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A NOTE ON POLYNOMIAL APPROXIMATION IN SOBOLEV SPACES

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Abstract. For domains which are star-shaped w.r.t. at least one point, we give new bounds on the constants in Jackson-inequalities in Sobolev spaces. For convex domains, these bounds do not depend on the eccentricity. For non-convex domains with a re-entrant corner, the bounds are uniform w.r.t. the exterior angle. The main tool is a new projection operator onto the space of polynomials.

Résumé. Pour des domaines étoilés on donne de nouvelles bornes sur les constantes dans les inégalités de Jackson pour les espaces de Sobolev. Pour des domaines convexes, les bornes ne dépendent pas de l'excentricité. Pour des domaines non-convexes ayant un point rentrant, les bornes sont uniformes par rapport à l'angle extérieur. L'outil central est un nouvel opérateur de projection sur l'espace des polynômes.

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1. Introduction and main results

In this note we derive new upper bounds on the constant $c_{m,j}$ in the Jackson-type inequality

$$\sup_{u\in H^{m+1}(\Omega)} \inf_{p\in \mathbb{P}_m} \frac{|u-p|_{H^j(\Omega)}}{|u|_{H^{m+1}(\Omega)}} \le c_{m,j} d^{m+1-j} \quad \forall\, 0\le j\le m.$$

Here, Ω is a bounded open domain in $\mathbb{R}^n$, $n\ge 2$, with diameter d. $H^k(\Omega)$, $k\in\mathbb{N}$, and $L^2(\Omega):=H^0(\Omega)$ denote the usual Sobolev- and Lebesgue-spaces equipped with the standard norms $\|.\|_{H^k(\Omega)}$ and semi-norms $|.|_{H^k(\Omega)}$. $\mathbb{P}_m$ is the space of all polynomials in n variables of degree at most m.

The interest in sharp explicit bounds on the constants $c_{m,j}$ originates from the a posteriori error analysis of finite element discretizations of partial differential equations. The correct calibration of many popular a posteriori error estimators depends, among others, on sharp explicit error estimates for suitable Clément-type interpolation operators. These estimates in turn employ the constants $c_{m,j}$ (*cf., e.g.*, [1, 4]).

Specifically, we consider two types of domains: (1) convex domains and (2) non-convex domains which are star-shaped w.r.t. at least one interior point. If Ω is convex, we obtain the bound

$$c_{m,j} \le \pi^{j-m-1} \binom{n+j-1}{j}^{1/2} \frac{((m+1-j)!)^{1/2}}{\left(\lceil \frac{m+1-j}{n} \rceil !\right)^{n/2}} \,. \tag{1.1}$$

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(As usual, $\lceil x \rceil$ denotes the largest integer less than or equal to x.) If Ω is not convex, but star-shaped w.r.t. the point $x \in \Omega$, we obtain the bound

$$c_{m,j} \le \max\left\{\frac{4}{\pi^2}K_1(\kappa) + K_2(\kappa), K_3(\kappa)\right\}^{(m+1-j)/2} \binom{n+j-1}{j}^{1/2} \frac{((m+1-j)!)^{1/2}}{\left(\lceil \frac{m+1-j}{n} \rceil !\right)^{n/2}}. \tag{1.2}$$

Here, the number κ is given by

$$\kappa := \max_{y\in\partial\Omega} |y-x|_2 \,/\, \min_{y\in\partial\Omega} |y-x|_2$$

($|.|_2$ denotes the Euclidean norm in $\mathbb{R}^n$) and the functions K_1, K_2, and K_3 are defined by

$$\begin{aligned} K_1(z) :=& 4z^{n-2} - 3z^{-2} \\ K_2(z) :=& \frac{4}{n(n+2)}[z^{n-2} - z^{-2}], \\ K_3(z) :=& \begin{cases} \ln z - \frac{1}{2} + \frac{1}{2}z^{-2}, & \text{if } n = 2, \\ \frac{2}{n(n-2)}z^{n-2} - \frac{1}{n-2} + \frac{1}{n}z^{-2}, & \text{if } n \ge 3. \end{cases} \end{aligned}$$

The main tool in establishing these bounds is a new projection operator of $H^m(\Omega)$ onto $\mathbb{P}_m$ which is defined in Section 2. In Sections 3 and 4 we will prove estimates (1.1) and (1.2), respectively. In Section 5 we will shortly comment on the generalization to L^q-norms.

The best estimates of $c_{m,j}$, which are known to us, are due to Durán [2]. When comparing his results with ours we observe two major differences.

First, if Ω is convex, the bound of [2] is proportional to $d^{n/2}|\Omega|^{-1/2}$, where $|\Omega|$ is the n-dimensional Lebesgue-measure of Ω. Thus, it is not uniformly bounded w.r.t. to the eccentricity $d^n|\Omega|^{-1}$, whereas our bound does not depend on this parameter.

Second, if Ω has a re-entrant corner, the bound of [2] tends to infinity when the exterior angle at the corner approaches zero. Our bounds, on the contrary, are uniform w.r.t. the exterior angle. To see this, we may assume that Ω is the intersection of the unit ball with the complement of a cone with base at the origin and angle $\alpha \in (0, \pi)$ having the positive x_1-axis as its axis of symmetry. An elementary geometrical argument than shows that Ω is star-shaped w.r.t. $(-1/2, 0, ..., 0)$ and that $\kappa = (1/2)\{5 + 4\cos(\alpha/2)\}^{1/2} \,/\, (1/2) \le 3$ for all $\alpha \in (0, \pi)$.

2. A projection operator

For any integer m and any measurable subset B of Ω with positive Lebesgue-measure $|B|$ we want to define a projection operator $P_{m,B}$ of $H^m(\Omega)$ onto $\mathbb{P}_m$ which has the following properties

$$D^\beta(P_{m,B}u) = P_{m-j,B}(D^\beta u), \tag{2.1}$$

$$\int_B D^\beta(u - P_{m,B}u) = 0 \tag{2.2}$$

for all $u \in H^m(\Omega)$, all $0 \le j \le m$, and all $\beta \in \mathbb{N}^n$ with $|\beta| := \beta_1 + \ldots + \beta_n = j$. To this end we denote by

$$\pi_B\varphi := \frac{1}{|B|}\int_B \varphi$$

the mean value of any integrable function φ on B. For any $u \in H^m(\Omega)$ we recursively define polynomials $p_{m,B}(u), ..., p_{0,B}(u)$ in $\mathbb{P}_m$ by

$$p_{m,B}(u) := \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=m}} \frac{1}{\alpha!} x^\alpha \pi_B(D^\alpha u) \tag{2.3}$$

and, for $k = m, m-1, ..., 1$,

$$p_{k-1,B}(u) = p_{k,B}(u) + \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=k-1}} \frac{1}{\alpha!} x^\alpha \pi_B[D^\alpha(u - p_{k,B}(u))]. \tag{2.4}$$

We set

$$P_{m,B}u := p_{0,B}(u).$$

Obviously, $P_{m,B}$ is a linear operator of $H^m(\Omega)$ onto $\mathbb{P}_m$. In order to see that it satisfies relations (2.1, 2.2) we first consider the case $j = 0$. Relation (2.1) then is a trivial identity and relation (2.2) immediately follows from equation (2.4) for $k = 1$. Next, assume that $m \geq 1$ and fix an arbitrary j with $1 \leq j \leq m$ and an arbitrary $\beta \in \mathbb{N}$ with $|\beta| = j$. Differentiating equation (2.3) we conclude that

$$\begin{aligned} D^\beta p_{m,B}(u) &= \sum_{\substack{\gamma\in\mathbb{N}^n \\ |\gamma|=m-j}} \frac{1}{\gamma!} x^\gamma \pi_B(D^\gamma D^\beta u) \\ &= p_{m-j,B}(D^\beta u). \end{aligned}$$

Differentiating equation (2.4) we recursively obtain for $k = m, m-1, ..., j+1$

$$\begin{aligned} D^\beta p_{k-1,B}(u) &= D^\beta p_{k,B}(u) + \sum_{\substack{\gamma\in\mathbb{N}^n \\ |\gamma|=k-1-j}} \frac{1}{\gamma!} x^\gamma \pi_B[D^\gamma D^\beta(u - p_{k,B}(u))] \\ &= p_{k-j,B}(D^\beta u) + \sum_{\substack{\gamma\in\mathbb{N}^n \\ |\gamma|=k-1-j}} \frac{1}{\gamma!} x^\gamma \pi_B[D^\gamma(D^\beta u - p_{k-j,B}(D^\beta u))] \\ &= p_{k-j-1,B}(D^\beta u). \end{aligned}$$

This proves that

$$\begin{aligned} D^\beta(P_{m,B}u) &= D^\beta(p_{j,B}(u)) \\ &= p_{0,B}(D^\beta u) = P_{m-j,B}(D^\beta u) \end{aligned}$$

and thus establishes relation (2.1). Relation (2.2) immediately follows from relation (2.1) and equation (2.4) for $k = 1$ with u replaced by $D^\beta u$.

3. Convex domains

Assume that Ω is convex. Payne and Weinberger [3] proved that

$$\|\varphi\|_{L^2(\Omega)} \leq \frac{d}{\pi} |\varphi|_{H^1(\Omega)} \tag{3.1}$$

holds for all $\varphi \in H^1(\Omega)$ having zero mean value.

Now, consider an arbitrary $u \in H^{m+1}(\Omega)$. Applying inequality (3.1) to $\varphi := u - P_{m,\Omega}u$ and using relation (2.2) with $\beta = 0$ yields that

$$\begin{aligned}\|u - P_{m,\Omega}u\|^2_{L^2(\Omega)} &\le \left(\frac{d}{\pi}\right)^2 |u - P_{m,\Omega}u|^2_{H^1(\Omega)} \\ &= \left(\frac{d}{\pi}\right)^2 \sum_{i=1}^{n} \|D_i(u - P_{m,\Omega}u)\|^2_{L^2(\Omega)}.\end{aligned}$$

Applying this estimate recursively to the derivatives of $u - P_{m,\Omega}u$ and using relations (2.1, 2.2) we arrive at

$$\begin{aligned}\|u - P_{m,\Omega}u\|^2_{L^2(\Omega)} &\le \left(\frac{d}{\pi}\right)^{2(m+1)} \sum_{1\le i_1,\dots,i_{m+1}\le n} \|D_{i_1}\dots D_{i_{m+1}}u\|^2_{L^2(\Omega)} \\ &= \left(\frac{d}{\pi}\right)^{2(m+1)} \sum_{\substack{\beta\in\mathbb{N}^n \\ |\beta|=m+1}} \frac{(m+1)!}{\beta!} \|D^\beta u\|^2_{L^2(\Omega)} \\ &\le \left(\frac{d}{\pi}\right)^{2(m+1)} \max\left\{\frac{(m+1)!}{\beta!} : \beta \in \mathbb{N}^n, |\beta| = m+1\right\} |u|^2_{H^{m+1}(\Omega)}.\end{aligned}$$

A simple symmetry argument shows that

$$\max\left\{\frac{(m+1)!}{\beta!} : \beta \in \mathbb{N}^n, |\beta| = m+1\right\} \le \frac{(m+1)!}{\left(\lceil \frac{m+1}{n} \rceil !\right)^n}$$

and thus completes the proof of estimate (1.1) for the case $j = 0$.

The case $1 \le j \le m$ follows from the case $j = 0$, relation (2.1), and the observation that

$$\sharp\left\{\alpha \in \mathbb{N}^n : |\alpha| = j\right\} = \binom{n+j-1}{j}.$$

4. Non-convex, star-shaped domains

Assume that Ω is star-shaped w.r.t. the point $x \in \Omega$ and not necessarily convex. Since the Lebesgue-integral is translation invariant we may from now on assume that x is the origin. Set

$$\rho := \min_{y\in\partial\Omega} |y|_2\,,\ R := \max_{y\in\partial\Omega} |y|_2\,,\ \kappa := \frac{R}{\rho}$$

and denote by B the ball with radius ρ and centre at the origin. Lemma 4.1 in [4] and its proof imply that

$$\|\varphi\|^2_{L^2(\Omega)} \le K_1(\kappa)\kappa^2 \|\varphi\|^2_{L^2(B)} + K_2(\kappa)R^2|\varphi|^2_{H^1(B)} + K_3(\kappa)R^2|\varphi|^2_{H^1(\Omega\setminus B)} \tag{4.1}$$

holds for all $\varphi \in H^1(\Omega)$.

Inserting $\varphi := u - P_{m,B}u$ in estimate (4.1) and invoking relation (2.2) for $\beta = 0$ and estimate (3.1) with Ω replaced by B, we conclude that

$$\begin{aligned}\|u - P_{m,B}u\|^2_{L^2(\Omega)} &\le K_1(\kappa)\kappa^2\rho^2\frac{4}{\pi^2}|u - P_{m,B}u|^2_{H^1(B)}\\ &\quad + K_2(\kappa)R^2|u - P_{m,B}u|^2_{H^1(B)}\\ &\quad + K_3(\kappa)R^2|u - P_{m,B}u|^2_{H^1(\Omega\setminus B)}\\ &\le \max\Big\{\frac{4}{\pi^2}K_1(\kappa) + K_2(\kappa), K_3(\kappa)\Big\}R^2\sum_{i=1}^{n}\|D_i(u - P_{m,B}u)\|^2_{L^2(\Omega)}.\end{aligned}$$

Applying this estimate recursively to the derivatives of $u - P_{m,B}u$ and recalling that $R \le d$ we arrive at

$$\begin{aligned}\|u - P_{m,B}u\|^2_{L^2(\Omega)} &\le \max\Big\{\frac{4}{\pi^2}K_1(\kappa) + K_2(\kappa), K_3(\kappa)\Big\}^{m+1} d^{2(m+1)} \sum_{1\le i_1,\dots,i_{m+1}\le n}\|D_{i_1\dots i_{m+1}}u\|^2_{L^2(\Omega)}\\ &\le \max\Big\{\frac{4}{\pi^2}K_1(\kappa) + K_2(\kappa), K_3(\kappa)\Big\}^{m+1} d^{2(m+1)}\frac{(m+1)!}{\big(\lceil\frac{m+1}{n}\rceil!\big)^n}|u|^2_{H^{m+1}(\Omega)}.\end{aligned}$$

This proves estimate (1.2) in the case $j = 0$. The case $1 \le j \le m$ again follows from the case $j = 0$ and relation (2.2).

5. Jackson-inequalities in $W^{1,q}$

With obvious modifications of the functions K_1, K_2, and K_3, the arguments of Sections 3 and 4 immediately carry over to general L^q-spaces with $1 < q < \infty$. The crucial ingredient is an explicit knowledge of a Poincaré-constant c_q such that

$$\|\varphi\|_{L^q(\omega)} \le c_q \operatorname{diam}(\omega)|\varphi|_{W^{1,q}(\omega)}$$

holds for all open convex subsets ω of $\mathbb{R}^n$ and all functions $\varphi \in W^{1,q}(\omega)$ having zero mean value on ω. (Note that the diameter is always computed using the Euclidean norm and that the natural semi-norm is $|\varphi|_{W^{1,q}} = \{\sum\|D_i\varphi\|^q_{L^q}\}^{1/q}$ with the obvious modification for $q = \infty$.)

From inequality (3.1) we know that $c_2 \le 1/\pi$. (This estimate is optimal, *cf.* [3].) Integration along lines, on the other hand, yields $c_\infty \le n^{1/2}$. (This estimate may not be optimal. The example $\Omega = (-1,1)^n$, $\varphi(x) = \sum x_i$, however, shows that it differs from the optimal estimate by a factor 2 at most.) Interpolating between L^2 and L^∞ we conclude that

$$c_q \le \pi^{-2/q}n^{1/2-1/q} \qquad \forall\, 2 \le q \le \infty.$$

For the range $1 < q < 2$ a similar estimate is not known to us.

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