Mathematical Modelling and Numerical Analysis M2AN, Vol. 33, N° 5, 1999, p. 923–938
Modélisation Mathématique et Analyse Numérique

OPTIMAL ERROR ESTIMATES FOR THE STOKES AND NAVIER–STOKES EQUATIONS WITH SLIP–BOUNDARY CONDITION

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Abstract. We consider a finite element discretization by the Taylor–Hood element for the stationary Stokes and Navier–Stokes equations with slip boundary condition. The slip boundary condition is enforced pointwise for nodal values of the velocity in boundary nodes. We prove optimal error estimates in the H^1 and L^2 norms for the velocity and pressure respectively.

AMS Subject Classification. 35Q30, 65N15, 65N30, 76D05, 76D07.

Received: February 9, 1998. Revised: November 27, 1998.

1. INTRODUCTION

We consider the stationary Navier–Stokes equations: given a bounded domain $\Omega \subseteq \mathbb{R}^d$, $d = 2$ or $d = 3$, find a velocity field u, a pressure p such that

$$\begin{aligned} -\frac{1}{\mathrm{Re}}\Delta u \;+\; (u\cdot\nabla)u \;+\; \nabla p &= f \qquad \text{in } \Omega \\ \operatorname{div} u &= 0 \qquad \text{in } \Omega \end{aligned} \tag{1.1}$$

as well as its linear counterpart, the Stokes equations:

$$\begin{aligned} -\frac{1}{\mathrm{Re}}\Delta u \;+\; \nabla p &= f \qquad \text{in } \Omega \\ \operatorname{div} u &= 0 \qquad \text{in } \Omega \end{aligned} \tag{1.2}$$

together with the slip boundary condition

$$u \cdot n = 0 \qquad \text{on } \partial\Omega. \tag{1.3}$$

To this boundary condition we have to add a condition on the tangential stresses, for instance

$$n \cdot \sigma(u,p)\tau_i = 0 \quad \text{ on } \partial\Omega, \quad i = 2,\ldots,d. \tag{1.4}$$

Keywords and phrases. Stokes equations, Navier–Stokes equations, finite elements, slip boundary condition, error estimates.

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Here, Re denotes the Reynolds number, n, τ_i the normal and tangential vectors on $\partial\Omega$ and

$$\sigma_{i,j} = \sigma(u,p)_{i,j} = \frac{1}{\mathrm{Re}} D(u)_{i,j} \; - \; p \; \delta_{i,j}$$

is the stress tensor with

$$D(u)_{i,j} = \Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)$$

the deformation tensor.

Boundary conditions (1.3, 1.4) play an important role in many physical situations, in particular for free boundary problems. We mention:

- coating flows, see for instance [11, 16],
- flow in semiconductor melts, see *e.g.* [10, 17].

In contrast to the Stokes and Navier–Stokes equations with Dirichlet boundary condition there seems to be rather few work concerned with numerical analysis of this problem with slip boundary condition. In [20] Verfürth proved error estimates for the discretization of (1.1–1.4) by the popular *Taylor–Hood* element and a polyhedral approximation Ω_h of the domain Ω. Verfürth gave a non optimal error bound of $\mathcal{O}(h^{1/2})$ in the H^1 and L^2 norm for the velocity and pressure respectively. In [21, 22] the same author proposed and analyzed a discretization where the slip boundary condition is enforced in a weak sense by Lagrange multipliers. Numerically, however, in most cases it is more convenient to use a discretization, where the slip boundary condition is incorporated in the ansatz space, see also [2].

The present article therefore analyzes a discretization of this type for the *Taylor–Hood* element. We improve the result from [20] to the optimal error bound $\mathcal{O}(h^{3/2})$ for the velocity and the pressure in H^1 and L^2 norms respectively for a polyhedral approximation of Ω.

The paper is organized as follows: In Section 2 the finite element formulation and some notations are given. In particular, we introduce a homeomorphism G_h which maps the discrete domain $\tilde{\Omega}_h$ onto Ω. We prove optimal error estimates for the Stokes problem in Section 3. The key idea is to transform the discrete solution *via* G_h onto the original domain Ω and to carry out the error analysis on Ω. We thereby avoid error terms which involve integration over the discrete boundary $\partial\tilde{\Omega}_h$. In Section 4 optimal error bounds are also obtained for the nonlinear Navier–Stokes equations in case of small Reynolds numbers. Section 5 concludes the article by presenting numerical results.

Remark 1.1. After having finished this paper we got to know about a related recent paper by Knobloch [12]. He considers the Stokes equations allowing both slip and no–slip boundary conditions. For a tetrahedral approximation of Ω and finite element spaces (including the Taylor–Hood element) satisfying suitable assumptions he obtains optimal orders of convergence. Our technique however is different from his in that we use the transformation G_h in order to carry out the analysis on Ω which simplifies the calculations considerably. A further difference is that we also treat the nonlinear problem and support our analysis by a numerical example.

2. Finite element formulation

In the following we assume for simplicity that Ω has no axis of symmetry. This will be needed to insure ellipticity of our problem, see (2.3) below.

In order to derive a variational formulation of (1.2–1.4) the momentum equation in (1.2) is multiplied by a function $v \in H^1(\Omega;\mathbb{R}^3)$ and integrated by parts:

$$\begin{aligned} \int_\Omega \Big(-\frac{1}{\mathrm{Re}}\Delta u + \nabla p\Big)\cdot v &= \frac{1}{2\mathrm{Re}}\int_\Omega D(u):D(v) \; - \; \int_\Omega p \operatorname{div} v - \int_{\partial\Omega} n\cdot\sigma(u,p)v \\ &= \frac{1}{2\mathrm{Re}}\int_\Omega D(u):D(v) \; - \; \int_\Omega p \operatorname{div} v \; - \; \int_{\partial\Omega} (n\cdot\sigma(u,p)n)v\cdot n, \end{aligned} \tag{2.1}$$

where the last equality follows from (1.4). Now it is natural to introduce the following bilinear forms:

$$\begin{aligned} a(u,v) &:= \frac{1}{2\mathrm{Re}}\int_\Omega D(u):D(v) && \text{for } u,v\in H^1(\Omega;\mathbb{R}^3)\\ b(v,q) &:= -\int_\Omega q\operatorname{div} v && \text{for } v\in H^1(\Omega;\mathbb{R}^3),\quad q\in L^2(\Omega)\end{aligned}$$

as well as the function spaces

$$\begin{aligned} X &:= \{v\in H^1(\Omega;\mathbb{R}^3)\mid v\cdot n = 0 \quad \text{on } \partial\Omega\}\\ M &:= \{q\in L^2(\Omega)\mid \int_\Omega q = 0\}\cdot\end{aligned}$$

The weak formulation of (1.2–1.4) then reads: find $(u,p)\in X\times M$ such that

$$\begin{aligned} a(u,v) + b(v,p) &= (f,v) && \text{for all } v\in X\\ b(u,q) &= 0 && \text{for all } q\in M\end{aligned} \tag{2.2}$$

where $(f,v)=\int_\Omega fv\mathrm{d}x$ denotes the L^2 inner product. Note that a weak solution which is smooth is also a solution of (1.2–1.4). Both existence and uniqueness of a weak solution follow from Korn's inequality, see (3.4) below

$$\|v\|^2_{H^1(\Omega)} \le C\int_\Omega D(v):D(v) \qquad \text{for all } v\in X, \tag{2.3}$$

the Babuška–Brezzi condition

$$\inf_{q\in M\setminus\{0\}}\ \sup_{v\in X\setminus\{0\}} \frac{b(v,q)}{\|v\|_{H^1}\|q\|_{L^2}} \ \ge\ \beta>0 \tag{2.4}$$

and the general theory of saddle point problems, *cf.* [5, 9].

Regularity properties of (u,p) were studied for instance in [18], in particular we have

$$\|u\|^2_{H^3(\Omega)} + \|p\|^2_{H^2(\Omega)} \le C\|f\|^2_{H^1(\Omega)} \qquad \text{if } \Omega \text{ is of class } C^4. \tag{2.5}$$

Next, let us denote by $\tilde{\mathcal{T}}_h$ a finite set of straight, closed d–simplices which triangulates a domain

$$\overline{\tilde\Omega}_h = \bigcup_{\tilde T\in\tilde{\mathcal{T}}_h}\tilde T$$

in such a way that all vertices on $\partial\tilde\Omega_h$ also lie on $\partial\Omega$. Denote by $h(\tilde T)=\mathrm{diam}(\tilde T)$ the diameter of $\tilde T$ and by $\rho(\tilde T)$ the radius of the largest ball inscribed $\tilde T$. We make the usual assumption of *shape regularity*, *i.e.* for a family of triangulations $(\tilde{\mathcal{T}}_h)_h$ we assume that

$$\sup_h \max_{\tilde T\in\tilde{\mathcal{T}}_h}\frac{h(\tilde T)}{\rho(\tilde T)}\le\kappa<\infty.$$

For every $\tilde{T} \in \tilde{\mathcal{T}}_h$ there exists an invertible affine mapping

$$\tilde{F}_{\tilde{T}} : \mathbb{R}^d \to \mathbb{R}^d, \qquad \tilde{F}_{\tilde{T}}(\hat{x}) = A_{\tilde{T}}\hat{x} + b_{\tilde{T}},$$

which maps the standard d–simplex $\hat{T}$ onto $\tilde{T}$. Besides the triangulation $\tilde{\mathcal{T}}_h$ which will be used to define the discrete problem and to carry out the practical computations we also introduce an *exact triangulation* $\mathcal{T}_h$ of Ω. The existence of such a triangulation together with the associated interpolation estimates is proved in [4, 13]. In essence, for every $\tilde{T} \in \tilde{\mathcal{T}}_h$ there is a mapping $\Phi_{\tilde{T}} \in C^3(\hat{T}; \mathbb{R}^d)$ such that $F_{\tilde{T}} := \tilde{F}_{\tilde{T}} + \Phi_{\tilde{T}}$ maps $\hat{T}$ onto a curved d–simplex $T \subseteq \overline{\Omega}$ and

$$\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} T.$$

Furthermore, the mapping G_h which is locally defined by

$$G_{h\,|\,\tilde{T}} := F_{\tilde{T}} \circ \tilde{F}_{\tilde{T}}^{-1}$$

(see Fig. 1) is a homeomorphism between $\tilde{\Omega}_h$ and Ω. The construction in [4, 13] also implies that $\Phi_{\tilde{T}} = 0$ if $\tilde{T}$ has at most one vertex on $\partial\tilde{\Omega}_h$, so that $G_h \equiv I$ on all simplices which are disjoint from $\partial\tilde{\Omega}_h$. Finally, we have the estimates

$$\begin{aligned}
&\sup_{x \in \tilde{T}} \|(DG_{h\,|\,\tilde{T}} - I)(x)\| \le Ch(\tilde{T}), \quad \|G_h\|_{H^{3,\infty}(\tilde{T})} \le C \\
&\sup_{\hat{x} \in \hat{T}} \|DF_{\tilde{T}}(\hat{x})\| \le C\|A_{\tilde{T}}\|, \quad \sup_{x \in T} \|DF_{\tilde{T}}^{-1}(x)\| \le C\|A_{\tilde{T}}^{-1}\| \\
&c_1 |\det A_{\tilde{T}}| \le |\det DF_{\tilde{T}}(\hat{x})| \le c_2 |\det A_{\tilde{T}}|, \quad \hat{x} \in \hat{T}
\end{aligned} \tag{2.6}$$

for all $\tilde{T} \in \tilde{\mathcal{T}}_h$. In particular, $v \in H^1(\Omega)$ if and only if $v \circ G_h \in H^1(\tilde{\Omega}_h)$ and

$$c_3\|v\|_{H^1(\Omega)} \le \|v \circ G_h\|_{H^1(\tilde{\Omega}_h)} \le c_4\|v\|_{H^1(\Omega)}.$$

Let us turn to the definition of the finite element spaces which we shall use. We denote by $\mathcal{N}_h$ the union of the set of all vertices of $\tilde{\mathcal{T}}_h$ with the set of all midpoints of edges of d–simplices in $\tilde{\mathcal{T}}_h$. Then we define

$$X_h := \{v_h \in C^0(\overline{\tilde{\Omega}}_h; \mathbb{R}^d) \mid v_{h\,|\,\tilde{T}} \in \mathcal{P}_2(\tilde{T})^d,\ v_h(p) \cdot n(G_h(p)) = 0\ \forall p \in \mathcal{N}_h \cap \partial\tilde{\Omega}_h\}$$

and

$$M_h := \{q_h \in C^0(\overline{\tilde{\Omega}}_h) \mid q_{h\,|\,\tilde{T}} \in \mathcal{P}_1(\tilde{T}),\ \int_{\tilde{\Omega}_h} q_h = 0\}$$

that is we use the so called *Taylor–Hood element* and enforce the slip boundary condition pointwise in all boundary vertices and midpoints. Note that the normal n appearing in the definition of X_h is the normal to the domain Ω. Defining $a_h : X_h \times X_h \to \mathbb{R}$, $b_h : X_h \times M_h \to \mathbb{R}$ by

$$\begin{aligned}
a_h(u_h, v_h) &:= \frac{1}{2\mathrm{Re}} \int_{\tilde{\Omega}_h} D(u_h) : D(v_h) \\
b_h(v_h, q_h) &:= -\int_{\tilde{\Omega}_h} q_h \operatorname{div} v_h
\end{aligned}$$

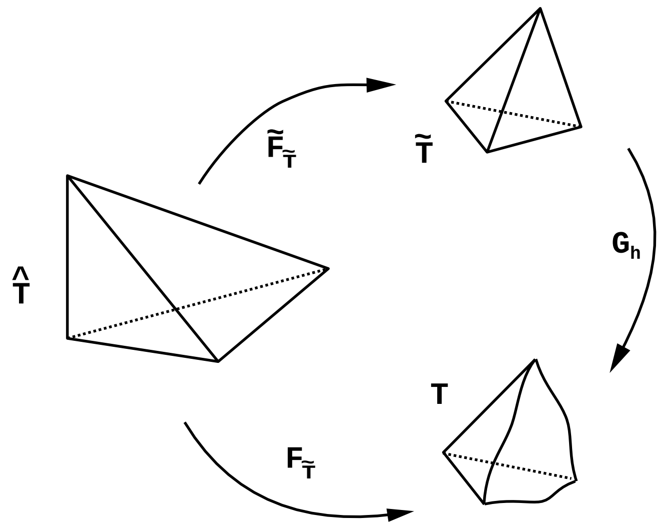

FIGURE 1. $\hat{T}, \tilde{T}$ and T at the boundary.

the discrete analogy of (2.2) reads: find $(u_h, p_h) \in X_h \times M_h$ such that

$$\begin{aligned} a_h(u_h, v_h) \; + \; b_h(v_h, p_h) \quad &= \quad \int_{\tilde{\Omega}_h} \tilde{f} \cdot v_h \qquad && \text{for all } v_h \in X_h \\ b_h(u_h, q_h) \quad &= \quad 0 && \text{for all } q_h \in M_h \end{aligned} \tag{2.7}$$

where $\tilde{f} := f \circ G_h$. It follows from [20] that (2.7) has a unique solution for all $0 < h \leq h_0$ with h_0 small enough.

Remark 2.1. Since the Babuška–Brezzi condition can be proved for the Taylor–Hood element without using an inverse estimate, see [15], we do not assume quasi uniformity for the triangulation $\tilde{\mathcal{T}}_h$. Thus our analysis is valid also in the case of adaptively refined meshes.

Remark 2.2. We have introduced $\tilde{f}$ above because we do not want to overburden the error analysis with the approximation of $\int_\Omega f \cdot v_h$. This can be done with the help of a suitable quadrature rule under appropriate regularity assumptions on f (*cf.* [7]: Chap. 4, Sect. 4.1).

3. Proof of the error estimate

One problem of the error analysis lies in the fact that (u, p) and (u_h, p_h) are defined on different spaces. To overcome this difficulty we assign to each $(v_h, q_h) \in X_h \times M_h$ the pair

$$(\bar{v}_h, \bar{q}_h) := \left(v_h \circ G_h^{-1}, q_h \circ G_h^{-1} - \frac{1}{|\Omega|} \int_\Omega q_h \circ G_h^{-1} \right) \in H^1(\Omega; \mathbb{R}^d) \times M.$$

Note that in general $\bar{v}_h \notin X$ since $v_h \cdot (n \circ G_h)$ only vanishes in the points of $\mathcal{N}_h$. Inserting $\bar{v}_h$ into (2.1) we get

$$a(u, \bar{v}_h) + b(\bar{v}_h, p) = \int_\Omega f \cdot \bar{v}_h + \int_{\partial\Omega} (n\sigma(u,p)n) \, \bar{v}_h \cdot n, \qquad v_h \in X_h.$$

Combining this identity with (2.7) we obtain the following error relation for $(u-\bar{u}_h, p-\bar{p}_h)$:

$$\begin{aligned} a(u-\bar{u}_h,\bar{v}_h)+b(\bar{v}_h,p-\bar{p}_h) &= a_h(u_h,v_h)-a(\bar{u}_h,\bar{v}_h)+b_h(v_h,p_h)-b(\bar{v}_h,\bar{p}_h) \\ &\quad +\int_\Omega f\cdot\bar{v}_h-\int_{\tilde{\Omega}_h}\tilde{f}\cdot v_h+\int_{\partial\Omega}(n\sigma(u,p)n)\,\bar{v}_h\cdot n \\ b(u-\bar{u}_h,\bar{q}_h) &= b_h(u_h,q_h)-b(\bar{u}_h,\bar{q}_h) \end{aligned} \tag{3.1}$$

for all $v_h\in X_h$, $q_h\in M_h$.

Before we can apply well–known results on the approximation of saddle point problems we need suitable discrete analogies of the ellipticity condition and the Babuška–Brezzi condition.

The following lemma will be very helpful in the subsequent analysis.

Lemma 3.1. *There exists $h_1>0$ such that for all $0<h\le h_1$ and all $v_h\in X_h$*

$$\|\bar{v}_h\cdot n\|_{L^2(\partial\Omega)}\le ch^{\frac{3}{2}}\|\bar{v}_h\|_{H^1(\Omega)}.$$

Proof. Let us denote by $\mathcal{T}_h^\partial$ the set of all simplices in $\mathcal{T}_h$ which have a $(d-1)$–face on $\partial\Omega$. The transformation rule and (2.6) imply

$$\int_{\partial\Omega}|\bar{v}_h\cdot n|^2=\sum_{T\in\mathcal{T}_h^\partial}\int_{\partial\Omega\cap T}|(v_h\circ G_h^{-1})\cdot n|^2\le c\sum_{T\in\mathcal{T}_h^\partial}\int_{\partial\tilde{\Omega}_h\cap\tilde{T}}|v_h\cdot(n\circ G_h)|^2.$$

Let us fix $T\in\mathcal{T}_h^\partial$ and put $\tilde{\Gamma}:=\partial\tilde{\Omega}_h\cap\tilde{T}$. In view of the definition of X_h we have $I_h^2\big(v_{h|\tilde{\Gamma}}\cdot(n\circ G_h)_{|\tilde{\Gamma}}\big)\equiv 0$ where I_h^2 denotes the Lagrange interpolation operator for polynomials of degree two. If we apply a well–known interpolation result (*cf.* [7]) we get

$$\begin{aligned} \int_{\tilde{\Gamma}}|v_h\cdot(n\circ G_h)|^2 &= \int_{\tilde{\Gamma}}|v_h\cdot(n\circ G_h)-I_h^2\big(v_h\cdot(n\circ G_h)\big)|^2 \\ &\le ch(\tilde{T})^6\int_{\tilde{\Gamma}}|D^3\big((v_h\cdot(n\circ G_h))_{|\tilde{\Gamma}}\big)|^2. \end{aligned}$$

Observing that $v_{h|\tilde{\Gamma}}\in P_2(\tilde{\Gamma})$ as well as $\partial\Omega\in C^4$ and using (2.6) we may estimate

$$\begin{aligned} \int_{\tilde{\Gamma}}|D^3\big(v_h\cdot(n\circ G_h)\big)|^2 &\le c\|v_h\|^2_{H^2(\tilde{\Gamma})}=c(\|v_h\|^2_{L^2(\tilde{\Gamma})}+\|Dv_h\|^2_{H^1(\tilde{\Gamma})}) \\ &\le ch(\tilde{T})^{-1}\|v_h\|^2_{L^2(\tilde{T})}+ch(\tilde{T})^{-2}\|v_h\|^2_{H^1(\tilde{\Gamma})} \\ &\le ch(\tilde{T})^{-1}\|v_h\|^2_{L^2(\tilde{T})}+ch(\tilde{T})^{-3}\|v_h\|^2_{H^1(\tilde{T})} \\ &\le ch(\tilde{T})^{-3}\|v_h\|^2_{H^1(\tilde{T})} \end{aligned}$$

in view of the inverse estimates

$$\|v_h\|_{L^2(\tilde{\Gamma})}\le ch(\tilde{T})^{-\frac{1}{2}}\|v_h\|_{L^2(\tilde{T})},\qquad \|Dv_h\|_{L^2(\tilde{\Gamma})}\le ch(\tilde{T})^{-1}\|v_h\|_{L^2(\tilde{\Gamma})}.$$

In conclusion

$$\int_{\partial\Omega} |\bar{v}_h \cdot n|^2 \leq c \sum_{T\in\mathcal{T}_h^\partial} h(\tilde{T})^3 \|v_h\|^2_{H^1(\tilde{T})} \leq ch^3 \sum_{T\in\mathcal{T}_h^\partial} \|\bar{v}_h\|^2_{H^1(T)} \leq ch^3 \|\bar{v}_h\|^2_{H^1(\Omega)}$$

and the lemma is proved. □

Note that we made use of the inverse estimate only locally, hence we do not need a quasiuniformity assumption on the triangulation.

The above result should be compared to Lemma 3.1 in [20] where the author obtained the boundary estimate

$$\|v_h \cdot n_h\|_{L^2(\partial\tilde{\Omega}_h)} \leq ch^{\frac{1}{2}} \|v_h\|_{H^1(\tilde{\Omega}_h)}, \quad v_h \in X_h$$

which then led to the suboptimal error bound $\mathcal{O}(h^{\frac{1}{2}})$. By employing the transformation G_h we are able to improve this estimate and this will turn out to be crucial in the proof of the optimal convergence rate.

The next lemma estimates the effect of the transformation on the bilinear forms.

Lemma 3.2. *There exists $h_2 > 0$ such that for all $0 < h \leq h_2$ and for all $v_h, w_h \in X_h,\ q_h \in M_h$*

$$(a) \quad |a(\bar{v}_h, \bar{w}_h) - a_h(v_h, w_h)| \leq ch\Big(\sum_{T\cap\partial\Omega\neq\emptyset} \|\bar{v}_h\|^2_{H^1(T)}\Big)^{\frac{1}{2}} \Big(\sum_{T\cap\partial\Omega\neq\emptyset} \|\bar{w}_h\|^2_{H^1(T)}\Big)^{\frac{1}{2}}$$

$$(b) \quad |b(\bar{v}_h, \bar{q}_h) - b_h(v_h, q_h)| \leq ch^{\frac{3}{2}} \|\bar{v}_h\|_{H^1(\Omega)} \|\bar{q}_h\|_{L^2(\Omega)} + ch\Big(\sum_{T\cap\partial\Omega\neq\emptyset} \|\bar{v}_h\|^2_{H^1(T)}\Big)^{\frac{1}{2}} \Big(\sum_{T\cap\partial\Omega\neq\emptyset} \|\bar{q}_h\|^2_{L^2(T)}\Big)^{\frac{1}{2}}.$$

Proof. Note first that $G_h \equiv I$ on all simplices which are disjoint from $\partial\tilde{\Omega}_h$. Then the first assertion can be proved in the same way as Lemma 8(ii) in [13].

In order to show (b) we first estimate $\int_\Omega q_h \circ G_h^{-1}$. Observing that $\int_{\tilde{\Omega}_h} q_h = 0$ we get

$$\begin{aligned}
\int_\Omega q_h \circ G_h^{-1} &= \sum_{T\cap\partial\Omega\neq\emptyset} \int_T q_h \circ G_h^{-1} + \sum_{T\cap\partial\Omega=\emptyset} \int_T q_h \circ G_h^{-1} \\
&= \sum_{T\cap\partial\Omega\neq\emptyset} \int_T q_h \circ G_h^{-1} + \sum_{T\cap\partial\Omega=\emptyset} \int_{\tilde{T}} q_h \\
&= \sum_{T\cap\partial\Omega\neq\emptyset} \Big(\int_{\tilde{T}} q_h\ |\mathrm{det} G_h| - \int_{\tilde{T}} q_h\Big)
\end{aligned}$$

in view of the transformation rule. From (2.6) and Hölder's inequality we obtain

$$\begin{aligned}
\Big|\int_\Omega q_h \circ G_h^{-1}\Big| &\leq c \sum_{T\cap\partial\Omega\neq\emptyset} h(\tilde{T}) \int_{\tilde{T}} |q_h| \leq c \sum_{T\cap\partial\Omega\neq\emptyset} h(\tilde{T})^{\frac{d}{2}+1} \|q_h\|_{L^2(\tilde{T})} \\
&\leq c\Big(\sum_{T\cap\partial\Omega\neq\emptyset} h(\tilde{T})^{d+2}\Big)^{\frac{1}{2}} \|q_h\|_{L^2(\tilde{\Omega}_h)} \leq ch^{\frac{3}{2}} \|q_h \circ G_h^{-1}\|_{L^2(\Omega)} \\
&\leq ch^{\frac{3}{2}} \|\bar{q}_h\|_{L^2(\Omega)} + ch^{\frac{3}{2}} \Big|\int_\Omega q_h \circ G_h^{-1}\Big|
\end{aligned}$$

since $q_h \circ G_h^{-1} = \bar{q}_h + \frac{1}{|\Omega|} \int_\Omega q_h \circ G_h^{-1}$ and

$$\sum_{T \cap \partial\Omega \neq \emptyset} h(\tilde{T})^{d-1} \leq c\mathcal{H}^{d-1}(\partial\tilde{\Omega}_h) \leq c\mathcal{H}^{d-1}(\partial\Omega) \tag{3.2}$$

with $\mathcal{H}^{d-1}$ the $(d-1)$-dimensional Hausdorff measure. Summarizing the above considerations we get for sufficiently small h_2

$$|\int_\Omega q_h \circ G_h^{-1}| \leq ch^{\frac{3}{2}} \|\bar{q}_h\|_{L^2(\Omega)} \tag{3.3}$$

and therefore

$$|b(\bar{v}_h, \bar{q}_h) - b_h(v_h, q_h)| \leq |b(\bar{v}_h, q_h \circ G_h^{-1}) - b_h(v_h, q_h)| + c\|\bar{v}_h\|_{H^1(\Omega)} \, |\int_\Omega q_h \circ G_h^{-1}|.$$

Estimating the first term again similarly as in Lemma 8(i) in [13] the result follows from (3.3). □

Next we obtain discrete versions of the conditions (2.3, 2.4).

Lemma 3.3. *There exists $h_3 > 0$ such that for all $0 < h \leq h_3$*

$$(a) \qquad a(\bar{v}_h, \bar{v}_h) \geq c_0 \|\bar{v}_h\|^2_{H^1(\Omega)}, \qquad v_h \in X_h$$

$$(b) \qquad \inf_{q_h \in M_h \setminus \{0\}} \sup_{v_h \in X_h \setminus \{0\}} \frac{b(\bar{v}_h, \bar{q}_h)}{\|\bar{v}_h\|_{H^1(\Omega)} \|\bar{q}_h\|_{L^2(\Omega)}} \geq \beta' > 0.$$

Proof. (a) Korn's second inequality (*cf.* [14]) together with the Poincaré–Morrey inequality (*cf.* [21]) implies

$$\|v\|^2_{H^1(\Omega)} \leq c\Big(\int_\Omega D(v) : D(v) + \|v\|^2_{L^2(\Omega)}\Big) \leq c\Big(\int_\Omega D(v) : D(v) + \int_{\partial\Omega} |v \cdot n|^2\Big) \tag{3.4}$$

for all $v \in H^1(\Omega; \mathbb{R}^d)$. Applying this estimate to $\bar{v}_h$ and using Lemma 3.1 we arrive at

$$\|\bar{v}_h\|^2_{H^1(\Omega)} \leq c\, a(\bar{v}_h, \bar{v}_h) + ch^3 \|\bar{v}_h\|^2_{H^1(\Omega)}$$

which gives (a).
(b) According to [20] the following condition is valid:

$$\inf_{q_h \in M_h \setminus \{0\}} \sup_{v_h \in X_h \setminus \{0\}} \frac{b_h(v_h, q_h)}{\|v_h\|_{H^1(\tilde{\Omega}_h)} \|q_h\|_{L^2(\tilde{\Omega}_h)}} \geq \beta > 0.$$

The assertion now follows from Lemma 3.2b if we observe that $\|\bar{v}_h\|_{H^1(\Omega)}, \|v_h\|_{H^1(\tilde{\Omega}_h)}$ and $\|\bar{q}_h\|_{L^2(\Omega)}, \|q_h\|_{L^2(\tilde{\Omega}_h)}$ are equivalent norms for v_h, q_h respectively. □

Now we are in position to prove the main result of this paper.

Theorem 3.4. *Let (u, p) be the solution of (1.2–1.4) and (u_h, p_h) the solution of (2.2). Then there exists $h_4 > 0$ such that for all $0 < h \leq h_4$*

$$\|u - \bar{u}_h\|_{H^1(\Omega)} + \|p - \bar{p}_h\|_{L^2(\Omega)} \leq ch^{\frac{3}{2}} \|f\|_{H^1(\Omega)}.$$

Proof. As we do not have $\{\bar{v}_h \mid v_h \in X_h\} \subset X$ we consider both spaces as subspaces of $H^1(\Omega, \mathbb{R}^d)$. From Lemma 3.3 we infer

$$\begin{aligned} a(\bar{v}_h, \bar{v}_h) &\geq c_0 \|\bar{v}_h\|^2_{H^1(\Omega)}, \qquad \text{for all } v_h \in X_h \\ \sup_{v_h \in X_h \setminus \{0\}} \frac{b(\bar{v}_h, \bar{q}_h)}{\|\bar{v}_h\|_{H^1(\Omega)}} &\geq \beta' \|\bar{q}_h\|_{L^2(\Omega)}, \qquad \text{for all } q_h \in M_h. \end{aligned} \tag{3.5}$$

Using the techniques in Section II.2 in [5] we conclude from the error relation (3.1, 3.5)

$$\|u - \bar{u}_h\|_{H^1(\Omega)} + \|p - \bar{p}_h\|_{L^2(\Omega)} \leq c\Big(\inf_{v_h \in X_h} \|u - \bar{v}_h\|_{H^1(\Omega)} + \inf_{q_h \in M_h} \|p - \bar{q}_h\|_{L^2(\Omega)} + \sum_{i=1}^{5} M_{i,h}\Big) \tag{3.6}$$

where

$$\begin{aligned} M_{1,h} &= \sup_{v_h \in X_h \setminus \{0\}} \frac{|a_h(u_h, v_h) - a(\bar{u}_h, \bar{v}_h)|}{\|\bar{v}_h\|_{H^1(\Omega)}} \\ M_{2,h} &= \sup_{v_h \in X_h \setminus \{0\}} \frac{|b_h(v_h, p_h) - b(\bar{v}_h, \bar{p}_h)|}{\|\bar{v}_h\|_{H^1(\Omega)}} \\ M_{3,h} &= \sup_{v_h \in X_h \setminus \{0\}} \frac{|\int_\Omega f \cdot \bar{v}_h - \int_{\tilde{\Omega}_h} \tilde{f} \cdot v_h|}{\|\bar{v}_h\|_{H^1(\Omega)}} \\ M_{4,h} &= \sup_{v_h \in X_h \setminus \{0\}} \frac{|\int_{\partial\Omega} (n\sigma(u,p)n)\, \bar{v}_h \cdot n|}{\|\bar{v}_h\|_{H^1(\Omega)}} \\ M_{5,h} &= \sup_{q_h \in M_h \setminus \{0\}} \frac{|b_h(u_h, q_h) - b(\bar{u}_h, \bar{q}_h)|}{\|\bar{q}_h\|_{L^2(\Omega)}}. \end{aligned}$$

In the following we estimate the various terms occurring on the right hand side of (3.6).

To begin, let us denote by I_h^2 the usual Lagrange interpolation operator on $\tilde{\Omega}_h$ for polynomials of degree two and define $w_h := I_h^2(u \circ G_h)$. Clearly, $w_h \in X_h$ and

$$\|u \circ G_h - w_h\|_{H^1(\tilde{T})} \leq ch(\tilde{T})^2 \|D^3(u \circ G_h)\|_{L^2(\tilde{T})}, \qquad \tilde{T} \in \tilde{\mathcal{T}}_h.$$

Transforming to Ω and using (2.5, 2.6) we get

$$\begin{aligned} \|u - \bar{w}_h\|_{H^1(\Omega)} &= \Big(\sum_{T \in \mathcal{T}_h} \|u - \bar{w}_h\|^2_{H^1(T)}\Big)^{\frac{1}{2}} \leq c\Big(\sum_{\tilde{T} \in \tilde{\mathcal{T}}_h} \|u \circ G_h - w_h\|^2_{H^1(\tilde{T})}\Big)^{\frac{1}{2}} \\ &\leq ch^2 \Big(\sum_{\tilde{T} \in \tilde{\mathcal{T}}_h} \|D^3(u \circ G_h)\|^2_{L^2(\tilde{T})}\Big)^{\frac{1}{2}} \leq ch^2 \|u\|_{H^3(\Omega)} \\ &\leq ch^2 \|f\|_{H^1(\Omega)}. \end{aligned}$$

Arguing similarly for the pressure we arrive at

$$\inf_{v_h \in X_h} \|u - \bar{v}_h\|_{H^1(\Omega)} + \inf_{q_h \in M_h} \|p - \bar{q}_h\|_{L^2(\Omega)} \leq ch^2 \|f\|_{H^1(\Omega)}.$$

Next, Lemma 3.2 gives

$$\begin{aligned} M_{1,h} &\leq ch\Big(\sum_{T\cap\partial\Omega\neq\emptyset}\|\bar{u}_h\|^2_{H^1(T)}\Big)^{\frac{1}{2}} \\ &\leq ch\|u-\bar{u}_h\|_{H^1(\Omega)} + ch\Big(\sum_{T\cap\partial\Omega\neq\emptyset}\|u\|^2_{H^1(T)}\Big)^{\frac{1}{2}} \\ &\leq ch\|u-\bar{u}_h\|_{H^1(\Omega)} + c\|u\|_{C^1(\bar{\Omega})}h\Big(\sum_{T\cap\partial\Omega\neq\emptyset}h(T)^3\Big)^{\frac{1}{2}} \\ &\leq ch\|u-\bar{u}_h\|_{H^1(\Omega)} + ch^{\frac{3}{2}}\|f\|_{H^1(\Omega)} \end{aligned}$$

where we used the continuous embedding $H^3(\Omega)\hookrightarrow C^1(\bar{\Omega})$, (3.2) and (2.5). Similarly

$$M_{2,h} \leq ch\|p-\bar{p}_h\|_{L^2(\Omega)} + ch^{\frac{3}{2}}\|f\|_{H^1(\Omega)}.$$

From the definition of $\tilde{f}$ and the fact that $G_h\equiv I$ on all simplices which are disjoint from $\partial\tilde{\Omega}_h$ we infer

$$\begin{aligned} |\int_\Omega f\cdot\bar{v}_h - \int_{\tilde{\Omega}_h}\tilde{f}\cdot v_h| &= |\sum_{\tilde{T}\cap\partial\tilde{\Omega}_h\neq\emptyset}\int_{\tilde{T}}(f\circ G_h)\cdot v_h(|\mathrm{det}DG_h|-1)\,| \\ &\leq ch\sum_{\tilde{T}\cap\partial\tilde{\Omega}_h\neq\emptyset}\|f\circ G_h\|_{L^2(\tilde{T})}\|v_h\|_{L^2(\tilde{T})} \\ &\leq ch\|f\|_{L^2(B_{\gamma h}(\partial\Omega)\cap\Omega)}\|\bar{v}_h\|_{L^2(B_{\gamma h}(\partial\Omega)\cap\Omega)}. \end{aligned}$$

In view of the embedding $H^1(\Omega)\hookrightarrow L^4(\Omega)$ we obtain

$$\|f\|_{L^2(B_{\gamma h}(\partial\Omega)\cap\Omega)} \leq \big(\mathcal{L}^d(B_{\gamma h}(\partial\Omega)\cap\Omega)\big)^{\frac{1}{4}}\|f\|_{L^4(\Omega)} \leq ch^{\frac{1}{4}}\|f\|_{H^1(\Omega)}$$

where $\mathcal{L}^d$ denotes the d–dimensional Lebesgue measure.

Using the same argument for $\bar{v}_h$ we arrive at

$$|\int_\Omega f\cdot\bar{v}_h - \int_{\tilde{\Omega}_h}\tilde{f}\cdot v_h| \leq ch^{\frac{3}{2}}\|f\|_{H^1(\Omega)}\|\bar{v}_h\|_{H^1(\Omega)}$$

for all $v_h\in X_h$ which implies

$$M_{3,h} \leq ch^{\frac{3}{2}}\|f\|_{H^1(\Omega)}.$$

Employing Lemma 3.1 we get

$$\begin{aligned} |\int_{\partial\Omega}(n\sigma(u,p)n)\,\bar{v}_h\cdot n| &\leq c\big(\|u\|_{C^1(\bar{\Omega})}+\|p\|_{C^0(\bar{\Omega})}\big)\|\bar{v}_h\cdot n\|_{L^2(\partial\Omega)} \\ &\leq ch^{\frac{3}{2}}\|f\|_{H^1(\Omega)}\|\bar{v}_h\|_{H^1(\Omega)} \end{aligned}$$

and therefore

$$M_{4,h} \leq ch^{\frac{3}{2}}\|f\|_{H^1(\Omega)}.$$

Finally, it follows in the same way as above

$$M_{5,h} \leq ch\|u - \bar{u}_h\|_{H^1(\Omega)} + ch^{\frac{3}{2}}\|f\|_{H^1(\Omega)}.$$

Combining the above estimates and choosing h_4 small enough the result follows. □

4. Error estimates for the nonlinear problem

In this section we extend our error analysis to the Navier–Stokes problem (1.1, 1.3, 1.4). The discrete problem now reads: find $(u_h, p_h) \in X_h \times M_h$ such that

$$\begin{aligned} a_h(u_h, v_h) + b_h(v_h, p_h) + N_h(u_h, u_h, v_h) &= \int_{\tilde{\Omega}_h} \tilde{f} \cdot v_h \quad \text{for all } v_h \in X_h \\ b_h(u_h, q_h) &= 0 \qquad\qquad \text{for all } q_h \in M_h. \end{aligned} \tag{4.1}$$

Here, X_h, M_h, a_h and b_h are the same as in Section 2 while N_h is defined by

$$N_h(u, v, w) := \frac{1}{2} \int_{\tilde{\Omega}_h} \Big(\big((u \cdot \nabla)v\big)w - \big((u \cdot \nabla)w\big)v \Big), \quad u, v, w \in H^1(\tilde{\Omega}_h; \mathbb{R}^d).$$

It is well–known that (1.1, 1.3, 1.4) has a unique solution $(u, p) \in H^3(\Omega; \mathbb{R}^d) \times H^2(\Omega)$ provided

$$(\mathrm{Re})^2 \|f\|_{L^2(\Omega)} < \bar{c}(\Omega) \tag{4.2}$$

with some constant $\bar{c}(\Omega) > 0$ depending on the domain Ω.

Furthermore,

$$\|u\|_{H^1(\Omega)} \leq c\mathrm{Re}\|f\|_{L^2(\Omega)}. \tag{4.3}$$

In what follows we assume that (4.2) holds. We shall use a quantified version of Newton's method in order to derive an error bound for the velocity.

Theorem 4.1. *Let X, Y be Banach spaces, $u_0 \in X$, $R > 0$ and $F \in C^1(B_R(u_0), Y)$. Assume that $DF(u_0)$ is an isomorphism of X onto Y with $\|DF(u_0)^{-1}\|_{L(Y,X)} \leq \gamma$, that $\|DF(u_0)^{-1}F(u_0)\|_X \leq \epsilon$ and that*

$$\|DF(u) - DF(v)\|_{L(X,Y)} \leq K\|u - v\|_X \quad \text{for all } u, v \in B_R(u_0).$$

If

$$2\epsilon < R, \quad 2K\epsilon\gamma < 1 \tag{4.4}$$

then the problem $F(u) = 0$ has a unique solution $u^* \in \overline{B_{2\epsilon}(u_0)}$.

Proof. See Theorem 15.6 of [8].

Let us now apply the above result to our situation. We set $X = Y = H^1(\Omega; \mathbb{R}^d)$. Let $T_h \in L(X_h', X_h)$ be the discrete Stokes operator which assigns to every $g_h \in X_h'$ the velocity $u_h \in H^1(\tilde{\Omega}_h, \mathbb{R}^d)$ of the unique solution $(u_h, p_h) \in X_h \times M_h$ of

$$\begin{aligned} a_h(u_h, v_h) + b_h(v_h, p_h) &= \langle g_h, v_h \rangle \quad \text{for all } v_h \in X_h \\ b_h(u_h, q_h) &= 0 \qquad\quad\ \text{for all } q_h \in M_h. \end{aligned}$$

Here, $\langle .,. \rangle$ denotes the duality between X_h' and X_h. We define the mapping $F_h : H^1(\Omega, \mathbb{R}^d) \to H^1(\Omega, \mathbb{R}^d)$ as follows: for a given $v \in H^1(\Omega, \mathbb{R}^d)$ let

$$w_h = T_h\Big(f \circ G_h - N_h(v \circ G_h, v \circ G_h, \cdot)\Big)$$

and set

$$F_h(v) := v - w_h \circ G_h^{-1} = v - \bar{w}_h.$$

Clearly, u_h is a solution of (4.1) if and only if $F_h(\bar{u}_h) = 0$.

Now we are in position to formulate the main result of this section.

Theorem 4.2. *Let (u,p) be the solution of (1.1, 1.3, 1.4). If $\mathrm{Re}\|f\|_{L^2(\Omega)}$ is sufficiently small, there exists $h_5 > 0$ such that for all $0 < h \le h_5$ (4.1) has a unique solution (u_h, p_h) which satisfies*

$$\|u - \bar{u}_h\|_{H^1(\Omega)} + \|p - \bar{p}_h\|_{L^2(\Omega)} \le ch^{\frac{3}{2}}.$$

Proof. Clearly $F_h \in C^1(B_1(u); H^1(\Omega; \mathbb{R}^d))$ and $DF_h(u) = I - S_h$ with $S_h \in L(H^1(\Omega; \mathbb{R}^d))$. From the uniform ellipticity of a_h (which follows from Lemma 3.2 and Lemma 3.3), the uniform continuity of N_h on $H^1(\Omega; \mathbb{R}^d)^3$ and (4.3) we conclude that $\|S_h\|_{L(H^1(\Omega))} \le 1/2$ provided $\mathrm{Re}\|f\|_{L^2(\Omega)}$ is small enough. Therefore, $DF_h(u)$ is an isomorphism of $H^1(\Omega; \mathbb{R}^d)$ and

$$\|DF_h(u)^{-1}\|_{L(H^1(\Omega))} \le \gamma \quad \text{uniformly in } h. \tag{4.5}$$

Next, we want to estimate $\epsilon_h := \|F_h(u)\|_{H^1}$. According to the definition of F_h we may write $F_h(u) = u - \bar{w}_h$ where $w_h \in X_h$ is the solution of the following problem:

$$\begin{aligned} a_h(w_h, v_h) + b_h(v_h, p_h) &= \int_{\tilde{\Omega}_h} f \circ G_h \cdot v_h - N_h(u \circ G_h, u \circ G_h, v_h) \quad &&\text{for all } v_h \in X_h \\ b_h(w_h, q_h) &= 0 &&\text{for all } q_h \in M_h. \end{aligned}$$

Since (u,p) is a solution of (1.1, 1.3, 1.4) we also have

$$\begin{aligned} a(u,v) + b(v,p) &= \int_\Omega f \cdot v - \int_\Omega (u \cdot \nabla) u \cdot v \quad &&\text{for all } v \in X \\ b(w,q) &= 0 &&\text{for all } q \in M. \end{aligned}$$

Just as in the proof for the linear case we may now deduce an error relation for $u - \bar{w}_h$:

$$\begin{aligned} a(u - \bar{w}_h, \bar{v}_h) + b(\bar{v}_h, p - \bar{p}_h) &= a_h(w_h, v_h) - a(\bar{w}_h, \bar{v}_h) + b_h(v_h, p_h) - b(\bar{v}_h, \bar{p}_h) \\ &\quad + \int_\Omega f \cdot \bar{v}_h - \int_{\tilde{\Omega}_h} f \circ G_h \cdot v_h + \int_{\partial\Omega} (n\sigma(u,p)n)\, \bar{v}_h \cdot n \\ &\quad - \int_\Omega (u \cdot \nabla) u \cdot \bar{v}_h + N_h(u \circ G_h, u \circ G_h, v_h) \qquad (4.6) \\ b(u - \bar{w}_h, \bar{q}_h) &= b_h(w_h, q_h) - b(\bar{w}_h, \bar{q}_h) \end{aligned}$$

for all $v_h \in X_h$, $q_h \in M_h$.

From Lemma 4.3 below for the case $w = u \circ G_h$ we infer

$$\sup_{v_h \in X_h \setminus \{0\}} \frac{|\int_\Omega (u \cdot \nabla) u \cdot \bar{v}_h - N_h(u \circ G_h, u \circ G_h, v_h)|}{\|\bar{v}_h\|_{H^1(\Omega)}} \le ch^{\frac{3}{2}}.$$

Combining this estimate with the argument in Theorem 3.4 we arrive at

$$\epsilon_h = \|F_h(u)\|_{H^1(\Omega)} = \|u - \bar{w}_h\|_{H^1(\Omega)} \le ch^{\frac{3}{2}}$$

which together with (4.5) implies

$$\|DF_h(u)^{-1}F_h(u)\|_{H^1(\Omega)} \le \gamma\|F_h(u)\|_{H^1(\Omega)} \le ch^{\frac{3}{2}}. \tag{4.7}$$

Furthermore, it is not hard to prove that

$$\|DF_h(v) - DF_h(w)\|_{L(H^1(\Omega))} \le K\|v - w\|_{H^1(\Omega)} \quad \text{for all } v, w \in B_1(u). \tag{4.8}$$

From (4.5, 4.7, 4.8) we see that (4.4) is satisfied provided $0 < h \le h_5$ and Theorem 4.1 implies that the equation $F_h(v) = 0$ has a unique solution $\bar{u}_h \in B_{ch^{\frac{3}{2}}}(u)$ or in other words

$$\|u - \bar{u}_h\|_{H^1(\Omega)} \le ch^{\frac{3}{2}}. \tag{4.9}$$

Furthermore, $u_h = \bar{u}_h \circ G_h$ solves (4.1).

Before we give a corresponding bound for the pressure we prove a lemma which deals with the nonlinearity and which has already been used above.

Lemma 4.3. *Let $(u, p) \in H^3(\Omega, \mathbb{R}^d) \times H^2(\Omega)$ be the solution of (1.1, 1.3, 1.4) and $w \in H^1(\tilde{\Omega}_h, \mathbb{R}^d)$. Then*

$$\sup_{v_h \in X_h \setminus \{0\}} \frac{|\int_\Omega (u\cdot\nabla)u\cdot\bar{v}_h - N_h(w, w, v_h)|}{\|\bar{v}_h\|_{H^1}} \le c\|u - w\circ G_h^{-1}\|_{H^1}\big(\|u\|_{H^1} + \|w\circ G_h^{-1}\|_{H^1}\big) + ch^{\frac{3}{2}}.$$

Proof. Using integration by parts together with (1.3) we get

$$\int_\Omega (u\cdot\nabla)u\cdot\bar{v}_h = \frac{1}{2}\int_\Omega (u\cdot\nabla)u\cdot\bar{v}_h - \frac{1}{2}\int_\Omega (u\cdot\nabla)\bar{v}_h\cdot u$$

so that

$$\begin{aligned}\int_\Omega (u\cdot\nabla)u\cdot\bar{v}_h - N_h(w,w,v_h) &= \frac{1}{2}\Big(\int_\Omega (u\cdot\nabla)u\cdot\bar{v}_h - \int_{\tilde{\Omega}_h}(w\cdot\nabla)w\cdot v_h\Big)\\ &\quad - \frac{1}{2}\Big(\int_\Omega (u\cdot\nabla)\bar{v}_h\cdot u - \int_{\tilde{\Omega}_h}(w\cdot\nabla)v_h\cdot w\Big).\end{aligned}$$

The transformation rule and (3.2) imply

$$\begin{aligned}\Big|\int_\Omega (u\cdot\nabla)u\cdot\bar{v}_h - \int_{\tilde{\Omega}_h}(w\cdot\nabla)w\cdot v_h\Big| &= \Big|\int_{\tilde{\Omega}_h}(u\circ G_h\cdot\nabla)u\circ G_h\cdot v_h|\mathrm{det}G_h| - \int_{\tilde{\Omega}_h}(w\cdot\nabla)w\cdot v_h\Big|\\ &\le \Big|\int_{\tilde{\Omega}_h}\Big((u\circ G_h\cdot\nabla)u\circ G_h\cdot v_h - (w\cdot\nabla)w\cdot v_h\Big)\Big|\\ &\quad + c\sum_{T\cap\partial\Omega\neq\emptyset}\|u\|_{C^1}^2 h(\tilde{T})\int_{\tilde{T}}|v_h|\\ &\le c\|u\circ G_h - w\|_{H^1}\big(\|u\circ G_h\|_{H^1} + \|w\|_{H^1}\big)\|v_h\|_{H^1}\\ &\quad + c\sum_{T\cap\partial\Omega\neq\emptyset} h(\tilde{T})^{\frac{d}{2}+1}\|v_h\|_{L^2(\tilde{T})}\\ &\le c\|u - w\circ G_h^{-1}\|_{H^1}\big(\|u\|_{H^1} + \|w\circ G_h^{-1}\|_{H^1}\big)\|\bar{v}_h\|_{H^1} + ch^{\frac{3}{2}}\|\bar{v}_h\|_{H^1}.\end{aligned}$$

Here we also employed the continuity of the trilinear form $(u, v, w) \mapsto \int_{\tilde{\Omega}_h} (u \cdot \nabla) v \cdot w$ on $\left(H^1(\tilde{\Omega}_h, \mathbb{R}^d)\right)^3$. The second term can be treated in the same way and the lemma follows. □

In order to obtain the pressure bound we now write down an error relation analogous to (4.6) with $u - \bar{w}_h$ replaced by $u - \bar{u}_h$ and $N_h(u \circ G_h, u \circ G_h, v_h)$ replaced by $N_h(u_h, u_h, v_h)$. Observing that $\|u_h\|_{H^1} \le c\mathrm{Re}\|f\|_{L^2}$ Lemma 4.3 implies

$$\sup_{v_h \in X_h \setminus \{0\}} \frac{\left|\int_\Omega (u \cdot \nabla) u \cdot \bar{v}_h - N_h(u_h, u_h, v_h)\right|}{\|\bar{v}_h\|_{H^1(\Omega)}} \le c\|u - \bar{u}_h\|_{H^1(\Omega)} + ch^{\frac{3}{2}}$$

so that the argument in Theorem 3.4 gives

$$\|p - \bar{p}_h\|_{L^2(\Omega)} \le ch^{\frac{3}{2}}.$$

This concludes the proof of Theorem 4.2. □

5. Numerical results

From a numerical point of view it is quite simple to treat the boundary condition incorporated in X_h, see also [2]. If one uses iterative methods to solve problem (2.7) or (4.1) (for example iterative solvers based on the Schur complement, see *e.g.* [2,6,19]), then the core method consists of solving the elliptic problem

$$a_h(u_h, v_h) = \langle \chi, v_h \rangle \qquad \text{for all } v_h \in X_h \tag{5.1}$$

with some right hand side χ. Now consider

$$\tilde{X}_h := \{ v_h \in C^0(\overline{\tilde{\Omega}}_h; \mathbb{R}^d) \mid v_{h\,|\,\tilde{T}} \in \mathcal{P}_2(\tilde{T})^d \}$$

the space of all piecewise quadratics (without enforced boundary condition). With $v_h \in \tilde{X}_h$ we associate its representation by a vector of nodal values $\underline{v} = \{v_p\}_p = \{v_p^1, v_p^2, v_p^3\}_p \subseteq \underline{\tilde{X}}_h$ such that

$$v_h = \sum_{p \in \mathcal{N}_h} v_p \varphi_p = \sum_{p \in \mathcal{N}_h} \left(v_p^1, v_p^2, v_p^3\right)^T \varphi_p$$

with $\{\varphi_p\}_p$ the nodal basis of $\tilde{X}_h$, *i.e.*

$$\varphi_p(q) = \delta_{pq} \qquad \text{for all } q \in \mathcal{N}_h.$$

Accordingly define

$$\underline{X}_h := \{ \underline{v} \in \underline{\tilde{X}}_h \,|\, v_p \cdot n_p = 0 \;\forall p \in \mathcal{N}_h \cap \partial\tilde{\Omega}_h \}$$

where $n_p = n(G_h(p))$. Define the projection $P : \underline{\tilde{X}}_h \to \underline{X}_h$ by

$$(P\underline{v})_p = \begin{cases} v_p - v_p \cdot n_p \, n_p & \text{if } p \in \mathcal{N}_h \cap \partial\tilde{\Omega}_h \\ v_p & \text{else} \end{cases}$$

and the operator $A_h : \underline{\tilde{X}}_h \to \underline{\tilde{X}}_h$ by

$$a_h(v_h, w_h) = (A_h \underline{v}, \underline{w})_{\mathbb{R}^{3N}} \qquad \text{for all } v_h, w_h \in \tilde{X}_h.$$

TABLE 1. Numerical results for 5×3 bisection steps.

level	$\sharp$ tetrahedra	$\sharp$ nodes	$\|u_h - u\|_{H^1}$	EOC	$\|p_h - p\|_{L^2}$	EOC
0	6	27	0.6985E+01	—	0.1976E+02	—
3	48	125	0.4607E+01	0.60	0.8430E+01	1.23
6	384	729	0.1566E+01	1.56	0.2543E+01	1.73
9	3072	4913	0.4530E+00	1.79	0.6862E+00	1.89
12	24576	35937	0.1224E+00	1.89	0.1791E+00	1.94
15	196608	274625	0.3301E−01	1.89	0.4599E−01	1.96

Here, $(\cdot,\cdot)_{\mathbb{R}^{3N}}$ denotes the Eucledian inner product in $\mathbb{R}^{3N}$, $N = \dim\tilde{X}_h$. Then (5.1) is equivalent to: Find $\underline{u} = P\underline{\tilde{u}}$, $\underline{\tilde{u}} \in \underline{\tilde{X}}_h$ such that

$$(A_h P\underline{\tilde{u}}, P\underline{v})_{\mathbb{R}^{3N}} = (P^T A_h P\underline{\tilde{u}}, \underline{v})_{\mathbb{R}^{3N}} = (PA_h P\underline{\tilde{u}}, \underline{v})_{\mathbb{R}^{3N}} = (P\underline{\chi}, \underline{v})_{\mathbb{R}^{3N}} \tag{5.2}$$

for all $\underline{v} \in \underline{\tilde{X}}_h$ and with $\chi_p := \langle \chi, \varphi_p \rangle$. Using an iterative procedure to solve (5.1) requires the evaluation of $a_h(v_h, w_h)$ which by (5.2) is equivalent to working on $\tilde{X}_h$ and projecting after each matrix times vector operation. Note that the projection P is a simple and numerically cheap operation. A similar consideration also holds in the case when the matrix corresponding to A_h is preconditioned by *e.g.* a multilevel procedure.

Since it is difficult to find a nontrivial explicit solution to (1.1, 1.3–1.4) we consider an example which is not entirely covered by our error analysis but in which the setting is even more complicated. Let $\Omega := B_1(0) \subseteq \mathbb{R}^3$ and choose the exterior force f in such a way that the pair (u, p) with

$$\begin{aligned} u(x,y,z) &= \begin{bmatrix} 2xz(x^2+y^2) \\ 2yz(x^2+y^2) \\ 4(x^2+y^2)(1-\frac{3}{2}(x^2+y^2)-z^2) \end{bmatrix}, \\ p(x,y,z) &= 16\Big(z - \frac{z^3}{3}\Big) \end{aligned}$$

solves (1.1) and (1.3). Instead of (1.4) we have nontrivial tangential stresses. The above solution is similar to *Hill's spherical vortex*, see for instance [3].

Note also that in view of the symmetry of Ω u is only unique up to a rigid body rotation. We discretize Ω by choosing a macro–triangulation and then refine this coarse grid by the *bisection method* introduced in [1]. The normal $n(G_h(p))$ is given by $n(G_h(p)) = p/|p|$. Table 1 shows the resulting errors and experimental orders of convergence (EOC) for successive refinements of the macro triangulation. "Level" denotes the number of refinement steps. Note that 3 refinement steps of the bisection method yield a triangulation with halved grid size.

Figure 2 shows the solution for refinement level 4×3.

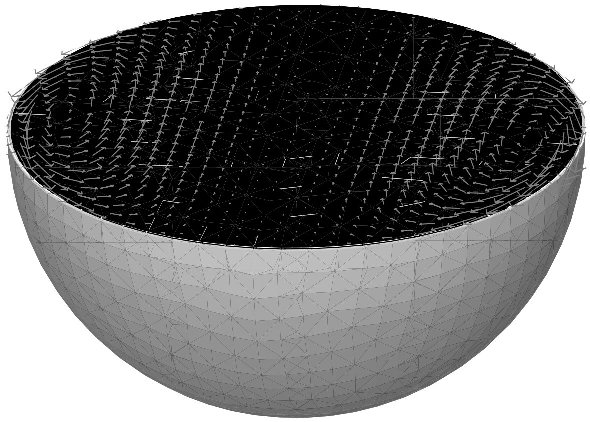

FIGURE 2. Solution on a clipping plane, refinement level 4×3.

The authors want to thank B. Höhn for performing the numerical example.

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