixed* finite element methods for fourth order problems are conspicuous by their absence in published research literature. Moreover, construction of estimates for these combined effects on mixed method solution for fourth order problems is associated with mathematical difficulties. The present paper contains new, original results in this direction. For other mixed/hybrid schemes for this fourth order elliptic problem, we refer to [6, 7, 10–12, 29].

2. Mixed variational problem

Let Ω be an open, **convex**, bounded domain in $\mathbb{R}^2$ with Lipschitz-continuous **curved** boundary Γ, piecewise of C^m class [1,17,21,32,38] $m \geq 3$, in which we consider the boundary value problem $(\mathbf{P})$: for given $f \in L^2(\Omega)$, find u such that:

$$(\mathbf{P}): \quad \Lambda u = f \text{ in } \Omega, \;\; u|_\Gamma = (\frac{\partial u}{\partial n})|_\Gamma = 0, \tag{2.1}$$

where

$$(\Lambda u)(x) \equiv \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2} \frac{\partial^2}{\partial x_k \partial x_l}(a_{ijkl}\frac{\partial^2 u}{\partial x_i \partial x_j})(x) \equiv (a_{ijkl}u_{,ij})_{,kl}(x) \quad \forall x \in \bar{\Omega}. \tag{2.2}$$

(In (2.2) and also in the sequel, Einstein's summation convention with respect to twice repeated indices $1 \leq i,j,k,l \leq 2$ **has been followed)**, coefficients a_{ijkl} satisfy the following conditions [5]:
$\forall i,j,k,l = 1,2$

- $(\mathbf{A1})$ $a_{ijkl} \in C^0(\bar{\Omega})$; $a_{ijkl} \geq 0$; $a_{ijkl}(x) = a_{klij}(x) = a_{jikl}(x) = a_{jilk}(x)$ $\forall x \in \bar{\Omega}$;
- $(\mathbf{A2})$ $\exists \alpha_0 > 0$ such that $\forall \underline{\xi} = (\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}) \in \mathbb{R}^4$ with $\xi_{21} = \xi_{12}$, $\quad a_{ijkl}(x)\xi_{ij}\xi_{kl} \geq \alpha_0 \|\underline{\xi}\|^2_{\mathbb{R}^4} \quad \forall x \in \bar{\Omega}$.

Then, under $(\mathbf{A1}$–$\mathbf{A2})$, the corresponding Galerkin variational problem $(\mathbf{P_G})$:
For given $f \in L^2(\Omega)$, find $u \in H^2_0(\Omega)$ [1, 17, 21, 26, 32] such that

$$(\mathbf{P_G}): \quad a(u,v) = l(v) \quad \forall v \in H^2_0(\Omega), \tag{2.3}$$

where

$$a(u,v) \;=\; \langle \Lambda u, v\rangle_{0,\Omega} \;=\; \int_\Omega a_{ijkl}u_{,ij}v_{,kl}\mathrm{d}\Omega \;=\; a(v,u) \qquad \forall u,\; v \in H^2_0(\Omega); \tag{2.4}$$

$$l(v) \;=\; \langle f, v\rangle_{0,\Omega} \;=\; \int_\Omega fv\mathrm{d}\Omega \qquad \forall v \in H^2_0(\Omega) \tag{2.5}$$

has a unique solution [4, 20].

Introducing Hilbert spaces $\mathbf{H}$ and $\mathbf{V}$ of admissible tensor-valued functions:

- $\mathbf{H} = \{\Phi : \Phi = (\phi_{ij})_{i,j=1,2};\;\; \phi_{ij} = \phi_{ji} \in L^2(\Omega)\;\; \forall i,j = 1,2\}$ (2.6)

with

$$\|\Phi\|^2_{\mathbf{H}} = \|\Phi\|^2_{0,\Omega} = \|\phi_{11}\|^2_{0,\Omega} + 2\|\phi_{12}\|^2_{0,\Omega} + \|\phi_{22}\|^2_{0,\Omega} \quad \forall \Phi \in \mathbf{H}; \tag{2.7}$$

- $\mathbf{V} = \{\Phi : \Phi \in \mathbf{H},\ \ \phi_{ij} \in H^1(\Omega)\ \forall i,j = 1,2\} \subset \mathbf{H}$ (2.8)

with

$$\|\Phi\|^2_{\mathbf{V}} = \|\Phi\|^2_{1,\Omega} = \|\phi_{11}\|^2_{1,\Omega} + 2\|\phi_{12}\|^2_{1,\Omega} + \|\phi_{22}\|^2_{1,\Omega} \quad \forall \Phi \in \mathbf{V},\ \mathbf{V} \hookrightarrow \mathbf{H};$$

and

- $W \equiv H^1_0(\Omega)$ with $\|\chi\|_W = \|\chi\|_{1,\Omega}\ \ \forall \chi \in W,$ (2.9)

we associate to ($\mathbf{P_G}$), the continuous **Mixed Variational Problem** ($\mathbf{Q}$) developed in [5] as follows: For given $f \in L^2(\Omega)$, find $(\Psi, u) \in \mathbf{V} \times W$ such that

$$(\mathbf{Q}) : \begin{array}{ll} A(\Psi,\Phi) + b(\Phi,u) = 0 & \forall\,\Phi \in \mathbf{V}, \\ -b(\Psi,v) = \langle f, v\rangle_{0,\Omega} & \forall\, v \in W, \end{array} \tag{2.10}$$

where $A(\cdot,\cdot) : \mathbf{V} \times \mathbf{V} \longrightarrow \mathbb{R},\ b(\cdot,\cdot) : \mathbf{V} \times W \longrightarrow \mathbb{R}$ are continuous bilinear forms defined by:

$$A(\Psi,\Phi) = \int_\Omega A_{ijkl}\psi_{ij}\phi_{kl}\,\mathrm{d}\Omega = A(\Phi,\Psi) \quad \forall\,\Psi,\Phi \in \mathbf{V} \subset \mathbf{H}; \tag{2.11}$$

$$b(\Phi,\chi) = \int_\Omega \phi_{ij,j}\chi_{,i}\,\mathrm{d}\Omega \qquad \forall \Phi \in \mathbf{V},\ \forall\,\chi \in W; \tag{2.12}$$

coefficients $A_{ijkl} = A_{ijkl}(x)$ are defined in terms of a_{ijkl} satisfying the following properties [5]:

- $A_{ijkl} \in C^0(\overline{\Omega}),\ \ A_{ijkl}(x) = A_{klij}(x) = A_{lkij}(x) = A_{lkji}(x)\ \ \forall i,j,k,l = 1,2,\ \ \forall x \in \bar{\Omega},$ (2.13)
- $\exists\,\alpha_0 > 0$ such that $\forall \underline{\xi} = (\xi_{11},\xi_{12},\xi_{21},\xi_{22}) \in \mathbb{R}^4$ with $\xi_{21} = \xi_{12}, A_{ijkl}(x)\xi_{ij}\xi_{kl} \geq \alpha_0\|\underline{\xi}\|^2_{\mathbb{R}^4}\ \ \forall x \in \bar{\Omega}.$ (2.14)
- $\forall x \in \bar{\Omega},\ \ \forall \underline{\xi} = (\xi_{11},\xi_{12},\xi_{21},\xi_{22}) \in \mathbb{R}^4$ with $\xi_{21} = \xi_{12},\ \ \forall \underline{\zeta} = (\zeta_{11},\zeta_{12},\zeta_{21},\zeta_{22}) \in \mathbb{R}^4$ with $\zeta_{21} = \zeta_{12},\qquad A_{ijkl}(x)a_{ijmn}(x)\xi_{mn}\zeta_{kl} = \xi_{ij}\zeta_{ij}.$ (2.15)

Proposition 2.1. [5]

(i) $\exists \alpha > 0$ *such that*

$$A(\Phi,\Phi) \geq \alpha\|\Phi\|^2_{\mathbf{H}} \quad \forall \Phi \in \mathbf{V} \hookrightarrow \mathbf{H}. \tag{2.16}$$

(ii) $\exists \beta > 0$ *such that*

$$\sup_{\Phi \in \mathbf{V}-\{\mathbf{0}\}} \frac{|b(\Phi,\chi)|}{\|\Phi\|_{\mathbf{V}}} \geq \beta\|\chi\|_{1,\Omega}\ \ \forall \chi \in W \tag{2.17}$$

(iii) **(Q)** *has* **at most** *one solution* $(\Psi,u) \in \mathbf{V} \times W$.

Remark 2.1. (2.17) is Babuška-Brezzi condition [2,13,14,30].

Since $A(\cdot,\cdot)$ is **not V**-elliptic, **(Q)** is **not well-posed** *a priori* in general. But we have

Theorem 2.1. [5] *If the solution $u \in H_0^2(\Omega)$ of Galerkin variational problem $(\mathbf{P_G})$ belongs to $H^3(\Omega)\cap H_0^2(\Omega)$ and $\psi_{ij} = a_{ijkl}u_{,kl} \in H^1(\Omega)$, $\forall i,j = 1,2$, then $(\mathbf{Q})$ has a unique solution $(\Psi,u) \in \mathbf{V}\times W$.*

Conversely, if $(\mathbf{Q})$ has a solution $(\Psi,u)\in\mathbf{V}\times W$, (which will be a unique one by virtue of Proposition 2.1), the second component u will be the unique solution of $(\mathbf{P_G})$ and

$$\Psi = (\psi_{ij})_{i,j=1,2} \text{ with } \psi_{ij} = a_{ijkl}u_{,kl} \text{ and } u_{,ij} = A_{ijkl}\psi_{kl} \;\; \forall i,j = 1,2. \tag{2.18}$$

Examples.

1. *Biharmonic problem*

For a_{ijkl} defined by: $a_{iiii} = 1$; $a_{1212} = a_{2121} = a_{2112} = a_{1221} = 1/2$; $a_{ijkl} = 0$ otherwise, which satisfy the assumptions (**A1–A2**) we get the Dirichlet problem of the biharmonic operator $\Lambda \equiv \Delta\Delta$. The coefficients A_{ijkl} are defined by: $A_{iiii} = 1$; $A_{1212} = A_{2121} = A_{2112} = A_{1221} = 1/2$; $A_{ijkl} = 0$ otherwise.

Then, the corresponding bilinear form $A(\cdot,\cdot)$ in **(Q)** is as follows:

$$A(\Psi,\Phi) = \int_\Omega \psi_{ij}\phi_{ij}\,\mathrm{d}\Omega \qquad \forall\Psi = (\psi_{ij})_{i,j=1,2},\; \Phi = (\phi_{ij})_{i,j=1,2} \in \mathbf{V}. \tag{2.19}$$

In this particular case, the algorithm **(Q)** reduces to the Hellan-Hermann-Miyoshi **(H-H-M)** algorithm [15,28] for the biharmonic equation, *i.e.* the solution $(\Psi,u)\in\mathbf{V}\times W$ of the problem **(Q)**:

$$\int_\Omega \psi_{ij}\phi_{ij}\,\mathrm{d}\Omega + \int_\Omega \phi_{ij,j}u_{,i}\,\mathrm{d}\Omega = 0 \qquad \forall\,\Phi\in\mathbf{V}, \tag{2.20}$$

$$\int_\Omega \psi_{ij,j}v_{,i}\,\mathrm{d}\Omega = -\langle f,v\rangle_{0,\Omega} \qquad \forall\, v\in W, \tag{2.21}$$

is given by: u, $\Psi = (\psi_{ij})_{i,j=1,2}$ with $\psi_{ij} = a_{ijkl}u_{,kl} = u_{,ij}$ $\forall i,j = 1,2$, where $u \in H_0^2(\Omega)\cap H^3(\Omega)$ is the solution of the problem $(\mathbf{P_G})$ corresponding to the biharmonic equation.

Remark 2.2. If u is the deflection of the bent elastic plate, then $\psi_{ij} = u_{,ij}$ $(i,j = 1,2)$ denote the components of the change in curvature tensor, but **not** the bending and twisting moments in the plate in general.

2. *Plate bending problems*

(i) Anisotropic case [4,25]:

$$\begin{aligned}&a_{iiii} = D_{ii},\; a_{1212} = a_{1221} = a_{2121} = a_{2112} = D_{66},\; a_{1112} = a_{1211} = a_{2111} = a_{1121} = D_{16},\\ &a_{1222} = a_{2122} = a_{2212} = a_{2221} = D_{26},\; a_{2211} = a_{1122} = D_{12}\end{aligned} \tag{2.22}$$

where $D_{ij} = D_{ij}(x_1,x_2)$ $\forall(x_1,x_2)\in\bar\Omega$ denote rigidities [25] defined by $D_{ij} = B_{ij}t^3/12$ $(i = 1,2;\; j = 1,2,6)$, the B_{ij}'s being expressions in terms of elastic constants of the generalized Hooke's Law for the anisotropic material of the thin plate, $t = t(x_1,x_2)$ being the thickness of the plate at the point $(x_1,x_2)\in\bar\Omega$, such that

$$\begin{aligned}&D_{11},\; D_{22},\; D_{66} > 0,\; D_{12} = \nu_1 D_{22} = \nu_2 D_{11} (0 \le \nu_i < 1/2),\\ &0 \le D_{i6} < (1-\nu_j)D_{ii} (i\ne j) 1 \le i,j \le 2,\; D_{16} + D_{26} < D_{66}.\end{aligned} \tag{2.23}$$

Define $A_{ijkl} = A_{ijkl}(x)\ \ \forall x = (x_1, x_2) \in \bar{\Omega}\ \ \forall\ i, j, k, l = 1, 2$ with the help of a_{ijkl} as follows:

$$\begin{aligned} &A_{iiii} = 4(D_{jj}D_{66} - D_{j6}^2)/|A(\cdot)|\ (i \neq j);\ A_{1212} = (D_{11}D_{22} - D_{12}^2)/|A(\cdot)|;\\ &A_{1112} = 2(D_{12}D_{26} - D_{16}D_{22})/|A(\cdot)|;\ A_{1122} = 4(D_{16}D_{26} - D_{12}D_{66})/|A(\cdot)|;\\ &A_{1222} = 2(D_{12}D_{16} - D_{11}D_{26})/|A(\cdot)| \end{aligned} \tag{2.24}$$

with $|A(\cdot)|$ defined by

$$|A(x)| = 4(D_{11}D_{22}D_{66} - D_{11}D_{26}^2 - D_{66}D_{12}^2 - D_{22}D_{16}^2 + D_{12}D_{16}D_{26})(x), \tag{2.25}$$

and other A_{ijkl} are determined with the symmetry property in (2.13). The corresponding bilinear form $A(\cdot,\cdot)$ in **(Q)** is given by:

$$\begin{aligned} A(\Psi,\Phi) = \int_\Omega \frac{4}{|A(x)|}\Big[&\{(D_{22}D_{66} - D_{26}^2)\psi_{11} + (D_{16}D_{26} - D_{12}D_{66})\psi_{22} + (D_{12}D_{26} - D_{16}D_{22})\psi_{12}\}\phi_{11}\\ &\{(D_{16}D_{26} - D_{12}D_{66})\psi_{11} + (D_{11}D_{66} - D_{16}^2)\psi_{22} + (D_{16}D_{12} - D_{11}D_{26})\psi_{12}\}\phi_{22}\\ &\{(D_{12}D_{26} - D_{16}D_{22})\psi_{11} + (D_{16}D_{12} - D_{11}D_{26})\psi_{22} + (D_{11}D_{22} - D_{12}^2)\psi_{12}\}\phi_{12}\Big]\,\mathrm{d}\Omega\\ &\forall\ \Psi,\Phi \in \mathbf{V}; \end{aligned} \tag{2.26}$$

$b(\cdot,\cdot)$ being the same bilinear form in (2.12).

The solution $(\Psi, u) \in \mathbf{V} \times W$ of (**Q**) is characterized by: u is the deflection of the bent plate, $\Psi = (\psi_{ij})_{1\le i,j\le 2}$ is the bending moment tensor with bending moments ψ_{ii} in the x_i-direction $(i = 1, 2)$ and twisting moment $\psi_{12} = \psi_{21}$, *i.e.* one obtains directly and simultaneously 'u' and $\psi_{ij}{}'s$.

(ii) The orthotropic case [4,25,37] can be obtained from the anisotropic case (i) by putting in (2.22–2.26),

$$\begin{aligned} &a_{iiii} = D_i;\ a_{1122} = a_{2211} = D_{12} = \nu_1 D_2 = \nu_2 D_1;\\ &a_{1212} = a_{2121} = a_{2112} = a_{1221} = D_t,\ a_{ijkl} = 0 \quad \text{otherwise}, \end{aligned} \tag{2.27}$$

where $D_i = E_i t^3/(12(1 - \nu_1\nu_2)) > 0,\ (i = 1, 2);\ D_\tau = Gt^3/12 > 0,\ H = D_1\nu_2 + 2D_\tau,\ G = E_1E_2/(E_1 + (1 + 2\nu_1)E_2) > 0,\ E_1\nu_2 = E_2\nu_1,\ E_i$ and $\nu_i,\ i = 1, 2$ being the Young's moduli and Poisson's coefficients respectively, and the thickness function $t \in C^0(\overline{\Omega})$ is such that $0 < t_0 \le t(x_1,\ x_2) \le t_1,\ \forall (x_1, x_2) \in \overline{\Omega}$. Then

$$A(\Psi,\Phi) = \int_\Omega \left[\frac{1}{D_1(1-\nu_1\nu_2)}(\psi_{11} - \nu_1\psi_{22})\phi_{11} + \frac{1}{D_2(1-\nu_1\nu_2)}(-\nu_2\psi_{11} + \psi_{22})\phi_{22} + \frac{1}{D_\tau}\psi_{12}\phi_{12}\right]\mathrm{d}\Omega\ \forall \Psi,\Phi \in \mathbf{V}, \tag{2.28}$$

and the solution $(\Psi, u) \in \mathbf{V} \times W$ of **(Q)** is such that u is the deflection of the bent plate, $\Psi = (\psi_{ij})_{i,j=1,2}$ with $\psi_{ij} = a_{ijkl}u_{,kl}\ \forall i, j = 1, 2$ giving the bending and twisting moments in the plate, *i.e.* $\psi_{11} = D_1(u_{,11} + \nu_2 u_{,22}),\ \psi_{22} = D_2(\nu_1 u_{,11} + u_{,22})$ are the bending moments in the x_1 and x_2 directions, the twisting moment being $\psi_{12} = \psi_{21} = 2D_t u_{,12}$.

(iii) The isotropic case is obtained from the orthotropic case by putting $E_1 = E_2 = E$, $\nu_1 = \nu_2 = \nu$ and consequently, $D_1 = D_2 = D$ in all formulae in (ii) for the orthotropic plate. In this case also, u is the deflection of the bent plate; $\psi_{11} = D(u_{,11} + \nu u_{,22}),\ \psi_{22} = D(\nu u_{,11} + u_{,22}),\ \psi_{12} = \psi_{21} = D(1 - \nu)u_{,12}$ are the bending moments in the x_1 and x_2 directions and twisting moment respectively.

Remark 2.3. For $D = 1,\ \nu = 0$, we get **H-H-M** mixed scheme in (2.20–2.21) [15].

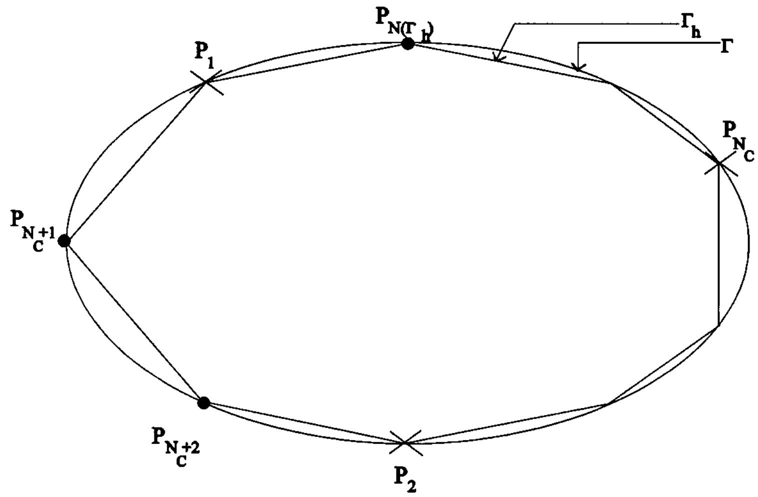

FIGURE 3.1. × Points $P_i \in \Gamma \cap \Gamma_h (1 \le i \le N_C)$ at which C^m - smoothness does not hold. • Points $P_i \in \Gamma \cap \Gamma_h (N_C + 1 \le i \le N(\Gamma_h))$ are additional vertices of Γ_h.

3. Mixed finite element problem (Q_h) with approximation of the curved boundary Γ and numerical integration

3.1. Triangulations τ_h and τ_h^{exact}

Let Γ_h be a (straight) polygonal boundary approximating Γ such that

$$\Gamma_h \subset \bar{\Omega},\ \Gamma_h \cap \Gamma = \{P_i\}_{i=1}^{N_c} \cup \{P_i\}_{i=N_c+1}^{N(\Gamma_h)} = V(\Gamma_h) \tag{3.1}$$

where $V(\Gamma_h)$ is the set of **all** vertices (corner points) of Γ_h with $\text{Card}(V(\Gamma_h)) = N(\Gamma_h)$, the set of all corner points $\{P_i\}_{i=1}^{N_c}$, at which C^m-smoothness $(m \ge 3)$ does **not** hold, being its proper subset.

Let $\Omega_h \subset \mathbb{R}^2$ the domain interior to Γ_h such that

$$\bar{\Omega}_h = \Omega_h \cup \Gamma_h \subset \bar{\Omega} \tag{3.2}$$

is the closed **convex polygonal** domain contained in $\bar{\Omega}$ (see Fig. 3.1).

Let τ_h be an exact, admissible, regular, quasi-uniform [3, 17] triangulation of $\bar{\Omega}_h$ such that

$$\bar{\Omega}_h = \cup_{T\in\tau_h} T \ \subset \bar{\Omega} \text{ with } \tau_h = \tau_h^b \cup \tau_h^0, \tag{3.3}$$

where

$$\begin{aligned} \tau_h^b \;&=\; \{T : T \in \tau_h,\ \text{exactly } \textbf{two} \text{ vertices } a_{1,T} \text{ and } a_{2,T} \text{ of } T \text{ lie on } \Gamma_h \cap \Gamma \text{ or} \\ &\quad\ \text{equivalently } \textbf{only one} \text{ side of } T \text{ is a part of } \Gamma_h\} \\ &=\; \text{set of all } \textbf{boundary} \text{ triangles of } \tau_h; \end{aligned} \tag{3.4}$$

$$\begin{aligned} \tau_h^0 \;&=\; \{T : T \in \tau_h \text{ is an interior triangle } i.e. \ \textbf{atmost} \text{ one of its vertices lie on } \Gamma_h\} \\ &=\; \text{set of all } \textbf{interior} \text{ triangles of } \tau_h. \end{aligned} \tag{3.5}$$

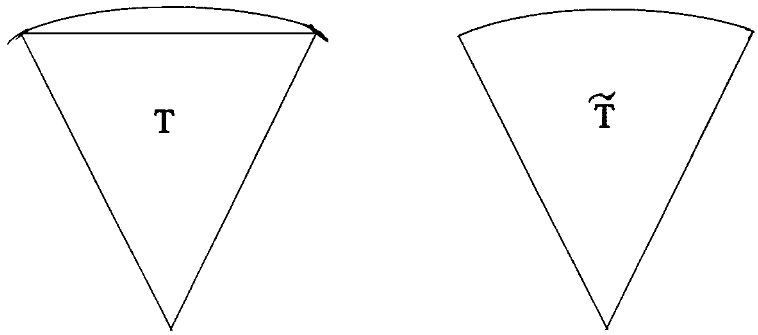

FIGURE 3.2

Let $\widetilde{\tau_h^b}$ denote the set of all curved boundary triangles $\tilde{T}$ obtained from the boundary triangles $T \in \tau_h^b$ by replacing the straight boundary side of T by a part of Γ joining the two boundary vertices on $\Gamma_h \cap \Gamma$, the other two sides being the same ones of the corresponding boundary triangle $T \in \tau_h^b$. (See Fig. 3.2.)

Then, $\tau_h^{\text{exact}} = \widetilde{\tau_h^b} \cup \tau_h^0$, $\tau_h^0 \subset \tau_h$ being the set of all **interior** triangles defined in (3.5), denotes an **exact** triangulation of $\bar{\Omega} = \Omega \cup \Gamma$. *i.e.*

$$\bar{\Omega} = \cup_{\tilde{T} \in \tau_h^{\text{exact}}} \tilde{T}, \qquad \bar{\Omega}_h = \cup_{T \in \tau_h} T, \qquad \text{Card}(\widetilde{\tau_h^b}) = \text{Card}(\tau_h^b). \tag{3.6}$$

3.2. Reference triangle $\hat{T}$ and affine mapping $F_T : \hat{T} \longrightarrow T$

Let $\hat{T}$ be the reference triangle with vertices $\hat{a}_1 = (1,0),\ \hat{a}_2 = (0,1),\ \hat{a}_3 = (0,0)$ and $\forall T \in \tau_h,\ F_T : \hat{T} \longrightarrow T$ be an invertible affine mapping from $\hat{T}$ onto $T \in \tau_h$ defined by:

$$\forall \hat{x} \in \hat{T},\ F_T(\hat{x}) = B_T\,\hat{x} + b_T = x \in T, \tag{3.7}$$

such that

$$F_T(\hat{a}_i) = a_{i,T},\ 1 \le i \le 3,\ \{a_{i,T}\}_{i=1}^3 \text{ being the vertices of } T \in \tau_h, \tag{3.8}$$

$$[J(F_T)] = B_T \text{ is the invertible } 2 \times 2 \text{ Jacobian matrix and Jacobian } J(F_T) = \det B_T > 0, \tag{3.9}$$

$\forall \hat{\phi} \in P_m(\hat{T}),\ \exists \phi \in P_m(T), P_m(K)$ being the linear space of polynomials of degree $\le m$ defined on $K = \hat{T}$ or T, such that $\forall x \in T$ with $x = F_T(\hat{x})$,

$$\phi(x) = \phi \cdot F_T(\hat{x}) = \hat{\phi}(\hat{x}) \text{ with } \hat{\phi} = \phi \cdot F_T,\ \phi = \hat{\phi} \cdot F_T^{-1}. \tag{3.10}$$

Thus, under the affine mapping F_T defined in (3.7), τ_h is affine-equivalent to $\hat{T}$, *i.e.* τ_h is an affine family of triangles and hence, an exact triangulation of $\bar{\Omega}_h = \Omega_h \cup \Gamma_h$.

3.3. Numerical integration formulae

Let

$$\int_{\hat{T}} \hat{\phi}(\hat{x})\,\mathrm{d}\hat{x} \approx \sum_{n=1}^{N_i} \hat{w_n}^i \hat{\phi}(\hat{b_n}^i) \ \ (i = 1, 2) \tag{3.11}$$

be **two** quadrature schemes with **positive** weights $\hat{w_n}^i > 0$ and evaluation points $\hat{b_n}^i \in \hat{T}$ $(i = 1, 2,\ 1 \leq n \leq N_i)$. The quadrature scheme (3.11) exact for $P_4(\hat{T})$ for $i = 1$ (resp. $P_2(\hat{T})$ for $i = 2$) will be used in the evaluation of the bilinear forms of the mixed finite element problem in the sequel. Then,

$$\int_T \phi(x)\mathrm{d}x = \int_{\hat{T}} \hat{\phi}(\hat{x}) \det(B_T)\mathrm{d}\hat{x} \approx \sum_{i=1}^{N_i} w^i_{n,T}\phi(b^i_{n,T}) \tag{3.12}$$

with $w^i_{n,T} = \det(B_T)\ \hat{w_n}^i > 0,\ b^i_{n,T} = F_T(\hat{b_n}^i) \in T,\ 1 \leq n \leq N_i,\quad i = 1, 2,$ is obtained from (3.11) under invertible affine mapping F_T in (3.7–3.10).

To each Ω_h, we associate auxiliary infinite dimensional Hilbert spaces $\mathbf{V}(\Omega_h)$ and $H^1_0(\Omega_h)$ defined by:

- $\mathbf{V}(\Omega_h) = \{\Phi : \Phi = (\phi_{ij})_{i,j=1,2},\ \phi_{ij} = \phi_{ji} \in H^1(\Omega_h)\ \ \forall i, j = 1, 2\}$ (3.13)

with

$$\|\Phi\|^2_{\mathbf{V}(\Omega_h)} = \|\Phi\|^2_{1,\Omega_h} = \sum_{i=1}^{2}\sum_{j=1}^{2} \|\phi_{ij}\|^2_{1,\Omega_h};$$

- $H^1_0(\Omega_h) = \{v : v \in H^1(\Omega_h),\ v|_{\Gamma_h} = 0\}$ with $\|v\|_{H^1_0(\Omega_h)} = \|v\|_{1,\Omega_h}$, (3.14)

and the auxiliary continuous bilinear forms

$$\tilde{A}(\cdot,\cdot) : \mathbf{V}(\Omega_h) \times \mathbf{V}(\Omega_h) \longrightarrow \mathbb{R}, \qquad \tilde{b}(\cdot,\cdot) : \mathbf{V}(\Omega_h) \times H^1(\Omega_h) \longrightarrow \mathbb{R}$$

defined by:

$$\tilde{A}(\Phi,\Psi) = \int_{\Omega_h} A_{ijkl}\phi_{ij}\psi_{kl}\ \mathrm{d}\Omega_h \text{ with } |\tilde{A}(\Psi,\Phi)| \leq \tilde{M}\|\Psi\|_{0,\Omega_h}\|\Phi\|_{0,\Omega_h} \qquad \forall\Phi,\ \Psi \in \mathbf{V}(\Omega_h); \tag{3.15}$$

$$\tilde{b}(\Phi,\chi) = \int_{\Omega_h} \phi_{ij,j}\chi_{,i}\ \mathrm{d}\Omega_h \text{ with } |\tilde{b}(\Phi,\chi)| \leq \tilde{m}\|\Phi\|_{1,\Omega_h}\|\chi\|_{1,\Omega_h} \qquad \forall\Phi \in \mathbf{V}(\Omega_h),\ \forall\chi \in H^1(\Omega_h). \tag{3.16}$$

And to each τ_h of $\bar{\Omega}_h$, we associate the following finite dimensional subspaces:

- $X_h = \{\phi_h : \phi_h \in C^0(\bar{\Omega}_h),\ \phi_h \downarrow_T \in P_2(T)\ \ \forall T \in \tau_h\} \subset H^1(\Omega_h)$; (3.17)

- $\mathbf{V}_h = \{\Phi_h : \Phi_h = (\phi_{hij})_{i,j=1,2},\ \phi_{hij} = \phi_{hji} \in X_h\ \ \forall i, j = 1, 2\} \subset \mathbf{V}(\Omega_h)$ (3.18)

with

$$\|\Phi_h\|_{\mathbf{V}_h} = \|\Phi_h\|_{\mathbf{V}(\Omega_h)};$$

- $W_h = \{\chi_h : \chi_h \in X_h,\ \chi_h \downarrow_{\Gamma_h} = 0\} \subset H^1_0(\Omega_h)$ with $\|\chi_h\|_{W_h} = \|\chi_h\|_{1,\Omega_h}$, (3.19)

in which we have replaced the essential boundary condition $\chi \downarrow_\Gamma$ in the definition of W in (2.9) by the boundary condition $\chi_h \downarrow_{\Gamma_h}$ in (3.19).

3.4. Extensions

Let $\tilde{T} \in \widetilde{\tau_h^b} \subset \tau_h{}^{\text{exact}}$ be a curved boundary triangle containing the corresponding boundary triangle $T \in \tau_h^b \subset \tau_h$ with $T \subset \tilde{T}$ (see Fig. 3.2). For $\phi_T = \phi_h \downarrow_T \in P_2(T)$ with $\phi_h \in X_h$, $\tilde{\phi}$ is the natural (polynomial) extension to $\tilde{T}$ of the polynomial $\phi_T \in P_2(T)$ defined by: $\tilde{\phi} \in P_2(\tilde{T})$ with $\tilde{\phi} \downarrow_T = \phi_T \in P_2(T)$.

Then, to X_h we associate $\tilde{X}_h$ as the linear space of *natural* (piecewise polynomial) extensions to $\bar{\Omega}$ of functions $\phi_h \in X_h$ defined in $\bar{\Omega}_h$:

- $$\tilde{X}_h = \{\tilde{\phi}_h : \tilde{\phi}_h \in C^0(\bar{\Omega}),\ \tilde{\phi}_h \downarrow_{\bar{\Omega}_h} = \phi_h \in X_h,\ \tilde{\phi}_h \downarrow_{\tilde{T}} \in P_2(\tilde{T})\ \forall \tilde{T} \in \widetilde{\tau_h^b} \subset \tau_h^{\text{exact}}\} \subset H^1(\Omega); \tag{3.20}$$

- $$\widetilde{\mathbf{V}}_h = \{\tilde{\Phi}_h : \tilde{\Phi}_h = (\widetilde{\phi_{hij}})_{i,j=1,2} \text{ with } \widetilde{\phi_{h12}} = \widetilde{\phi_{h21}} \text{ such that } \widetilde{\phi_{hij}} \in \tilde{X}_h\ \ \forall i,j = 1,2\}; \tag{3.21}$$

- $$\tilde{W}_h = \{\tilde{\chi}_h : \tilde{\chi}_h \downarrow_{\Omega_h} \in W_h,\ \tilde{\chi}_h \downarrow_{\Omega - \Omega_h} = 0\} \subset H_0^1(\Omega). \tag{3.22}$$

With the help of numerical integration formulae in (3.12), we define new continuous, bilinear forms

$$A_h^{NI}(\cdot,\cdot) : \mathbf{V}_h \times \mathbf{V}_h \longrightarrow \mathbb{R},\ b_h^{NI}(\cdot,\cdot) : \mathbf{V}_h \times W_h \longrightarrow \mathbb{R}$$

by

$$A_h^{NI}(\Phi_h, \Psi_h) = \sum_{T \in \tau_h} \sum_{n=1}^{N_1} w_{n,T}^1 (A_{ijkl}\phi_{hij}\psi_{hkl})(b_{n,T}^1) = A_h^{NI}(\Psi_h, \Phi_h) \qquad \forall \Phi_h, \Psi_h \in \mathbf{V}_h, \tag{3.23}$$

and $\exists M_0 > 0$ such that

$$\begin{aligned} |A_h^{NI}(\Psi_h, \Phi_h)| &\leq M_0 \|\Psi_h\|_{0,\Omega_h} \|\Phi_h\|_{0,\Omega_h} && \forall \Psi_h,\ \Phi_h \in \mathbf{V}_h; \\ b_h^{NI}(\Phi_h, \chi_h) &= \sum_{T \in \tau_h} \sum_{n=1}^{N_2} w_{n,T}^2 (\phi_{hij,j}\chi_{h,i})(b_{n,T}^2) && \forall \Phi_h \in \mathbf{V}_h, \forall \chi_h \in W_h, \end{aligned} \tag{3.24}$$

and $\exists m_0 > 0$ such that

$$|b_h^{NI}(\Phi_h, \chi_h)| \leq m_0 \|\Phi_h\|_{1,\Omega_h} \|\chi_h\|_{1,\Omega_h} \qquad \forall \Phi_h \in \mathbf{V}_h,\ \chi_h \in W_h.$$

Now, to the problem $(\mathbf{Q})$ in (2.10), we associate the following **'Affine' Mixed Finite Element Problem** $(\mathbf{Q_h})$ as follows: Find $(\Psi_h, u_h) \in \mathbf{V}_h \times W_h$ such that

$$(\mathbf{Q_h}) : \begin{array}{ll} A_h^{NI}(\Psi_h, \Phi_h) + b_h^{NI}(\Phi_h, u_h) = 0 & \forall \Phi_h \in \mathbf{V}_h, \\ -b_h^{NI}(\Psi_h, \chi_h) = \langle f, \chi_h \rangle_{0,\Omega_h} & \forall \chi_h \in W_h, \end{array} \tag{3.25}$$

where $A_h^{NI}(\cdot,\cdot),\ b_h^{NI}(\cdot,\cdot)$ are defined by (3.23) and (3.24) respectively,

$$\langle f, \chi_h \rangle_{0,\Omega_h} = \int_{\Omega_h} f \chi_h \, \mathrm{d}\Omega_h \ \ \forall \chi_h \in W_h. \tag{3.26}$$

Remark 3.1. We are considering the important situations in which exact integration of (3.26) is possible.

Lemma 3.1. *Let the quadrature schemes (3.11) with $i = 1$ and 2 correspond to the definitions of $A_h^{NI}(\cdot,\cdot)$ and $b_h^{NI}(\cdot,\cdot)$ in (3.23) and (3.24) respectively.*
Then, (a) $\exists\alpha_0 > 0$, independent of h, such that

$$A_h^{NI}(\Phi_h, \Phi_h) \geq \alpha_0 \|\Phi_h\|^2_{0,\Omega_h} \quad \forall \Phi_h \in \mathbf{V}_h; \tag{3.27}$$

(b) $\exists\beta_1 > 0$, independent of h, such that

$$\sup_{\Phi_h \in \mathbf{V}_h - \{0\}} \frac{|b_h^{NI}(\Phi_h, \chi_h)|}{\|\Phi_h\|_{\mathbf{V}_h}} \geq \beta_1 \|\chi_h\|_{W_h} \quad \forall \chi_h \in W_h. \tag{3.28}$$

Proof. (a) For $i = 1$, the quadrature scheme (3.11) used in (3.23) is exact for $P_4(\hat{T})$. Then, using (2.14), we have:

$$\begin{aligned}
\forall T \in \tau_h, \quad \sum_{n=1}^{N_1} w^1_{n,T}(A_{ijkl}\phi_{hij}\phi_{hkl})(b^1_{n,T}) &\geq \alpha_0 \sum_{n=1}^{N_1} w^1_{n,T}(\phi^2_{h11}(b^1_{n,T}) + 2\phi^2_{h12}(b^1_{n,T}) + \phi^2_{h22}(b^1_{n,T})), \\
&= \alpha_0 \sum_{n=1}^{N_1} \hat{w_n}^1 (det B_T)(\hat{\phi}^2_{11}(\hat{b_n}^1) + 2\hat{\phi}^2_{12}(\hat{b_n}^1) + \hat{\phi}^2_{22}(\hat{b_n}^1)) \\
&= \alpha_0 \int_T (\phi^2_{h11} + 2\phi^2_{h12} + \phi^2_{h22})\, \mathrm{d}T = \alpha_0 \|\Phi_h\|^2_{0,T}.
\end{aligned}$$

(b) For $i = 2$, the quadrature scheme (3.11) used in (3.24) is exact for $P_2(\hat{T})$. Choose $\Phi_h^* = (\chi_h, 0, 0, \chi_h)$ with $\chi_h \in W_h$.
Then

$$\Phi_h^* \in \mathbf{V}_h \ \text{ with } \ \|\Phi_h^*\|_{1,\Omega_h} = \sqrt{2}\|\chi_h\|_{1,\Omega_h} \tag{3.29}$$

and

$$\sup_{\Phi_h \in \mathbf{V}_h - \{0\}} \frac{|b_h^{NI}(\Phi_h, \chi_h)|}{\|\Phi_h\|_{1,\Omega_h}} \geq \frac{b_h^{NI}(\Phi_h^*, \chi_h)}{\|\Phi_h^*\|_{1,\Omega_h}} = \frac{b_h^{NI}(\Phi_h^*, \chi_h)}{\sqrt{2}\|\chi_h\|_{1,\Omega_h}} \quad [\text{using } (3.29)] \tag{3.30}$$

where

$$b_h^{NI}(\Phi_h^*, \chi_h) = \sum_{T\in\tau_h}\sum_{n=1}^{N_2} w^2_{n,T}\left[(\chi_{h,1})^2 + (\chi_{h,2})^2\right](b^2_{n,T}) \geq \gamma \sum_{T\in\tau_h} |\chi_h|^2_{1,T} \text{ with } \gamma > 0 \ [17].$$

$$\Longrightarrow b_h^{NI}(\Phi_h^*, \chi_h) \geq \gamma \sum_{T\in\tau_h} |\chi_h|^2_{1,T} = \gamma|\chi_h|^2_{1,\Omega_h} = \gamma|\tilde{\chi_h}|^2_{1,\Omega}. \tag{3.31}$$

Applying Friedrichs' inequality in (3.31), we have

$$b_h^{NI}(\Phi_h^*, \chi_h) \geq \gamma C(\Omega)\|\tilde{\chi_h}\|^2_{1,\Omega} = \gamma C(\Omega)\|\chi_h\|^2_{1,\Omega_h}. \tag{3.32}$$

From (3.30) and (3.32), we get

$$\sup_{\Phi_h \in \mathbf{V}_h - \{0\}} \frac{|b_h^{NI}(\Phi_h, \chi_h)|}{\|\Phi_h\|_{\mathbf{V}_h}} \geq \beta_1 \|\chi_h\|_{1,\Omega_h} \quad \forall \chi_h \in W_h \text{ with } \beta_1 = \gamma C(\Omega)/\sqrt{2} > 0.$$

Remark 3.2. The inequality (3.28) is the discrete Babuška-Brezzi condition [13, 14, 30].

Theorem 3.1. *The 'affine' mixed finite element problem* $(\mathbf{Q_h})$ *defined by (3.25) has a unique solution* $(\Psi_h, u_h) \in \mathbf{V}_h \times W_h$.

Proof. Since the linear problem $(\mathbf{Q_h})$ is defined on $\mathbf{V}_h \times W_h$ which is a finite dimensional vector space, the uniqueness of its solution in $\mathbf{V}_h \times W_h$ implies its existence in $\mathbf{V}_h \times W_h$. The homogeneous problem corresponding to $(\mathbf{Q_h})$:

$$A_h^{NI}(\Psi_h, \Phi_h) + b_h^{NI}(\Phi_h, u_h) = 0 \qquad \forall \Phi_h \in \mathbf{V}_h,$$
$$-b_h^{NI}(\Psi_h, \chi_h) = 0 \qquad \forall \chi_h \in W_h,$$

has a unique solution $\Psi_h = \mathbf{0}, u_h = 0$ by virtue of (3.27) and (3.28), from which the result follows.

4. Error estimates

4.1. Auxiliary interpolation operator $\mathcal{P}_h$

Since functions in $H^s(\Omega) \cap H_0^1(\Omega)$ with $s \geq 2$ are continuous in $\bar{\Omega}$ with $\bar{\Omega}_h \subset \bar{\Omega}$ and $\Gamma \cap \Gamma_h = V(\Gamma_h) =$ set of boundary vertices of $\tau_h = \{a_{i,T}\}_{i=1, T \in \tau_h^b}^2$ [see (3.1)], we can define an auxiliary interpolation operator $\mathcal{P}_h$ as follows: $\forall \chi \in H^s(\Omega) \cap H_0^1(\Omega),\ s = 2, 3,$

$$\mathcal{P}_h \chi \in C^0(\bar{\Omega}_h),\ \mathcal{P}_h \chi \downarrow_T \in P_2(T),\ \mathcal{P}_h \chi(a_{i,T}) = \chi(a_{i,T}),\ 1 \leq i \leq 6,\ \ \forall T \in \tau_h, \tag{4.1}$$

$\{a_{i,T}\}_1^3$ and $\{a_{i,T}\}_4^6$ being the vertices and midside nodes of $T \in \tau_h$ respectively such that $\partial T_1 = [a_{1,T}, a_{2,T}]$ is the boundary side of $T \in \tau_h^b$. Then, from (4.1) it follows that $\forall$ boundary triangle $T \in \tau_h^b$, $\mathcal{P}_h \chi(a_{i,T}) = 0$ $(i = 1, 2)$, but $\mathcal{P}_h \chi(a_{4,T}) = \chi(a_{4,T}) \neq 0$ in general for $a_{4,T} = (a_{1,T} + a_{2,T})/2$. Hence, $\forall \chi \in H^s(\Omega) \cap H_0^1(\Omega),\ s = 2, 3,$

$$\mathcal{P}_h \chi \in \{\chi_h : \chi_h \in H^1(\Omega_h) \cap C^0(\bar{\Omega}_h),\ \ \chi_h(a_{i,T}) = 0\ \ \forall T \in \tau_h^b,\ i = 1, 2\}, \tag{4.2}$$

$$\chi \downarrow_{\Omega_h} \in \{\chi : \chi \in H^s(\Omega_h),\ \chi(a_{i,T}) = 0\ \ \forall T \in \tau_h^b,\ i = 1, 2\},$$

and the classical estimate [17] holds: $\exists C > 0$, independent of h, such that

$$\|\chi - \mathcal{P}_h \chi\|_{r, \Omega_h} \leq C h^{s-r} |\chi|_{s, \Omega_h} \quad (s = 2, 3; r = 0, 1). \tag{4.3}$$

(In (4.3) and also in the sequel the same C has been used to denote a generic strictly positive constant, independent of h, having different values at different steps of the proofs.)

But $\mathcal{P}_h \chi \notin W_h \subset H_0^1(\Omega_h)$. Hence, we introduce W_h- interpolation operator $\mathcal{P}_{0h}$ defined by:

$$\begin{aligned} &\forall \chi \in H^s(\Omega) \cap H_0^1(\Omega),\ s = 2, 3,\ \ \mathcal{P}_{0h} \chi \in C^0(\bar{\Omega}_h),\ \ \mathcal{P}_{0h} \chi \downarrow_T \in P_2(T)\ \ \forall T \in \tau_h, \\ &\mathcal{P}_{0h} \chi(a_{i,T}) = \chi(a_{i,T})\ \ \forall \textbf{ interior node } a_{i,T} \in \Omega_h,\ \mathcal{P}_{0h} \chi \downarrow_{\Gamma_h} = 0. \end{aligned} \tag{4.4}$$

From (4.4), it follows that $\mathcal{P}_{0h} \chi \in W_h \subset H_0^1(\Omega_h)$ and we have

Proposition 4.1. *Let* $\tau_h = \tau_h^b \cup \tau_h^0$ *be the triangulation defined in (3.1–3.5).* $\forall \chi \in H^s(\Omega) \cap H_0^1(\Omega),\ s = 2, 3,$ *let* $\mathcal{P}_{0h} \chi \in W_h$ *be defined by (4.4). Then, the following estimates hold:*

$$For\ s = 2, \qquad \|\chi - \mathcal{P}_{0h} \chi\|_{r, \Omega_h} \leq C h^{2-r} \|\chi\|_{2, \Omega}\ \ (r = 0, 1); \tag{4.5}$$

$$For\ s = 3, \qquad \|\chi - \mathcal{P}_{0h} \chi\|_{r, \Omega_h} \leq C h^{3-r-1/2} \|\chi\|_{3, \Omega}\ \ (r = 0, 1). \tag{4.6}$$

Proof. $\forall \chi \in H^s(\Omega) \cap H^1_0(\Omega),\ s = 2, 3,$

$$\|\chi - \mathcal{P}_{0h}\chi\|_{0,\Omega_h} \ \leq\ \|\chi - \mathcal{P}_h\chi\|_{0,\Omega_h} + \|\mathcal{P}_h\chi - \mathcal{P}_{0h}\chi\|_{0,\Omega_h} \tag{4.7}$$

and

$$|\chi - \mathcal{P}_{0h}\chi|_{1,\Omega_h} \ \leq\ |\chi - \mathcal{P}_h\chi|_{1,\Omega_h} + |\mathcal{P}_h\chi - \mathcal{P}_{0h}\chi|_{1,\Omega_h}, \tag{4.8}$$

where $\mathcal{P}_h\chi$ is defined by (4.1). Then, from (4.3),

$$\|\chi - \mathcal{P}_h\chi\|_{0,\Omega_h} \leq Ch^s|\chi|_{s,\Omega_h}; \quad \|\chi - \mathcal{P}_h\chi\|_{1,\Omega_h} \leq Ch^{s-1}|\chi|_{s,\Omega_h}. \tag{4.9}$$

From (4.1, 4.3, 4.4), we have: $\forall$ **interior** triangle $T \in \tau_h^0,\ (\mathcal{P}_h\chi - \mathcal{P}_{0h}\chi)\downarrow_T = 0$, and $\forall$ **boundary** triangle $T \in \tau_h^b,\ (\mathcal{P}_h\chi - \mathcal{P}_{0h}\chi)\downarrow_T = \chi(a_{4,T})\phi_{4,T}$ with $\phi_{4,T} \in P_2(T),\ \phi_{4,T}(a_{4,T}) = 1,\ \phi_{4,T}(a_{i,T}) = 0,\quad 1 \leq i \neq 4 \leq 6,\ a_{4,T} = (a_{1,T} + a_{2,T})/2$ being the midpoint of the boundary side ∂T_1 of $T \in \tau_h^b$.

Hence,

$$\|\mathcal{P}_h\chi - \mathcal{P}_{0h}\chi\|^2_{0,\Omega_h} = \sum_{T\in\tau_h^b} \|\mathcal{P}_h\chi - \mathcal{P}_{0h}\chi\|^2_{0,T} = \sum_{T\in\tau_h^b} |\chi(a_{4,T})|^2\|\phi_{4,T}\|^2_{0,T} \tag{4.10}$$

and

$$|\mathcal{P}_h\chi - \mathcal{P}_{0h}\chi|^2_{1,\Omega_h} = \sum_{T\in\tau_h^b} |\chi(a_{4,T})|^2|\phi_{4,T}|^2_{1,T}. \tag{4.11}$$

But

$$\|\phi_{4,T}\|^2_{0,T} \leq Ch_T^2\|\hat{\phi}_4\|^2_{0,\hat{T}} \leq Ch_T^2;\ \ |\phi_{4,T}|^2_{1,T} \leq C|\hat{\phi}_4|^2_{1,\hat{T}} \leq C \quad [16, 38]. \tag{4.12}$$

Now, we will find estimate for $|\chi(a_{4,T})|$ in (4.10) and (4.11), for which we are to consider the cases $s = 2$ and $s = 3$ separately.

Case $s = 2$. From imbedding results [1] $H^2(\Omega) \hookrightarrow C^{0,\lambda}(\bar{\Omega})$ with $\lambda \in [0, 1[,\ C^{0,\lambda}(\bar{\Omega})$ being the linear space of λ-Holder continuous functions in Ω. Hence, $\forall\chi \in H^2(\Omega) \cap H^1_0(\Omega) \subset C^{0,\lambda}(\bar{\Omega})$, with $\lambda \in [0, 1[,\ |\chi(a_{4,T}) - \chi(\tilde{a}_{4,T})| \leq C\|a_{4,T} - \tilde{a}_{4,T}\|^{\lambda}_{\mathbb{R}^2}\|\chi\|_{2,\Omega}$, where $\tilde{a}_{4,T} \in \Gamma$ is the point of intersection of the perpendicular bisector of the boundary side $\partial T_1 = [a_{1,T}, a_{2,T}]$ of $T \in \tau_h^b$ with the boundary $\partial\tilde{T} \cap \Gamma$ such that $\|a_{4,T} - \tilde{a}_{4,T}\|_{\mathbb{R}^2} \leq Ch_T^2$ and $\chi(\tilde{a}_{4,T}) = 0$.

Hence

$$|\chi(a_{4,T})| \leq Ch_T^{2.\lambda}\|\chi\|_{2,\Omega}\ \ \forall\lambda \subset [0, 1[. \tag{4.13}$$

From (4.10) and (4.12), for $h = \max_{T\in\tau_h}\{h_T\}$, we have

$$\begin{aligned}\|\mathcal{P}_h\chi - \mathcal{P}_{0h}\chi\|^2_{0,\Omega_h} \ &\leq\ C\left[\sum_{T\in\tau_h^b} h_T^{4\lambda}\|\chi\|^2_{2,\Omega}h_T^2\right] \leq Ch^{4\lambda+1}\left(\sum_{T\in\tau_h^b} h_T\right)\|\chi\|^2_{2,\Omega}\\ &\leq\ Ch^{4\lambda+1}\mathrm{meas}(\Gamma_h)\|\chi\|^2_{2,\Omega} \leq Ch^{4\lambda+1}\mathrm{meas}(\Gamma)\|\chi\|^2_{2,\Omega} \leq Ch^{4\lambda+1}\|\chi\|^2_{2,\Omega},\end{aligned}$$

where $\sum_{T\in\tau_h^b} h_T \leq C$ meas (Γ_h) for some $C > 0$, independent of 'h', since τ_h is a regular triangulation.

$$\Longrightarrow \|\mathcal{P}_h\chi - \mathcal{P}_{0h}\chi\|_{0,\Omega_h} \quad\leq\quad Ch^{2\lambda+1/2}\|\chi\|_{2,\Omega} \text{ with } \lambda \in [0, 1[. \tag{4.14}$$

Similarly, from (4.11–4.13), we get

$$\begin{aligned}|\mathcal{P}_h\chi-\mathcal{P}_{0h}\chi|^2_{1,\Omega_h} &\le C\sum_{T\in\tau_h^b} h_T^{4\lambda}\|\chi\|^2_{2,\Omega}\\ &\le Ch^{4\lambda-1}\Big(\sum_{T\in\tau_h^b} h_T\Big)\|\chi\|^2_{2,\Omega}\\ &\le Ch^{4\lambda-1}\text{ meas }(\Gamma)\|\chi\|^2_{2,\Omega}\text{ with }\lambda\in]1/4,1[\\ \Longrightarrow |\mathcal{P}_h\chi-\mathcal{P}_{0h}\chi|_{1,\Omega_h} &\le Ch^{2\lambda-1/2}\|\chi\|_{2,\Omega}\text{ with }\lambda\in]1/4,1[. \end{aligned} \tag{4.15}$$

Hence, from (4.7–4.9, 4.14, 4.15), we get: for $\lambda\in[3/4,1[$,

$$\begin{aligned}\|\chi-\mathcal{P}_{0h}\chi\|_{0,\Omega_h} &\le C\Big[h^2|\chi|_{2,\Omega_h}+h^{2\lambda+1/2}\|\chi\|_{2,\Omega}\Big]\le Ch^2\|\chi\|_{2,\Omega}\ ;\\ |\chi-\mathcal{P}_{0h}\chi|_{1,\Omega_h} &\le C\Big[h|\chi|_{2,\Omega_h}+h^{2\lambda-1/2}\|\chi\|_{2,\Omega}\Big]\le Ch\|\chi\|_{2,\Omega}\end{aligned}$$

and

$$\|\chi-\mathcal{P}_{0h}\chi\|^2_{1,\Omega_h} = |\chi-\mathcal{P}_{0h}\chi|^2_{1,\Omega_h}+\|\chi-\mathcal{P}_{0h}\chi\|^2_{0,\Omega_h}\le Ch^2\|\chi\|^2_{2,\Omega}$$

which implies the result.

Case $s=3$. Since $H^3(\Omega)\hookrightarrow C^1(\bar\Omega)\equiv C^{0,1}(\bar\Omega)$, we have $\|\chi\|_{1,\infty,\Omega}\le C\|\chi\|_{3,\Omega}\forall\chi\in H^3(\Omega)\cap H^1_0(\Omega)$. Since $\|a_{4,T}-\tilde a_{4,T}\|_{\mathbb{R}^2}\le Ch_T^2$, using the mean-value theorem along the line segment $[\tilde a_{4,T},a_{4,T}]$ we have:

$$\forall\chi\in H^3(\Omega)\cap H^1_0(\Omega),\ |\chi(a_{4,T})|\le Ch_T^2\sup_{\vec\xi\in]\tilde a_{4,T},a_{4,T}[}|\frac{\partial\chi}{\partial s}(\vec\xi)|\le Ch_T^2|\chi|_{1,\infty,\Omega}\le Ch_T^2\|\chi\|_{3,\Omega}. \tag{4.16}$$

Hence, from (4.10–4.12, 4.16),

$$\begin{aligned}\|\mathcal{P}_h\chi-\mathcal{P}_{0h}\chi\|^2_{0,\Omega_h} &\le C\Big[\sum_{T\in\tau_h^b}h_T^4\|\chi\|^2_{3,\Omega}h_T^2\Big]\le Ch^5(\sum_{T\in\tau_h^b}h_T)\|\chi\|^2_{3,\Omega}\\ \Longrightarrow \|\mathcal{P}_h\chi-\mathcal{P}_{0h}\chi\|_{0,\Omega_h} &\le Ch^{5/2}(\text{meas}(\Gamma))^{1/2}\|\chi\|_{3,\Omega}\le Ch^{5/2}\|\chi\|_{3,\Omega}\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}|\mathcal{P}_h\chi-\mathcal{P}_{0h}\chi|^2_{1,\Omega_h} &\le C\Big[\sum_{T\in\tau_h^b}h_T^4\|\chi\|^2_{3,\Omega}\Big]\le Ch^3(\sum_{T\in\tau_h^b}h_T)\|\chi\|^2_{3,\Omega}\\ \Longrightarrow |\mathcal{P}_h\chi-\mathcal{P}_{0h}\chi|_{1,\Omega_h} &\le Ch^{3/2}(\text{meas}(\Gamma))^{1/2}\|\chi\|_{3,\Omega}\le Ch^{3/2}\|\chi\|_{3,\Omega}.\end{aligned} \tag{4.18}$$

Thus, from (4.7–4.9) and (4.17–4.18), we get (4.5–4.6):

$$\begin{aligned} \|\chi - \mathcal{P}_{0h}\chi\|_{0,\Omega_h} &\leq C\left[h^3|\chi|_{3,\Omega_h} + h^{5/2}\|\chi\|_{3,\Omega}\right] \leq Ch^{5/2}\|\chi\|_{3,\Omega}; \\ |\chi - \mathcal{P}_{0h}\chi|_{1,\Omega_h} &\leq C\left[h^2|\chi|_{3,\Omega_h} + h^{3/2}\|\chi\|_{3,\Omega}\right] \leq Ch^{3/2}\|\chi\|_{3,\Omega} \end{aligned} \tag{4.19}$$

and

$$\|\chi - \mathcal{P}_{0h}\chi\|^2_{1,\Omega_h} = |\chi - \mathcal{P}_{0h}\chi|^2_{1,\Omega_h} + \|\chi - \mathcal{P}_{0h}\chi\|^2_{0,\Omega_h} \leq Ch^3\|\chi\|^2_{3,\Omega},$$

and we get (4.6).

Remark 4.1. There is a loss of exponent of h by $1/2$ in (4.6) due to a 'crude' polygonal approximation of the curved boundary Γ. Moreover, from the proof of the Case $s = 3$, we find that it can **not** be improved upon even by assuming additional regularity of χ *i.e.* $\|\chi - \mathcal{P}_{0h}\chi\|_{r,\Omega_h} \leq Ch^{3-r-1/2}\|\chi\|_{3,\Omega}$ $\forall \chi \in H^s(\Omega) \cap H^1_0(\Omega)$ with $s > 3$. Hence it suggests to improve the boundary approximation, for example, by isoparametric mapping [9].

We will need the inverse inequalities [14, 17, 18]: $\forall \phi_h \in X_h$ (resp. $\Phi_h \in \mathbf{V}_h$), $\exists \gamma^* > 0$ (resp. $\exists \gamma > 0$) independent of h, such that

$$|\phi_h|_{1,\Omega_h} \leq \frac{\gamma^*}{h}|\phi_h|_{0,\Omega_h} \quad (\text{resp. } |\Phi_h|_{1,\Omega_h} \leq \frac{\gamma}{h}|\Phi_h|_{0,\Omega_h}) \tag{4.20}$$

and the following important well known estimates:

Proposition 4.2. [38] *For domains Ω and Ω_h defined earlier such that $\omega_h = \Omega - \Omega_h$ with $h \in]0, h_0[$, $0 < h_0 < 1, \forall \chi \in H^1(\Omega)$,*

$$\|\chi\|_{0,\omega_h} \leq Ch\|\chi\|_{1,\Omega} \quad \text{for some } C > 0. \tag{4.21}$$

Lemma 4.1 (p. 199 [36])**.** *Let $T \in \tau_h^b$ and $\tilde{T} \in \widetilde{\tau_h^b}$ be any pair of boundary triangles such that $T \subset \tilde{T}$, $\tilde{T} \in \tau_h^{\text{exact}}$ being the curved boundary triangle constructed from the boundary triangle $T \in \tau_h$ [see (3.1–3.6)]. Suppose that $\rho = \text{meas}\,(\tilde{T} - T)/\text{meas}\,T$. Let $\tilde{p}$ be a polynomial on $\tilde{T}$, which is a natural (polynomial) extension to $\tilde{T}$ of the polynomial 'p' defined on T. Then, $\exists C > 0$, depending only on the degree of p, such that*

$$\|\tilde{p}\|^2_{1,\tilde{T}-T} \leq C\rho(T)\|p\|^2_{1,T} \ \ \forall T \in \tau_h^b \subset \tau_h. \tag{4.22}$$

Corollary 4.1. *Let $\tilde{\phi}_h \in \tilde{X}_h$ be the natural extension to $\bar{\Omega}$ of the function $\phi_h \in X_h$ defined in (3.20). Then, $\exists C > 0$, independent of h, such that*

$$\|\tilde{\phi}_h\|^2_{1,\omega_h} = \sum_{T \subset \tilde{T} \in \tilde{\tau}_h^b} \|\tilde{\phi}_h\|^2_{1,\tilde{T}-T} \leq Ch\|\phi_h\|^2_{1,\Omega_h} \ \ \forall \phi_h \in X_h \text{ with } \tilde{\phi}_h \in \tilde{X}_h \tag{4.23}$$

and

$$\omega_h = \Omega - \Omega_h, \ \text{meas}(\omega_h) = O(h^2) \ [38]. \tag{4.24}$$

Proof. The result (4.23) is obtained from (4.22) by summing over all boundary triangles $T \in \tau_h^b$ with $T \subset \tilde{T} \in \widetilde{\tau_h^b}$ and increasing the right-hand side to include all interior triangles $T \in \tau_h^0$ and considering the fact that $\rho = O(h)$ $\forall \tau_h$ [36].

Proposition 4.3.

- *Let* $A_{ijkl} \in W^{1,\infty}(\Omega)\ \ \forall i,j,k,l = 1,2.$ (4.25)

- *Let the quadrature scheme (3.11) with $i = 1$, which is exact for $P_4(\hat{T})$, correspond to the definition (3.23) of $A_h^{NI}(\cdot,\cdot)$. Then, $\exists C > 0$, independent of h, such that $\forall \underline{\sigma}_h,\ \Phi_h \in \mathbf{V}_h$,*

$$|\tilde{A}(\underline{\sigma}_h, \Phi_h) - A_h^{NI}(\underline{\sigma}_h, \Phi_h)| \quad \leq \quad Ch\|A\|_{1,\infty,\Omega}\|\underline{\sigma}_h\|_{0,\Omega_h}\|\Phi_h\|_{0,\Omega_h}, \tag{4.26}$$

where $\tilde{A}(\cdot,\cdot)$ and $A_h^{NI}(\cdot,\cdot)$ are defined by (3.15) and (3.23) respectively,

$$\|A\|_{1,\infty,\Omega} = \sup_{\tilde{T}\in\tau_h^{\text{exact}}} \sum_{i,j,k,l=1}^{2} \|A_{ijkl}\|_{1,\infty,\tilde{T}}. \tag{4.27}$$

Proof. For fixed $i,j,k,l = 1,2$ (*i.e.* **no summation is to be understood with respect to twice repeated indices** i,j,k,l), $\forall T \in \tau_h$, set

$$E_T(A_{ijkl}\sigma_{hij}\phi_{hkl}) = \int_T A_{ijkl}\sigma_{hij}\phi_{hkl}\mathrm{d}T - \sum_{n=1}^{N_1} w_{n,T}^1 (A_{ijkl}\sigma_{hij}\phi_{hkl})(b_{n,T}^1), \tag{4.28}$$

$$\hat{E}(\widehat{A_{ijkl}}\hat{\sigma_{ij}}\hat{\phi_{kl}}) = \int_{\hat{T}} \widehat{A_{ijkl}}(\hat{x})\hat{\sigma_{ij}}(\hat{x})\hat{\phi_{kl}}(\hat{x})\mathrm{d}\hat{T} - \sum_{n=1}^{N_1} \hat{w_n^1}(\widehat{A_{ijkl}}\hat{\sigma_{ij}}\hat{\phi_{kl}})(\hat{b_n^1}) \tag{4.29}$$

with

$$E_T(A_{ijkl}\sigma_{hij}\phi_{hkl}) \quad = \quad (\det B_T)\hat{E}(\widehat{A_{ijkl}}\hat{\sigma_{ij}}\hat{\phi_{kl}}),\ b_{n,T}^1 = F_T(\hat{b_n^1}). \tag{4.30}$$

Then

$$\begin{aligned} |\tilde{A}(\underline{\sigma}_h, \Phi_h) - A_h^{NI}(\underline{\sigma}_h, \Phi_h)| \quad &= \quad |\int_{\Omega_h} A_{ijkl}\sigma_{hij}\phi_{hkl}\mathrm{d}\Omega_h - \sum_{T\in\tau_h}\sum_{n=1}^{N_1} w_{n,T}^1(A_{ijkl}\sigma_{hij}\phi_{hkl})(b_{n,T}^1)| \\ &\leq \quad \sum_{T\in\tau_h}\sum_{i,j,k,l=1}^{2} |E_T(A_{ijkl}\sigma_{hij}\phi_{hkl})|. \end{aligned} \tag{4.31}$$

$\forall$ fixed $i,j,k,l = 1,2,\ \hat{\sigma_{ij}},\ \hat{\phi_{kl}} \in P_2(\hat{T}),\ \widehat{A_{ijkl}} \in W^{1,\infty}(\hat{T})$ and hence

$$\begin{aligned} |\hat{E}(\widehat{A_{ijkl}}\hat{\sigma_{ij}}\hat{\phi_{kl}})| \quad &\leq \quad C\|\widehat{A_{ijkl}}\hat{\sigma_{ij}}\hat{\phi_{kl}}\|_{0,\infty,\hat{T}} \leq C\|\widehat{A_{ijkl}}\|_{0,\infty,\hat{T}}\|\hat{\sigma_{ij}}\hat{\phi_{kl}}\|_{0,\infty,\hat{T}} \\ &\leq \quad C\|\widehat{A_{ijkl}}\|_{1,\infty,\hat{T}}\|\hat{\sigma_{ij}}\hat{\phi_{kl}}\|_{0,\infty,\hat{T}}. \end{aligned} \tag{4.32}$$

$\forall$ fixed $i,j,k,l = 1,2,$ and for fixed $\hat{\sigma_{ij}},\ \hat{\phi_{kl}} \in P_2(\hat{T})$, define

$$\hat{\mathcal{E}}(\cdot) : W^{1,\infty}(\hat{T}) \longrightarrow \mathbb{R} \text{ by } \hat{\mathcal{E}}(\widehat{A_{ijkl}}) = \hat{E}(\widehat{A_{ijkl}}\hat{\sigma_{ij}}\hat{\phi_{kl}}). \tag{4.33}$$

From (4.32), (4.33) $\hat{\mathcal{E}}(\cdot)$ is a linear bounded functional on $W^{1,\infty}(\hat{T})$ with $\|\hat{\mathcal{E}}(\cdot)\| \leq C\|\hat{\sigma_{ij}}\hat{\phi_{kl}}\|_{0,\infty,\hat{T}}$ and $\hat{\mathcal{E}}(\hat{p_0}) = \hat{E}(\hat{p_0}\hat{\sigma_{ij}}\hat{\phi_{kl}}) = 0\ \ \forall \hat{p_0} \in P_0(\hat{T})$, since the quadrature formula in (4.29) is exact for $P_4(\hat{T})$.

Hence, by Bramble-Hilbert lemma, we have: $\forall$ fixed $i,j,k,l=1,2$ (with no summation),

$$|\hat{\mathcal{E}}(\widehat{A_{ijkl}})| = |\hat{E}(\widehat{A_{ijkl}}\hat{\sigma_{ij}}\hat{\phi_{kl}})| \leq C\|\hat{\sigma_{ij}}\hat{\phi_{kl}}\|_{0,\infty,\hat{T}}|\widehat{A_{ijkl}}|_{1,\infty,\hat{T}}. \tag{4.34}$$

But $\forall$ fixed $T \in \tau_h$,

$$|\widehat{A_{ijkl}}|_{1,\infty,\hat{T}} \leq Ch_T\|A_{ijkl}\|_{1,\infty,T} \quad \forall i,j,k,l=1,2 \ \ [16] \tag{4.35}$$

and

$$\begin{aligned}\|\hat{\sigma_{ij}}\hat{\phi_{kl}}\|_{0,\infty,\hat{T}} &\leq \|\hat{\sigma_{ij}}\|_{0,\infty,\hat{T}}\|\hat{\phi_{kl}}\|_{0,\infty,\hat{T}} \leq C\|\hat{\sigma_{ij}}\|_{0,\hat{T}}\|\hat{\phi_{kl}}\|_{0,\hat{T}} \quad \text{(norm equivalence in a f.d.v.s.)}\\ &\leq C(\det B_T)^{-1}\|\sigma_{hij}\|_{0,T}\|\phi_{hkl}\|_{0,T} \ \ [16].\end{aligned} \tag{4.36}$$

Hence, $\forall$ fixed $i,j,k,l=1,2$,

$$|\hat{E}((\widehat{A_{ijkl}})(\hat{\sigma_{ij}}\hat{\phi_{kl}}))| \leq Ch_T(\det B_T)^{-1}\|\sigma_{hij}\|_{0,T}\|\phi_{hkl}\|_{0,T}\|A_{ijkl}\|_{1,\infty,T} \quad \forall T \in \tau_h. \tag{4.37}$$

$$\Longrightarrow |E_T(A_{ijkl}\sigma_{hij}\phi_{hkl})| \leq Ch_T\|\sigma_{hij}\|_{0,T}\|\phi_{hkl}\|_{0,T}\|A_{ijkl}\|_{1,\infty,T} \ \ \forall T \in \tau_h \ \ \text{[using (4.30)]} \tag{4.38}$$

$$\begin{aligned}\Longrightarrow \sum_{T\in\tau_h}\sum_{i,j,k,l=1}^{2}|E_T(A_{ijkl}\sigma_{hij}\phi_{hkl})| &\leq \sum_{T\in\tau_h}\sum_{i,j,k,l=1}^{2} Ch_T\|\sigma_{hij}\|_{0,T}\|\phi_{hkl}\|_{0,T}\|A_{ijkl}\|_{1,\infty,T}\\ &\leq Ch\Bigg(\sum_{T\in\tau_h}\sum_{i,j,k,l=1}^{2}\|A_{ijkl}\|_{1,\infty,T}\Bigg)\Bigg(\sum_{T\in\tau_h}\sum_{i,j,k,l=1}^{2}\|\sigma_{hij}\|_{0,T}\|\phi_{hkl}\|_{0,T}\Bigg)\\ &\leq Ch\|A\|_{1,\infty,\Omega}\|\underline{\sigma}_h\|_{0,\Omega_h}\|\Phi_h\|_{0,\Omega_h}\end{aligned} \tag{4.39}$$

where

$$\|A\|_{1,\infty,\Omega} \geq \|A\|_{1,\infty,\Omega_h} = \sup_{T\in\tau_h}\sum_{i,j,k,l=1}^{2}\|A_{ijkl}\|_{1,\infty,T} \tag{4.40}$$

and the result (4.26) follows from (4.31, 4.39, 4.40).

Proposition 4.4. *Suppose that the conditions of Theorem 2.1 hold. Then,*

$$|\tilde{A}(\Psi,\Phi_h) + \tilde{b}(\Phi_h,u)| \leq Ch^{\frac{3}{2}}(1+\sqrt{h})\|u\|_{3,\Omega}\|\Phi_h\|_{1,\Omega_h}, \tag{4.41}$$

where $\tilde{A}(\cdot,\cdot)$ and $\tilde{b}(\cdot,\cdot)$ are defined by (3.15) and (3.16) respectively.

Proof. From the conditions of Theorem 2.1, $u \in H^3(\Omega)\cap H_0^2(\Omega)$ is the solution of $(\mathbf{P_G})$ with $\Psi = (\psi_{ij})_{i,j=1,2}$, $\psi_{ij} = a_{ijkl}u_{,kl} \in H^1(\Omega) \ \ \forall i,j=1,2$.

Then, $\forall\ \Phi_h \in \mathbf{V}_h$

$$\begin{aligned}\left|\tilde{A}(\Psi,\Phi_h)+\tilde{b}(\Phi_h,u)\right| &= \Big|\int\limits_{\Omega_h} A_{ijkl}\psi_{ij}\phi_{hkl}\mathrm{d}x+\int\limits_{\Omega_h}\phi_{hij,j}u_{,i}\mathrm{d}x\Big| \\ &= \Big|\int\limits_{\Omega_h} A_{ijkl}\psi_{ij}\phi_{hkl}\mathrm{d}x+\int\limits_{\Omega_h}\phi_{hij,j}u_{,i}\mathrm{d}x-\int\limits_{\Omega} A_{ijkl}\psi_{ij}\widetilde{\phi_{hkl}}\mathrm{d}x-\int\limits_{\Omega}\widetilde{\phi_{hij,j}}u_{,i}\mathrm{d}x\Big|\end{aligned}$$

[by virtue of (2.10)]

$$\leq \Big|\int\limits_{\omega_h} A_{ijkl}\psi_{ij}\widetilde{\phi_{hkl}}dx\Big|+\Big|\int\limits_{\omega_h}\widetilde{\phi_{hij,j}}u_{,i}dx\Big| \ \text{ with } \ \omega_h=\Omega-\Omega_h, \tag{4.42}$$

where $\tilde{\Phi}_h=(\tilde{\phi}_{hij})_{i,j=1,2}\in\tilde{\mathbf{V}}_h$ is a natural extension to $\bar{\Omega}$ of $\Phi_h\in\mathbf{V}_h$ defined in (3.21).

• Estimate for the first term on the right-hand side of (4.42)

$$\Big|\int\limits_{\omega_h} A_{ijkl}\psi_{ij}\widetilde{\phi_{hkl}}\mathrm{d}x\Big| \leq \sum_{i,j,k,l=1}^{2}\Big|\int\limits_{\omega_h} A_{ijkl}\psi_{ij}\widetilde{\phi_{hkl}}\mathrm{d}x\Big| = \sum_{k,l=1}^{2}\Big|\int\limits_{\omega_h} u_{,kl}\widetilde{\phi_{hkl}}\mathrm{d}x\Big|,$$

since $A_{ijkl}\psi_{ij}=A_{ijkl}a_{ijmn}u_{,mn}=\delta_{km}\delta_{ln}u_{,mn}=u_{,kl}$ [see (2.15)].

Then, since $u\in H^3(\Omega)$, $\widetilde{\phi_{hkl}}\in H^1(\Omega)$, we can use (4.21).

Hence, for fixed $k,l=1,2$

$$\begin{aligned}\Big|\int\limits_{\omega_h} u_{,kl}\widetilde{\phi_{hkl}}\ \mathrm{d}\Omega\Big| &\leq \|u_{,kl}\|_{0,\omega_h}\|\widetilde{\phi_{hkl}}\|_{0,\omega_h} \\ &\leq (Ch\|u_{,kl}\|_{1,\Omega})(Ch\|\widetilde{\phi_{hkl}}\|_{1,\Omega}) \leq Ch^2\|u\|_{3,\Omega}\|\widetilde{\phi_{hkl}}\|_{1,\Omega} \leq Ch^2\|u\|_{3,\Omega}\|\tilde{\Phi}_h\|_{1,\Omega}\end{aligned}$$

$$\Longrightarrow \Big|\int\limits_{\omega_h} A_{ijkl}\psi_{ij}\widetilde{\phi_{hkl}}\mathrm{d}x\Big| \leq Ch^2\|u\|_{3,\Omega}\|\Phi_h\|_{1,\Omega_h}. \tag{4.43}$$

• Estimate for the second term on the right-hand side of (4.42)

For fixed $i,j=1,2$,

$$\Big|\int\limits_{\omega_h}\widetilde{\phi_{hij,j}}u_{,i}\mathrm{d}x\Big| \leq (\text{meas }\omega_h)^{\frac{1}{2}}\|u_{,i}\widetilde{\phi_{hij,j}}\|_{0,\omega_h} \leq Ch\|u_{,i}\widetilde{\phi_{hij,j}}\|_{0,\omega_h} \tag{4.44}$$

[since $u_{,i}\in H^2(\Omega)\hookrightarrow C^0(\bar{\Omega})$, $\widetilde{\phi_{hij,j}}\in L^2(\Omega)\Longrightarrow u_{,i}\widetilde{\phi_{hij,j}}\in L^2(\Omega)$ and $\text{meas}(\omega_h)=O(h^2)$ (see (4.24))].

But for fixed $i,j=1,2$

$$\begin{aligned}\|u_{,i}\widetilde{\phi_{hij,j}}\|^2_{0,\omega_h} &\leq (\max_{x\in\overline{\Omega}}|u_{,i}(x)|)^2\left|\widetilde{\phi_{hij,j}}\right|^2_{0,\omega_h} \leq \|u_{,i}\|^2_{0,\infty,\Omega}\left|\widetilde{\phi_{hij,j}}\right|^2_{0,\omega_h} \\ &\leq C\|u_{,i}\|^2_{2,\Omega}\|\widetilde{\phi_{hij}}\|^2_{1,\omega_h} \leq C\|u\|^2_{3,\Omega}\|\widetilde{\phi_{hij}}\|^2_{1,\omega_h} \leq Ch\|u\|^2_{3,\Omega}\|\phi_{hij}\|^2_{1,\Omega_h},\end{aligned} \tag{4.45}$$

since the third inequality in (4.45) follows from $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ and the last inequality follows from (4.23). Then from (4.44) and (4.45),

$$\sum_{i,j=1}^{2} \Big| \int_{\omega_h} \widetilde{\phi_{hij,j}} u_{,i} \mathrm{d}x \Big| \quad \leq \quad \sum_{i,j=1}^{2} Ch^{\frac{3}{2}} \|\phi_{hij}\|_{1,\Omega_h} \|u\|_{3,\Omega} \quad \leq \quad Ch^{\frac{3}{2}} \|u\|_{3,\Omega} \|\Phi_h\|_{1,\Omega_h}. \tag{4.46}$$

Finally, from (4.42, 4.43, 4.46), we get the result (4.41).

Lemma 4.2. *Let the quadrature scheme (3.11) with $i = 2$, which is exact for $P_2(\hat{T})$, correspond to the definition (3.24) of $b_h^{NI}(\cdot,\cdot)$ and $\tilde{b}(\cdot,\cdot)$ be defined by (3.16).*

Then, $\forall \Phi \in \mathbf{V}(\Omega_h)$, $\exists$ a tensor-valued function $\Theta_h \in \mathbf{V}_h$, for which the following hold:

$$\tilde{b}(\Phi, \chi_h) = b_h^{NI}(\Theta_h, \chi_h) \;\; \forall \chi_h \in W_h \subset H_0^1(\Omega_h) \tag{4.47}$$

and $\exists C > 0$, independent of h, such that

$$\|\Phi - \Theta_h\|_{r,\Omega_h} \leq Ch^{1-r} \|\Phi\|_{1,\Omega}, \; (r = 0, 1). \tag{4.48}$$

Proof. For $\phi \in H^1(\Omega_h)$, we can associate a $\phi_h \in \mathbf{X}_h$ such that

$$\|\phi - \phi_h\|_{r,\Omega_h} \leq Ch^{1-r} \|\phi\|_{1,\Omega_h} \; (r = 0, 1) \tag{4.49}$$

for some $C > 0$ independent of h.
$\Rightarrow \forall \Phi = (\phi_{ij})_{i,j=1,2} \in \mathbf{V}(\Omega_h), \; \exists \Phi_h \in \mathbf{V}_h$ such that

$$\|\Phi - \Phi_h\|_{r,\Omega_h}^2 = \sum_{i,j=1}^{2} \|\phi_{ij} - \phi_{hij}\|_{r,\Omega_h}^2 \leq \sum_{i,j=1}^{2} C^2 h^{2-2r} \|\phi_{ij}\|_{1,\Omega_h}^2 = C^2 h^{2-2r} \|\Phi\|_{1,\Omega_h}^2 \; (r = 0, 1)$$
$$\Rightarrow \|\Phi - \Phi_h\|_{r,\Omega_h} \leq Ch^{1-r} \|\Phi\|_{1,\Omega_h} \;\; (r = 0, 1). \tag{4.50}$$

Define an auxiliary bilinear form $\mathcal{B}_h(\cdot,\cdot) : W_h \times W_h \longrightarrow \mathcal{R}$ by:

$$\mathcal{B}_h(z_h, \mu_h) = \sum_{T \in \tau_h} \sum_{n=1}^{N_2} w_{n,T}^2 (\nabla z_h \cdot \nabla \mu_h)(b_{n,T}^2) \quad \forall z_h, \; \mu_h \in W_h \tag{4.51}$$

with $w_{n,T}^2 > 0, \; 1 \leq n \leq N_2,$ which corresponds to the quadrature scheme 3.11) with $i = 2$ exact for $P_2(\hat{T})$, and a linear form $l_h(\cdot) : W_h \longrightarrow \mathbb{R}$ by:

$$l_h(\mu_h) = \tilde{b}(\Phi, \mu_h) - b_h^{NI}(\Phi_h, \mu_h) \quad \forall \mu_h \in W_h \tag{4.52}$$

for fixed elements $\Phi \in \mathbf{V}(\Omega_h), \; \Phi_h \in \mathbf{V}_h$ satisfying (4.50).

$\mathcal{B}_h(\cdot,\cdot)$ is continuous on $W_h \times W_h$ and W_h-elliptic.

In fact,

$$\begin{aligned}
\mathcal{B}_h(\mu_h,\mu_h) &= \sum_{T\in\tau_h}\sum_{n=1}^{N_2} w_{n,T}^2(\nabla\mu_h\cdot\nabla\mu_h)(b_{n,T}^2)\\
&\geq C|\mu_h|_{1,\Omega_h}^2 = C|\tilde{\mu_h}|_{1,\Omega}^2 \ \text{(since } \tilde{\mu}_h=0 \text{ outside } \Omega_h)\\
&\geq C(\Omega)\|\tilde{\mu_h}\|_{1,\Omega}^2 \ \text{(by virtue of Friedrichs' inequality)}\\
&= C(\Omega)\|\mu_h\|_{1,\Omega_h}^2 \quad \text{(since } \tilde{\mu}_h=\mu_h \text{ in } \Omega_h,\ \tilde{\mu}_h=0 \text{ outside } \Omega_h)\\
\Longrightarrow \mathcal{B}_h(\mu_h,\mu_h) &\geq C(\Omega)\|\mu_h\|_{1,\Omega_h}^2 \quad \forall\mu_h\in W_h. \qquad (4.53)
\end{aligned}$$

$l_h(\cdot)$ is continuous onW_h.

Hence, from Lax-Milgram lemma, $\exists$ a unique $z_h\in W_h$ such that

$$\mathcal{B}_h(z_h,\mu_h)=\tilde{b}(\Phi,\mu_h)-b_h^{NI}(\Phi_h,\mu_h)\ \ \forall\mu_h\in W_h \qquad (4.54)$$

for fixed $\Phi\in\mathbf{V}(\Omega_h)$ and $\Phi_h\in\mathbf{V}_h$ satisfying (4.50).

Choose $\underline{\sigma}_h=(z_h\delta_{ij})_{i,j=1,2}$ with $z_h\in W_h$. Then

$$\underline{\sigma}_h\in\mathbf{V}_h \text{ with } \|\underline{\sigma}_h\|_{1,\Omega_h}=\sqrt{2}\|z_h\|_{1,\Omega_h}, \qquad (4.55)$$

and

$$\begin{aligned}
b_h^{NI}(\underline{\sigma}_h,\mu_h)=\mathcal{B}_h(z_h,\mu_h)=\tilde{b}(\Phi,\mu_h)-b_h^{NI}(\Phi_h,\mu_h) \qquad &\text{[using (4.54)]}\\
\Longrightarrow b_h^{NI}(\underline{\sigma}_h,\mu_h)+b_h^{NI}(\Phi_h,\mu_h)=b_h^{NI}(\underline{\sigma}_h+\Phi_h,\mu_h)=\tilde{b}(\Phi,\mu_h) \qquad &\forall\mu_h\in W_h\\
\Longrightarrow \text{the result (4.47) holds with } \Theta_h=(\underline{\sigma}_h+\Phi_h)\in\mathbf{V}_h,\Phi_h \text{ satisfying (4.50).} \qquad &(4.56)
\end{aligned}$$

• **Estimate for** $\|\Phi-\Theta_h\|_{1,\Omega_h}$

$\forall$ fixed elements $\Phi\in\mathbf{V}(\Omega_h),\ \Phi_h\in\mathbf{V}_h$ satisfying (4.50), we get from (4.53, 4.54) and the continuity of $l_h(\cdot)$:

$$\begin{aligned}
C\|z_h\|_{1,\Omega_h}^2\leq\mathcal{B}_h(z_h,z_h) &\leq M\|z_h\|_{1,\Omega_h}(\|\Phi\|_{1,\Omega_h}+\|\Phi_h\|_{1,\Omega_h})\\
&\leq M\|z_h\|_{1,\Omega_h}(2\|\Phi\|_{1,\Omega_h}+\|\Phi-\Phi_h\|_{1,\Omega_h})\\
&\leq CM\|z_h\|_{1,\Omega_h}\|\Phi\|_{1,\Omega_h} \text{ [by virtue of (4.50)]}\\
\Longrightarrow \|z_h\|_{1,\Omega_h} &\leq C\|\Phi\|_{1,\Omega_h}. \qquad (4.57)
\end{aligned}$$

Hence from (4.55) and (4.57), and the definition of Θ_h, we have

$$\|\underline{\sigma}_h\|_{1,\Omega_h}=\sqrt{2}\|z_h\|_{1,\Omega_h}\leq C\|\Phi\|_{1,\Omega_h}. \qquad (4.58)$$

$$\Longrightarrow \|\Theta_h-\Phi_h\|_{1,\Omega_h}\leq C\|\Phi\|_{1,\Omega_h}. \qquad (4.59)$$

$$\begin{aligned}
\Longrightarrow \|\Phi-\Theta_h\|_{1,\Omega_h} &\leq \|\Phi-\Phi_h\|_{1,\Omega_h}+\|\Phi_h-\Theta_h\|_{1,\Omega_h}\\
&\leq C\|\Phi\|_{1,\Omega_h}\leq C\|\Phi\|_{1,\Omega}. \qquad (4.60)
\end{aligned}$$

• **Estimate for** $\|\Phi-\Theta_h\|_{0,\Omega_h}$

Since Ω is convex, $\forall g\in L^2(\Omega),$ define $\chi\in H^2(\Omega)\cap H_0^1(\Omega)$ as the unique solution of:

$$-\Delta\chi=g \text{ in } \Omega,\ \ \chi|_\Gamma=0 \qquad \text{with } \ \|\chi\|_{2,\Omega}\leq C\|g\|_{0,\Omega}. \qquad (4.61)$$

$\forall z_h \in W_h$ with $\tilde{z_h} \in \tilde{W}_h$, we have

$$\|z_h\|_{0,\Omega_h} = \|\tilde{z}_h\|_{0,\Omega} = \sup_{g\in L^2(\Omega)} \frac{|\int_\Omega \tilde{z}_h g \mathrm{d}\Omega|}{\|g\|_{0,\Omega}}\,. \tag{4.62}$$

Then from (4.61),

$$\begin{aligned} -\int_\Omega (\Delta\chi).\tilde{z}_h\ \mathrm{d}\Omega &= \int_\Omega g\tilde{z}_h\ \mathrm{d}\Omega \quad \forall \tilde{z}_h \in \tilde{W}_h \subset H_0^1(\Omega) \\ \Longrightarrow \int_{\Omega_h} (\nabla\chi).\nabla z_h\ \mathrm{d}\Omega_h &= \int_{\Omega_h} g z_h\ \mathrm{d}\Omega_h \quad \forall z_h \in W_h,\ \text{since } \tilde{z}_h = 0\ \text{ in } \Omega - \Omega_h. \end{aligned} \tag{4.63}$$

Hence, using (4.54), we get:

$$\begin{aligned} |\int_{\Omega_h} g z_h\ \mathrm{d}\Omega_h| = |\int_{\Omega_h} \nabla\chi.\nabla z_h\ \mathrm{d}\Omega_h| \le |\int_{\Omega_h} \nabla(\chi-\chi_h).\nabla z_h\ \mathrm{d}\Omega_h| \ + \ |\int_{\Omega_h} \nabla\chi_h.\nabla z_h\ \mathrm{d}\Omega_h \\ - \sum_{T\in\tau_h}\sum_{n=1}^{N_2} w_{n,T}^2(\nabla\chi_h.\nabla z_h)(b_{n,T}^2)| + |\tilde{b}(\Phi,\chi_h) - b_h^{NI}(\Phi_h,\chi_h)| \end{aligned}$$

$\forall \chi_h \in W_h$ and for fixed elements $\Phi \in \mathbf{V}(\Omega_h),\ \Phi_h \in \mathbf{V}_h$ satisfying (4.50).

Then, for $\chi_h = \mathcal{P}_{0h}\chi \in W_h$ with $\chi \in H^2(\Omega)\cap H_0^1(\Omega)$ defined in (4.4), we have:

$$\int_{\Omega_h} \nabla(\mathcal{P}_{0h}\chi)\cdot\nabla z_h\ \mathrm{d}\Omega_h = \sum_{T\in\tau_h}\sum_{n=1}^{N_2} w_{n,T}^2((\nabla\mathcal{P}_{0h}\chi)\cdot\nabla z_h)(b_{n,T}^2),$$

and consequently,

$$\begin{aligned} |\int_{\Omega_h} g z_h\ \mathrm{d}\Omega_h| \quad &\le \quad |\int_{\Omega_h} \nabla(\chi-\mathcal{P}_{0h}\chi).\nabla z_h\ \mathrm{d}\Omega_h| + |\tilde{b}(\Phi-\Phi_h,\mathcal{P}_{0h}\chi-\chi)| \\ &\quad + |\tilde{b}(\Phi-\Phi_h,\chi) - b(\Phi-\tilde{\Phi_h},\chi)| + |b(\Phi-\tilde{\Phi_h},\chi)|. \end{aligned} \tag{4.64}$$

• **Estimate for the first term on the right-hand side of (4.64)**

Using (4.5) and (4.57),

$$\begin{aligned} \left|\int_{\Omega_h} \nabla(\chi-\mathcal{P}_{0h}\chi).\nabla z_h\ \mathrm{d}\Omega_h\right| \quad &\le \quad |\chi-\mathcal{P}_{0h}\chi|_{1,\Omega_h}|z_h|_{1,\Omega_h} \le Ch\|g\|_{0,\Omega}\|z_h\|_{1,\Omega_h} \\ &\le \quad Ch\|g\|_{0,\Omega}\|\Phi\|_{1,\Omega_h} \quad [\text{by (4.61)}]. \end{aligned} \tag{4.65}$$

• **Estimate for the second term on the right-hand side of (4.64)**

Using the continuity of $\tilde{b}(\cdot,\cdot)$, (4.5, 4.50) and (4.61), we have

$$|\tilde{b}(\Phi-\Phi_h,\mathcal{P}_{0h}\chi-\chi)| \le \tilde{m}\|\Phi-\Phi_h\|_{1,\Omega_h}\|\mathcal{P}_{0h}\chi-\chi\|_{1,\Omega_h} \le Ch\|\Phi\|_{1,\Omega_h}\|g\|_{0,\Omega}. \tag{4.66}$$

• **Estimate for the third term on the right-hand side of (4.64)**

$$\begin{aligned}|\tilde{b}(\Phi-\Phi_h,\chi)-b(\Phi-\tilde{\Phi}_h,\chi)| &= |\int_{\Omega_h}(\phi_{ij}-\phi_{hij})_{,j}\chi_{,i}\mathrm{d}\Omega_h-\int_{\Omega}(\phi_{ij}-\tilde{\phi}_{hij})_{,j}\chi_{,i}\mathrm{d}\Omega| \\ &\le |\int_{\omega_h}(\phi_{ij}-\tilde{\phi}_{hij})_{,j}\chi_{,i}\mathrm{d}x| \le \|(\phi_{ij}-\tilde{\phi}_{hij})_{,j}\|_{0,\omega_h}\|\chi_{,i}\|_{0,\omega_h} \\ &\le Ch\|\chi\|_{2,\Omega}\sum_{i,j=1}^{2}\|(\phi_{ij}-\tilde{\phi}_{hij})_{,j}\|_{0,\omega_h} \quad \text{[using (4.21)]}. \end{aligned} \tag{4.67}$$

For fixed $i,j=1,2$ $\|(\phi_{ij}-\tilde{\phi}_{hij})_{,j}\|_{0,\omega_h} \le \|\phi_{ij}-\tilde{\phi}_{hij}\|_{1,\omega_h} \le \|\phi_{ij}\|_{1,\omega_h}+\|\tilde{\phi}_{hij}\|_{1,\omega_h}$.

From (4.23), $\forall i,j=1,2,\ \tilde{\phi}_{hij}\in\tilde{X}_h$ defined in (3.20),

$$\|\tilde{\phi}_{hij}\|_{1,\omega_h} \le Ch^{1/2}\|\phi_{hij}\|_{1,\Omega_h} \le Ch^{1/2}\|\Phi_h\|_{1,\Omega_h}. \tag{4.68}$$

Then, using (4.50),

$$\|\tilde{\phi}_{hij}\|_{1,\omega_h} \le Ch^{1/2}(\|\Phi-\Phi_h\|_{1,\Omega_h}+\|\Phi\|_{1,\Omega_h}) \le Ch^{1/2}(\|\Phi\|_{1,\Omega_h}+\|\Phi\|_{1,\Omega_h}) \le Ch^{1/2}\|\Phi\|_{1,\Omega}. \tag{4.69}$$

From (4.67–4.69), we have

$$\sum_{i,j=1}^{2}\|(\phi_{ij}-\tilde{\phi}_{hij})_{,j}\|_{0,\omega_h} \le \|\Phi\|_{1,\Omega}+Ch^{1/2}\|\Phi\|_{1,\Omega} \le C\|\Phi\|_{1,\Omega}. \tag{4.70}$$

Hence, from (4.67),

$$|\tilde{b}(\Phi-\Phi_h,\chi)-b(\Phi-\tilde{\Phi}_h,\chi)| \le Ch\|\chi\|_{2,\Omega}\|\Phi\|_{1,\Omega} \le Ch\|g\|_{0,\Omega}\|\Phi\|_{1,\Omega}. \tag{4.71}$$

• **Estimate for the fourth term on the right-hand side of (4.64)**
For $\chi\in H^2(\Omega)\cap H_0^1(\Omega)$,

$$|b(\Phi-\tilde{\Phi}_h,\chi)| \le C\|\Phi-\tilde{\Phi}_h\|_{0,\Omega}|\chi|_{2,\Omega} \tag{4.72}$$

with

$$\|\Phi-\tilde{\Phi}_h\|_{0,\Omega}^2 = \|\Phi-\Phi_h\|_{0,\Omega_h}^2+\|\Phi-\Phi_h\|_{0,\omega_h}^2 \quad (\omega_h=\Omega-\Omega_h). \tag{4.73}$$

For fixed $i, j = 1, 2,\ (\phi_{ij} - \tilde{\phi}_{hij}) \in H^1(\Omega)$ and hence from (4.21),

$$\|\phi_{ij} - \tilde{\phi}_{hij}\|^2_{0,\omega_h} \leq C^2 h^2 \|\phi_{ij} - \tilde{\phi}_{hij}\|^2_{1,\Omega}. \tag{4.74}$$

But

$$\begin{aligned} \|\phi_{ij} - \tilde{\phi}_{hij}\|^2_{1,\Omega} &= \|\phi_{ij} - \phi_{hij}\|^2_{1,\Omega_h} + \|\phi_{ij} - \tilde{\phi}_{hij}\|^2_{1,\omega_h} \\ &\leq C\|\phi_{ij}\|^2_{1,\Omega_h} + (\|\phi_{ij}\|_{1,\omega_h} + \|\tilde{\phi}_{hij}\|_{1,\omega_h})^2 \leq C\left[\|\phi_{ij}\|^2_{1,\Omega} + \|\phi_{ij}\|^2_{1,\Omega} + \|\tilde{\phi}_{hij}\|^2_{1,\omega_h}\right] \\ \Longrightarrow \|\phi_{ij} - \tilde{\phi}_{hij}\|^2_{1,\Omega} &\leq C\left[\|\Phi\|^2_{1,\Omega} + Ch\|\Phi\|^2_{1,\Omega}\right] \ \text{[using (4.23)]} \\ \Longrightarrow \sum_{i,j=1}^{2} \|\phi_{ij} - \tilde{\phi}_{hij}\|^2_{1,\Omega} &\leq C\|\Phi\|^2_{1,\Omega} \Longrightarrow \|\Phi - \tilde{\Phi}_h\|_{1,\Omega} \leq C\|\Phi\|_{1,\Omega}. \end{aligned}$$

Hence

$$\|\Phi - \tilde{\Phi}_h\|_{0,\omega_h} \leq Ch\|\Phi\|_{1,\Omega} \qquad \text{[from (4.74)]}.$$

Then from (4.73)

$$\|\Phi - \tilde{\Phi}_h\|^2_{0,\Omega} \leq Ch^2\|\Phi\|^2_{1,\Omega} + C^2h^2\|\Phi\|^2_{1,\Omega} \leq C^2h^2\|\Phi\|^2_{1,\Omega}.$$

Hence,

$$|b(\Phi - \tilde{\Phi}_h, \chi)| \quad \leq \quad Ch\|\Phi||_{1,\Omega}\|\chi\|_{2,\Omega} \leq Ch\|g\|_{0,\Omega}\|\Phi\|_{1,\Omega} \ \text{[using (4.61)]}. \tag{4.75}$$

Substituting the estimates (4.65–4.66, 4.71, 4.75) in (4.64) and using the result $\|\Phi\|_{1,\Omega_h} \leq \|\Phi\|_{1,\Omega}$, we have

$$\begin{aligned} |\int_\Omega g\tilde{z}_h \,\mathrm{d}\Omega| &= |\int_{\Omega_h} gz_h \,\mathrm{d}\Omega_h| \leq Ch\|g\|_{0,\Omega}(\|\Phi\|_{1,\Omega} + \|\Phi\|_{1,\Omega} + \|\Phi\|_{1,\Omega} + \|\Phi\|_{1,\Omega}) \\ &\leq Ch\|g\|_{0,\Omega}\|\Phi\|_{1,\Omega}. \end{aligned} \tag{4.76}$$

Then, from (4.62, 4.76), we have

$$\|z_h\|_{0,\Omega_h} = \|\tilde{z}_h\|_{0,\Omega} = \sup_{g\in L^2(\Omega)} \frac{|\int_\Omega \tilde{z}_h g \mathrm{d}\Omega|}{\|g\|_{0,\Omega}} \leq Ch\|\Phi\|_{1,\Omega}.$$

Since, $\underline{\sigma}_h = (z_h\delta_{ij})$ with $z_h \in W_h$ and $\underline{\Theta}_h = \underline{\sigma}_h + \Phi_h \in \mathbf{V}_h$, [see (4.56)], we have

$$\|\Theta_h - \Phi_h\|_{0,\Omega_h} = \|\underline{\sigma}_h\|_{0,\Omega_h} = \sqrt{2}\|z_h\|_{0,\Omega_h} \leq Ch\|\Phi\|_{1,\Omega}. \tag{4.77}$$

Hence

$$\begin{aligned} \|\Phi - \Theta_h\|_{0,\Omega_h} &\leq \|\Phi - \Phi_h\|_{0,\Omega_h} + \|\Phi_h - \Theta_h\|_{0,\Omega_h} \\ &\leq Ch\|\Phi\|_{1,\Omega_h} + Ch\|\Phi\|_{1,\Omega} \ \text{[from (4.50) and (4.77)]} \\ \Longrightarrow \|\Phi - \Theta_h\|_{0,\Omega_h} &\leq Ch\|\Phi\|_{1,\Omega}. \end{aligned} \tag{4.78}$$

Thus, (4.60) and (4.78) establish the result (4.48).

Theorem 4.1.

- *Suppose that the assumptions of Theorem 2.1, Propositions 4.3 and 4.4 hold.*

- *Let* $\{\tau_h\}$ *(resp.* $\{\tau_h^{\text{exact}}\}$*) be a family of quasi-uniform, regular, admissible triangulations* [17] *of* $\bar{\Omega}_h = \Omega_h \cup \Gamma_h$ *(resp.* $\bar{\Omega} = \Omega \cup \Gamma$ *) defined in (3.3) with* $0 < h < h_0,\ h_0 \in]0, 1[$.

- *Let the quadrature scheme (3.11) with* $i = 1$ *[resp.* $i = 2$*], which is exact for* $P_4(\hat{T})$ *[resp.* $P_2(\hat{T})$*] correspond to the definition (3.23) of* $A_h^{NI}(\cdot,\cdot)$ *[resp. (3.24) of* $b_h^{NI}(\cdot,\cdot)$*].*
Then, $\exists C > 0$*, independent of* h*, such that*

$$\|\Psi - \Psi_h\|_{0,\Omega_h} \le C\sqrt{h}\Big[\|u\|_{3,\Omega} + h^{1/2}\|\Psi\|_{1,\Omega}\Big]; \tag{4.79}$$

$$\|u - u_h\|_{1,\Omega_h} \le C\sqrt{h}\Big[\|u\|_{3,\Omega} + h^{1/2}\|\Psi\|_{1,\Omega}\Big], \tag{4.80}$$

where $(\Psi, u) \in \mathbf{V} \times W$ *[resp.* $(\Psi_h, u_h) \in \mathbf{V}_h \times W_h$*] is the unique solution of* $(\mathbf{Q})$ [*resp.* $(\mathbf{Q_h})$].

Proof. Since $\Psi \downarrow_{\Omega_h} \in \mathbf{V}(\Omega_h)$, from Lemma 4.2, $\exists \Theta_h \in \mathbf{V}_h$ such that

$$b_h^{NI}(\Theta_h, \chi_h) = \tilde{b}(\Psi, \chi_h) = \int\limits_{\Omega_h} \psi_{ij,j}\chi_{h,i}\,\mathrm{d}\Omega_h \quad \forall \chi_h \in W_h \tag{4.81}$$

and

$$\|\Psi - \Theta_h\|_{r,\Omega_h} \le Ch^{1-r}\|\Psi\|_{1,\Omega} \quad (r = 0, 1). \tag{4.82}$$

Then, from (4.81), the definition of $\tilde{W}_h$, and the second equation of (2.10),

$$b_h^{NI}(\Theta_h, \chi_h) = \int\limits_{\Omega} \psi_{ij,j}\tilde{\chi}_{h,i}\,\mathrm{d}\Omega = b(\Psi, \tilde{\chi}_h) = -\langle f, \tilde{\chi}_h\rangle_{0,\Omega} = -\langle f, \chi_h\rangle_{0,\Omega_h} \quad \forall \chi_h \in W_h \text{ with } \tilde{\chi}_h \in \tilde{W}_h. \tag{4.83}$$

Hence using the second equation of (3.25) and (4.83), we have

$$b_h^{NI}(\Psi_h - \Theta_h, \chi_h) = -\langle f, \chi_h\rangle_{0,\Omega_h} + \langle f, \chi_h\rangle_{0,\Omega_h} = 0 \quad \forall \chi_h \in W_h. \tag{4.84}$$

From the ellipticity of $A_h^{NI}(\cdot,\cdot)$ in (3.27), we have for $\Theta_h \in \mathbf{V}_h$ corresponding to $\Psi \in \mathbf{V}(\Omega_h)$ satisfying (4.81–4.83),

$$\begin{aligned}
\alpha\|\Psi_h-\Theta_h\|^2_{0,\Omega_h} \le A_h^{NI}(\Psi_h-\Theta_h,\Psi_h-\Theta_h) &= \Big[\tilde{A}(\Psi-\Theta_h,\Psi_h-\Theta_h)-\tilde{A}(\Psi,\Psi_h-\Theta_h)\Big]\\
&\quad+\Big[\tilde{A}(\Theta_h,\Psi_h-\Theta_h)-A_h^{NI}(\Theta_h,\Psi_h-\Theta_h)\Big]\\
&\quad-b_h^{NI}(\Psi_h-\Theta_h,u_h) \quad \text{[using the first equation of (3.25)]}\\
&=\tilde{A}(\Psi-\Theta_h,\Psi_h-\Theta_h)-\Big[\tilde{A}(\Psi,\Psi_h-\Theta_h)+\tilde{b}(\Psi_h-\Theta_h,u)\Big]\\
&\quad+\tilde{b}(\Psi_h-\Theta_h,u)+\Big[\tilde{A}(\Theta_h,\Psi_h-\Theta_h)-A_h^{NI}(\Theta_h,\Psi_h-\Theta_h)\Big]\\
&\quad-b_h^{NI}(\Psi_h-\Theta_h,u_h)\\
&=\tilde{A}(\Psi-\Theta_h,\Psi_h-\Theta_h)+\tilde{b}(\Psi_h-\Theta_h,u-\mathcal{P}_{0h}u)\\
&\quad-\Big[\tilde{A}(\Psi,\Psi_h-\Theta_h)+\tilde{b}(\Psi_h-\Theta_h,u)\Big]\\
&\quad+\Big[\tilde{A}(\Theta_h,\Psi_h-\Theta_h)-A_h^{NI}(\Theta_h,\Psi_h-\Theta_h)\Big]\\
&\quad+\Big[\tilde{b}(\Psi_h-\Theta_h,\mathcal{P}_{0h}u)-b_h^{NI}(\Psi_h-\Theta_h,\mathcal{P}_{0h}u)\Big], \qquad (4.85)
\end{aligned}$$

which has been obtained by using (4.84) and the definition (4.4) of $\mathcal{P}_{0h}u \in W_h$. Since the quadrature scheme (3.11) with $i = 2$ corresponding to the definition of $b_h^{NI}(\cdot,\cdot)$ is exact for $P_2(\hat{T})$,

$$\tilde{b}(\Psi_h-\Theta_h,\mathcal{P}_{0h}u)-b_h^{NI}(\Psi_h-\Theta_h,\mathcal{P}_{0h}u)=0. \qquad (4.86)$$

Hence, applying the triangular inequality, the continuity of $\tilde{A}(\cdot,\cdot)$ and $\tilde{b}(\cdot,\cdot)$ and finally dividing both sides by $\alpha\|\Psi_h-\Theta_h\|_{0,\Omega_h}$, we get from (4.85–4.86):

$$\begin{aligned}
\|\Psi_h-\Theta_h\|_{0,\Omega_h} \le C\Bigg\{&\left[\frac{\tilde{M}\|\Psi-\Theta_h\|_{0,\Omega_h}\|\Psi_h-\Theta_h\|_{0,\Omega_h}}{\|\Psi_h-\Theta_h\|_{0,\Omega_h}}+\frac{\tilde{m}\|\Psi_h-\Theta_h\|_{1,\Omega_h}\|u-\mathcal{P}_{0h}u\|_{1,\Omega_h}}{\|\Psi_h-\Theta_h\|_{0,\Omega_h}}\right] \qquad (4.87)\\
&+\left[\frac{\tilde{A}(\Psi,\Psi_h-\Theta_h)+\tilde{b}(\Psi_h-\Theta_h,u)|}{\|\Psi_h-\Theta_h\|_{0,\Omega_h}}\right]+\left[\frac{|\tilde{A}(\Theta_h,\Psi_h-\Theta_h)-A_h^{NI}(\Theta_h,\Psi_h-\Theta_h)|}{\|\Psi_h-\Theta_h\|_{0,\Omega_h}}\right]\Bigg\}.
\end{aligned}$$

Using $\|\Psi-\Theta_h\|_{0,\Omega_h} \le Ch\|\Psi\|_{1,\Omega}$ [from (4.82)]

$$\begin{aligned}
&\|u-\mathcal{P}_{0h}u\|_{1,\Omega_h} \le Ch^{3/2}\|u\|_{3,\Omega} \text{ for } u\in H^3(\Omega)\cap H_0^2(\Omega), \text{ [from (4.6)]},\\
&\|\Psi_h-\Theta_h\|_{1,\Omega_h} \le \frac{C}{h}\|\Psi_h-\Theta_h\|_{0,\Omega_h}, \text{ [from (4.20)]}, \qquad (4.88)
\end{aligned}$$

and [from Propositions 4.3 and 4.4],

$$|\tilde{A}(\Theta_h,\Psi_h-\Theta_h)-A_h^{NI}(\Theta_h,\Psi_h-\Theta_h)| \le Ch\|A\|_{1,\infty,\Omega}\|\Theta_h\|_{0,\Omega_h}\|\Psi_h-\Theta_h\|_{0,\Omega_h} \qquad (4.89)$$

$$|\tilde{A}(\Psi,\Psi_h-\Theta_h)+\tilde{b}(\Psi_h-\Theta_h,u)| \le Ch^{3/2}(1+\sqrt{h})\|u\|_{3,\Omega}\|\Psi_h-\Theta_h\|_{1,\Omega_h}, \qquad (4.90)$$

we have

$$\|\Psi_h - \Theta_h\|_{0,\Omega_h} \le C\Big[h\|\Psi\|_{1,\Omega} + \sqrt{h}\|u\|_{3,\Omega} + \sqrt{h}(1+\sqrt{h})\|u\|_{3,\Omega} + h\|A\|_{1,\infty,\Omega}\|\Theta\|_{0,\Omega_h}\Big] \tag{4.91}$$

with $C > 0$, independent of h.

Since $\|\Theta_h\|_{0,\Omega_h} \le \|\Psi - \Theta_h\|_{0,\Omega_h} + \|\Psi\|_{0,\Omega_h} \le Ch\|\Psi\|_{1,\Omega} + \|\Psi\|_{1,\Omega} \le C\|\Psi\|_{1,\Omega}$, we get from (4.91):

$$\|\Psi_h - \Theta_h\|_{0,\Omega_h} \quad \le \quad C\Big[h^{1/2}(2+\sqrt{h})\|u\|_{3,\Omega} + h\|\Psi\|_{1,\Omega}\Big] \le Ch^{1/2}\Big[\|u\|_{3,\Omega} + h^{1/2}\|\Psi\|_{1,\Omega}\Big].$$

Hence

$$\begin{aligned}
\|\Psi - \Psi_h\|_{0,\Omega_h} &\le \|\Psi - \Theta_h\|_{0,\Omega_h} + \|\Psi_h - \Theta_h\|_{0,\Omega_h} \\
&\le C\Big[h\|\Psi\|_{1,\Omega} + h^{1/2}(\|u\|_{3,\Omega} + h^{1/2}\|\Psi\|_{1,\Omega})\Big] \\
\Longrightarrow \|\Psi - \Psi_h\|_{0,\Omega_h} &\le Ch^{1/2}\Big[\|u\|_{3,\Omega} + h^{1/2}\|\Psi\|_{1,\Omega}\Big] \text{ with } C > 0, \text{ independent of } h. \qquad (4.92)
\end{aligned}$$

Now, we will prove (4.80).

- **Estimate for** $\|u - u_h\|_{1,\Omega_h}$

From the discrete Brezzi-Babuška condition (3.28) for $b_h^{NI}(\cdot,\cdot)$, we have

$$\beta_1\|u_h - \mathcal{P}_{0h}u\|_{1,\Omega_h} \le \sup_{\Phi_h \in \mathbf{V}_h - \{0\}} \frac{|b_h^{NI}(\Phi_h, u_h - \mathcal{P}_{0h}u)|}{\|\Phi_h\|_{1,\Omega_h}}\,. \tag{4.93}$$

But

$$\begin{aligned}
b_h^{NI}(\Phi_h, u_h - \mathcal{P}_{0h}u) &= \tilde{b}(\Phi_h, u - \mathcal{P}_{0h}u) + b_h^{NI}(\Phi_h, u_h) - \tilde{b}(\Phi_h, u) + \Big[\tilde{b}(\Phi_h, \mathcal{P}_{0h}u) - b_h^{NI}(\Phi_h, \mathcal{P}_{0h}u)\Big] \\
&\quad + \tilde{A}(\Psi, \Phi_h) - \tilde{A}(\Psi, \Phi_h),
\end{aligned}$$

where

$$\tilde{b}(\Phi_h, \mathcal{P}_{0h}u) - b_h^{NI}(\Phi_h, \mathcal{P}_{0h}u) = 0 \ \ \forall \Phi_h \in \mathbf{V}_h$$

$$\begin{aligned}
\Longrightarrow b_h^{NI}(\Phi_h, u_h - \mathcal{P}_{0h}u) &= \tilde{b}(\Phi_h, u - \mathcal{P}_{0h}u) - A_h^{NI}(\Psi_h, \Phi_h) - \Big[\tilde{A}(\Psi, \Phi_h) + \tilde{b}(\Phi_h, u)\Big] \\
&\quad + \tilde{A}(\Psi - \Psi_h, \Phi_h) + \tilde{A}(\Psi_h, \Phi_h) \ \ \text{[using (3.25)]} \\
\Longrightarrow b_h^{NI}(\Phi_h, u_h - \mathcal{P}_{0h}u) &= \tilde{A}(\Psi - \Psi_h, \Phi_h) + \tilde{b}(\Phi_h, u - \mathcal{P}_{0h}u) + \Big[\tilde{A}(\Psi_h, \Phi_h) - A_h^{NI}(\Psi_h, \Phi_h)\Big] \\
&\quad - \Big[\tilde{A}(\Psi, \Phi_h) + \tilde{b}(\Phi_h, u)\Big]. \qquad (4.94)
\end{aligned}$$

Applying the triangular inequality and the continuity of the bilinear forms $\tilde{A}(\cdot,\cdot)$ and $\tilde{b}(\cdot,\cdot)$ in (4.94), we get from (4.93):

$$\|u_h - \mathcal{P}_{0h}u\|_{1,\Omega_h} \leq 1/\beta \Bigg\{ \Bigg[\tilde{M}\|\Psi - \Psi_h\|_{0,\Omega_h} + \tilde{m}\|u - \mathcal{P}_{0h}u\|_{1,\Omega_h} \Bigg] + \Bigg[\sup_{\Phi_h \in \mathbf{V}_h - \{0\}} \frac{|\tilde{A}(\Psi,\Phi_h) + \tilde{b}(\Phi_h,u)|}{\|\Phi_h\|_{1,\Omega_h}} \Bigg] + \Bigg[\sup_{\Phi_h \in \mathbf{V}_h - \{0\}} \frac{|\tilde{A}(\Psi_h,\Phi_h) - A_h^{NI}(\Psi_h,\Phi_h)|}{\|\Phi_h\|_{1,\Omega_h}} \Bigg] \Bigg\}. \tag{4.95}$$

Then, applying (4.92) and Propositions 4.3 and 4.4, we have

$$\|u_h - \mathcal{P}_{0h}u\|_{1,\Omega_h} \leq C\Big[h^{1/2}(\|u\|_{3,\Omega} + \sqrt{h}\|\Psi\|_{1,\Omega}) + h^{3/2}\|u\|_{3,\Omega} + h\|A\|_{1,\infty,\Omega}\|\Psi_h\|_{0,\Omega_h} h^{3/2}(1+\sqrt{h})\|u\|_{3,\Omega} \Big]. \tag{4.96}$$

But

$$\begin{aligned} \|\Psi_h\|_{0,\Omega_h} &\leq \|\Psi - \Psi_h\|_{0,\Omega_h} + \|\Psi\|_{0,\Omega_h} \leq Ch^{1/2}(\|u\|_{3,\Omega} + \sqrt{h}\|\Psi\|_{1,\Omega}) + \|\Psi\|_{0,\Omega} \\ &\leq C(\|u\|_{3,\Omega} + \|\Psi\|_{1,\Omega}). \end{aligned} \tag{4.97}$$

From (4.96) and (4.97),

$$\begin{aligned} \|u_h - \mathcal{P}_{0h}u\|_{1,\Omega_h} &\leq C\Big[h^{1/2}(\|u\|_{3,\Omega} + h^{1/2}\|\Psi\|_{1,\Omega}) + (h^{3/2} + h + h\sqrt{h}(1+\sqrt{h}))\|u\|_{3,\Omega} + h\|\Psi\|_{1,\Omega} \Big] \\ &\leq Ch^{1/2}\Big[\|u\|_{3,\Omega} + h^{1/2}\|\Psi\|_{1,\Omega} \Big]. \end{aligned} \tag{4.98}$$

Then,

$$\begin{aligned} \|u - u_h\|_{1,\Omega_h} &\leq \|u - \mathcal{P}_{0h}u\|_{1,\Omega_h} + \|\mathcal{P}_{0h}u - u_h\|_{1,\Omega_h} \;\leq\; C(h^{3/2}\|u\|_{3,\Omega} + h^{1/2}\|u\|_{3,\Omega} + h\|\Psi\|_{1,\Omega}) \\ \Longrightarrow \|u - u_h\|_{1,\Omega_h} &\leq Ch^{1/2}(\|u\|_{3,\Omega} + h^{1/2}\|\Psi\|_{1,\Omega}) \end{aligned} \tag{4.99}$$

with $C > 0$ independent of h.

Remark 4.2. Error estimate (4.79) [resp. (4.80)] depends on the estimates of the **three** terms occurring due to the errors involved with

(i) interpolation;

(ii) approximation of the curved boundary Γ by the polygon Γ_h;

(iii) non-exact integration
i.e. the terms in the second and third square brackets on the right-hand side of (4.87) (resp. (4.95)) correspond to (ii) and (iii) respectively, and the terms in the first square bracket correspond to (i) and also indirectly to (ii) [see (4.6)]. Hence, it will be interesting to study the two particular cases:
Case 1: there is **no** approximation of boundary, in other words Γ is a polygon, but numerical integration is performed, *i.e.* error due to (iii) is present, but an error due to (ii) is **absent**;
Case 2: polygonal boundary approximation is made,but **no** numerical integration is necessary and hence, it is not performed *i.e.* error due to (ii) is present, but an error due to (iii) is **absent**.

Case 1. Γ is a (straight) polygonal boundary of the convex polygonal domain Ω which is considered in all papers [4, 5, 15, 28, 33] etc., *i.e.*

$$\Gamma = \Gamma_h,\ \Omega = \Omega_h, \bar{\Omega} = \bar{\Omega}_h = \cup_{T\in\tau_h} T \ \ \forall h > 0 \tag{4.100}$$

$\Longrightarrow$ error due to (ii) is **absent**. Moreover, using higher order elements *i.e.* P_m-elements with $m > 2$, to construct finite element spaces, a **remarkable improvement** in the error estimates, *i.e.* $\|\Psi - \Psi_h\|_{0,\Omega} = O(h^{m-1})$, $\|u - u_h\|_{1,\Omega} = O(h^{m-1}), m \geq 2$ can be obtained under some additional assumptions on the regularity of solution and the use of quadrature schemes with higher degree of accuracy. In fact, (4.100) holds, and P_m-elements with $m \geq 2$ can be used to define $X_h \subset H^1(\Omega), \mathbf{V}_h \subset \mathbf{V},\ W_h \subset H_0^1(\Omega)$, *i.e.*

$$\begin{aligned} X_h &= \{\chi_h : \chi_h \in C^0(\bar{\Omega}),\ \chi_h \downarrow_T \in P_m(T) \qquad \forall T \in \tau_h\} \subset H^1(\Omega), \\ \mathbf{V}_h &= \{\Phi_h : \Phi_h = (\phi_{hij})_{1\leq i,j\leq 2} \text{ with } \phi_{hij} = \phi_{hji} \in X_h\} \subset \mathbf{V}, \\ W_h &= \{\chi_h : \chi_h \in X_h, \chi_h \downarrow_\Gamma = 0\} \subset H_0^1(\Omega). \end{aligned} \tag{4.101}$$

Then, we use quadrature schemes (3.11) with higher degrees of accuracy:

(**A3**) $A_h^{NI}(\cdot,\cdot)$ (resp. $b_h^{NI}(\cdot,\cdot)$) defined by (3.23) [resp. (3.24)] corresponds to the quadrature scheme (3.11) with $i = 1$ [resp. $i = 2$] which is exact for $P_{3m-2}(\hat{T})$ [resp. $P_{2m-2}(\hat{T})$].

Following the steps of the proofs of (3.27) and (3.28), we have: $\exists\alpha_0 > 0$, independent of h such that

$$A_h^{NI}(\Phi_h, \Phi_h) \geq \alpha_0 \|\Phi_h\|_{0,\Omega}^2 \qquad \forall \Phi_h \in \mathbf{V}_h \subset \mathbf{V}; \tag{4.102}$$

$\exists\beta_1 > 0$, independent of h such that

$$\sup_{\Phi_h\in\mathbf{V}_h-\{0\}} \frac{|b_h^{NI}(\Phi_h, \chi_h)|}{\|\Phi_h\|_{\mathbf{V}}} \geq \beta_1 \|\chi_h\|_{1,\Omega} \qquad \forall \chi_h \in W_h \subset H_0^1(\Omega) \tag{4.103}$$

and the corresponding ($\mathbf{Q_h}$) has a unique solution $(\Psi_h, u_h) \in \mathbf{V}_h \times W_h$.

Moreover,

$$(4.100) \implies \tilde{A}(\cdot,\cdot) = A(\cdot,\cdot),\ \tilde{b}(\cdot,\cdot) = b(\cdot,\cdot) \tag{4.104}$$

and

$$|\tilde{A}(\Psi, \Phi_h) + \tilde{b}(\Phi_h, u)| = |A(\Psi, \Phi_h) + b(\Phi_h, u)| = 0 \ \ \forall \Phi_h \in \mathbf{V}_h \subset \mathbf{V} \tag{4.105}$$

in Proposition 4.4 by virtue of the first equation (2.10);

$$|b(\Phi_h, \chi_h) - b_h^{NI}(\Phi_h, \chi_h)| = 0 \ \ \forall \Phi_h \in \mathbf{V}_h,\ \forall \chi_h \in W_h \tag{4.106}$$

with $\mathbf{V}_h$ and W_h defined by (4.101), since the quadrature scheme used in $b_h^{NI}(\cdot,\cdot)$ is exact for $P_{2m-2}(\hat{T}),\ m \geq 2$. Proposition 4.3 is replaced by the following result, whose proof is analogous.

Under (**A3**), for $A_{ijkl} \in W^{m-1,\infty}(\Omega)$ with $m \geq 2 \ \ \forall i,j,k,l = 1,2$, we have

$$|A(\Phi_h, \underline{\sigma}_h) - A_h^{NI}(\Phi_h, \underline{\sigma}_h)| \leq Ch^{m-1} \|\Phi_h\|_{0,\Omega} \|\underline{\sigma}_h\|_{0,\Omega} \ \ \forall \Phi_h, \underline{\sigma}_h \in \mathbf{V}_h \subset \mathbf{V}. \tag{4.107}$$

By virtue of (4.100), the interpolation operators $\mathcal{P}_h$ and $\mathcal{P}_{0h}$ defined in (4.1) and (4.4) respectively are now identical, *i.e.*

$$\mathcal{P}_h\chi = \mathcal{P}_{0h}\chi \in W_h \subset H_0^1(\Omega) \ \ \ \forall \chi \in H^s(\Omega) \cap H_0^1(\Omega) \qquad \text{with } \ s \geq 2, \tag{4.108}$$

and estimates (4.5) and (4.6) are replaced by the classical estimates:

$$\forall \chi \in H^s(\Omega) \cap H_0^1(\Omega) \text{ with } s \geq 2, \quad \|\chi - \mathcal{P}_{0h}\chi\|_{r,\Omega} \leq Ch^{s-r}|\chi|_{s,\Omega} \;\; (s \geq 2; r = 0, 1). \tag{4.109}$$

Lemma 4.2 holds with $b_h^{NI}(\cdot,\cdot)$ corresponding to the quadrature scheme (3.11) with $i = 2$, which is exact for $P_{2m-2}(\hat{T})$. Then, $\forall \Phi = (\phi_{ij})_{1\leq i,j\leq 2}$ with $\phi_{ij} = \phi_{ji} \in H^{m-1}(\Omega)$, $m \geq 2$, $\exists \Theta_h \in \mathbf{V}_h \subset \mathbf{V}$ satisfying $b(\Phi, \chi_h) = b_h^{NI}(\Theta_h, \chi_h)$ $\forall \chi_h \in W_h \subset H_0^1(\Omega)$ such that

$$\|\Phi - \Theta_h\|_{r,\Omega} \leq Ch^{m-r-1}\|\Phi\|_{m-1,\Omega} \;\; (r = 0, 1). \tag{4.110}$$

Now, following the steps of the proof of Theorem 4.1, using assumption (**A3**) and (4.100–4.110), assuming that $u \in H^{m+1}(\Omega) \cap H_0^2(\Omega)$ with $m \geq 2$ is the unique solution of ($\mathbf{P_G}$) in (2.3–2.4) and (Ψ, u) is the unique solution of (**Q**) with $\Psi = (\psi_{ij})_{1\leq i,j\leq 2}$, $\psi_{ij} = \psi_{ji} \in H^{m-1}(\Omega)$ $\forall i, j = 1, 2$,

$$\|\Psi - \Psi_h\|_{0,\Omega} \leq Ch^{m-1}\Big[\|u\|_{m+1,\Omega} + \|\Psi\|_{m-1,\Omega}\Big], \tag{4.111}$$

$$\|u - u_h\|_{1,\Omega} \leq Ch^{m-1}\Big[\|u\|_{m+1,\Omega} + \|\Psi\|_{m-1,\Omega}\Big]. \tag{4.112}$$

Estimates (4.111) and (4.112) are of the same order $O(h^{m-1}), m \geq 2$ as obtained in [5] (resp. for **H-H-M** mixed scheme for the biharmonic problem (2.20–2.21) by Brezzi-Raviart in [15], pages 16–17) under the same regularity assumptions, when errors due to (ii) and (iii) are **absent**, *i.e.* when Γ is a polygon and exact integration is performed.

For $m = 1$, **neither** the estimates of [5] **nor** those of Brezzi-Raviart in [15] hold (see Remark 2 of [15], page 20), but Miyoshi obtained estimates of order $O(h^{1/2})$ for $m = 1$ in [28], in which the elegant, systematic mixed method analysis of Babuška-Brezzi-Raviart has **not** been followed (see also [34])!

Hence, based on Babuška-Brezzi-Raviart mixed method analysis, *best available* error estimates for this problem using P_2 elements are of order $O(h)$ [5], when errors due to (ii) and (iii) are absent. Moreover, when quadrature schemes with higher degrees of accuracy $P_{3m-2}(\hat{T})$ for $A_h^{NI}(\cdot,\cdot)$ and $P_{2m-2}(\hat{T})$ for $b_h^{NI}(\cdot,\cdot)$ $(m \geq 2)$ are used, the error due to **only** (iii) is of the order $O(h^{m-1})$ [see (4.107)].

Case 2. Curved boundary Γ is approximated by a polygon Γ_h as in (3.1–3.6), but exact integration is possible and performed *i.e.* only error due to (ii) is present. Since exact integration is performed, $\tilde{A}(\cdot,\cdot) = A_h^{NI}(\cdot,\cdot)$, $\tilde{b}(\cdot,\cdot) = b_h^{NI}(\cdot,\cdot)$ and the term in the third square bracket on the right-hand side of (4.87) [resp. (4.95)] vanishes.

For polygonal approximation Γ_h to Γ, we have $\text{meas}(\omega_h) = O(h^2)$ with $\omega_h = \Omega - \Omega_h$. Consequently, from (4.42, 4.44, 4.45) in the proof of Proposition 4.4,

$$|\tilde{A}(\Psi, \Psi_h - \Phi_h) + \tilde{b}(\Psi_h - \Phi_h, u)| = O(h^{3/2}), \tag{4.113}$$

even if P_m-elements with $m \geq 2$ are used to define finite element spaces $\mathbf{V}_h$ and W_h [see (4.101)]. In other words, by using P_m-elements with $m > 2$, the estimate (4.113) can **not** be improved unless better approximation of Γ is made. Incidentally, this is exactly the reason for using P_2-elements in the definition of $\mathbf{V}_h$ and W_h in (3.18) and (3.19) respectively.

Again, from Remark 4.1, we find that for

$$u \in H^{m+1}(\Omega) \cap H_0^2(\Omega) \subset H^{m+1}(\Omega) \cap H_0^1(\Omega), \;\; \|u - \mathcal{P}_{0h}u\|_{1,\Omega_h} = O(h^{3/2}) \;\; \forall m \geq 2, \tag{4.114}$$

which can **not** be improved upon unless boundary approximation is improved. Hence, for polygonal boundary approximation, we find from (4.113) and (4.114) that the estimates of order $O(h^{3/2})$ can **not** be improved upon by any choice of $m > 2$

(i) in the definition of $\mathbf{V}_h$ and W_h in (4.101) and

(ii) in the regularity of solution $u \in H^{m+1}(\Omega) \cap H_0^2(\Omega)$ of $(\mathbf{P_G})$, *i.e.* **the optimal case is** $m = 2$.

Finally, the use of the inverse inequality (4.88) in (4.91) is necessary (see [5,14,15,18]) and gives the estimate:

$$\|\Psi - \Psi_h\|_{0,\Omega_h} = O(h^{1/2}), \tag{4.115}$$

which is used to get the estimate:

$$\|u - u_h\|_{1,\Omega_h} = O(h^{1/2}). \tag{4.116}$$

Thus, for this crude **but most important and commonly used polygonal approximation** Γ_h to Γ, there is a loss in the exponent of h **by only '1/2'** in the estimates (4.115–4.116), the best available estimates [5], [15] based on Babuška-Brezzi-Raviart mixed method analysis being $\|\Psi - \Psi_h\|_{0,\Omega} = O(h)$, $\|u - u_h\|_{1,\Omega} = O(h)$ for $m = 2$, when there is *neither* boundary approximation *nor* non-exact integration (see also Case 1 above for $m = 2$). In fact, in [9], the estimates $\|\Psi - \Psi_h\|_{0,\Omega_h} = O(h)$, $\|u - u_h\|_{1,\Omega_h} = O(h)$ have been obtained when Γ has been approximated by a curved boundary Γ_h constructed with the help of isoparametric mapping, for which $\bar{\Omega}_h \not\subset \bar{\Omega}$, $\bar{\Omega} \not\subset \bar{\Omega}_h$ and $\bar{\Omega}_h$ is **no** longer convex in general. Consequently, a completely different, independent analysis has been developed in [9].

Hence, it is obvious from the facts explained above that for polygonal approximation Γ_h, the estimates $\|\Psi - \Psi_h\|_{0,\Omega_h} = O(h^{1/2})$ and $\|u - u_h\|_{1,\Omega_h} = O(h^{1/2})$ are the 'best' ones based on Babuška-Brezzi-Raviart mixed method analysis for fourth order problems.

The authors express their special thanks to the referee for all his suggestions for the improvement of the paper and particularly for pointing out the necessity of substantiation of the interpolation estimates in (4.5–4.6).

References

[1] R.A. Adams, *Sobolev Spaces.* Academic Press, NewYork (1975).
[2] I. Babuška, Error Bounds for Finite Element Method. *Numer. Math.* **16** (1971) 322–333.
[3] I. Babuška and J.E. Osborn, Eigenvalue Problems, in *Handbook of Numerical Analysis*, Vol. 2, Part 1, J.L. Lions and P.G. Ciarlet Eds., North-Holland, Amsterdam (1991) 641–787.
[4] S. Balasundaram and P.K. Bhattacharyya, On Existence of Solution of the Dirichlet Problem of Fourth Order Partial Differential Equations with Variable Coefficients. *Quart. Appl. Math.* **39** (1983) 311–317.
[5] S. Balasundaram and P.K. Bhattacharyya, A Mixed Finite Element Method for Fourth Order Elliptic Equations with Variable Coefficients. *Comput. Math. Appl.* **10** (1984) 245–256.
[6] S. Balasundaram and P.K. Bhattacharyya, A Mixed Finite Element Method for Fourth Order Elliptic Operators with Variable Coefficients, 4th Int. Symp. on Finite Element Methods in Flow Problems, Chuo University, Tokyo (1982), in *Finite Element Analysis of Flow Problems,* T. Kawai Ed., Tokyo University Press (1982) 995–1001.
[7] S. Balasundaram and P.K. Bhattacharyya, A Mixed Finite Element Method for Fourth Order Partial Differential Equations. *ZAMM* **66** (1986) 489–499.
[8] M. Bernadou, Méthodes d'éléments finis pour les problèmes de coques minces. *Collection Recherches en Mathematiques Appliquees,* Masson, Paris (1994).
[9] P.K. Bhattacharyya and N. Nataraj, Error Estimates for Isoparametric Mixed Finite Element Solution of 4th Order Elliptic Problems with Variable Coefficients (submitted).
[10] P.K. Bhattacharyya, A Mixed Finite Element Method for Fourth Order Elliptic Operators with Variable Coefficients, in *Proc. of 6th Joint France-Italy-USSR Symp. on Numerical Solution of Nonlinear Problems*, INRIA, France (1983) 127–144; Russian Translation, in *Methods of Computational Mathematics and Mathematical Modelling*, Computational Mathematics Section, Academy of Sciences, Moscow, USSR (1985) 76–99.

[11] P.K. Bhattacharyya, S. Gopalsamy and S. Balasundaram, On a Mixed Finite Element Method for Clamped Anisotropic Plate Bending Problems. *Internat. J. Numer. Methods Engrg.* **28** (1989) 1351–1370.

[12] P.K. Bhattacharyya and S. Balasundaram, A Mixed Finite Element Method for Fourth Order Elliptic Problems with Variable Coefficients. *J. Comput. Appl. Math.* **22** (1988) 1–24.

[13] F. Brezzi, On the Existence, Uniqueness and Approximation of Saddle-Point Problems Arising from Lagrangian Multipliers. *RAIRO Anal. Numér.* **8** (1974) 129–151.

[14] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*. Springer-Verlag (1991).

[15] F. Brezzi and P.A. Raviart, Mixed Finite Element Methods for 4th Order Elliptic Equations, in *Topics in Numerical Analysis III*, J. Miller Ed., Academic Press, New York (1978) 33–56.

[16] P.G. Ciarlet, Basic Error Estimates for Elliptic Problems, in *Handbook of Numerical Analysis,* Vol. 2, Part 1, J.L. Lions and P.G. Ciarlet Eds., North-Holland, Amsterdam (1991).

[17] P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*. North-Holland, Amsterdam (1978).

[18] P.G. Ciarlet and P.A. Raviart, A Mixed Finite Element Method for the Biharmonic Equation, in *Symp. on Mathematical Aspects of Finite Elements in Partial Differential Equations*, C. de Boor Ed., Academic Press, New York (1974) 125–145.

[19] P.G. Ciarlet and P.A Raviart, The Combined Effect of Curved Boundaries and Numerical Integration in Isoparametric Finite Element Methods, in *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations*, A.K. Aziz Ed., Academic Press, New York (1972) 409–474.

[20] S. Gopalsamy and P.K. Bhattacharyya, On Existence and Uniqueness of Solution of Boundary Value Problems of Fourth Order Elliptic Partial Differential Equations with Variable Coefficients. *J. Math. Anal. Appl.* **136** (1988) 589–608.

[21] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*. Pitman, Boston (1985).

[22] K. Hellan, Analysis of Elastic Plates in Flexure by a Simplified Finite Element Method. *Acta Polytech. Scand., Civil Engineering Series,* Trondheim, 46 (1967).

[23] L. Herrmann, Finite Element Bending Analysis for Plates. *J. Eng. Mech. Div. ASCE* **93**, EM 5 (1967) 49–83.

[24] V.A. Kondratev, Boundary Value Problems for Elliptic Equations in Domains with Conical or Angular Points. *Trudy Moskov. Mat. Obshch.* **16** (1967) 7209–292.

[25] S.G. Leknitskii, *Anisotropic Plates*. Gordon and Breach Science Publishers, New York (1968).

[26] J.L. Lions, *Problèmes aux Limites dans les Équations aux Derivées Partielles.* Les Presses de l'Université de Montréal, Montréal (1965).

[27] L. Mansfield, Approximation of the Boundary in the Finite Element solution of fourth order problems. *SIAM J. Numer. Anal.* **15** (1978) 568–579.

[28] T. Miyoshi, A finite element method for the solution of fourth order partial differential equations. *Kumamoto J. Sci. (Math.)* **9** (1973) 87–116.

[29] N. Nataraj, P.K. Bhattacharyya, S. Balasundaram and S. Gopalsamy, On a Mixed-Hybrid Finite Element Method for Anisotropic Plate Bending Problems. *Internat. J. Numer. Methods Engrg.* **39** (1996) 4063–4089.

[30] J.T. Oden and G.F. Carey, *Finite Elements-Mathematical Aspects IV, The Texas Finite Element Series*. Prentice Hall, Eaglewood Cliffs, New Jersey (1983).

[31] A.B. Ouaritini, *Méthodes d'éléments finis mixtes pour des problèmes de coques minces*. Ph.D. thesis, University of Pau et Pays de L'Adour, France (1984).

[32] P.A. Raviart and J.M. Thomas, *Introduction à l'Analyse Numérique des Équations aux Dérivées Partielles*. Mason, Paris (1983).

[33] J.E. Roberts and J.M. Thomas, Mixed and Hybrid Methods, in *Handbook of Numerical Analysis,* Vol. 2, Part 1, J.L Lions and P.G. Ciarlet Eds., North-Holland, Amsterdam (1991) 523–633.

[34] R. Scholtz, A Mixed Method for 4th order Problems Using Linear Finite Elements. *RAIRO Anal. Numér.* **12** (1978) 85–90.

[35] R. Scott, *Finite Element Techniques for Curved Boundaries*. Ph.D. thesis, M.I.T. (1973).

[36] G. Strang and G.J. Fix, *An Analysis of the Finite Element Method*. Prentice-Hall, New York (1973).

[37] S. Timoshenko and S. Woinowsky-Kreiger, *Theory of Plates and Shells*. McGraw-Hill Book Company, New York (1959).

[38] A. Zeníšek, *Nonlinear Elliptic and Evolution Problems and their Finite Element Approximations*. Academic Press, New York (1990).

[39] M. Zlamal, Curved elements in the finite element method, I. *SIAM J. Numer. Anal.* **10** (1973) 229–240.

[40] M. Zlamal, Curved elements in the finite element method, II. *SIAM J. Numer. Anal.* **11** (1974) 347–362.