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NONCONFORMING GALERKIN METHODS BASED ON QUADRILATERAL ELEMENTS FOR SECOND ORDER ELLIPTIC PROBLEMS

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Abstract. Low-order nonconforming Galerkin methods will be analyzed for second-order elliptic equations subjected to Robin, Dirichlet, or Neumann boundary conditions. Both simplicial and rectangular elements will be considered in two and three dimensions. The simplicial elements will be based on $\mathcal{P}_1$, as for conforming elements; however, it is necessary to introduce new elements in the rectangular case. Optimal order error estimates are demonstrated in all cases with respect to a broken norm in $H^1(\Omega)$ and in the Neumann and Robin cases in $L^2(\Omega)$.

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1. Introduction

In the first part (Sect. 2) of this paper, low-order nonconforming Galerkin methods will be defined and analyzed for second-order elliptic equations subjected to Robin, Dirichlet, or Neumann boundary conditions. The object is to introduce a new nonconforming element over rectangles or quadrilaterals in two dimensions and rectangles in three dimensions. Simplicial elements based on $\mathcal{P}_1$ will be analyzed first, and that analysis will be used to motivate the choice of the rectangular elements. Optimal order error estimates are demonstrated in all cases with respect to a broken norm in $H^1(\Omega)$ and in the Neumann and Robin cases in $L^2(\Omega)$. Since the Robin condition leads to a somewhat more complicated analysis, this case will be presented in detail.

Rannacher and Turek [11], in the setting of the Stokes problem, analyzed two forms of nonconforming elements based on simply rotating the usual bilinear element to employ Span $\{1, x, y, x^2 - y^2\}$ as the local basis. On rectangles, they construct a very clever argument that uses a cancellation property on each rectangle, plus a serious application of an inverse property, to show optimal order approximation of the solution of the Stokes problem; however, if the usual definition of the global nonconforming space by requiring continuity at interfacial midpoints is adopted, there is a loss of optimality for truly quadrilateral partitions of the domain. (Their argument covers higher dimensions, and an obvious simplification of it covers the second order elliptic

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problem.) We shall offer several modifications to the rotated bilinear local basis and avoid this loss, while reducing the analysis to the exact analogue of the classical analysis of the simplicial nonconforming procedure, as given in [5,8,12] and textbooks such as [3,4,13] for second order elliptic problems, the Stokes problem, and plate bending. There is no essential difference in either programming effort or in computer run time between our two- or three-dimensional elements and the rotated bilinear or trilinear element.

The Robin boundary problem is stated in Section 2.1 and the corresponding nonconforming Galerkin problem described in Section 2.2 for a simplicial partition of the domain; this method is analyzed in the following two sections. A form of Strang's Second Lemma is employed in Section 2.3 to give the well-known proof of the convergence of the Galerkin approximation in an energy norm at an optimal rate; this short argument is repeated here to illustrate the rôle of an orthogonality condition that will motivate our selection of a basis for nonconforming methods on rectangular elements. The duality argument applied in Section 2.4 to obtain an optimal rate for convergence in $L^2(\Omega)$ again demonstrates the value of this same orthogonality. As a result of the use of a quadrature to impose the boundary condition, additional regularity on the boundary is required over that which would be needed if the boundary condition were imposed exactly.

Rectangular elements are treated in Section 2.5, along with an extension in the two-dimensional case to quadrilaterals. The local spaces, which as stated above differ from the local spaces in conforming procedures, are described; as stated above, the related convergence analysis is reduced to that for the simplicial case. In Section 2.6 the implementation of these methods by means of local interpolation of the coefficients in the differential equation is discussed. Then, in the next two sections, Section 2.7 and Section 2.8, the simpler problems when either Neumann or Dirichlet boundary conditions are prescribed are treated briefly. The error estimates for the Neumann problem are again of optimal order in both norms, but the L^2 estimate is suboptimal in the Dirichlet case as a result of an inability to enter the Dirichlet data into the finite element method with sufficient accuracy. Some specific, technical estimates related to the quadratures used in approximating the Robin boundary condition are derived at the end of this part in Section 2.9.

In the second part of the paper (Sect. 3), a domain decomposition iterative procedure based on the use of Robin transmission conditions to pass information from a subdomain to its neighbors will be introduced for these methods. Quite analogous iterative procedures for conforming methods for second order elliptic problems were introduced first by Lions [9,10] and then applied to the more difficult Helmholtz problem by Després [6]; later [7], a more precise convergence argument was established for the second order elliptic problem as approximated by mixed finite element methods. We shall analyze the convergence of the iteration for the nonconforming Galerkin method based on rectangular elements, using arguments related to those of [7]. Both two-dimensional and three-dimensional problems are discussed. The analysis would apply equally to nonconforming methods based on $\mathcal{P}_1$-elements over simplices.

The two-dimensional case of the finite element method is hybridized in Section 3.1, and the domain decomposition procedure is defined in Section 3.2. A simple, but imprecise, convergence analysis for the iteration is also presented in Section 3.2. Estimates of the spectral radius of the iteration operator are derived in the next two sections under different hypotheses. In Section 3.5, the three-dimensional problem is treated quite briefly. Some technical lemmas needed in this part of the analysis are found in the last section.

2. Formulation and convergence analysis

2.1. The elliptic problem with Robin boundary conditions

Let us consider the second order elliptic boundary problem given by

$$-\nabla \cdot (a\nabla u) + cu = f, \qquad x \in \Omega, \tag{2.1a}$$

$$a\frac{\partial u}{\partial \nu} + du = g, \qquad x \in \partial\Omega, \tag{2.1b}$$

where

- $\overline{\Omega} = \cup_{j=1}^{J} \overline{\Omega}_j \subset \mathbb{R}^n$, $n = 2$ or 3; Ω_j simplicial and the partition quasiregular; $\operatorname{diam}(\Omega_j) \le h$.
- The coefficients a, c, and d are smooth and $0 < a_0 \le a(x) \le a_1$, $0 \le c(x) \le c_1$, $0 < d_0 \le d(x) \le d_1$.

The weak form of (2.1) that we consider is given by seeking $u \in H^1(\Omega)$ such that

$$a(u, v) = F(v), \quad v \in H^1(\Omega), \tag{2.2}$$

where

$$\begin{aligned} a(u, v) &= (a\nabla u, \nabla v) + (cu, v) + \langle du, v\rangle, && (2.3a)\\ F(v) &= (f, v) + \langle g, v\rangle; && (2.3b) \end{aligned}$$

$(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ denote the $L^2(\Omega)$ and $L^2(\partial\Omega)$ inner products, respectively.

2.2. The simplicial nonconforming Galerkin method

Let us turn to the approximation of the solution of (2.3) through a nonconforming Galerkin method.
Let

$$\Gamma_j = \partial\Omega \cap \partial\Omega_j, \quad \Gamma_{jk} = \Gamma_{kj} = \partial\Omega_j \cap \partial\Omega_k,$$

and denote the centers of Γ_j and Γ_{jk} by ξ_j and ξ_{jk}, respectively. Let $\mathcal{P}_\ell(E)$ denote the class of polynomials of degree ℓ on the set E, and set

$$\mathcal{NC}_j^h = \mathcal{P}_1(\Omega_j), \quad n = 2 \text{ or } 3.$$

Let

$$\mathcal{NC}^h = \left\{ v \,|\, v_j = v|_{\Omega_j} \in \mathcal{NC}_j^h, j = 1, \dots, J;\ v_j(\xi_{jk}) = v_k(\xi_{jk}), \forall\{j,k\} \right\}.$$

For convenience in the analysis below, let

$$\Lambda^h = \left\{ \lambda \,|\, \lambda_{jk} = \operatorname{tr}_{\Gamma_{jk}}(\lambda\,|_{\Omega_j}) \in \mathcal{P}_0(\Gamma_{jk});\ \lambda_{jk} + \lambda_{kj} = 0;\ \lambda_j = \operatorname{tr}_{\Gamma_j}(\lambda\,|_{\Omega_j}) \in \mathcal{P}_0(\Gamma_j) \right\}.$$

Define projections Π and P_0, by

$$\begin{aligned} &\Pi : H^2(\Omega) \to \mathcal{NC}^h : && (v - \Pi v)(\xi) = 0, && \xi = \xi_{jk} \text{ or } \xi_j;\\ &P_0 : H^2(\Omega) \to \Lambda^h : && \left\langle a\frac{\partial v_j}{\partial \nu_j} - P_0 v_j, z \right\rangle_\Gamma = 0, && z \in \mathcal{P}_0(\Gamma),\ \Gamma = \Gamma_{jk} \text{ or } \Gamma_j. \end{aligned}$$

Since Π reproduces linear functions on elements and P_0 reproduces constants on faces, it follows from standard polynomial approximation results that

$$\begin{aligned} \|v - \Pi v\| &+ h\left(\sum_j \|v - \Pi v\|_{1,j}^2\right)^{\frac12} + h^2\left(\sum_j \|v - \Pi v\|_{2,j}^2\right)^{\frac12} + h^{\frac12}\left(\sum_j |v - \Pi v|_j^2\right)^{\frac12}\\ &+ h^{\frac32}\left(\sum_j \left|\frac{\partial}{\partial \nu_j}(v - \Pi v)\right|_j^2\right)^{\frac12} + h^{\frac32}\left(\sum_j \left|a\frac{\partial v_j}{\partial \nu_j} - P_0 v\right|_j^2\right)^{\frac12} \le C\|v\|_2 h^2, \quad v \in H^2(\Omega), \end{aligned} \tag{2.4}$$

where $\|z\|_{m,j}^2 = \|z\|_{H^m(\Omega_j)}^2$, $|z|_{m,j}^2 = \sum_k \|z\|_{H^m(\Gamma_{jk})}^2$, with Γ_j replacing Γ_{jk} for boundary faces.

The integral $\langle\cdot,\cdot\rangle$ will be approximated as $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ by means of quadrature rules, which will be discussed in detail in Section 2.9.

Let $(\cdot,\cdot)_j = (\cdot,\cdot)_{\Omega_j}$, and set

$$\begin{aligned} a_h(z,w) &= \sum_j (a\nabla z, \nabla w)_j + (cz,w) + \langle\!\langle dz, w\rangle\!\rangle, \\ F_h(z) &= (f,z) + \langle\!\langle g, z\rangle\!\rangle. \end{aligned}$$

Then, the *nonconforming Galerkin approximation* of (2.2) is defined as the solution $u_h \in \mathcal{NC}^h$ of the equations

$$a_h(u_h, v) = F_h(v), \qquad v \in \mathcal{NC}^h. \tag{2.5}$$

The uniqueness of u_h is trivial; if f and g vanish, the boundary term, at least for any quadrature method admitted in Section 2.9, forces $u_h(\xi_j)$ to vanish, the $(a\nabla u_h, \nabla v)$-term insures that u_h is constant on each Ω_j, and $u_h \in \mathcal{NC}^h$ requires continuity at the ξ_{jk}-points, so that u_h vanishes. Existence follows from finite dimensionality.

2.3. **The second Strang lemma and the energy error estimate**

Strang [4, 12, 13] provided the following lemma to characterize the error for nonconforming methods in the (broken) energy norm

$$\|z\|_{1,h} = a_h(z,z)^{\frac{1}{2}}.$$

Lemma 2.1. *If $u_h \in \mathcal{NC}^h$ is the solution of* (2.5) *and $u \in H^1(\Omega)$ the solution of* (2.2), *then*

$$\|u-u_h\|_{1,h} \le C\left\{\inf_{v\in\mathcal{NC}^h} \|u-v\|_{1,h} + \sup_{w\in\mathcal{NC}^h} \frac{|a_h(u,w)-F_h(w)|}{\|w\|_{1,h}}\right\}.$$

Proof (as given in [4]). For $z_h \in \mathcal{NC}^h$,

$$\begin{aligned} \|u_h - z_h\|_{1,h}^2 &= a_h(u-z_h, u_h-z_h) + a_h(u_h-u, u_h-z_h) \\ &= a_h(u-z_h, u_h-z_h) + [F_h(u_h-z_h) - a_h(u, u_h-z_h)] \\ &\le \|u-z_h\|_{1,h}\|u_h-z_h\|_{1,h} + |a_h(u,u_h-z_h) - F_h(u_h-z_h)|, \end{aligned}$$

and the lemma follows from this inequality and the triangle inequality.

Let us apply the lemma to the simplicial nonconforming method. First, (2.4) implies that

$$\inf_{v\in\mathcal{NC}^h} \|u-v\|_{1,h} \le C\|u\|_2 h.$$

Next, let $w \in \mathcal{NC}^h$. Denote by $E(G,w)$, $w \in \mathcal{NC}^h$, the boundary quadrature error

$$E(G,w) = \sum_j \left\{\langle G, w\rangle_{\Gamma_j} - \langle\!\langle G, w\rangle\!\rangle_{\Gamma_j}\right\}.$$

Then, a short calculation shows that

$$a_h(u,w) - F_h(w) = \sum_j \left\langle a\frac{\partial u_j}{\partial \nu_j}, w\right\rangle_{\partial\Omega_j\setminus\Gamma_j} + E(g-du, w). \tag{2.6}$$

The following orthogonalities are useful:

$$\langle P_0 u_j, w_j\rangle_{\Gamma_{jk}} + \langle P_0 u_k, w_k\rangle_{\Gamma_{kj}} = \langle P_0 u_j, w_j - w_k\rangle_{\Gamma_{jk}} = 0, \quad w \in \mathcal{NC}^h, \tag{2.7a}$$

$$\left\langle a\frac{\partial u_j}{\partial \nu_j} - P_0 u_j, 1\right\rangle_\Gamma = 0, \qquad \Gamma = \Gamma_j \quad \text{or} \quad \Gamma_{jk}. \tag{2.7b}$$

Thus, it follows that, for $m_j \in \mathcal{P}_0(\Omega_j)$,

$$\sum_j \left\langle a\frac{\partial u_j}{\partial \nu_j}, w\right\rangle_{\partial\Omega_j\setminus\Gamma_j} = \sum_j \left\langle a\frac{\partial u_j}{\partial \nu_j} - P_0 u_j, w_j - m_j\right\rangle_{\partial\Omega_j\setminus\Gamma_j},$$

so that

$$a_h(u,w) - F_h(w) = \sum_j \left\langle a\frac{\partial u_j}{\partial \nu_j} - P_0 u_j, w_j - m_j\right\rangle_{\partial\Omega_j\setminus\Gamma_j} + E(g - du, w). \tag{2.8}$$

By (2.4), a standard trace theorem, and approximation of w_j locally by a properly chosen constant (its average over Ω_j),

$$\begin{aligned}\left|\sum_j \left\langle a\frac{\partial u_j}{\partial \nu_j} - P_0 u_j, w_j - m_j\right\rangle_{\partial\Omega_j}\right| &\le C\|u\|_2 h^{\frac12} \cdot \left(\sum_j \|w - m_j\|_j \|\nabla(w - m_j)\|_j\right)^{\frac12} \\ &\le C\|u\|_2 \left(\sum_j \|\nabla w_j\|_j^2\right)^{\frac12} h \le C\|u\|_2 \|w\|_{1,h} h.\end{aligned} \tag{2.9}$$

Two quadrature rules are discussed in Section 2.9: the midpoint rule, which is first-order correct, and a second-order correct rule, the two-point Gauss rule for $n = 2$ and a triangle rule for $n = 3$. If the subscript ℓ is used to indicate the order of the rule, it is shown [see (2.24, 2.26, 2.27)] that

$$|E_\ell(g - du, w)| \le C(|g|_{1,\partial\Omega} + |u|_{1,\partial\Omega})|w|_{\partial\Omega} h, \qquad w \in \mathcal{NC}^h,\ \ell = 1 \text{ or } 2, \tag{2.10a}$$

$$|E_2(g - du, w)| \le C(|g|_{2,\partial\Omega} + |u|_{2,\partial\Omega})|w|_{\partial\Omega} h^2, \qquad w \in \mathcal{NC}^h. \tag{2.10b}$$

For the midpoint rule, it is easy to see that

$$\int_{\Gamma_j} w^2\, \mathrm{d}s \le Kh(a\nabla w, \nabla w)_{\Omega_j}$$

when $w(\xi_j) = 0$; for the second-order rules, the boundary quadrature of the square of an element in $\mathcal{P}_1(\Omega_j)$ is exact. Thus, $|w|_{\partial\Omega} \le C\|w\|_{1,h}$, so that combining (2.9) and (2.10a) with Strang's lemma gives us the following *energy error estimate.*

Theorem 2.1. *Let u and u_h be the solutions of* (2.2) *and* (2.5), *respectively. Then, the error satisfies the estimate*

$$\|u - u_h\|_{1,h} \le C(\|u\|_2 + |g|_{1,\partial\Omega} + |u|_{1,\partial\Omega})h. \tag{2.11}$$

The boundary norms can be omitted if exact quadrature is employed on the boundary integrals in (2.5)*, and only minimal regularity is then required of the solution.*

The estimate (2.11) is optimal with respect to rate, but not with respect to regularity of the solution u of (2.2). The $|u|_{1,\partial\Omega}$-term can be omitted if problem (2.1) is H^2-regular. Also, all boundary norms can be considered to be broken over the collection of boundary faces Γ_j.

The bound (2.10b) will be useful in the next section, where an L^2 error estimate will be derived.

The critical part of the analysis above is the application of the orthogonalities given in (2.7); these two properties for the piecewise linear nonconforming elements will also be critical in the duality argument in the next section and are fundamental in defining nonconforming elements over rectangles. They were used in energy norm estimates earlier; see [3, 4], for example. Céa, in an unpublished manuscript dating to 1976, discussed the rôle of such orthogonalities in nonconforming methods in general.

2.4. Duality and the L^2 error estimate

The duality argument introduced by Aubin and Nitsche (see [2–4]) can be applied to the nonconforming method to deduce an $L^2(\Omega)$ error estimate; see, *e.g.*, [5, 11]. In order to do so, we require that the differential problem (2.1) be H^2-regular; as will appear in the development below, it will also be necessary to assume additional regularity of the boundary data and for the trace of the solution there. It will become clear that an optimal rate of convergence will result if a quadrature rule that is exact for polynomials on a face of degree at least two is used on the boundary integrals and that a nonoptimal rate would result from the midpoint rule, which was seen to be adequate to obtain an optimal rate in the energy norm.

Let

$$\eta = \Pi u - u_h,$$

and let $\psi \in H^2(\Omega)$ be the solution of

$$\begin{aligned} L\psi = -\nabla \cdot (a\nabla\psi) + c\psi &= \eta, \quad x \in \Omega, \\ a\frac{\partial\psi}{\partial\nu} + d\psi &= 0, \quad x \in \partial\Omega; \end{aligned}$$

thus, $\|\psi\|_2 \le C\|\eta\|$. Note that, by (2.4), $\|\eta\|_{1,h}$ satisfies an inequality of the same form, (2.11), as $u - u_h$.

Then, since $\eta \in \mathcal{NC}^h$,

$$\begin{aligned} \|\eta\|^2 &= (L\psi, \eta) = a_h(\psi, \eta) + E(d\psi, \eta) - \sum_j \left\langle a\frac{\partial\psi_j}{\partial\nu_j}, \eta_j \right\rangle_{\partial\Omega_j\setminus\Gamma_j} \\ &= a_h(\psi, \eta) + E(d\psi, \eta) - \sum_j \left\langle a\frac{\partial\psi_j}{\partial\nu_j} - P_0\psi_j, \eta_j - q_j \right\rangle_{\partial\Omega_j\setminus\Gamma_j}, \end{aligned}$$

whenever $q_j \in \mathcal{P}_0(\Omega_j)$. Next, for $v \in \mathcal{NC}^h$,

$$\begin{aligned} a_h(\eta, v) &= a_h(u, v) - a_h(u_h, v) - a_h(u - \Pi u, v) \\ &= E(g - du, v) + \sum_j \left\langle a\frac{\partial u_j}{\partial\nu_j} - P_0 u_j, v_j \right\rangle_{\partial\Omega_j\setminus\Gamma_j} - a_h(u - \Pi u, v). \end{aligned}$$

Since $\psi_j = \psi\,|_{\Gamma_{jk}} = \psi_k$,

$$\left\langle a\frac{\partial u_j}{\partial\nu_j} - P_0 u_j, \psi_j \right\rangle_{\Gamma_{jk}} + \left\langle a\frac{\partial u_k}{\partial\nu_k} - P_0 u_k, \psi_k \right\rangle_{\Gamma_{kj}} = 0$$

and

$$\|\eta\|^2 = a_h(\eta, \psi - v) - a_h(u - \Pi u, v) + E(d\psi, \eta) - E(du - g, v) \quad (2.12)$$
$$- \sum_j \left\langle a\frac{\partial \psi_j}{\partial \nu_j} - P_0\psi_j, \eta_j - q_j \right\rangle_{\partial\Omega_j\setminus\Gamma_j} + \sum_j \left\langle a\frac{\partial u_j}{\partial \nu_j} - P_0 u_j, v_j - \psi_j \right\rangle_{\partial\Omega_j\setminus\Gamma_j}.$$

Let

$$\|v\|_{2,h} = \left(\sum_j \|v\|^2_{2,\Omega_j} \right)^{\frac{1}{2}}.$$

Then, there exists $v \in \mathcal{NC}^h$ such that

$$\|\psi - v\| + h\|\psi - v\|_{1,h} + h^2\|v\|_{2,h} \le C\|\psi\|_2 h^2 \le C\|\eta\| h^2.$$

Now, let us bound each of the terms on the right-hand side of (2.12). First,

$$|a_h(\eta, \psi - v)| \le C\|\eta\|_{1,h}\|\eta\| h.$$

As in the previous section, it follows that, for properly chosen q,

$$\left| \sum_j \left\langle a\frac{\partial \psi_j}{\partial \nu_j} - P_0\psi_j, \eta_j - q_j \right\rangle_{\partial\Omega_j\setminus\Gamma_j} \right| \le C\|\eta\| \, \|\eta\|_{1,h} h$$

and

$$\left| \sum_j \left\langle a\frac{\partial u_j}{\partial \nu_j} - P_0 u_j, v_j - \psi_j \right\rangle_{\partial\Omega_j\setminus\Gamma_j} \right| \le C\|u\|_2\|\eta\| h^2.$$

Before looking at the boundary integral terms, let us consider $a_h(u - \Pi u, v)$:

$$a_h(u - \Pi u, v) = \sum_j (u - \Pi u, Lv)_j + \sum_j \left\langle u - \Pi u, a\frac{\partial v}{\partial \nu_j} \right\rangle_{\partial\Omega_j} + \sum_j \langle\!\langle u - \Pi u, dv \rangle\!\rangle_{\Gamma_j}$$
$$= \sum_j (u - \Pi u, Lv)_j - \sum_j \left\langle u - \Pi u, \left(a\frac{\partial}{\partial \nu_j} + d \right) (\psi - v) \right\rangle_{\Gamma_j}$$
$$- E(u - \Pi u, v) + \sum_j \left\langle u - \Pi u, a\frac{\partial v_j}{\partial \nu_j} \right\rangle_{\partial\Omega_j\setminus\Gamma_j},$$

since $a\partial\psi/\partial\nu + d\psi = 0$ on $\partial\Omega$. First,

$$\left| \sum_j (u - \Pi u, Lv)_j \right| \le C\|u\|_2 h^2 \|v\|_{2,h} \le C\|u\|_2\|\eta\| h^2.$$

Next,

$$\left| \sum_j \left\langle u - \Pi u, \left(a\frac{\partial}{\partial \nu} + d \right) (\psi - v) \right\rangle_{\Gamma_j} \right| \le C|u - \Pi u|_{\partial\Omega}\|\psi - v\|^{\frac{1}{2}}_{1,h}\|\psi - v\|^{\frac{1}{2}}_{2,h}$$
$$\le C\|u\|_2\|\eta\| h^2.$$

Since, by (2.24, 2.27, 2.26),

$$|E_\ell(u-\Pi u, dv)| \le \begin{cases} C|u-\Pi u|_{1,\partial\Omega}|v|_{\partial\Omega}h, & \ell = 1 \text{ or } 2, \\ C|u-\Pi u|_{2,\partial\Omega}|v|_{\partial\Omega}h^2, & \ell = 2, \end{cases}$$

then

$$|E_\ell(u-\Pi u, dv)| \le \begin{cases} C\|u\|_2\|\eta\|h^{\frac{3}{2}}, & \ell = 1 \text{ or } 2, \\ C|u|_{2,\partial\Omega}\|\eta\|h^2, & \ell = 2. \end{cases}$$

Now, note that

$$\langle u_j - \Pi u_j, P_0\psi_j\rangle_{\Gamma_{jk}} + \langle u_k - \Pi u_k, P_0\psi_k\rangle_{\Gamma_{kj}} = -\langle \Pi u_j - \Pi u_k, P_0\psi_j\rangle_{\Gamma_{jk}} = 0,$$

since $(\Pi u_j - \Pi u_k) \perp 1$ on Γ_{jk}, as it is a linear function vanishing at ξ_{jk} for simplicial elements; this orthogonality will be imposed in defining the basis over rectangular elements. Thus,

$$\sum_j \left\langle u-\Pi u, a\frac{\partial v_j}{\partial \nu_j}\right\rangle_{\partial\Omega_j\setminus\Gamma_j} = \sum_j \left\langle u-\Pi u, a\frac{\partial (v-\psi)_j}{\partial \nu_j}\right\rangle_{\partial\Omega_j\setminus\Gamma_j} + \sum_j \left\langle u-\Pi u, a\frac{\partial \psi_j}{\partial \nu_j} - P_0\psi_j\right\rangle_{\partial\Omega_j\setminus\Gamma_j},$$

and

$$\begin{aligned} \left|\sum_j \left\langle u-\Pi u, a\frac{\partial v_j}{\partial \nu_j}\right\rangle_{\partial\Omega_j\setminus\Gamma_j}\right| &\le C|u-\Pi u|_{\partial\Omega}\left(\|\psi - v\|_{1,h}^{\frac{1}{2}}\|\psi\|_2^{\frac{1}{2}} + \|\psi\|_2 h^{\frac{1}{2}}\right) \\ &\le C\|u\|_2\|\eta\|h^2. \end{aligned}$$

We have $E(d\psi,\eta)$ and $E(du-g,v)$ left to bound:

$$|E_\ell(d\psi,\eta)| \le C|\psi|_{1,\partial\Omega}|\eta|_{\partial\Omega}h \le C\|\eta\|_{1,h}\|\eta\|h, \qquad \ell = 1 \text{ or } 2.$$

Finally,

$$|E_\ell(du-g,v)| \le \begin{cases} C(|u|_{1,\partial\Omega} + |g|_{1,\partial\Omega})|v|_{\partial\Omega}h, & \ell = 1 \text{ or } 2, \\ C(|u|_{2,\partial\Omega} + |g|_{2,\partial\Omega})|v|_{\partial\Omega}h^2, & \ell = 2. \end{cases}$$

Thus,

$$|E_\ell(du-g,v)| \le \begin{cases} C(\|u\|_2 + |g|_{1,\partial\Omega})\|\eta\|h, & \ell = 1 \text{ or } 2, \\ C(|u|_{2,\partial\Omega} + |g|_{2,\partial\Omega})\|\eta\|h^2, & \ell = 2. \end{cases}$$

Combining this collection of bounds gives the estimate

$$\|\eta\| \le C\left\{\|\eta\|_{1,h}h + \|u\|_2h^2 + \varepsilon_\ell\right\}, \tag{2.13}$$

where

$$\varepsilon_\ell \le \begin{cases} (\|u\|_2 + |g|_{1,\partial\Omega})\,h, & \ell = 1, \\ (|u|_{2,\partial\Omega} + |g|_{2,\partial\Omega})\,h^2, & \ell = 2; \end{cases} \tag{2.14}$$

the ε_ℓ-term is missing in (2.13) if exact quadrature is used on the boundary integrals. Since $\|u-\Pi u\| \le C\|u\|_2 h^2$, we have shown that

$$\|u-u_h\| \le C\left\{(\|u\|_2 + |g|_{1,\partial\Omega})\, h^2 + \varepsilon_\ell\right\}\cdot \tag{2.15}$$

In both (2.14) and (2.15), the boundary norms can be interpreted as broken over the boundary partition.

The bounds for ε_ℓ given in (2.14) appear to imply that the application of the midpoint quadrature rule, while leading to an optimal order convergence rate in the energy norm, gives an $\mathcal{O}(h)$ convergence rate in L^2; *i.e.*, no improvement over the energy rate. However, applying the quadrature rules associated with $\ell = 2$ gives the optimal $\mathcal{O}(h^2)$ rate on L^2, provided that the solution has the regularity demanded in (2.15). We state the main result regarding L^2-convergence in the following theorem.

Theorem 2.2. *Let the Robin boundary problem* (2.1) *be $H^2(\Omega)$-regular, and let u denote its solution. If u_h is the solution of* (2.5) *and if a second-order correct quadrature method is used in the evaluation $\langle\langle\cdot,\cdot\rangle\rangle$ of boundary integrals, then*

$$\|u-u_h\| \le C\,(\|u\|_2 + |u|_{2,\partial\Omega} + |g|_{2,\partial\Omega})\, h^2.$$

The boundary norm terms can be omitted if exact quadrature is applied on $\partial\Omega$.

2.5. Rectangular nonconforming methods

Consider the two-dimensional case first, and take as reference element the square $\hat{R} = [-1,1]^2$. The usual bilinear basis for conforming Galerkin procedures over rectangular elements is based on $\text{Span}\,\{1,x,y,xy\}$ on the reference element. In the nonconforming method, we wish to impose continuity at the midpoints of the faces just as for simplicial nonconforming methods and to use values at these points as the degrees of freedom; however, interpolation at these nodes fails. The first thought is to rotate the basis through 45 degrees; *i.e.*, try a basis built on $\mathcal{R} = \text{Span}\,\{1,x,y,x^2-y^2\}$. Now, unique interpolation is valid over the desired nodes. However, a look back at the convergence proofs for the simplicial nonconforming method shows that a critical role in defining the projection P_0 (and in the proof) was played by the property

$$\langle 1, w_j - w_k\rangle_{\Gamma_{jk}} = 0, \quad w \in \mathcal{NC}^h. \tag{2.16}$$

Since

$$\mathcal{R}\mid_{\{y=1\}} = \text{Span}\,\{1,x,x^2\},$$

restricting a function in $\mathcal{R}\mid_{\{y=1\}}$ to vanish at $x=0$ leaves $\text{Span}\,\{x,x^2\}$, so that (2.16) fails. This failure is easily remedied by modifying x^2 to $x^2-\frac{5}{3}x^4$, which is orthogonal to linear functions. This function does not vanish at the Gauss points $\pm 1/\sqrt{3}$, a property that will be useful in order to apply two-point Gauss quadrature on the boundary Γ so that an optimal order error estimate in $L^2(\Omega)$ can be derived for either Neumann or Robin boundary conditions. Now, the function $x^2-\frac{25}{6}x^4+\frac{7}{2}x^6$ both is orthogonal to linear functions on $[-1,1]$ and vanishes at the Gauss points. So, let

$$\theta_\ell(x) = \begin{cases} x^2-\frac{5}{3}x^4, & \ell = 1,\\ x^2-\frac{25}{6}x^4+\frac{7}{2}x^6, & \ell = 2, \end{cases} \tag{2.17}$$

and define two reference bases by

$$\mathcal{Q}_\ell = \text{Span}\,\{1,x,y,\theta_\ell(x)-\theta_\ell(y)\}, \quad \ell = 1,2. \tag{2.18}$$

It is easy to see that unique interpolation over the nodes is retained for either basis; also, we now have the orthogonality property (2.16) and $\theta_2(x)$ vanishes for $x = \pm 1/\sqrt{3}$. A nodal basis is easily found; the basis function corresponding to the node $(1,0)$ is given by

$$w^{(\ell)}_{1,0} = \frac{1}{4} + \frac{1}{2}x + \frac{\theta_\ell(x) - \theta_\ell(y)}{4\theta_\ell(1)}, \quad \ell = 1, 2. \tag{2.19}$$

An extension to quadrilateral elements is immediate. If Q is a quadrilateral, there is a unique (up to rotation in the order of the vertices) bilinear map $F : \hat{R} \to Q$ and F is affine on the edges of $\hat{R}$. Thus, if

$$\mathcal{Q}_\ell(Q) = \{v \, : \, v = \hat{v} \circ F^{-1}, \ \hat{v} \in \mathcal{Q}_\ell(\hat{R})\}, \quad \ell = 1, 2,$$

then the orthogonality property (2.16) remains valid for $\ell = 1$ or 2 and the desired vanishing at Gauss points holds for $\ell = 2$. Moreover, the two affine maps induced on a common edge between adjacent quadrilateral elements coincide, so that requiring continuity at midpoints of edges is consistent with the mappings. If shape quasiregularity is enforced on a partition into quadrilaterals, then the approximation properties (2.4) also remain valid.

The properties listed above will allow us to observe that the entire convergence argument for the simplicial case remains valid. We delay stating the results until after deriving a useful three-dimensional basis.

When $n = 3$, the minimum dimension of $\mathcal{Q}_\ell$ is six, and the choices

$$\begin{aligned} \mathcal{Q}_\ell &= \text{Span}\,\{1, x, y, z, \theta_\ell(x) - \theta_\ell(y), \theta_\ell(x) - \theta_\ell(z)\} \\ &= \text{Span}\,\{1, x, y, z, \theta_\ell(y) - \theta_\ell(z), \theta_\ell(y) - \theta_\ell(x)\} \\ &= \text{Span}\,\{1, x, y, z, \theta_\ell(z) - \theta_\ell(x), \theta_\ell(z) - \theta_\ell(y)\}, \quad \ell = 1, 2, \end{aligned} \tag{2.20}$$

have that dimension; moreover, $\mathcal{Q}_\ell$ is invariant under both reflection and permutation of the coordinates. It also has the critical orthogonality property (2.16). The nodal basis element associated with the node $(1,0,0)$ is given by

$$w^{(\ell)}_{1,0,0} = \frac{1}{6} + \frac{1}{2}x - \frac{1}{6\theta_\ell(1)}\left(2\theta_\ell(x) - \theta_\ell(y) - \theta_\ell(z)\right), \quad \ell = 1, 2;$$

the other five nodal basis functions can be obtained by reflection and permutation. Thus, this choice for a local basis is completely acceptable for $\ell = 1$ or 2.

Two other acceptable choices are given by

$$\begin{aligned} \mathcal{Q}_\ell &= \text{Span}\,\{1, x, y, z, \theta_\ell(x), \theta_\ell(y), \theta_\ell(z)\}, \qquad \ell = 1, 2, \\ &= \text{Span}\left\{\frac{1}{2}x \pm \frac{\theta_\ell(x)}{2\theta_\ell(1)}, \frac{1}{2}y \pm \frac{\theta_\ell(y)}{2\theta_\ell(1)}, \frac{1}{2}z \pm \frac{\theta_\ell(z)}{2\theta_\ell(1)}, 1 - \frac{1}{\theta_\ell(1)}\left(\theta_\ell(x) + \theta_\ell(y) + \theta_\ell(z)\right)\right\}. \end{aligned} \tag{2.21}$$

The seven degrees of freedom associated with (2.21) are the values at the centers of the faces and at the center of the element; for computational purposes, the basis element associated to the origin is a bubble function (as shown above) and can be eliminated without serious cost over what would be required with the corresponding basis consisting of six functions.

Either of these elements can be extended to parallelepipeds trivially by means of a trilinear map; unfortunately, it can also be shown that the desired orthogonalities are lost on a flat, quadrilateral face that is not a parallelogram.

As a consequence of the requirement of the orthogonality (2.16) and the analyses of the boundary quadrature procedures given in Section 2.9, the analyses in Section 2.3 and Section 2.4 of the error $u - u_h$ apply without modification in the broken H^1-norm for $\ell = 1$ and 2 and in $L^2(\Omega)$ for $\ell = 2$. Thus, Theorems 2.1 and 2.2

are valid for our nonconforming Galerkin method over rectangular elements. Also, an inspection of the proofs shows that these theorems hold when simplicial and rectangular elements are mixed in the partition of Ω.

2.6. **Interpolation of coefficients**

The implementation of the finite element procedure depends on approximating the integrals in a_h; this can be done either through the use of quadrature formulae on individual elements or by interpolating the coefficients a, c, and d and then computing exact quadratures. The second of these procedures will be discussed in this section.

We shall consider the perturbation of the approximate solution caused by perturbing the coefficients $a(x)$ and $c(x)$, since quadrature has already been applied on the boundary $\partial\Omega$ and its effect on the approximation error has been taken into account. Let the perturbed (interpolated) coefficients be denoted by $\overline{a}(x)$ and $\overline{c}(x)$, and set

$$\overline{a}_h(z,w) = \sum_j (\overline{a}\nabla z, \nabla w)_j + (\overline{c}z, w) + \langle\langle dz, w\rangle\rangle.$$

Two cases cover most of the occurrences of (2.1): $c(x) \equiv 0$ and

$$0 < c_0 \le c(x) \le c_1.$$

If $c \equiv 0$, the obvious choice of $\overline{c}$ is also zero. Assume that $\overline{a}$ and $\overline{c}$ are chosen so as to satisfy the same bounds as a and c; *i.e.*, let $a_0 \le \overline{a}(x) \le a_1$ and, if $c_0 > 0$, $c_0 \le \overline{c}(x) \le c_1$.

Let $\overline{u}_h \in \mathcal{NC}^h$ be the solution of

$$\overline{a}_h(\overline{u}_h, v) = F_h(v) = (f, v) + \langle\langle g, v\rangle\rangle, \quad v \in \mathcal{NC}^h.$$

If

$$e_h = u_h - \overline{u}_h,$$

then

$$\begin{aligned} \overline{a}_h(e_h, v) &= a_h(u_h, v) - \overline{a}_h(\overline{u}_h, v) + (\overline{a}_h - a_h)(u_h, v) \\ &= \sum_j ((\overline{a} - a)\nabla u_h, \nabla v)_j + ((\overline{c} - c)u_h, v), \quad v \in \mathcal{NC}^h. \end{aligned}$$

For the high-order term in a_h, consider the simplicial nonconforming method first. Then, for any $v \in \mathcal{NC}^h$, ∇v is constant on any element Ω_j; consequently, taking

$$\overline{a}(x) = \frac{1}{|\Omega_j|}\int_{\Omega_j} a(y)\,\mathrm{d}y, \quad x \in \Omega_j,$$

has no effect on the a-integral and, so, does not alter the approximate solution; this, of course, is the same in the conforming Galerkin procedure over simplices. Next, let us find an interpolation of $a(x)$ in the rectangular case which will not affect the approximate solution. We will look at the two-dimensional case when $\ell = 1$; the $\{\ell = 2\}$-case and the three-dimensional cases can be treated analogously. For w and z in $\mathcal{NC}^h$ on $[-1,1]^2$,

$$\frac{\partial w}{\partial x}\frac{\partial z}{\partial x} \in \text{Span}\left\{1, x - \frac{10}{3}x^3, \left(x - \frac{10}{3}x^3\right)^2\right\}.$$

Thus, to define $\overline{a}\,|_{\Omega_j}$, carry $a\,|_{\Omega_j}$ to the reference square, take $\overline{a}$ in the form

$$\overline{a}(x,y) = \frac{1}{|\Omega_j|}\int_{\Omega_j} a(\alpha)\,d\alpha + Ax + B\left(x^2 - \frac{2}{3}\right) + Cy + D\left(y^2 - \frac{2}{3}\right),$$

and orthogonalize $a - \overline{a}$ against

$$\text{Span}\left\{1, x - \frac{10}{3}x^3, y - \frac{10}{3}y^3, \left(x - \frac{10}{3}x^3\right)^2, \left(y - \frac{10}{3}y^3\right)^2\right\};$$

then, return this projection to Ω_j as $\overline{a}$ on Ω_j. Again, there is no induced modification in the approximate solution. If, in addition, $c \equiv 0$, $\overline{u}_h = u_h$.

If $c \not\equiv 0$, projection of c into $\mathcal{P}_2(\Omega_j)$, $j = 1, \ldots, J$, gives $\overline{c}$ for which the solution remains unchanged for a simplicial partition of Ω. For a rectangular element, it seems impractical to project c onto the 8-dimensional space $\mathcal{Q}_\ell \otimes \mathcal{Q}_\ell$, but projecting c into $\mathcal{P}_1(\Omega_j)$ will give a bound of the form

$$\|u_h - \overline{u}_h\|_{1,h} + \|u_h - \overline{u}_h\| \le C\|u_h\|_{1,h}h^2 \le C\|u\|_1 h^2. \tag{2.22}$$

The bound (2.22) applies in all cases.

2.7. Neumann boundary conditions

The Neumann problem is obtained from the Robin problem by setting the coefficient d equal to zero. If $c(x) \ge c_0 > 0$, then the analysis for the Robin case applies with the only change being the elimination of the norm of u on $\partial\Omega$ in the error bounds in Theorems 2.1 and 2.2. If $c \equiv 0$, then the consistency condition

$$(f,1) + \langle g,1\rangle = 0$$

translates to

$$(f,1) + \langle\langle g,1\rangle\rangle = 0$$

for the Galerkin procedure. This can force a trivial shift by the addition of a small, $\mathcal{O}(h^3)$, constant in the boundary data function g when one of the $\ell = 2$ quadrature formulae discussed above is applied in the discretization of the boundary condition; otherwise, the error bounds remain valid.

2.8. Dirichlet boundary conditions

Let us consider briefly the application of the analogous nonconforming Galerkin methods to the Dirichlet problem

$$\begin{aligned} Lu = -\nabla\cdot(a\nabla u) + cu &= f, \quad x \in \Omega,\\ u &= g, \quad x \in \partial\Omega. \end{aligned}$$

Redefine a_h to be

$$a_h(z,w) = \sum_j (a\nabla z, \nabla w)_j + (cz, w),$$

and seek $u_h \in \mathcal{NC}^h$ (here, for rectangular elements, there is no advantage in using the $\{\ell = 2\}$-basis in place of the $\{\ell = 1\}$-basis) such that

$$\begin{aligned} a_h(u_h, v) &= (f, v), \quad v \in \mathcal{NC}_0^h = \{z \in \mathcal{NC}^h \mid z(\xi_j) = 0, \forall \text{ midpoints } \xi_j \in \Gamma_j\}, \\ u_h(\xi_j) &= g(\xi_j). \end{aligned}$$

A simple calculation shows that

$$a_h(u, v) = (f, v) + \sum_j \left\langle a\frac{\partial u}{\partial \nu_j}, v_j \right\rangle_{\partial\Omega_j}, \quad v \in \mathcal{NC}_0^h.$$

Since $v_j \perp 1$ on Γ_j for $v \in \mathcal{NC}_0^h$, the analysis in Section 2.3 can be repeated to give

$$a_h(u, v) = (f, v) + \sum_j \left\langle a\frac{\partial u}{\partial \nu_j} - P_0 u_j, v_j - m_j \right\rangle_{\partial\Omega_j}, \quad v \in \mathcal{NC}_0^h,$$

where $m_j \in \mathcal{P}_0(\partial\Omega_j)$. Thus,

$$|a_h(u, v) - (f, v)| \leq K\|u\|_2 h^{\frac{1}{2}} \left(\sum_j \|v_j - m_j\|_j \|\nabla(v_j - m_j)\|_j \right)^{\frac{1}{2}} \leq K\|u\|_2 \|v\|_{1,h} h,$$

and it follows that the optimal order energy error estimate

$$\|u - u_h\|_{1,h} \leq C\|u\|_2 h \tag{2.23}$$

holds under minimal regularity.

Let $\Pi u \in \mathcal{NC}^h$ be defined as before and set

$$\eta = \Pi u - u_h \in \mathcal{NC}_0^h.$$

Now, let us indicate the duality argument that leads to an L^2 error bound. Shift the auxiliary problem to

$$\begin{aligned} L\psi = -\nabla \cdot (a\nabla\psi) + c\psi &= \eta, \quad x \in \Omega, \\ \psi &= 0, \quad x \in \partial\Omega. \end{aligned}$$

The analogue of (2.12) is given by

$$\|\eta\|^2 = a_h(\eta, \psi - v) - \sum_j \left\langle a\frac{\partial \psi}{\partial \nu_j} - P_0 \psi_j, \eta_j - r_j \right\rangle_{\partial\Omega_j} - \sum_j \left\langle a\frac{\partial u}{\partial \nu_j} - P_0 u_j, v_j - \psi_j \right\rangle_{\partial\Omega_j} + a_h(u - \Pi u, v), \quad v \in \mathcal{NC}_0^h,$$

where $r_j \in \mathcal{P}_0(\Omega_j)$. The remainder of the argument parallels that given in Section 2.4 and will not be repeated, except to note that the boundary term

$$\sum \left\langle u - \Pi u, a\frac{\partial \psi}{\partial \nu_j} \right\rangle_{\Gamma_j},$$

which does not appear in the Robin argument, can be bounded as follows:

$$\left|\sum \left\langle u-\Pi u, a\frac{\partial\psi}{\partial\nu_j}\right\rangle_{\Gamma_j}\right| \le C|u-\Pi u|_{\partial\Omega}\|\eta\| \le C\|u\|_2\|\eta\|h^{\frac{3}{2}}.$$

As a consequence of this term, the error cannot be bounded in $L^2(\Omega)$ by $\mathcal{O}(h^2)$ and it is necessary to settle for the bound

$$\|u-u_h\| \le C\|u\|_2 h^{\frac{3}{2}}.$$

In contrast with the L^2 error estimate for either Neumann or Robin boundary conditions, we are left with a suboptimal convergence rate. In the other two cases, the boundary information, data in both cases plus the solution in the Robin case, enter through integrals on the boundary. Consequently, there is control over the discretization accuracy associated with the boundary condition in these cases, while the Dirichlet data must be represented by a single parameter per boundary face. We could have imposed the average value in place of the midpoint value; however, this merely shifts terms for losing $h^{\frac{1}{2}}$. We were able to insure an optimal rate for the other cases by applying a quadrature rule of greater accuracy than a single parameter rule.

2.9. **Some quadrature lemmata**

Some technical lemmata related to the approximation of the boundary condition as a result of the application of quadrature formulae will be collected in this section. We wish to estimate

$$E(g,w) = \langle g,w\rangle - \langle\!\langle g,w\rangle\!\rangle = \sum_j \left\{\langle g,w\rangle_{\Gamma_j} - \langle\!\langle g,w\rangle\!\rangle_{\Gamma_j}\right\}, \quad w\in\mathcal{NC}^h,$$

where g will be assumed to be in $H^s(\partial\Omega)$ for $s=1$ or 2. The midpoint rule will be treated on both simplicial and rectangular elements simultaneously, but it will be convenient to consider the simplicial and rectangular cases separately for higher order quadratures.

Let Γ be a face of a boundary element, simplicial or rectangular with $n=2$ or 3, and let ξ be its midpoint. The midpoint rule is given, as always, by

$$\langle\!\langle g,w\rangle\!\rangle_\Gamma = (gw)(\xi)|\Gamma|.$$

For the restriction to any boundary face Γ of any of the bases discussed for a nonconforming Galerkin method,

$$\langle 1,w\rangle_\Gamma = w(\xi)|\Gamma| = \langle\!\langle 1,w\rangle\!\rangle_\Gamma, \qquad w\in\mathcal{NC}^h,$$

so that

$$E_1(\Gamma;g,w) = \langle g,w\rangle_\Gamma - \langle\!\langle g,w\rangle\!\rangle_\Gamma = \langle g-g(\xi),w\rangle_\Gamma$$

and

$$|E_1(\Gamma;g,w)| \le |g-g(\xi)|_\Gamma|w|_\Gamma \le C|g|_{1,\Gamma}|w|_\Gamma h;$$

here, the subscript 1 indicates that the first-order, midpoint quadrature rule has been applied. Hence,

$$|E_1(g,w)| \le C|g|_{1,\partial\Omega}|w|_{\partial\Omega}h, \qquad w\in\mathcal{NC}^h. \tag{2.24}$$

Next, let Γ be the boundary face of a simplicial element. If $n = 2$, apply two-point Gaussian quadrature on Γ. Let $I_1 g$ be the linear interpolant of g over the Gauss points. Since this rule is exact on $\mathcal{P}_3(\Gamma)$ and $w \in \mathcal{P}_1(\Gamma)$,

$$E_2(\Gamma; g, w) = \langle g - I_1 g, w\rangle_\Gamma, \tag{2.25}$$

so that

$$|E_2(g, w)| \le C|g|_{2,\partial\Omega}|w|_{\partial\Omega}h^2, \qquad w \in \mathcal{NC}^h. \tag{2.26}$$

If $n = 3$ and Γ is a boundary triangle, let ζ_i, $i = 1, 2, 3$, be the midpoints of the edges of Γ and set

$$\langle g, w\rangle_\Gamma = \sum_{i=1}^{3}(gw)(\zeta_i)\frac{|\Gamma|}{3}.$$

This quadrature rule (p. 183 of [4]) is exact on polynomials of degree 2; consequently, if $I_2 g$ denotes linear interpolation over the three quadrature points,

$$E_2(\Gamma; g, w) = \langle g - I_2 g, w\rangle_\Gamma, \qquad w \in \mathcal{NC}^h,$$

and (2.26) holds again.

Now, turn to rectangular elements, where we will consider only product quadrature rules. It suffices to consider $\Gamma = [-1, 1]^2$ as the top face of a cube; the two-dimensional case follows similarly. Then, for either of the two choices of $\mathcal{Q}_2$ (note that $\mathcal{Q}_1$ is treated above and is excluded here) offered in Section 2.5,

$$\mathcal{R} = \mathcal{Q}_2 \,|_\Gamma = \text{ Span}\left\{1, x, y, x^2 - \frac{25}{6}x^4 + \frac{7}{2}x^6,\ y^2 - \frac{25}{6}y^4 + \frac{7}{2}y^6\right\}.$$

Apply 2×2 Gauss quadrature on Γ; it is exact on $\mathcal{P}_3 \otimes \mathcal{P}_3$, which does not automatically allow us to reduce $E_2(\Gamma; g, w)$ to a form $\langle g - Ig, w\rangle_\Gamma$ for some simple interpolation of g. However, let I_3 denote bilinear interpolation over the four Gauss points as nodes. Let, for $w \in \mathcal{NC}^h$,

$$w = w_1 + w_2, \quad w_1 \in \text{ Span}\,\{1, x, y\}, \quad w_2 \in \text{ Span}\left\{x^2 - \frac{25}{6}x^4 + \frac{7}{2}x^6,\ y^2 - \frac{25}{6}y^4 + \frac{7}{2}y^6\right\}.$$

Then,

$$\langle I_3 g, w\rangle_\Gamma - \langle\!\langle I_3 g, w\rangle\!\rangle_\Gamma = \langle I_3 g, w_2\rangle_\Gamma - \langle\!\langle I_3 g, w_2\rangle\!\rangle_\Gamma = 0,$$

since $I_3 g \perp w_2$ and w_2 vanishes at the Gauss points. Thus,

$$E_2(\Gamma; g, w) = \langle g - I_3 g, w\rangle_\Gamma$$

and $|g - I_3 g|_\Gamma \le Ch^2|g|_{2,\Gamma}$, so that (2.26) follows for the application of the 2×2 Gauss rule by scaling Γ to size h.

Of course, it is also true that

$$|E_2(g, w)| \le C|g|_{1,\partial\Omega}|w|_{\partial\Omega}h, \qquad w \in \mathcal{NC}^h, \tag{2.27}$$

for any of the higher order rules mentioned.

3. A domain decomposition iterative procedure

3.1. The hybridized nonconforming finite element method

We shall discuss a domain decomposition iterative procedure for the rectangular nonconforming method in this part of the paper. Occasional trivial modifications in the presentation suffice to cover the simplicial case. We shall treat only decomposition into individual elements here; as in earlier work [6, 7] utilizing Robin transmission conditions, we begin by hybridizing [1] the finite element method.

First note that $\partial v/\partial \nu_{jk}$ is constant on Γ_{jk} for any $v \in Q(\Omega_j)$. Thus, it is reasonable to define a hybridization of (2.5) by associating a space of Lagrange multipliers $\widetilde{\lambda}^h \in \Lambda^h$ associated with $-a(\xi_{jk})\partial p/\partial \nu_{jk}$ on Γ_{jk}. Also, localize the nonconforming Galerkin space $\mathcal{NC}^h$ by removing the midpoint continuity constraints on the interfaces between elements:

$$\mathcal{NC}^h_{-1} = \{\widetilde{v} \in L^2(\Omega) : \widetilde{v}\,|_{\Omega_j} \in Q(\Omega_j)\}\cdot$$

The hybridized procedure corresponding to (2.5) is defined in the following fashion: find $(\widetilde{p}^h, \widetilde{\lambda}^h) \in \mathcal{NC}^h_{-1} \times \Lambda^h$ such that

$$\sum_j (a\nabla \widetilde{p}^h, \nabla v)_j + (c\widetilde{p}^h, v) + \sum_j \langle\!\langle \widetilde{\lambda}^h, v\rangle\!\rangle_{\partial\Omega_j} + \langle\!\langle d\widetilde{p}^h, v\rangle\!\rangle = (f, v) + \langle\!\langle g, v\rangle\!\rangle, \qquad v \in \mathcal{NC}^h_{-1}, \tag{3.1a}$$

$$\sum_j \langle\!\langle \theta, \widetilde{p}^h\rangle\!\rangle_{\partial\Omega_j} = 0, \qquad \theta \in \Lambda^h; \tag{3.1b}$$

in the above equations, we have implicitly set the Lagrange multiplier $\widetilde{\lambda}$ to zero on boundary faces to shorten notation and below we consider any element of Λ^h to vanish on Γ. Assume that the two-point Gauss rule has been applied to the integral over $\partial\Omega$.

The following lemma is immediate.

Lemma 3.1. *If $\widetilde{p}^h \in \mathcal{NC}^h_{-1}$, then $\widetilde{p}^h \in \mathcal{NC}^h$ if and only if*

$$\sum_j \langle\!\langle \theta, \widetilde{p}^h\rangle\!\rangle_{\partial\Omega_j} = 0, \qquad \theta \in \Lambda^h. \tag{3.2}$$

Let us demonstrate the uniqueness (and, consequently, existence) of the solution of (3.1). Set $f = g = 0$ and note that the choice $\theta = \widetilde{\lambda}^h$ in (3.1b) yields

$$\sum_j \langle\!\langle \widetilde{\lambda}^h, \widetilde{p}^h\rangle\!\rangle_{\partial\Omega_j} = 0.$$

Then, choose $v = \widetilde{p}^h$ in (3.1a) and use the above equation to obtain

$$\sum_j (a\nabla \widetilde{p}^h, \nabla \widetilde{p}^h)_j + (c\widetilde{p}^h, \widetilde{p}^h) + \langle\!\langle d\widetilde{p}^h, \widetilde{p}^h\rangle\!\rangle = 0. \tag{3.3}$$

Since $c \geq 0$ and $d > 0$,

$$\sum_j (a\nabla \widetilde{p}^h, \nabla \widetilde{p}^h)_j = 0 \quad \text{and} \quad \widetilde{p}^h(\xi_j^\ell) = 0 \text{ if } \xi_j^\ell \in \Gamma; \tag{3.4}$$

ξ_j^ℓ, $\ell = 1, 2$, are the two Gauss points on an edge $\Gamma_j = \partial\Omega_j \cap \Gamma$ for a boundary element Ω_j. The first of these relations implies that $\widetilde{p}^h$ is constant on each Ω_j. (If $\dim(\Omega) = 3$, there are four Gauss points, at all of which $\widetilde{p}^h$ vanishes.)

We wish to show that $\widetilde{p}^h \equiv 0$ in Ω. If Ω_j has a face contained in Γ, then it follows from (3.4) that $\widetilde{p}^h$ vanishes on Ω_j. Then, we can choose the test function v in (3.1a) to be supported on Ω_j and to vanish at all but one of the nodal points on Ω_j; in this manner, we see that the Lagrange multiplier $\widetilde{\lambda}^h$ vanishes on Γ_{jk} if Ω_k is adjacent to Ω_j. Note that the continuity of $\widetilde{p}^h$ at the midpoint of Γ_{jk} implies that the same argument shows that $\widetilde{p}^h$ and $\widetilde{\lambda}^h$ vanish on Ω_k. Since any element is connected to a boundary element in a finite number of steps, uniqueness is established. Thus, if we combine the above with Lemma 3.1, we have demonstrated the following theorem.

Theorem 3.1. *Problem* (3.1) *has a unique solution. Moreover,* $\widetilde{p}^h$ *is a solution of* (2.5) *and the error estimates derived in Section 2 hold.*

3.2. The domain decomposition procedure

Consider decomposing the solution of (2.1) into the solution of the local problems

$$-\nabla\cdot(a(x)\nabla p_j)+c(x)p_j = f, \qquad x\in\Omega_j, \tag{3.5a}$$

$$a(x)\frac{\partial p_j}{\partial\nu_j}+d(x)p_j = g, \qquad x\in\Gamma_j, \tag{3.5b}$$

subject to the natural consistency conditions

$$a\frac{\partial p_j}{\partial\nu_{jk}}+a\frac{\partial p_k}{\partial\nu_{kj}} = 0, \qquad \text{on } \Gamma_{jk}, \tag{3.6a}$$

$$p_j = p_k, \qquad \text{on } \Gamma_{jk}, \tag{3.6b}$$

where ν_{jk} denotes the unit outward normal to Γ_{jk} directed toward Ω_k. Instead of requiring (3.6), we will impose the equivalent Robin transmission conditions

$$a\frac{\partial p_j}{\partial\nu_{jk}}+\beta p_j=-a\frac{\partial p_k}{\partial\nu_{kj}}+\beta p_k, \qquad \Gamma_{jk}\subset\partial\Omega_j, \tag{3.7a}$$

$$a\frac{\partial p_k}{\partial\nu_{kj}}+\beta p_k=-a\frac{\partial p_j}{\partial\nu_{jk}}+\beta p_j, \qquad \Gamma_{kj}\subset\partial\Omega_k, \tag{3.7b}$$

with β being a positive constant. Using (3.7), we can state a weak formulation of (3.5) as follows: Find $p_j\in H^1(\Omega_j)$ such that

$$(a\nabla p_j,\nabla v)_j+(cp_j,v)_j+\sum_k\left\langle a\frac{\partial p_k}{\partial\nu_{kj}}+\beta(p_j-p_k),v\right\rangle_{\Gamma_{jk}}+\langle dp_j,v\rangle_{\Gamma_j}=(f,v)_j+\langle g,v\rangle_{\Gamma_j},\quad v\in H^1(\Omega_j). \tag{3.8}$$

We localize the calculations by defining an iterative procedure at the differential level as follows: given $p_j^0\in H^1(\Omega_j)$, $j=1,\ldots,J$, find p_j^n as the solution of

$$(a\nabla p_j^n,\nabla v)_j+(cp_j^n,v)_j+\sum_k\langle\beta p_j^n,v\rangle_{\Gamma_{jk}}+\langle dp_j^n,v\rangle_{\Gamma_j}=$$
$$-\sum_k\left\langle a\frac{\partial p_k^{n-1}}{\partial\nu_{kj}}-\beta p_k^{n-1},v\right\rangle_{\Gamma_{jk}}+(f,v)_j+\langle g,v\rangle_{\Gamma_j},\ v\in H^1(\Omega_j). \tag{3.9}$$

Next, we will define a discretized version of (3.9). For that purpose, let I^h denote the set of all internal interfaces Γ_{jk} and introduce a new set Λ_*^h of Lagrange multipliers λ_{jk}^h associated with the flux $-a\partial p_j/\partial\nu_j$ on Γ_{jk} (*i.e.*, $\lambda\sim -a\partial p_j/\partial\nu_j$) as follows:

$$\Lambda_*^h=\left\{\lambda^h:\lambda^h\,|_{\Gamma_{jk}}=\lambda_{jk}\in P_0(\Gamma_{jk})\equiv\Lambda_{jk},\ \Gamma_{jk}\in I^h\right\}.$$

Here, we wish to distinguish between Λ_{jk} and Λ_{kj}; we define two Lagrange multipliers on the point set $\Gamma_{jk} = \Gamma_{kj}$ independently and do not impose the constraint $\lambda_{jk} + \lambda_{kj} = 0$. Also, let

$$\mathcal{NC}_j^h = \mathcal{NC}_{-1}^h \mid_{\Omega_j} = \mathcal{Q}_1(\Omega_j).$$

The domain-decomposition iterative procedure for the hybridized, nonconforming Galerkin method is defined in the following manner. Let

$$(p_j^{h,0}, \lambda_{jk}^{h,0}) \in \mathcal{NC}_j^h \times \Lambda_{jk}$$

be given for all j and k. Then, compute $(p_j^{h,n}, \lambda_{jk}^{h,n}) \in \mathcal{NC}_j^h \times \Lambda_{jk}$ as the solution of the equations

$$(a\nabla p_j^{h,n}, \nabla v)_j + (cp_j^{h,n}, v)_j + \sum_k \langle\langle \beta p_j^{h,n}, v\rangle\rangle_{\Gamma_{jk}} + \langle\langle dp_j^{h,n}, v\rangle\rangle_{\Gamma_j} =$$
$$\sum_k \langle\langle \lambda_{kj}^{n-1} + \beta p_k^{h,n-1}, v\rangle\rangle_{\Gamma_{jk}} + (f,v)_j + \langle\langle g, v\rangle\rangle_{\Gamma_j}, \; v \in \mathcal{NC}_j^h, \quad (3.10a)$$

$$\lambda_{jk}^{h,n} = -\lambda_{kj}^{h,n-1} + \beta\left(p_j^{h,n}(\xi_{jk}) - p_k^{h,n-1}(\xi_{jk})\right). \quad (3.10b)$$

In the sections to follow, we will show the convergence of $(p_j^{h,n}, \lambda_{jk}^{h,n})$ to $(\widetilde{p}_j^h, \widetilde{\lambda}_{jk}^h)$, where $\widetilde{p}_j^h = \widetilde{p}^h \mid_{\Omega_j}$ and $\widetilde{\lambda}_{jk}^h = \widetilde{\lambda}^h \mid_{\Gamma_{jk}}$, first without assuming c_0 to be positive and then, with a better rate, when $c_0 > 0$. Let us do some preliminaries here before turning to the proofs.

Substituting (3.10b) into (3.10a) leads to the equation

$$(a\nabla p_j^{h,n}, \nabla v)_j + (cp_j^{h,n}, v)_j + \sum_k \langle\langle \lambda_{jk}^{h,n}, v\rangle\rangle_{\Gamma_{jk}} + \langle\langle dp_j^{h,n}, v\rangle\rangle_{\Gamma_j} = (f,v)_j + \langle\langle g, v\rangle\rangle_{\Gamma_j}, \quad v \in \mathcal{NC}_j^h. \quad (3.11)$$

Then, note that $\widetilde{p}_j^h$ satisfies the local equation

$$(a\nabla \widetilde{p}_j^h, \nabla v)_j + (c\widetilde{p}_j^h, v)_j + \sum_k \langle\langle \widetilde{\lambda}_{jk}^h, v\rangle\rangle_{\Gamma_{jk}} + \langle\langle d\widetilde{p}_j^h, v\rangle\rangle_{\Gamma_j} = (f,v)_j + \langle\langle g, v\rangle\rangle_{\Gamma_j}, \quad v \in \mathcal{NC}_j^h. \quad (3.12)$$

Also, since $\widetilde{\lambda}_{jk}^h = -\widetilde{\lambda}_{kj}^h$, (3.1b) is equivalent to

$$\widetilde{\lambda}_{jk}^h = -\widetilde{\lambda}_{kj}^h + \beta\left(\widetilde{p}_j^h(\xi_{jk}) - \widetilde{p}_k^h(\xi_{jk})\right). \quad (3.13)$$

Set

$$e_j^n = p_j^{h,n} - \widetilde{p}_j^h, \qquad \mu_{jk}^n = \lambda_{jk}^{h,n} - \widetilde{\lambda}_{jk}^h.$$

Then, (3.10b–3.13) imply the error equations

$$(a\nabla e_j^n, \nabla v)_j + (ce_j^n, v)_j + \sum_k \langle\langle \mu_{jk}^n, v\rangle\rangle_{\Gamma_{jk}} + \langle\langle de_j^n, v\rangle\rangle_{\Gamma_j} = 0, \; v \in \mathcal{NC}_j^h, \quad (3.14a)$$
$$\mu_{jk}^n = -\mu_{kj}^{n-1} + \beta\left(e_j^n(\xi_{jk}) - e_k^{n-1}(\xi_{kj})\right). \quad (3.14b)$$

The choice $v = e_j^n$ in (3.14a) gives

$$(a\nabla e_j^n, \nabla e_j^n)_j + (ce_j^n, e_j^n)_j + \sum_k \langle\langle \mu_{jk}^n, e_j^n\rangle\rangle_{\Gamma_{jk}} + \langle\langle de_j^n, e_j^n\rangle\rangle_{\Gamma_j} = 0,$$

which leads to the useful relations

$$\sum_k |\mu_{jk}^n \mp \beta e_j^n(\xi_{jk})|^2_{0,\Gamma_{jk}} = \sum_k \left(|\mu_{jk}^n|^2_{0,\Gamma_{jk}} + |\beta|^2 |e_j^n(\xi_{jk})|^2_{0,\Gamma_{jk}}\right) \mp 2\beta \sum_k \langle\langle \mu_{jk}^n, e_j^n \rangle\rangle_{\Gamma_{jk}}$$
$$= \sum_k \left(|\mu_{jk}^n|^2_{0,\Gamma_{jk}} + \beta^2 |e_j^n(\xi_{jk})|^2_{0,\Gamma_{jk}}\right) \pm 2\beta \left\{(a\nabla e_j^n, \nabla e_j^n)_j + (ce_j^n, e_j^n)_j + \langle\langle de_j^n, e_j^n\rangle\rangle_{\Gamma_j}\right\} \cdot \tag{3.15}$$

Following Lions [9, 10] and Després [6], set

$$R \equiv R(e,\mu) = \sum_{\Gamma_{jk}\in I^h} |\mu_{jk} - \beta e_j(\xi_{jk})|^2_{0,\Gamma_{jk}}. \tag{3.16}$$

Then, from (3.14b) and (3.15) we see that

$$R^n = \sum_{\Gamma_{jk}\in I^h} |\mu_{jk}^n - \beta e_j^n(\xi_{jk})|^2_{0,\Gamma_{jk}} = \sum_{\Gamma_{jk}\in I^h} |\mu_{kj}^{n-1} + \beta e_k^{n-1}(\xi_{jk})|^2_{0,\Gamma_{jk}}$$
$$= R^{n-1} - 4\beta \sum_j \left\{(a\nabla e_j^{n-1}, \nabla e_j^{n-1})_j + (ce_j^{n-1}, e_j^{n-1})_j + \langle\langle de_j^{n-1}, e_j^{n-1}\rangle\rangle_{\Gamma_j}\right\} \cdot \tag{3.17}$$

Since R^n is a decreasing sequence of nonnegative numbers,

$$\sum_{n=1}^{\infty} \sum_j \left\{(a\nabla e_j^n, \nabla e_j^n)_j + (ce_j^n, e_j^n)_j + \langle\langle de_j^n, e_j^n\rangle\rangle_{\Gamma_j}\right\} < \infty, \tag{3.18}$$

and a rather weak convergence theorem can be proved for the iteration. We shall, instead, discuss the spectral radius of the iteration operator.

3.3. The convergence of the iteration when $c_0 = 0$

Let $T_{f,g} : \mathcal{NC}_{-1}^h \times \Lambda^h \to \mathcal{NC}_{-1}^h \times \Lambda^h$ be the affine map such that for any $(u,\theta) \in \mathcal{NC}_{-1}^h \times \Lambda^h$, $(p,\lambda) \equiv T_{f,g}(u,\theta)$ is the solution, for all j, of

$$(a\nabla p_j, \nabla v)_j + (cp_j, v)_j + \sum_k \langle\langle \beta p_j, v\rangle\rangle_{\Gamma_{jk}} + \langle\langle dp_j, v\rangle\rangle_{\Gamma_j} =$$
$$\sum_k \langle\langle \theta_{kj} + \beta u_k, v\rangle\rangle_{\Gamma_{jk}} + (f,v)_j + \langle\langle g, v\rangle\rangle_{\Gamma_j}, \quad v \in \mathcal{NC}_j^h, \tag{3.19}$$

$$\lambda_{jk} = -\theta_{kj} + \beta\left(p_j(\xi_{jk}) - u_k(\xi_{jk})\right). \tag{3.20}$$

Lemma 3.2. *The pair $(p,\lambda) \in \mathcal{NC}_{-1}^h \times \Lambda^h$ is a solution of* (3.12, 3.13) *if and only if it is a fixed point of the operator $T_{f,g}$. If (p,λ) is a fixed point of $T_{f,g}$, then $p_j(\xi_{jk}) = p_k(\xi_{kj})$ and $\lambda_{jk} = -\lambda_{kj}$ for all $\Gamma_{jk} \in I^h$, so that $p \in \mathcal{NC}^h$ and is the solution of* (2.5).

Proof. Let (p,λ) be a fixed point of $T_{f,g}$. Then, substituting (3.20) into (3.19) gives

$$(a\nabla p_j, \nabla v)_j + (cp_j, v)_j + \sum_k \langle\langle \lambda_{jk}, v\rangle\rangle_{\Gamma_{jk}} + \langle\langle dp_j, v\rangle\rangle_{\Gamma_j} = (f,v)_j + \langle\langle g, v\rangle\rangle_{\Gamma_j}, \tag{3.21}$$

so that (p, λ) satisfies (3.12). Also, from (3.20),

$$\lambda_{jk} = -\lambda_{kj} + \beta\left(p_j(\xi_{jk}) - p_k(\xi_{jk})\right),$$

so that (3.13) is satisfied. Since it also follows from (3.20) that $\lambda_{kj} = -\lambda_{jk} + \beta\left(p_k(\xi_{jk}) - p_j(\xi_{jk})\right)$, it is clear that

$$p_j(\xi_{jk}) = p_k(\xi_{jk}) \quad \text{and} \quad \lambda_{jk} = -\lambda_{kj}.$$

Thus, we have shown that any fixed point of $T_{f,g}$ is a solution of (3.12, 3.13) and that $\lambda_{jk} = -\lambda_{kj}$ for all $\Gamma_{jk} \in I^h$. It is obvious that any solution of (3.12, 3.13) is a fixed point of $T_{f,g}$. This completes the proof. $\square$

Since $T_{f,g}(u, \theta)$ can be decomposed as the sum of $T_{0,0}(u, \theta)$ and $T_{f,g}(0, 0)$, (u, θ) is a fixed point of $T_{f,g}$ if and only if

$$(u, \theta) = T_{f,g}(u, \theta) = T_{0,0}(u, \theta) + T_{f,g}(0, 0).$$

Thus, a fixed point (u, θ) of $T_{f,g}$ is a solution of the equation

$$(I - T_{0,0})(u, \theta) = T_{f,g}(0, 0).$$

Our object is to show that the spectral radius $\rho(T_{0,0})$ of $T_{0,0}$ is strictly smaller than one, thereby ensuring the convergence of the iterative procedure (3.10) at a linear rate.

Lemma 3.3. $\rho(T_{0,0}) < 1$.

Proof. Let γ be an eigenvalue of $T_{0,0}$ and let (p, λ) be an associated eigenvector, so that

$$T_{0,0}(p, \lambda) = \gamma(p, \lambda). \tag{3.22}$$

It follows from (3.16) that

$$R(T_{0,0}(p, \lambda)) = |\gamma|^2 R(p, \lambda), \tag{3.23}$$

and, by (3.17),

$$R(T_{0,0}(p, \lambda)) = R(p, \lambda) - 4\beta \sum_j \left\{(a\nabla p_j, \nabla p_j)_j + (cp_j, p_j)_j + \langle\langle dp_j, p_j \rangle\rangle_{\Gamma_j}\right\}\cdot \tag{3.24}$$

Hence,

$$|\gamma|^2 = 1 - \frac{4\beta}{R(p, \lambda)} \sum_j \left\{(a\nabla p_j, \nabla p_j)_j + (cp_j, p_j) + \langle\langle dp_j, p_j \rangle\rangle_{\Gamma_j}\right\}\cdot \tag{3.25}$$

Thus, $|\gamma| \le 1$ and $|\gamma| = 1$ if and only if

$$\sum_j \left\{(a\nabla p_j, \nabla p_j)_j + (cp_j, p_j)_j + \langle\langle dp_j, p_j \rangle\rangle_{\Gamma_j}\right\} = 0, \tag{3.26}$$

so that it suffices to demonstrate that $|\gamma| = 1$ implies that the associated eigenvector (p, λ) is trivial. Clearly, if $c_0 > 0$, $p_j = 0$, $j = 1, \ldots, J$, and it follows from (3.21) that $\lambda_{jk} = 0$ for all j and k. If not, we first observe that it follows from (3.26) that

$$\nabla p_j = 0 \text{ in } \Omega_j,\ j = 1, \ldots, J, \qquad \text{and} \qquad p_j(\xi_j) = 0 \text{ if } \xi_j \in \Gamma_j. \tag{3.27}$$

Then, for any boundary element Ω_j, $p_j = 0$ in Ω_j. It also follows that

$$\gamma \sum_k \langle\langle \lambda_{jk}, v \rangle\rangle_{\Gamma_{jk}} = 0, \quad v \in \mathcal{NC}_j^h,$$

for boundary elements, from which it follows that $\lambda_{jk} = 0$ at nodes of boundary elements.

Next, take an element Ω_j with a face in common with a boundary element Ω_k. If $c(x) > 0$, it follows from (3.26) that $p_j = 0$ in Ω_j. If $c(x) \equiv 0$ on Ω_j, note that

$$\lambda_{kj} = \lambda_{jk} = 0 \text{ and } p_k(\xi_{kj}) = p_j(\xi_{jk}) = 0 \text{ on } \Gamma_{jk}.$$

Thus, we again have $p_j = 0$ and $\lambda_{j\ell} = 0$. Working inward from the boundary element-by-element shows that (p, λ) vanishes, so that $\rho(T_{0,0}) < 1$, as we set out to show.

This argument does not establish a bound for $\rho(T_{0,0}) < 1$ in terms of the discretization parameter h; we show in the next section that having $c_0 > 0$ allows such an estimate.

3.4. An estimate for the spectral radius of the iteration operator when $c_0 > 0$

We assume in this section that $c(x) \geq c_0 > 0$ and show that there exists a positive constant M such that, for any eigenvector (p, λ) of $T_{0,0}$,

$$R(p, \lambda) \leq 4M\beta \sum_j \left\{ (a\nabla p_j, \nabla p_j)_j + (cp_j, p_j)_j + \langle\langle dp_j, p_j \rangle\rangle_{\Gamma_j} \right\}, \tag{3.28}$$

from which, by (3.28) and (3.25), it follows that

$$|\gamma|^2 \leq 1 - \frac{1}{M} \cdot \tag{3.29}$$

That, in turn, will imply an estimate for the rate of convergence of the iterative procedure (3.10).

First, if (p, λ) is an eigenvector of $T_{0,0}$, then substituting (3.20) in (3.19) leads to

$$(a\nabla p_j, \nabla v)_j + (cp_j, v)_j + \sum_k \langle\langle \lambda_{jk}, v \rangle\rangle_{\Gamma_{jk}} + \langle\langle dp_j, v \rangle\rangle_{\Gamma_j} = 0, \quad v \in \mathcal{NC}_j^h, \ \forall j. \tag{3.30}$$

Then, let Ω_j be an arbitrary element and choose $v = \widetilde{v} \in \mathcal{NC}_j^h$ in (3.30) such that

$$\widetilde{v}(\xi_{jk}) = \lambda_{jk},$$

with the convention that $\lambda_{jk} = 0$ if the corresponding face is in Γ. Then, the bound (3.43) derived in Section 3.6 implies that

$$h_{\max}(\Omega_j)^{-1} \left(\|\widetilde{v}\|_{0,\Omega_j}^2 + h_{\min}(\Omega_j)^2 \|\nabla \widetilde{v}\|_{0,\Omega_j}^2 \right) \leq K \langle\langle \lambda_{jk}, \lambda_{jk} \rangle\rangle_{\partial\Omega_j}. \tag{3.31}$$

Then, by (3.30) and (3.31),

$$\begin{aligned} \langle\langle \lambda_{jk}, \lambda_{jk} \rangle\rangle_{\partial\Omega_j} &= -(a\nabla p_j, \nabla \widetilde{v})_j - (cp_j, \widetilde{v})_j \\ &\leq C \left(\|\nabla p_j\|_{0,\Omega_j} h_{\max}(\Omega_j)^{\frac{1}{2}} h_{\min}(\Omega_j)^{-1} + \|p_j\|_{0,\Omega_j} h_{\max}(\Omega_j)^{\frac{1}{2}} \right) \langle\langle \lambda_{jk}, \lambda_{jk} \rangle\rangle_{\partial\Omega_j}^{\frac{1}{2}}, \end{aligned}$$

and

$$\langle\langle \lambda_{jk}, \lambda_{jk} \rangle\rangle_{\partial\Omega_j} \leq C h_{\max}(\Omega_j) \left[\|p_j\|_{0,\Omega_j}^2 + h_{\min}(\Omega_j)^{-2} \|\nabla p_j\|_{0,\Omega_j}^2 \right]. \tag{3.32}$$

Also, by (3.41),

$$\langle\!\langle p_j, p_j \rangle\!\rangle_{\partial\Omega_j} \le \frac{C}{h_{\min}(\Omega_j)} \|p_j\|^2_{0,\partial\Omega_j}. \tag{3.33}$$

Set

$$\zeta = \max_j \frac{h_{\max}(\Omega_j)}{h_{\min}(\Omega_j)}, \quad h_{\max} = \max_j h_{\max}(\Omega_j), \quad h_{\min} = \min_j h_{\min}(\Omega_j). \tag{3.34}$$

Combining (3.32) and (3.33), we see that

$$\begin{aligned} R(p,\lambda) &= \sum_{jk} |\lambda_{jk} - \beta p_j(\xi_{jk})|^2_{0,\Gamma_{jk}} \le 2\sum_j \left\{ \langle\!\langle \lambda_{jk}, \lambda_{jk} \rangle\!\rangle_{\partial\Omega_j} + \beta^2 \langle\!\langle p_j, p_j \rangle\!\rangle_{\Omega_j} \right\} \\ &\le C \sum_j \left(\frac{h_{\max}(\Omega_j)}{h_{\min}(\Omega_j)^2} \|\nabla p_j\|^2_{0,\Omega_j} + \left(h_{\max}(\Omega_j) + \frac{\beta^2}{h_{\min}(\Omega_j)} \right) \|p_j\|^2_{0,\Omega_j} \right) \\ &\le C \sum_j \left(\frac{\zeta}{h_{\min}} \|\nabla p_j\|^2_{0,\Omega_j} + \left(\frac{\beta^2}{h_{\min}} + h_{\max} \right) \|p_j\|^2_{0,\Omega_j} \right) \\ &\le 4M(\beta)\beta \sum_j \left((a\nabla p_j, \nabla p_j)_j + (cp_j, p_j)_j + \langle\!\langle dp_j, p_j \rangle\!\rangle_{\Gamma_j} \right), \end{aligned} \tag{3.35}$$

where

$$M(\beta) = \frac{1}{4} C \max\left(\frac{\zeta}{a_0 h_{\min} \beta}, \frac{1}{c_0} \left(\frac{\beta}{h_{\min}} + \frac{h_{\max}}{\beta} \right) \right). \tag{3.36}$$

The function $M(\beta)$ is minimized by choosing the two terms in (3.36) to be equal; hence the optimal β satisfies the equation

$$\beta^2 = a_0^{-1} c_0 \zeta - c_0 h_{\min} h_{\max} \sim a_0^{-1} c_0 \zeta,$$

so that

$$\beta \sim \sqrt{a_0^{-1} c_0 \zeta}. \tag{3.37}$$

Then,

$$M \sim K \sqrt{(a_0 c_0)^{-1} \zeta} \frac{1}{h_{\min}}, \tag{3.38}$$

and

$$|\gamma|^2 \le 1 - K \sqrt{a_0 c_0 \zeta^{-1}} h_{\min}.$$

Thus, it follows that

$$\rho(T_{0,0}) \le 1 - K \sqrt{a_0 c_0 \zeta^{-1}} h_{\min}, \tag{3.39}$$

with a different K.

Theorem 3.2. *Let $a(x) \geq a_0 > 0$ and $c(x) \geq c_0 > 0$, and let ζ be the maximum aspect ratio for the partition Ω_j, $j = 1, \ldots, J$. Let β be chosen as in* (3.37)*. Then, the spectral radius of the operator $T_{0,0}$ satisfies the bound* (3.39)*.*

If the partition is quasiregular, then $\zeta = \mathcal{O}(1)$ as $h_{\max} \to 0$, $\beta = \mathcal{O}(1)$ and $\rho(T_{0,0}) \leq 1 - Kh$ as $h \to 0$. This is the best rate of convergence that can be expected in a domain decomposition iteration based on subdomains at the element level.

3.5. The three-dimensional problem

Let us consider the nonconforming finite element space based on the reference cubic element $\hat{R} = [-1, 1]^3$ given by either choice of $\mathcal{Q}_\ell$ as given in (2.20) or (2.21) in Section 2. The hybridization procedure and localizations can be carried out in exactly the same manner as for the two-dimensional problem, so that a domain decomposition iteration can be defined in a completely analogous fashion to that above. Moreover, the analysis of convergence of the iteration is unchanged, except for modifying the values of the constants in the technical lemmata.

3.6. Some calculus

Consider the element $E = (-\frac{1}{2}h_x, \frac{1}{2}h_x) \times (-\frac{1}{2}h_y, \frac{1}{2}h_y)$, and set

$$h_{\min}(E) = \min(h_x, h_y), \qquad h_{\max}(E) = \max(h_x, h_y),$$

and consider the basis $\mathcal{Q}_1$. It is easy to see that the basis element that is one at $(-\frac{1}{2}h_x, 0)$ and vanishes at the other three nodes is given by

$$v = \frac{1}{4} + \frac{x}{h_x} - \frac{3}{2}\left(\frac{x^2}{h_x^2} - \frac{20x^4}{3h_x^4} - \frac{y^2}{h_y^2} - \frac{20y^4}{3h_y^4}\right).$$

Thus,

$$\iint_E v^2 \, \mathrm{d}x \, \mathrm{d}y = \frac{781}{5040} h_x h_y,$$

and it follows that

$$\|z\|_{0,\infty,E} \leq \frac{K}{\sqrt{h_x h_y}} \|z\|_{0,E}, \quad z \in \mathcal{NC}^h(E), \tag{3.40}$$

where K will be a generic constant in this section. From (3.40), it is easy to see that

$$\langle\!\langle z, z \rangle\!\rangle_E \leq 4h_{\max}(E)\|z\|_{0,\infty,E}^2 \leq \frac{K}{h_{\min}(E)} \|z\|_{0,E}^2, \quad z \in \mathcal{NC}^h(E). \tag{3.41}$$

Another simple calculation shows that

$$\iint_E |\nabla v|^2 \, \mathrm{d}x \, \mathrm{d}y = \left(\frac{65h_y}{28h_x} + \frac{37h_x}{28h_y}\right) \leq K \frac{h_{\max}(E)}{h_{\min}(E)},$$

so that

$$\|\nabla z\|_{0,E} \leq \frac{K}{h_{\min}(E)} \|z\|_{0,E}, \quad z \in \mathcal{NC}^h(E). \tag{3.42}$$

Conversely, a scaling argument shows that

$$h_{\max}(E)^{-1}\left(\|z\|_{0,E}^2 + h_{\min}(E)^2\|\nabla z\|_{0,E}^2\right) \le K\langle\langle z, z\rangle\rangle_{\partial E}, \quad z \in \mathcal{NC}^h(E). \tag{3.43}$$

Completely analogous calculations can be made when $\mathcal{Q}_2$ is considered and for either basis suggested in the three-dimensional case.

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