Mathematical Modelling and Numerical Analysis M2AN, Vol. 33, N° 6, 1999, p. 1149–1185
Modélisation Mathématique et Analyse Numérique

INTERPOLATION OF NON-SMOOTH FUNCTIONS ON ANISOTROPIC FINITE ELEMENT MESHES

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Abstract. In this paper, several modifications of the quasi-interpolation operator of Scott and Zhang [30] are discussed. The modified operators are defined for non-smooth functions and are suited for application on anisotropic meshes. The anisotropy of the elements is reflected in the local stability and approximation error estimates. As an application, an example is considered where anisotropic finite element meshes are appropriate, namely the Poisson problem in domains with edges.

AMS Subject Classification. 65D05, 65N30, 65N50.

Received: October 9, 1997. Revised: April 3, 1998 and March 26, 1999.

1. Introduction

The solution of elliptic boundary value problems may have *anisotropic behaviour* near certain manifolds $M \subset \overline{\Omega}$. That means that the solution varies significantly only perpendicularly to M. Examples include the Poisson problem in domains with edges M and singularly perturbed convection diffusion reaction problems where M is part of the boundary or an internal manifold. In such cases it is an obvious idea to reflect this anisotropy in the discretization by using *anisotropic meshes* with a small mesh size in the direction of the rapid variation of the solution and a larger mesh size in the perpendicular direction.

In order to describe the elements of anisotropic meshes mathematically, consider an elliptic boundary value problem posed over a polyhedral domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. We study the discretization error of the finite element method on a family of meshes $\mathcal{T}_h = \{e\}$ with the usual admissibility conditions (see, for example, Conditions ($\mathcal{T}_h1$–$\mathcal{T}_h5$) in Chapter 2 of [18]). Denote by h_e the diameter of the finite element e, and by ϱ_e the supremum of the diameters of all balls contained in e. Then it is assumed in the classical finite element theory that $h_e \lesssim \varrho_e$, for the definition of $\lesssim$ see Section 2. This assumption is no longer valid in the case of anisotropic meshes. Conversely, anisotropic elements e are characterized by

$$\frac{h_e}{\varrho_e} \to \infty$$

where the limit can be considered as $h \to 0$ (see the application to the Poisson equation in [4,9] or Section 7) or $\varepsilon \to 0$ where ε is some (small perturbation) parameter of the problem (see the singularly perturbed problems in [6,7]).

Keywords and phrases. Anisotropic finite elements, interpolation error estimate, quasi-interpolation, non-smooth functions, edge singularity, reaction diffusion problem.

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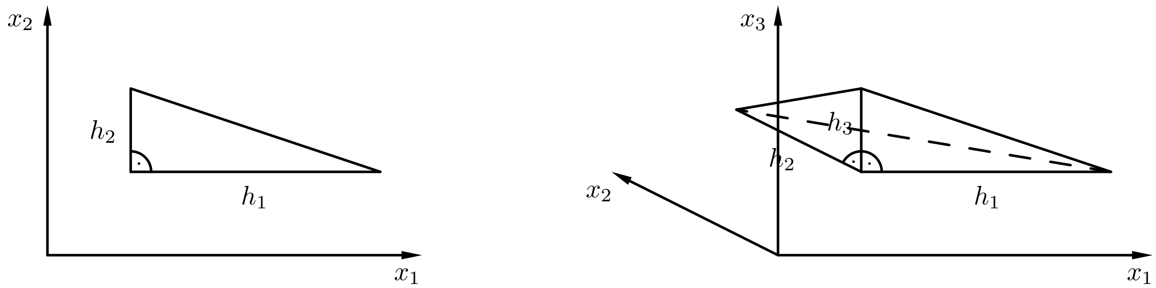


FIGURE 1. Illustration of the simplest anisotropic finite elements.

Local estimates of the interpolation error are basic ingredients for *a priori* estimates of the finite element error, for proving the equivalence of error estimators and the exact error, and for investigating multi-level algorithms for the solution of the system of algebraic equations which arise in the finite element method. For Lagrangian finite elements, the simplest approximation is the *nodal interpolant* $\mathrm{I}_h : C(\overline{\Omega}) \to V_h := \operatorname{span}\{\varphi_i, i \in I\}$,

$$(\mathrm{I}_h u)(x) := \sum_{i \in I} u(X_i)\,\varphi_i(x), \tag{1.1}$$

where X_i are the nodes and $\varphi_i(x)$ are the *nodal basis functions*, $\varphi_i(X_j) = \delta_{ij}$ for all $i, j \in I$. Because I_h is defined locally on every element the interpolation error $u - \mathrm{I}_h u$ can be estimated elementwise. Before we discuss the drawback of the nodal interpolant we shall recall some *anisotropic interpolation error estimates*. We denote error estimates as anisotropic if they are sharp enough to reflect the different element sizes and not only the diameter.

For simplicity in this Introduction consider a triangle or a tetrahedron $e \subset \mathbb{R}^d$ with element sizes $h_1, \ldots, h_d$ as given in Figure 1. That means that the element e has d edges of length h_i which are parallel to the corresponding coordinate axes. Then for linear elements the following estimates hold [4, 7]:

$$\|u - \mathrm{I}_h u; L^p(e)\| \lesssim \sum_{|\alpha|=\ell} h^\alpha \|D^\alpha u; L^p(e)\|, \quad \text{if } \begin{cases} \ell = 1 \text{ and } p \in (d, \infty], \\ \ell = 2 \text{ and } p \in [1, \infty], \end{cases} \tag{1.2}$$

$$|u - \mathrm{I}_h u; W^{1,p}(e)| \lesssim \sum_{|\alpha|=1} h^\alpha |D^\alpha u; W^{1,p}(e)|, \quad \text{if } d = 2 \text{ or } p \in (2, \infty]. \tag{1.3}$$

For the notation see Section 2. The necessity of the condition $p > 2$ in the three-dimensional case is discussed at several places [4, 22, 31]. In the sequel, we will call an *estimate* to be *of type* (m, n) if certain mth derivatives (left-hand side) are estimated against nth derivatives of the solution. In this sense estimate (1.3) is of type $(1, 2)$.

For some applications, the nodal interpolant is not appropriate. First, the main drawback is that nodal values of u have to be well-defined for the definition of $\mathrm{I}_h u$. For example, the solution of the Poisson equation with mixed boundary conditions can be of such poor regularity in the neighbourhood of edges that $u \notin W^{s,2}(\Omega)$ for any $s > 3/2$. This causes the interpolation theory with I_h to fail. Second, estimate (1.3) holds only for $p > 2$ in the three-dimensional case. But $p = 2$ is the natural choice in the investigation of the finite element approximation error. Using $p > 2$ and the Hölder inequality leads to sub-optimal results, see the discussion in Section 7. Third, there is no estimate of type $(1, 1)$ for the nodal interpolant. Such estimates are of advantage for the investigation of multi-grid/multi-level methods for the solution of the system of algebraic equations which arise in the finite element method.

As a remedy, other approximation operators Q_h with $\mathrm{Q}_h u \in V_h$ can be considered. They are sometimes called *quasi-interpolants* and should preserve the following favourable properties of I_h.

1. $\mathrm{Q}_h u$ shall be defined locally. This means, that $(\mathrm{Q}_h u)(x)$ with $x \in e$ shall depend only on the values of u in a small neighbourhood S_e of e, where S_e consists of a finite number (independent of h) of elements of $\mathcal{T}_h$. (For the nodal interpolant I_h we had in particular $S_e = e$.)
2. If possible, Q_h shall reproduce piecewise polynomials: $\mathrm{Q}_h u_h = u_h$ for all $u_h \in V_h$.

For *isotropic meshes* such operators have been studied in the literature. For an introduction, define by a generalization of (1.1)

$$(\mathrm{Q}_h u)(x) := \sum_{i \in I} a_i \, \varphi_i(x) \tag{1.4}$$

with real numbers a_i still to be specified. Note that $\mathrm{Q}_h = \mathrm{I}_h$ if $a_i = u(X_i)$ for all $i \in I$.

In order to treat non-smooth functions the idea is to consider subdomains $\sigma_i \subset \overline{\Omega}$ (their choice will be discussed later), to define an L^2-projection operator

$$\Pi_{\sigma_i} : L^2(\sigma_i) \to \mathcal{P}_{k,\sigma_i}, \tag{1.5}$$

and to choose

$$a_i := (\Pi_{\sigma_i} u)(X_i), \tag{1.6}$$

for the notation see Section 2, for more details see (3.1–3.3). The numbers a_i can be considered as averaged values of u in X_i. Different authors chose different σ_i resulting in different quasi-interpolation operators. We will now introduce three of them. For unambiguous reference we distinguish them by different symbols, C_h, O_h, and Z_h.

Clément [19] uses $\overline{\sigma_i} := \bigcup_{\overline{e} \ni X_i} \overline{e}$. The resulting operator C_h,

$$(\mathrm{C}_h u)(x) := \sum_{i \in I} (\Pi_{\sigma_i} u)(X_i) \cdot \varphi_i(x),$$

is even defined for $u \in L^1(\Omega)$ and allows estimates of type (m, ℓ) for all $0 \le m \le \ell \le k+1$, $k \ge 1$ is defined in Section 2. However, the operator C_h in this original form does not satisfy Property 2 above, but this can be corrected by defining

$$\Pi_{\sigma_i} : L^2(\sigma_i) \to V_h|_{\sigma_i}. \tag{1.7}$$

A modification of the Clément operator is discussed by Oswald [28]. For defining σ_i, he fixes just one (arbitrary) element $e =: \sigma_i$ with $X_i \in \overline{e}$. The resulting operator O_h allows the same estimates as C_h, but we have $V_h|_{\sigma_i} = \mathcal{P}_{k,\sigma_i}$. Some more details on C_h and O_h are given at the end of Section 3 when more notation has been introduced and more ideas have been developed.

The disadvantage of both C_h and O_h is that they do not preserve Dirichlet boundary conditions. For this reason, Scott and Zhang [30] modified again the choice of σ_i and used not only d-dimensional subdomains σ_i but also $(d-1)$-dimensional ones. In particular, they chose $\sigma_i \subset \partial\Omega$ if $X_i \in \partial\Omega$. Because we exploit this idea in this paper we will introduce the resulting operator Z_h in more detail in Section 3. In particular, we derive some anisotropic estimates of type $(0, \ell)$, $1 \le \ell \le k+1$, and show that the operator Z_h has to be modified for error estimates of type $(1, \ell)$.

The aim of the paper is to define and to investigate quasi-interpolation operators which do not have the disadvantages of the Lagrange interpolation operator (see above) and which allow for proving anisotropic estimates of type (m, ℓ), with $m \ge 0$, for anisotropic meshes. Using the idea of lower-dimensional subdomains σ_i we define in Sections 4–6 three operators of that type, S_h, L_h, and E_h. There are differences in the applicability of these operators concerning the types of elements and the ability to preserve Dirichlet boundary conditions.

We will summarize this in Section 8. Before, in Section 7, we shall apply the operators S_h and E_h and derive finite element error estimates for the Poisson problem in certain domains with edges. The result can not be obtained using the nodal interpolation operator I_h or the original quasi-interpolation operators C_h, O_h, and Z_h. This underlines the importance of this study.

Nevertheless, some questions need further research. First, the investigation in this paper is limited to domains of tensor product type. It is not straightforward to drop this assumption. Second, estimates of type $(1,1)$ are derived only for L_h. This means, such an estimate is not available for three-dimensional "needle elements" ($h_1 \sim h_2 \ll h_3$).

2. Notation and auxiliary results

The notation $a \lesssim b$ and $a \sim b$ means the existence of positive constants C_1 and C_2 (which are independent of $\mathcal{T}_h$ and of the function under consideration) such that $a \le C_2 b$ and $C_1 b \le a \le C_2 b$, respectively.

Let d be the space dimension and $x = (x_1, \dots, x_d)$ the global Cartesian coordinate system. We use a multi-index notation with $\alpha := (\alpha_1, \dots, \alpha_d)$, α_i non-negative integers,

$$|\alpha| := \sum_{i=1}^{d} \alpha_i, \quad x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \text{ and } D^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}.$$

$W^{\ell,p}(e)$ ($\ell \in \mathbb{N}_0$, $p \in [1,\infty]$) are the Sobolev spaces with

$$\|v; W^{\ell,p}(e)\|^p := \sum_{|\alpha| \le \ell} \int_e |D^\alpha v|^p, \quad |v; W^{\ell,p}(e)|^p := \sum_{|\alpha| = \ell} \int_e |D^\alpha v|^p$$

for $p < \infty$ and the usual modification for $p = \infty$.

Finite elements $e \subset \mathbb{R}^d$ are defined *via* (a finite number of) reference element(s) $\hat{e} \subset \mathbb{R}^d$. In the cases of triangles ($\hat{e} := \{(\hat{x}_1, \hat{x}_2) \in \mathbb{R}^2 : 0 < \hat{x}_1 < 1, 0 < \hat{x}_2 < 1 - \hat{x}_1\}$), rectangles ($\hat{e} := \{(\hat{x}_1, \hat{x}_2) \in \mathbb{R}^2 : 0 < \hat{x}_1, \hat{x}_2 < 1\}$), pentahedra ($\hat{e} := \{(\hat{x}_1, \hat{x}_2, \hat{x}_3) \in \mathbb{R}^3 : 0 < \hat{x}_1, \hat{x}_3 < 1, 0 < \hat{x}_2 < 1 - \hat{x}_1\}$), and hexahedra ($\hat{e} := \{(\hat{x}_1, \hat{x}_2, \hat{x}_3) \in \mathbb{R}^3 : 0 < \hat{x}_1, \hat{x}_2, \hat{x}_3 < 1\}$) it is sufficient to consider one unique $\hat{e}$. Only for tetrahedra we consider two reference elements: $\hat{e} := \{(\hat{x}_1, \hat{x}_2, \hat{x}_3) \in \mathbb{R}^3 : 0 < \hat{x}_1 < 1, 0 < \hat{x}_2 < 1 - \hat{x}_1, 0 < \hat{x}_3 < 1 - \hat{x}_1 - \hat{x}_2\}$ for elements with a face parallel to the x_1, x_2-plane and $\hat{e} := \{(\hat{x}_1, \hat{x}_2, \hat{x}_3) \in \mathbb{R}^3 : 0 < \hat{x}_1 < 1, 0 < \hat{x}_2 < 1 - \hat{x}_1, \hat{x}_1 < \hat{x}_3 < 1 - \hat{x}_2\}$ for elements without such a face.

In this paper, we treat mainly *meshes of tensor product type* and *tensor product meshes*. The elements of these meshes are defined as follows.

Definition 1. An affine finite element is called *element of tensor product type*, when the transformation of a reference element $\hat{e}$ to the element e has (block) diagonal form,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \pm h_{1,e} & 0 \\ 0 & \pm h_{2,e} \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} + b_e \qquad \text{for } d = 2, \tag{2.1}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} B_e & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & \pm h_{d,e} \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} + b_e \qquad \text{for } d = 3, \tag{2.2}$$

where $b_e \in \mathbb{R}^d$ and $B_e \in \mathbb{R}^{2\times 2}$ with

$$|\det B_e| \sim h_{1,e}^2, \quad \|B_e\| \sim h_{1,e}, \quad \|B_e^{-1}\| \sim h_{1,e}^{-1}. \tag{2.3}$$

In this way the element sizes $h_{1,e}, \dots, h_{d,e}$ are implicitly defined. Note that (2.3) yields $h_{1,e} \sim h_{2,e}$ for three-dimensional elements. Up to now we did not assume a relation between $h_{1,e}$ and $h_{d,e}$. But in Sections 4 and 6

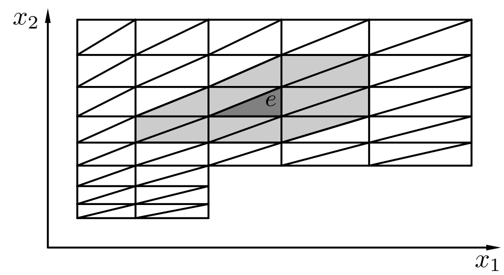


FIGURE 2. Illustration of a mesh of tensor product type in two dimensions and of the patch S_e.

we will consider the case $h_{1,e} \lesssim h_{d,e}$ (interesting is $h_{1,e} = o(h_{d,e})$) and in Section 5 we will examine $h_{d,e} \lesssim h_{1,e}$. Note further that under these assumptions the triangles/tetrahedra can be grouped into pairs/triples which form a rectangle/pentahedron of tensor product type. We will use this property in Section 4.

Definition 2. An affine finite element $e \subset \mathbb{R}^d$ is called *tensor product element*, when transformation (2.2) is reduced to

$$x_i = h_{i,e}\hat{x}_i + b_{i,e}, \quad i = 1, \ldots, d. \tag{2.4}$$

In two dimensions there is no difference between tensor product elements and elements of tensor product type. But in three dimensions we admit independent mesh sizes $h_{1,e}$, $h_{2,e}$, and $h_{3,e}$, so that a tensor product element is not necessarily a special case of an element of tensor product type.

We demand that there is no abrupt change in the element sizes, that means, the relation

$$h_{i,e} \sim h_{i,e'} \quad \text{for all } e' \text{ with } \quad \overline{e} \cap \overline{e'} \neq \emptyset \tag{2.5}$$

holds for $i = 1, \ldots, d$. In view of (2.5) and because most considerations in this paper are local, we will often omit the second subscript.

The set of shape functions $\mathcal{P}_{k,e}$,

$$\mathcal{P}_{k,e} \supset \mathcal{P}_k^d := \left\{ \sum_{|\alpha| \le k} a_\alpha x^\alpha; \quad x = (x_1, \ldots, x_d), \ a_\alpha \in \mathbb{R} \right\}, \tag{2.6}$$

is defined as usual, that means, $\mathcal{P}_{k,e} = \mathcal{P}_k^d$ for the simplicial elements, and

$$\mathcal{P}_{k,e} := \mathcal{Q}_k^d := \left\{ \sum_{0 \le \alpha_1, \alpha_2, \alpha_3 \le k} a_\alpha x^\alpha, \ a_\alpha \in \mathbb{R} \right\}, \qquad \mathcal{P}_{k,e} := \left\{ \sum_{\substack{0 \le \alpha_1 + \alpha_2 \le k \\ 0 \le \alpha_3 \le k}} a_\alpha x^\alpha, \ a_\alpha \in \mathbb{R} \right\}$$

for quadrilateral/hexahedral elements and for pentahedral elements, respectively. Moreover, for a simple notation later on we define $\mathcal{P}_{-1}^d := \{0\}$.

Let $V_h := \{v_h \in W^{1,2}(\Omega) : v_h|_e \in \mathcal{P}_{k,e} \text{ for all } e \in \mathcal{T}_h\}$ be the finite element space, a space of piecewise polynomial functions on the family of meshes under consideration.

Finally, denote by

$$S_e := \operatorname{int} \bigcup \{ \overline{e'} : e' \in \mathcal{T}_h, \overline{e'} \cap \overline{e} \neq \emptyset \} \tag{2.7}$$

the patch of elements around e, see also the illustration for a general mesh in Figure 2. Moreover, we denote uniformly in the whole paper by

X_i	the nodes of the mesh, $i \in I$,
φ_i	the nodal shape functions, $\varphi_i(X_j) = \delta_{ij}$,
σ_i	a subdomain related to X_i (different for C_h, O_h, Z_h, S_h, L_h, and E_h),
k	the degree of the shape functions in the sense of (2.6),
Π_{σ_i}	the projection operator $L^2(\sigma_i) \to \mathcal{P}_{k,\sigma_i}$,
I_h	the nodal interpolation operator,
Q_h	a general quasi-interpolation operator,
C_h	the Clément operator,
O_h	the quasi-interpolation operator introduced by Oswald,
Z_h	the original Scott-Zhang operator,
S_h	the modified Scott-Zhang operator using small edges(2D)/faces(3D),
L_h	the modified Scott-Zhang operator using large edges(2D)/faces(3D),
E_h	the modified Scott-Zhang operator using long edges (3D).

We will prove now a lemma which is useful in several proofs of this paper. The lemma has similarities to the Bramble-Hilbert theory which was developed in [16,17] for isotropic elements and extended in [4] to anisotropic elements. Here, the difference is that (in general) S_e can not be transformed by an affine mapping to a reference configuration $\hat{S}$. The isotropic version of Lemma 1 is proved in [30] using results from [21] and can easily be generalized to our case.

Lemma 1. *For any $u \in W^{\ell,p}(S_e)$ there exists a polynomial $w \in \mathcal{P}^d_{\ell-1}$ such that*

$$\sum_{|\alpha| \leq \ell - m} h^\alpha |D^\alpha(u-w); W^{m,p}(S_e)| \lesssim \sum_{|\alpha| = \ell - m} h^\alpha |D^\alpha u; W^{m,p}(S_e)|,$$

for all $m = 0, \ldots, \ell$.

Proof. By the change of variables $x_i = \tilde{x}_i h_i$ we transform S_e to $\tilde{S}_e$. According to (2.5) and the tensor product character of our mesh we realize that $\tilde{S}_e$ has a diameter of order one. Moreover, $\tilde{S}_e$ is star-shaped with respect to a ball B_1 with $\operatorname{diam} B_1 \sim 1$, or $\tilde{S}_e$ is at least the union of a finite collection of (overlapping) domains $\tilde{S}_{e,j}$ that are star-shaped with respect to a balls B_j with $\operatorname{diam} B_j \sim 1$. Let $B \subset \tilde{S}_e$ be any ball with $\operatorname{diam} B \sim 1$, choose a function $\phi \in C_0^\infty(B)$ with integral one, and define

$$\tilde{w}(\tilde{x}) := \sum_{|\alpha| \leq \ell - 1} \int_B \phi(\tilde{y}) \cdot (\tilde{D}^\alpha \tilde{u})(\tilde{y}) \cdot \frac{(\tilde{x} - \tilde{y})^\alpha}{\alpha!} \, \mathrm{d}\tilde{y} \ \in \mathcal{P}^d_{\ell-1},$$

$\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_d)$, $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_d)$, $\alpha! = \alpha_1! \cdots \alpha_d!$. We can now apply Theorem 4.2 of [21] with $\mathcal{A} = \{\alpha \in \mathbb{N}_0^d : |\alpha| = \ell\}$, and obtain for all β with $|\beta| = m$, $0 \leq m \leq \ell - 1$,

$$\|\tilde{D}^\beta(\tilde{u} - \tilde{w}); W^{\ell-m-1,p}(\tilde{S}_e)\| \lesssim |\tilde{D}^\beta \tilde{u}; W^{\ell-m,p}(\tilde{S}_e)|.$$

By transforming this estimate to S_e and summing up over all β we conclude

$$\sum_{|\alpha|\le\ell-m-1} h^\alpha \|D^{\alpha+\beta}(u-w);L^p(S_e)\| \lesssim \sum_{|\alpha|=\ell-m} h^\alpha\|D^{\alpha+\beta}u;L^p(S_e)\|,$$
$$\sum_{|\alpha|\le\ell-m-1} h^\alpha |D^{\alpha}(u-w);W^{m,p}(S_e)| \lesssim \sum_{|\alpha|=\ell-m} h^\alpha|D^{\alpha}u;W^{m,p}(S_e)|.$$

Because $D^\gamma w=0$ for $|\gamma|=\ell$ the sum on the left-hand side can be extended to $|\alpha|\le\ell-m$. □

Corollary 2. *Let $m_1+m_2=m\le\ell$. For any $u\in W^{\ell,p}(S_e)$ there exists a polynomial $w\in\mathcal{P}^d_{m-1}$ such that*

$$\sum_{|\alpha|\le m_2}\sum_{|\beta|\le\ell-m} h^{\alpha+\beta}|D^{\alpha+\beta}(u-w);W^{m_1,p}(S_e)| \lesssim \sum_{|\alpha|=m_2}\sum_{|\beta|\le\ell-m} h^{\alpha+\beta}|D^{\alpha+\beta}u;W^{m_1,p}(S_e)|.$$

Proof. We reformulate the left-hand side and split it in two terms.

$$\sum_{|\alpha|\le m_2}\sum_{|\beta|\le\ell-m} h^{\alpha+\beta}|D^{\alpha+\beta}(u-w);W^{m_1,p}(S_e)| \sim \sum_{|\delta|\le\ell-m_1} h^{\delta}|D^{\delta}(u-w);W^{m_1,p}(S_e)|$$
$$= \sum_{|\delta|\le m_2} h^{\delta}|D^{\delta}(u-w);W^{m_1,p}(S_e)| + \sum_{m_2<|\delta|\le\ell-m_1} h^{\delta}|D^{\delta}(u-w);W^{m_1,p}(S_e)|.$$

In view of $m_2=m-m_1$, the first term can be estimated *via* Lemma 1. The second term contains only derivatives of order higher than m, that means that w plays no role. Consequently, w can be chosen such that

$$\sum_{|\alpha|\le m_2}\sum_{|\beta|\le\ell-m} h^{\alpha+\beta}|D^{\alpha+\beta}(u-w);W^{m_1,p}(S_e)|$$
$$\lesssim \sum_{|\delta|=m_2} h^{\delta}|D^{\delta}u;W^{m_1,p}(S_e)| + \sum_{m_2<|\delta|\le\ell-m_1} h^{\delta}|D^{\delta}u;W^{m_1,p}(S_e)|$$
$$\lesssim \sum_{|\alpha|=m_2} h^{\alpha}|D^{\alpha}u;W^{m_1,p}(S_e)| + \sum_{|\alpha|=m_2}\sum_{1\le|\beta|\le\ell-m} h^{\alpha+\beta}|D^{\alpha+\beta}u;W^{m_1,p}(S_e)|,$$

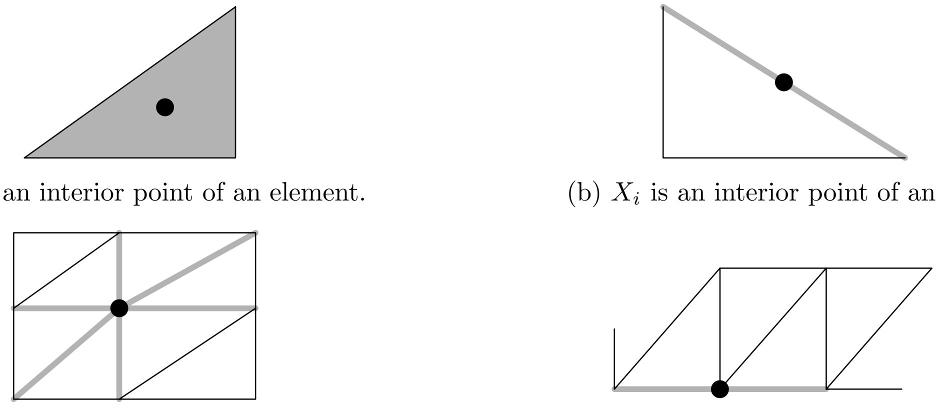
and the corollary is proved. □

3. The original Scott-Zhang operator Z_h

In this section we will recall the operator Z_h defined by Scott and Zhang [30] and examine to what extent anisotropic error estimates can be derived by simply carrying out the transformations more carefully. We will see that estimates of type $(0,\ell)$ are valid, but modifications of the operator are necessary for estimates of derivatives of the approximation error.

As introduced in Section 1 we define $\mathrm{Z}_h u$ *via* numbers $a_i=(\Pi_{\sigma_i}u)(X_i)$, where Π_{σ_i} is a projection operator with respect to a certain subdomain σ_i, $i\in I$. The subdomains σ_i are chosen by the following rules (see also Fig. 3 for the case of triangles).

- If the node X_i is an *interior point* of an element $e\subset\mathcal{T}_h$ then $\sigma_i:=e$.
- Otherwise X_i is a *boundary point* of one or more elements $e\subset\mathcal{T}_h$, and σ_i is chosen as some $(d-1)$-dimensional edge/face ς of one of these elements:
  - If there is an edge/face ς so that X_i is an *interior point* of ς, then σ_i is uniquely determined by $\sigma_i:=\varsigma$.
  - If not, then σ_i is taken as one of the edges/faces with $X_i\in\bar\varsigma$. However, we restrict this choice in the case $X_i\in\partial\Omega$ by demanding $\sigma_i\subset\partial\Omega$ then.



(a) X_i is an interior point of an element.

(b) X_i is an interior point of an edge.

(c) X_i is a vertex within the domain (here: 6 possibilities for σ_i).

(d) X_i is a vertex at the boundary (2 possibilities for σ_i).

FIGURE 3. Choice of σ_i in dependence on X_i for the definition of Z_h.

The $L^2(\sigma_i)$-projection $\Pi_{\sigma_i} u \in V_h|_{\sigma_i}$ is defined by

$$\|u - \Pi_{\sigma_i} u; L^2(\sigma_i)\| = \min_{v \in V_h|_{\sigma_i}} \|u - v; L^2(\sigma_i)\|. \tag{3.1}$$

An explicit representation of $(\Pi_{\sigma_i} u)(X_i)$ can be given by introducing the (unique) function $\psi_i \in V_h|_{\sigma_i}$ with

$$\int_{\sigma_i} \psi_i \varphi_j = \delta_{ij} \quad \text{for all } j \in I. \tag{3.2}$$

Then one finds easily that

$$(\Pi_{\sigma_i} u)(X_i) = \int_{\sigma_i} u \psi_i. \tag{3.3}$$

To see this recall that a projection operator $P : X \to Y \subset X$ can be defined *via* $Pu = \sum_j (u, \psi_j)_X \, \varphi_j$ where $\{\varphi_j\}$ is a basis in Y and $\{\psi_j\}$ is the corresponding biorthogonal basis with respect to the scalar product $(.,.)_X$ in X. As already mentioned in Section 1, see (1.4) and (1.6), the Scott-Zhang operator Z_h is now defined as

$$\mathrm{Z}_h u := \sum_i (\Pi_{\sigma_i} u)(X_i) \cdot \varphi_i = \sum_i \left(\int_{\sigma_i} u \psi_i \right) \cdot \varphi_i. \tag{3.4}$$

Though Π_{σ_i} is defined by (3.1) for $u \in L^2(\sigma_i)$, this approach can be extended to functions $u \in L^1(\sigma_i)$ because the polynomial function ψ_i is from $L^\infty(\sigma_i)$ so the integral in (3.3) is finite. That means that the approximation operator $\mathrm{Z}_h : W^{\ell,p}(\Omega) \to V_h$ can be defined for

$$\ell \geq 1 \quad \text{for } p = 1, \qquad \ell > \frac{1}{p} \quad \text{otherwise}. \tag{3.5}$$

The restrictions to ℓ and p in (3.5) follow from a trace theorem and guarantee that $u|_{\sigma_i} \in L^1(\sigma_i)$ also for $(d-1)$-dimensional σ_i. In this paper, we consider only integer ℓ, therefore (3.5) is equivalent to

$$\ell \geq 1, \quad p \in [1, \infty].$$

Note further that the approximation operator Z_h does not only preserve homogeneous Dirichlet boundary conditions but also inhomogeneous conditions $u = g$ on $\partial\Omega$ (at least in the sense of $L^1(\partial\Omega)$) if $g \in V_h|_{\partial\Omega}$.

For isotropic *simplicial* elements e ($h_1 \sim \ldots \sim h_d$) Scott and Zhang proved the following stability and approximation result [30]: If $1 \le \ell \le k+1$ and $p \in [1,\infty]$ then the estimates

$$|\mathrm{Z}_h u; W^{m,q}(e)| \lesssim (\mathrm{meas}\, e)^{1/q-1/p} \sum_{j=0}^{\ell} h_1^{j-m} |u; W^{j,p}(S_e)| \tag{3.6}$$

$$|u - \mathrm{Z}_h u; W^{m,p}(e)| \lesssim h_1^{\ell-m} |u; W^{\ell,p}(S_e)| \tag{3.7}$$

hold for $0 \le m \le \ell$. Recall that k corresponds to the degree of the polynomials, see (2.6). Recall also the definition of S_e from (2.7) and note that $\sigma_i \subset S_e$ for all i with $X_i \in \overline{e}$.

The anisotropic estimate corresponding to (3.7) would be

$$|u - \mathrm{Z}_h u; W^{m,p}(e)| \lesssim \sum_{|\alpha|=\ell-m} h^\alpha |D^\alpha u; W^{m,p}(S_e)|. \tag{3.8}$$

We prove now that this estimate is valid for $m = 0$. This result is restricted here to meshes of tensor product type but it is not restricted to simplicial elements.

Theorem 3. *On anisotropic meshes of tensor product type the Scott-Zhang approximation operator* Z_h *satisfies the following stability and approximation error estimates of type* $(0,\ell)$*:*

$$\|\mathrm{Z}_h u; L^q(e)\| \lesssim (\mathrm{meas}\, e)^{1/q-1/p} \sum_{|\alpha|\le\ell} h^\alpha \|D^\alpha u; L^p(S_e)\|, \tag{3.9}$$

$$\|u - \mathrm{Z}_h u; L^q(e)\| \lesssim (\mathrm{meas}\, e)^{1/q-1/p} \sum_{|\alpha|=\ell} h^\alpha \|D^\alpha u; L^p(S_e)\|, \tag{3.10}$$

$\ell = 1, \ldots, k+1$, *provided that* $u \in W^{\ell,p}(S_e)$*. For* (3.10) *the numbers* $p, q \in [1,\infty]$ *and* $\ell \in \mathbb{N}$ *must be such that* $W^{\ell,p}(e) \hookrightarrow L^q(e)$*.*

Proof. We start by concluding from $\int_{\sigma_i} \varphi_i \psi_i = 1$ and $\|\varphi_i; L^\infty(\sigma_i)\| = 1$ that

$$\|\psi_i; L^\infty(\sigma_i)\| \sim (\mathrm{meas}\, \sigma_i)^{-1}. \tag{3.11}$$

Using the definition of $\mathrm{Z}_h u$ we find with (3.11) that

$$\begin{aligned} \|\mathrm{Z}_h u; L^q(e)\| &\le \sum_{i\in I_e} \left\| \varphi_i \int_{\sigma_i} u\psi_i; L^q(e) \right\| \\ &\le (\mathrm{meas}\, e)^{1/q} \sum_{i\in I_e} \left| \int_{\sigma_i} u\psi_i \right| \\ &\lesssim (\mathrm{meas}\, e)^{1/q} \sum_{i\in I_e} (\mathrm{meas}\, \sigma_i)^{-1} \|u; L^1(\sigma_i)\|, \end{aligned}$$

where I_e is the set of nodes contained in $\overline{e}$. If σ_i has the same dimension as e (that means X_i is an inner node of e and $\sigma_i = e$) then we use the Hölder inequality and find

$$\begin{aligned} \|u; L^1(\sigma_i)\| &\le (\mathrm{meas}\, e)^{1-1/p} \|u; L^p(\sigma_i)\| \\ &\lesssim \mathrm{meas}\, \sigma_i \, (\mathrm{meas}\, e)^{-1/p} \|u; L^p(S_e)\|. \end{aligned} \tag{3.12}$$

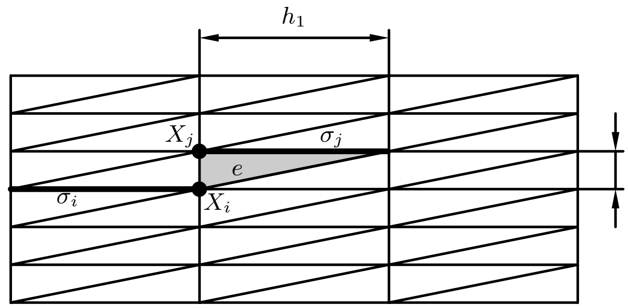


FIGURE 4. Illustration of the counterexample.

If σ_i has lower dimension we use the trace theorem $W^{\ell,p}(S_e) \hookrightarrow W^{\ell,p}(e') \hookrightarrow L^1(\sigma_i)$ ($e' \subset S_e$ is an element with $\sigma_i \subset \overline{e'}$) in the form

$$\|u; L^1(\sigma_i)\| \lesssim \operatorname{meas}\sigma_i (\operatorname{meas} e)^{-1/p} \sum_{|\alpha|\le\ell} h^\alpha \|D^\alpha u; L^p(S_e)\| \tag{3.13}$$

which holds for $\ell \ge 1$. Combining the last three estimates we obtain the stability estimate (3.9). From this we derive for any $w \in \mathcal{P}^d_{\ell-1} \subset \mathcal{P}^d_k$

$$\begin{aligned} \|u - \mathrm{Z}_h u; L^q(e)\| &\le \|u - w; L^q(e)\| + \|\mathrm{Z}_h(u-w); L^q(e)\| \\ &\lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|\le\ell} h^\alpha \|D^\alpha(u-w); L^p(S_e)\| \end{aligned}$$

where we used the embedding $W^{\ell,p}(e) \hookrightarrow L^q(e)$. With Lemma 1 we conclude (3.10). □

By the following example we show that estimate (3.8) does *not* hold for $m \ge 1$ in the general setting of σ_i as introduced above.

Example 1. In this example we will show that (3.8) does in general not hold in the case $m = k = 1$ and the whole range of ℓ, namely $\ell = 1, 2$. Consider the situation as illustrated in Figure 4, and let $u = u(x_1)$ be any function which is independent of the variable x_2. This leads to $a_i \ne a_j$, where a_i and a_j are independent of h_2, that means

$$\left.\frac{\partial \mathrm{Z}_h u}{\partial x_2}\right|_e = h_2^{-1} f(u, x_1, h_1)$$

with a certain function f. In view of $\partial u/\partial x_2 = 0$ we obtain

$$\begin{aligned} |u - \mathrm{Z}_h u; W^{1,p}(e)| &\ge \left\| \frac{\partial \mathrm{Z}_h u}{\partial x_2}, L^p(e) \right\| = h_2^{-1+1/p} F(u, h_1), \\ \sum_{|\alpha|=\ell-1} h^\alpha |D^\alpha u; W^{1,p}(S_e)| &= h_1^{\ell-1} \left\| \frac{\partial^\ell u}{\partial x_1^\ell}, L^p(S_e) \right\| = h_2^{1/p} G(u, h_1). \end{aligned}$$

Consequently, for $f(u, x_1, h_1) \ne 0$ (which is the case in general) and $h_2 = h_1^s$ with sufficiently large s (depending on u) estimate (3.8) can not be satisfied.

For this example the following points were essential:

1. *Long edges* are chosen for σ_i.

2. X_i and X_j have the same x_1-coordinate but the projections of σ_i and σ_j on the x_1-axis are *different*.

Since we have some freedom in the choice of σ_i we will investigate in the next two sections the operator in the cases where one of these points is avoided. In Section 4 we will use short edges (2D) or small faces (3D) as σ_i. Large sides with identical projection are chosen in Section 5. The resulting operators will be denoted by S_h (small sides) and L_h (large sides).

Having now an idea which choice of σ_i could work, we want to point out that the desired error estimate cannot be obtained with the original proof of [30]. We can see this from the following two examples.

Example 2. The proof of Theorem 3 followed essentially the steps of the proof in [30]. Let us see which result we obtain for a derivative. Consider an element $e \subset \mathbb{R}^2$ of a mesh of tensor product type, a function $u \in W^{\ell,p}(S_e)$, $\ell \in \{1,2\}$, the polynomial degree $k = 1$ and a multi-index γ with $|\gamma| = 1$. Let all σ_i be defined as short edges. Then we get by following the proof of Theorem 3

$$\begin{aligned}
\|D^\gamma \mathrm{S}_h u; L^q(e)\| &\leq \sum_{i\in I_e} \|D^\gamma \varphi_i; L^q(e)\| \left| \int_{\sigma_i} u\psi_i \right| \\
&\lesssim h^{-\gamma} (\operatorname{meas} e)^{1/q} \sum_{i\in I_e} (\operatorname{meas} \sigma_i)^{-1} \|u; L^1(\sigma_i)\| \\
&\lesssim h^{-\gamma} (\operatorname{meas} e)^{1/q} \sum_{i\in I_e} (\operatorname{meas} e)^{-1/p} \sum_{|\alpha|\leq \ell} h^\alpha \|D^\alpha u; L^p(S_e)\| \\
&\sim h^{-\gamma} (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|\leq \ell} h^\alpha \|D^\alpha u; L^p(S_e)\|.
\end{aligned}$$

For estimating the error $D^\gamma(u - \mathrm{S}_h u)$ we apply this estimate to $u - w$ instead of u, with $w \in \mathcal{P}^d_{\ell-1}$. By applying Lemma 1 we get

$$\|D^\gamma(u - \mathrm{S}_h u); L^q(e)\| \lesssim h^{-\gamma} (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|=\ell} h^\alpha \|D^\alpha u; L^p(S_e)\|. \tag{3.14}$$

Let $h_1 \ll h_2$, $\gamma = (1,0)$, then one term at the right-hand side is $h_1^{-1} h_2^\ell \|D^{(0,2)} u; L^p(S_e)\|$ which may become arbitrary large. Therefore we do not obtain estimate (3.8) with the original proof, but only a sub-optimal right-hand side as in (3.14).

Example 3. Let us perform a backward analysis. Assume that (3.8) is the appropriate estimate for an element e of a mesh of tensor product type with an arbitrary $h_1 \ll h_2$. For $m = 1$, $\ell = 2$, we have in particular

$$\begin{aligned}
\|D^{(1,0)}(u - \mathrm{S}_h u); L^p(e)\| &\lesssim \sum_{|\alpha|=1} h^\alpha |D^\alpha u; W^{1,p}(S_e)| \\
&\sim \sum_{|\alpha|=1} h^\alpha \|D^{\alpha+(1,0)} u; L^p(S_e)\| + h_2 \|D^{(0,2)} u; L^p(S_e)\|.
\end{aligned} \tag{3.15}$$

Change the variables *via* $x_i = \tilde{x}_i h_i$, $i = 1,2$, to obtain an estimate for an element $\tilde{e}$ with $\operatorname{diam} \tilde{e} \sim \varrho_{\tilde{e}} \sim 1$. The estimate (3.15) transforms to

$$\begin{aligned}
h_1^{-1} \|\tilde{D}^{(1,0)}(\tilde{u} - \mathrm{S}_h \tilde{u}); L^p(\tilde{e})\| &\lesssim \sum_{|\alpha|=1} h_1^{-1} \|\tilde{D}^{\alpha+(1,0)} \tilde{u}; L^p(\tilde{S}_e)\| + h_2^{-1} \|\tilde{D}^{(0,2)} \tilde{u}; L^p(\tilde{S}_e)\|, \\
\|\tilde{D}^{(1,0)}(\tilde{u} - \mathrm{S}_h \tilde{u}); L^p(\tilde{e})\| &\lesssim \sum_{|\alpha|=1} \|\tilde{D}^{\alpha+(1,0)} \tilde{u}; L^p(\tilde{S}_e)\| + h_2^{-1} h_1 \|\tilde{D}^{(0,2)} \tilde{u}; L^p(\tilde{S}_e)\|.
\end{aligned}$$

For this estimate to be satisfied for arbitrary $h_1 = o(h_2)$ we have to show

$$\begin{aligned} \|\tilde{D}^\gamma(\tilde{u} - \mathrm{S}_h\tilde{u}); L^p(\tilde{e})\| &\lesssim |\tilde{D}^\gamma\tilde{u}; W^{1,p}(\tilde{S}_e)|, \\ \|D^\gamma(u - \mathrm{S}_h u); L^p(e)\| &\lesssim \sum_{|\alpha|=1} h^\alpha \|D^{\alpha+\gamma}u; L^p(S_e)\|, \end{aligned}$$

at least for $\gamma = (1,0)$. Otherwise the estimate is not invariant with respect to scaling. If we want to derive the error estimate by using the stability estimate as in the proof of Theorem 3, we must prove

$$\|D^\gamma \mathrm{S}_h u; L^q(e)\| \lesssim (\mathrm{meas}\, e)^{1/q-1/p} \sum_{|\alpha|\le \ell-|\gamma|} h^\alpha |D^\alpha u; W^{|\gamma|,p}(S_e)|.$$

We have seen in the examples that choosing appropriate σ_i is not enough. We need also a refined proof for obtaining anisotropic estimates for derivatives of the interpolation error. We will develop such refined proofs for general k, ℓ, m, in the next sections. However, we need in all cases that all σ_i, $i \in I$, are parallel. Therefore we are restricted to meshes of tensor product type (introduced in Def. 1 and investigated in Sects. 4 and 5) or to tensor product meshes (introduced in Def. 2 and investigated in Sect. 6). The proof for more general meshes is still open.

In the remaining part of this section we will discuss to what extent the previous results carry over to the operators C_h and O_h which were considered by Clément [19] and Oswald [28] for isotropic meshes. Recall from the Introduction that the difference between Z_h, C_h, and O_h is only in the definition of the subdomains σ_i. In particular, σ_i is d-dimensional for C_h and O_h and for all $i \in I$.

For O_h one can verify easily that all results in this section remain true, except that Dirichlet boundary conditions are not satisfied. Moreover, Condition (3.5) can even be omitted; the operator is defined for all $u \in L^1(\Omega)$. Therefore estimates (3.6, 3.7, 3.9, 3.10) hold for $\ell = 0$ as well. Example 1 can be modified in the obvious way. (Z_h has to be substituted by O_h in all relations.)

For the Clément operator C_h, one has to decide whether Π_{σ_i} should be defined as in (1.5) or (1.7). In both cases the same estimates as for O_h can be proved. Note that we used in the proof only $\mathrm{C}_h w = w$ for $w \in \mathcal{P}_k^d$ which is satisfied. As discussed already in the Introduction, $\mathrm{C}_h v_h = v_h$ is in general not satisfied for $v_h \in V_h$.

Siebert [32] and Kunert [24] derived also some results for the operator C_h for anisotropic meshes. However, they considered only the case $k = 1$, $p = 2$, and only subsets $H^1_{\mathcal{T}}(\Omega) \subset W^{1,2}(\Omega)$ of so-called mesh adapted functions. This allows them to prove global results of the form

$$\begin{aligned} \sum_e \varrho_e^{-1} \|v - \mathrm{C}_h v, L^2(e)\| &\lesssim |v; W^{1,2}(\Omega)|, \\ \sum_e h_{i,e}\varrho_e^{-1} \left\| \frac{\partial}{\partial x_i}(v - \mathrm{C}_h v)), L^2(e) \right\| &\lesssim |v; W^{1,2}(\Omega)|, \quad i = 1, \dots, d, \end{aligned}$$

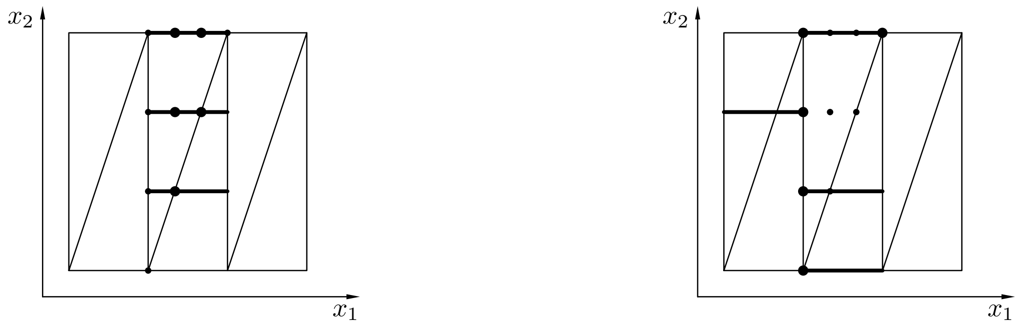
where $\varrho_e \sim \min_{j=1,\dots,d} h_{j,e}$. Using these estimates they prove asymptotic properties of *a posteriori* error estimators. For v they insert the (exact) finite element error $u - u_h$. Unfortunately, the condition $u - u_h \in H^1_{\mathcal{T}}(\Omega)$ can not be proved/tested in general.

To satisfy Dirichlet boundary conditions all the authors [19, 24, 32] considered a modification of C_h near the boundary which is small enough to keep the approximation order.

4. The operator S_h: A modification of Z_h by choosing small sides

4.1. Stability and approximation in classical Sobolev spaces

In this section we will investigate the operator S_h which was first introduced in Section 3, after Example 1. Throughout the section we assume that e is an element of tensor product type, see Definition 1 in Section 2.



(a) Points where σ_i is uniquely determined.

(b) Points where σ_i can be chosen (here one choice).

FIGURE 5. Choice of σ_i in dependence of X_i in the case of operator S_h, $k = 3$.

Since the definition of the σ_i is different from that of Z_h in Section 3 we will clarify this here: σ_i is (not necessarily uniquely) determined according to the following three properties, compare Figure 5.

(P1) σ_i is parallel to the x_1-axis/x_1, x_2-plane.
(P2) $X_i \in \overline{\sigma_i}$.
(P3) There exists an edge/face ς of some element e such that the projection of ς on the x_1-axis/x_1, x_2-plane is identical with the projection of σ_i.

In connection with (P3) we have to note that σ_i is not necessary an edge/face of one element, see also Figure 5. Nevertheless, σ_i together with $\mathcal{P}_k^{d-1}$ or $\mathcal{Q}_k^{d-1}$ is a Lagrangian finite element of dimension $d-1$, which follows from the tensor-product character of the elements e. For simplicity, we will use the terminology "σ_i is an edge/face". We remark in particular that in the case of simplicial elements and $k \geq 2$ there is no d-dimensional finite element $e' \subset S_e$ such that $\sigma_i \subset \overline{e'}$. This implies that $\mathcal{P}_{k,\sigma_i} \neq V_h|_{\sigma_i}$ and in general $\Pi_{\sigma_i} v_h \neq v_h|_{\sigma_i}$ for $v_h \in V_h$. That means that we lose Property 2 in Section 1. However, we need in the proofs only $\Pi_{\sigma_i} w = w$ for $w \in \mathcal{P}_{k,\sigma_i}$ which is of course satisfied.

Because σ_i is said to be a *small* edge/face this implies

$$h_j \leq h_d \quad \text{in } S_e \quad (j = 1, \dots, d). \tag{4.1}$$

Note that in three dimensions and according to (2.2, 2.3), only elements with $h_1 \sim h_2 \lesssim h_3$ can be treated. But this is sufficient to handle edge singularities, see Section 7.

We will see that for the operator S_h anisotropic estimates of type (m, ℓ), $m < \ell \leq k+1$, can be derived. The main difficulty is to prove the stability estimate. The approximation property follows then easily using Lemma 1 from Section 2. To elucidate the different techniques for derivatives in x_1- and x_d-direction we first formulate and prove two lemmata. Then we establish the main theorem of this section. Finally, we give an example which shows that estimates of type (m, m), $1 \leq m \leq k+1$, are impossible.

Lemma 4. *Consider an element e of a mesh of tensor product type and assume that* (4.1) *is valid. Then the derivative of $\mathrm{S}_h u$ in x_d-direction satisfies an* $(1,1)$*-estimate. The relation*

$$\left\| \frac{\partial}{\partial x_d} \mathrm{S}_h u; L^q(e) \right\| \lesssim (\operatorname{meas} e)^{1/q - 1/p} \, |u; W^{1,p}(S_e)|$$

holds for $u \in W^{1,p}(S_e)$ and all $p, q \in [1, \infty]$.

Proof. Using the definition of the operator S_h (in analogy to (3.4)), the Hölder inequality, estimate (3.11), and the trace Theorem (3.13), we obtain for all $w \in \mathcal{P}_0^d$

$$\begin{aligned}
\left\| \frac{\partial}{\partial x_d} \mathrm{S}_h u; L^q(e) \right\| &= \left\| \frac{\partial}{\partial x_d} \mathrm{S}_h (u-w); L^q(e) \right\| \leq \sum_{i \in I_e} \left\| \frac{\partial \varphi_i}{\partial x_d}; L^q(e) \right\| \left| \int_{\sigma_i} (u-w) \psi_i \right| \\
&\lesssim h_d^{-1} (\operatorname{meas} e)^{1/q} \sum_{i \in I_e} \| u-w; L^1(\sigma_i) \| \, \| \psi_i; L^\infty(\sigma_i) \| \\
&\lesssim h_d^{-1} (\operatorname{meas} e)^{1/q} \sum_{i \in I_e} (\operatorname{meas} \sigma_i) (\operatorname{meas} e)^{-1/p} \sum_{|\alpha| \leq 1} h^\alpha \| D^\alpha (u-w); L^p(S_e) \| (\operatorname{meas} \sigma_i)^{-1} \\
&\lesssim h_d^{-1} (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha| \leq 1} h^\alpha \| D^\alpha (u-w); L^p(S_e) \|.
\end{aligned}$$

Using Lemma 1 with $m = 0$, $\ell = 1$, and relying on (4.1) we obtain the assertion. ☐

Lemma 5. *Consider an element e of a mesh of tensor product type and assume that* (4.1) *is valid. Then the derivative of $\mathrm{S}_h u$ in x_1-direction satisfies an* $(1,2)$*-estimate. The relation*

$$\left\| \frac{\partial}{\partial x_1} \mathrm{S}_h u; L^q(e) \right\| \lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha| \leq 1} h^\alpha | D^\alpha u; W^{1,p}(S_e) |$$

holds for $u \in W^{2,p}(S_e)$ and all $p, q \in [1, \infty]$.

Proof. Let $w = w(x_d) \in \mathcal{P}_k^1$. Then we get in analogy to the proof of Lemma 4

$$\left\| \frac{\partial}{\partial x_1} \mathrm{S}_h u; L^q(e) \right\| \lesssim h_1^{-1} (\operatorname{meas} e)^{1/q} \sum_{i \in I_e} (\operatorname{meas} \sigma_i)^{-1} \| u-w; L^1(\sigma_i) \|.$$

Denote by σ the smallest of the domains σ_i, $i \in I_e$. Introduce now $k+1$ (simply connected) $(d-1)$-dimensional domains $\zeta_j \subset S_e$ such that for all σ_i $(i \in I_e)$ there exists a $\zeta_j \supset \sigma_i$. Note that, due to (2.5), ζ_j $(j = 0, \ldots, k)$ is isotropic with a diameter of order h_1, and therefore $\operatorname{meas} \sigma_i \sim \operatorname{meas} \zeta_j \sim \operatorname{meas} \sigma$ for all i and j. Consequently, we obtain

$$\begin{aligned}
\left\| \frac{\partial}{\partial x_1} \mathrm{S}_h u; L^q(e) \right\| &\lesssim h_1^{-1} (\operatorname{meas} e)^{1/q} (\operatorname{meas} \sigma)^{-1} \sum_{j=0}^{k} \| u-w; L^1(\zeta_j) \| \\
&\leq h_1^{-1} (\operatorname{meas} e)^{1/q} (\operatorname{meas} \sigma)^{-1} \sum_{j=0}^{k} \sum_{\substack{|\alpha| \leq 1 \\ \alpha_d = 0}} h^\alpha \| D^\alpha (u-w); L^1(\zeta_j) \|.
\end{aligned}$$

Observe now that $w = w_j = \text{const.}$ on ζ_j. On the other hand, because the ζ_j have different x_d-coordinate, we can define w from given w_j $(j = 0, \ldots, k)$. So we can use Lemma 1 for dimension $d-1$ to choose $w_j \in \mathcal{P}_0^{d-1}$ such that

$$\sum_{\substack{|\alpha| \leq 1 \\ \alpha_d = 0}} h^\alpha \| D^\alpha (u-w_j); L^1(\zeta_j) \| \lesssim \sum_{\substack{|\alpha| = 1 \\ \alpha_d = 0}} h^\alpha \| D^\alpha u; L^1(\zeta_j) \|$$

and to conclude with the trace Theorem (3.13) (applied for each ζ_j)

$$\left\| \frac{\partial}{\partial x_1} \mathrm{S}_h u; L^q(e) \right\| \lesssim (\operatorname{meas} e)^{1/q} (\operatorname{meas} \sigma)^{-1} \sum_{j=0}^{k} \sum_{\substack{|\alpha|=1 \\ \alpha_d=0}} \| D^\alpha u; L^1(\zeta_j) \| \tag{4.2}$$

$$\lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{\substack{|\alpha|=1 \\ \alpha_d=0}} \sum_{|\beta|\le 1} h^\beta \| D^{\alpha+\beta} u; L^p(S_e) \|. \tag{4.3}$$

Thus the proposition is proved. □

By analogy we can treat the derivative with respect to x_2 in the three-dimensional case.

Theorem 6. *Assume that* (4.1) *is valid. Then the modified Scott-Zhang operator* S_h *satisfies on anisotropic meshes of tensor-product type the following estimates of type* (m, ℓ)*:*

$$|\mathrm{S}_h u; W^{m,q}(e)| \lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha| \le \ell-m} h^\alpha |D^\alpha u; W^{m,p}(S_e)|, \tag{4.4}$$

$$|u - \mathrm{S}_h u; W^{m,q}(e)| \lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha| = \ell-m} h^\alpha |D^\alpha u; W^{m,p}(S_e)|, \tag{4.5}$$

$0 \le m \le \ell - 1 \le k$, *provided that* $u \in W^{\ell,p}(S_e)$. *For* (4.5) *the numbers* $p, q \in [1, \infty]$ *must be such that* $W^{\ell,p}(e) \hookrightarrow W^{m,q}(e)$. *For* $m \ge 2$ *we exclude triangular and tetrahedral elements.*

Proof. Consider first the stability estimate (4.4). For $m = 0$, (4.4) can be proved as (3.9). For $m = 1$, (4.4) is proved in Lemmata 4 and 5. Let $m \ge 2$. Consider a multi-index γ with $|\gamma| = m$ and define $m_2 := \gamma_d$, $m_1 = m - m_2$. For arbitrary $\omega_1 = \omega_{1,1}(x_1, \ldots, x_{d-1})\omega_{1,2}(x_d)$, $\omega_{1,1} \in \mathcal{P}^{d-1}_{m_1-1}$, $\omega_{1,2} \in \mathcal{P}^1_k$, (that is why we exclude simplicial elements) and $\omega_2 \in \mathcal{P}^d_{m-1}$ we obtain in analogy to the proof of Lemma 5

$$\begin{aligned}
\| D^\gamma \mathrm{S}_h u; L^q(e) \| &= \| D^\gamma \mathrm{S}_h ((u - \omega_2) - \omega_1); L^q(e) \| \\
&\lesssim h^{-\gamma} (\operatorname{meas} e)^{1/q} (\operatorname{meas} \sigma)^{-1} \sum_{i \in I_e} \| u - \omega_2 - \omega_1; L^1(\sigma_i) \| \\
&\lesssim h^{-\gamma} (\operatorname{meas} e)^{1/q} (\operatorname{meas} \sigma)^{-1} \sum_{j=0}^{k} \sum_{\substack{|\alpha| \le m_1 \\ \alpha_d=0}} h^\alpha \| D^\alpha (u - \omega_2 - \omega_1); L^1(\zeta_j) \|.
\end{aligned}$$

Then we determine $w_j \in \mathcal{P}^{d-1}_{m_1-1}$ $(j = 0, \ldots, k)$ such that

$$\sum_{\substack{|\alpha| \le m_1 \\ \alpha_d=0}} h^\alpha \| D^\alpha (u - \omega_2 - w_j); L^1(\zeta_j) \| \lesssim \sum_{\substack{|\alpha| = m_1 \\ \alpha_d=0}} h^\alpha \| D^\alpha (u - \omega_2); L^1(\zeta_j) \|.$$

Note that the w_j depend on $(u - \omega_2)$ and ω_2 is still to be chosen. The polynomial ω_1 is now determined by the w_j $(j = 0, \ldots, k)$ such that the estimate can be continued by

$$\| D^\gamma \mathrm{S}_h u; L^q(e) \| \lesssim h_d^{-m_2} (\operatorname{meas} e)^{1/q} (\operatorname{meas} \sigma)^{-1} \sum_{j=0}^{k} \sum_{\substack{|\alpha| = m_1 \\ \alpha_d=0}} \| D^\alpha (u - \omega_2); L^1(\zeta_j) \|. \tag{4.6}$$

Thus the factor $h_1^{-m_1}$ is eliminated. We proceed now as in the proof of Lemma 4. Using the trace Theorem (3.13) for all j, α and with $\ell - m_1 \geq \ell - m \geq 1$ instead of ℓ we conclude

$$\begin{aligned}\|D^\gamma \mathrm{S}_h u; L^q(e)\| &\lesssim h_d^{-m_2} (\mathrm{meas}\, e)^{1/q-1/p} \sum_{\substack{|\alpha|=m_1\\ \alpha_d=0}} \sum_{|\beta|\leq \ell-m_1} h^\beta \|D^{\alpha+\beta}(u-\omega_2); L^p(S_e)\| \\ &\lesssim h_d^{-m_2} (\mathrm{meas}\, e)^{1/q-1/p} \sum_{|\delta|\leq \ell-m} \sum_{|\beta|\leq m_2} h^{\beta+\delta} |D^{\beta+\delta}(u-\omega_2); W^{m_1,p}(S_e)|.\end{aligned}$$

Using Corollary 2 (Section 2) we obtain

$$\begin{aligned}\|D^\gamma \mathrm{S}_h u; L^q(e)\| &\lesssim h_d^{-m_2} (\mathrm{meas}\, e)^{1/q-1/p} \sum_{|\delta|\leq \ell-m} \sum_{|\beta|=m_2} h^{\beta+\delta} |D^{\beta+\delta} u; W^{m_1,p}(S_e)| \\ &\lesssim (\mathrm{meas}\, e)^{1/q-1/p} \sum_{|\delta|\leq \ell-m} h^\delta |D^\delta u; W^{m,p}(S_e)|.\end{aligned}$$

Here we used $h^\beta \leq h_d^{m_2}$ for $|\beta| = m_2$ which follows from (4.1). Thus (4.4) is proved.

For proving estimate (4.5) we need (4.4) and the assumptions on p and q. Since these parameters were chosen such that $W^{\ell,p}(e) \hookrightarrow W^{m,q}(e)$, we have also $W^{\ell-m,p}(e) \hookrightarrow L^q(e)$, this means

$$\|v; L^q(e)\| \lesssim (\mathrm{meas}\, e)^{1/q-1/p} \sum_{|\alpha|\leq \ell-m} h^\alpha \|D^\alpha v; L^p(e)\|$$

for all $v \in W^{\ell-m,p}(e)$. Applying this estimate for all derivatives D^α with $|\alpha| = m$ and summing up the resulting inequalities, we obtain for $v \in W^{\ell,p}(e)$

$$|v; W^{m,q}(e)| \lesssim (\mathrm{meas}\, e)^{1/q-1/p} \sum_{|\alpha|\leq \ell-m} h^\alpha |D^\alpha v; W^{m,p}(e)|. \tag{4.7}$$

Together with (4.4) we conclude that for all $w \in \mathcal{P}_{\ell-1}^d$ the following estimate holds,

$$\begin{aligned}|u - \mathrm{S}_h u; W^{m,q}(e)| &\leq |u - w; W^{m,q}(e)| + |\mathrm{S}_h(u-w); W^{m,q}(e)| \\ &\lesssim (\mathrm{meas}\, e)^{1/q-1/p} \sum_{|\alpha|\leq \ell-m} h^\alpha |D^\alpha (u-w); W^{m,p}(S_e)|.\end{aligned}$$

With Lemma 1 the proposition is proved. □

Finally, we want to give an example which shows that

$$|\mathrm{S}_h u; W^{1,2}(e)| \lesssim \|u; W^{1,2}(S_e)\| \tag{4.8}$$

does not hold for general $u \in W^{1,2}(S_e)$.

Example 4. Consider $k = 1$ and a triangle with the vertices $X_1 = (0, 0)$, $X_2 = (h, 0)$, and $X_3 = (0, 1)$, and let $\sigma_1 = (-h, 0) \times \{0\}$, $\sigma_2 = (0, h) \times \{0\}$, compare Figure 6. For $u = r^\varepsilon \sin(\theta/2)$ (r, θ are here polar coordinates) we obtain

$$\begin{aligned}u|_{\sigma_1} = |x_1|^\varepsilon \quad &\Rightarrow \quad (\Pi_{\sigma_1} u)(X_1) = \int_0^h x^\varepsilon \left(-\frac{6x}{h^2} + \frac{4}{h}\right) \sim h^\varepsilon, \\ u|_{\sigma_2} = 0 \quad &\Rightarrow \quad (\Pi_{\sigma_2} u)(X_2) = 0.\end{aligned}$$

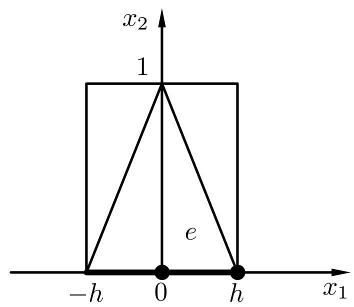


FIGURE 6. Illustration of Example 4.

Consequently,

$$\frac{\partial \mathrm{S}_h u}{\partial x_1} \sim h^{\varepsilon-1}, \qquad |\mathrm{S}_h u; W^{1,2}(e)| \gtrsim h^{\varepsilon-1}(\operatorname{meas} e)^{1/2} = h^{\varepsilon-1/2} \to \infty$$

for $h \to 0$, $\varepsilon < 1/2$. But

$$|u; W^{1,2}(S_e)|^2 \lesssim \int_0^1 \int_0^\pi \left(r^{\varepsilon-1} \sin\frac{\theta}{2} \right)^2 r \mathrm{d}\theta \mathrm{d}r \sim \int_0^1 r^{2(\varepsilon-1)+1}\, \mathrm{d}r < \infty$$

for $\varepsilon > 0$. Thus (4.8) does not hold.

4.2. **Stability in weighted Sobolev spaces**

We have seen in Example 4 that $\mathrm{S}_h u$ does not satisfy an estimate of type $(1,1)$. However, S_h can be applied in some situations where $u \notin W^{2,p}(S_e)$ for some p we are interested in.

We restrict ourselves to the three-dimensional case, consider an arbitrary bounded domain $G \subset \mathbb{R}^3$ with zero distance to the x_3-axis (the x_3-axis may intersect G but this is not typical), and introduce cylindrical coordinates *via* $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. Define for $\ell \in \mathbb{N}_0$, $p \in [1,\infty]$, $\beta \in \mathbb{R}$, the weighted Sobolev space

$$V_\beta^{\ell,p}(G) \quad := \quad \{v \in \mathcal{D}'(G) : \|v; V_\beta^{\ell,p}(G)\| < \infty\}, \tag{4.9}$$

$$\|v; V_\beta^{\ell,p}(G)\|^p \quad := \quad \sum_{|\alpha|\le \ell} \int_G |r^{\beta-\ell+|\alpha|} D^\alpha v|^p. \tag{4.10}$$

Such spaces are relevant in the treatment of singular functions of the type $v = r^\lambda \sin\lambda\theta$ or $v = r^\lambda \cos\lambda\theta$, $\lambda \in (0,1)$. Notice that

$$\begin{aligned} &v \in W^{s,2}(G) && \iff \quad s < 1+\lambda, \\ &v \in V_\beta^{s,2}(G) \quad \forall s \ge 0 && \iff \quad \beta > s-1-\lambda. \end{aligned}$$

For our application in Section 7 we need the stability of the modified Scott-Zhang operator in these weighted spaces.

Lemma 7. *Consider an element e of a mesh of tensor product type and assume that* (4.1) *is valid. Let m be an integer and β, p, q be real numbers with $0 \le m \le k$, $\beta < 2 - 2/p$, $\beta \le 1$, $p, q \in [1,\infty]$, and assume that the x_3-axis proceeds through S_e. Then for $u \in W^{m,p}(S_e) \cap V_\beta^{m+1,p}(S_e)$ the stability estimate*

$$|\mathrm{S}_h u; W^{m,q}(e)| \lesssim (\operatorname{meas} e)^{1/q-1/p} h_1^{-\beta} \sum_{|\alpha|=m-1} \sum_{|t|=1} h^t \|D^{\alpha+t} u; V_\beta^{1,p}(S_e)\| \tag{4.11}$$

holds. For $m \geq 2$ we exclude tetrahedral elements.

Proof. We start with estimate (4.6) which was obtained in the proof of Theorem 6. Let γ be a multi-index with $|\gamma| = m$, $m_1 = m - \gamma_3$, and $\omega_2 \in \mathcal{P}^d_{m-1}$. Then there holds

$$\|D^\gamma \mathrm{S}_h u; L^q(e)\| \lesssim h_3^{-\gamma_3} (\operatorname{meas} e)^{1/q} (\operatorname{meas} \sigma)^{-1} \sum_{j=0}^{k} \sum_{\substack{|\alpha|=m-\gamma_3 \\ \alpha_3=0}} \|D^\alpha (u - \omega_2); L^1(\zeta_j)\|. \tag{4.12}$$

Let $\gamma_3 > 0$, then we can continue, similar to the proof of Theorem 6, with the trace theorem because we assumed $u \in W^{m,p}(S_e)$.

$$\|D^\gamma \mathrm{S}_h u; L^q(e)\| \lesssim h_3^{-\gamma_3} (\operatorname{meas} e)^{1/q-1/p} \sum_{\substack{|\alpha|=m-\gamma_3 \\ \alpha_3=0}} \sum_{|\delta|\leq\gamma_3} h^\delta \|D^{\alpha+\delta}(u - \omega_2); L^p(S_e)\|.$$

Using Corollary 2 we obtain

$$\begin{aligned} \|D^\gamma \mathrm{S}_h u; L^q(e)\| &\lesssim h_3^{-\gamma_3} (\operatorname{meas} e)^{1/q-1/p} \sum_{\substack{|\alpha|=m-\gamma_3 \\ \alpha_3=0}} \sum_{|\delta|=\gamma_3} h^\delta \|D^{\alpha+\delta} u; L^p(S_e)\| \\ &\lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|=m} \|D^\alpha u; L^p(S_e)\|. \end{aligned} \tag{4.13}$$

We estimate the right-hand side *via* the trivial embeddings $V^{1,p}_\beta(S_e) \hookrightarrow V^{0,p}_{\beta-1}(S_e) \hookrightarrow L^p(S_e)$, $\beta \leq 1$, which leads with (4.1) to

$$\begin{aligned} \sum_{|\alpha|=m} \|D^\alpha u; L^p(S_e)\| &\sim \sum_{|\alpha|=m-1} \sum_{|t|=1} \|D^{\alpha+t} u; L^p(S_e)\| \\ &\lesssim h_1^{-\beta+1} \sum_{|\alpha|=m-1} \sum_{|t|=1} \|r^{\beta-1} D^{\alpha+t} u; L^p(S_e)\| \\ &\lesssim h_1^{-\beta} \sum_{|\alpha|=m-1} \sum_{|t|=1} h^t \|D^{\alpha+t} u; V^{1,p}_\beta(S_e)\|, \end{aligned} \tag{4.14}$$

which is the desired result.

For $\gamma_3 = 0$ we use (4.12) with $\omega_2 = 0$ and estimate the $L^1(\zeta_j)$-norms against weighted norms *via* the Hölder inequality:

$$\|v; L^1(\zeta_j)\| \leq \|r^{-\beta}; L^{p'}(\zeta_j)\| \cdot \|r^\beta v; L^p(\zeta_j)\| \tag{4.15}$$

with p' from $1/p + 1/p' = 1$. The $L^{p'}(\zeta_j)$-norm of $r^{-\beta}$ is finite if and only if $p'\beta < 2$ which is equivalent to $\beta < 2 - 2/p$. Using $\operatorname{meas} \sigma \sim \operatorname{meas} \zeta_j \sim h_1^2$ for all j, and $r \lesssim h_1$ we get

$$\|r^{-\beta}; L^{p'}(\zeta_j)\| \lesssim h_1^{(-\beta p'+2)/p'} \sim (\operatorname{meas} \sigma)^{1-1/p} h_1^{-\beta}. \tag{4.16}$$

The application of $W^{1,p}(S_e) \hookrightarrow L^p(\zeta_j)$ to $r^\beta v$ implies the trace theorem $V^{1,p}_\beta(S_e) \hookrightarrow V^{0,p}_\beta(\zeta_j)$ which leads to

$$\|r^\beta v; L^p(\zeta_j)\| \lesssim (\operatorname{meas} \sigma)^{1/p} (\operatorname{meas} e)^{-1/p} \sum_{|s|\leq 1} h_1^{1-|s|} h^s \|r^{\beta-1+|s|} D^s v; L^p(S_e)\|.$$

Combining these estimates we obtain

$$\|v; L^1(\zeta_j)\| \le \operatorname{meas}\sigma\,(\operatorname{meas}e)^{-1/p} h_1^{-\beta} \sum_{|s|\le 1} h_1^{1-|s|} h^s \|r^{\beta-1+|s|} D^s v; L^p(S_e)\|$$

and thus with (4.12)

$$\begin{aligned} \|D^\gamma \mathrm{S}_h u; L^q(e)\| &\lesssim (\operatorname{meas}e)^{1/q} (\operatorname{meas}\sigma)^{-1} \sum_{j=0}^{k} \sum_{|\alpha|=m} \|D^\alpha u; L^1(\zeta_j)\| \\ &\lesssim (\operatorname{meas}e)^{1/q-1/p} h_1^{-\beta} \sum_{|\alpha|=m} \sum_{|s|\le 1} h_1^{1-|s|} h^s \|r^{\beta-1+|s|} D^{\alpha+s} u; L^p(S_e)\|. \end{aligned} \tag{4.17}$$

The last step to derive (4.11) is done by a rearrangement of the terms at the right-hand side, namely

$$\begin{aligned} \sum_{|t|=1} \sum_{|s|\le 1} h_1^{1-|s|} h^s \|r^{\beta-1+|s|} D^{t+s} u; L^p(S_e)\| &= \sum_{|t|=1} \sum_{|s|=1} h^s \|r^\beta D^{t+s} u; L^p(S_e)\| + \sum_{|t|=1} h_1 \|r^{\beta-1} D^t u; L^p(S_e)\| \\ &\lesssim \sum_{|t|=1} \sum_{|s|=1} h^s \|r^\beta D^{t+s} u; L^p(S_e)\| + \sum_{|s|=1} h^s \|r^{\beta-1} D^s u; L^p(S_e)\| \\ &\sim \sum_{|s|=1} h^s \|D^s u; V_\beta^{1,p}(S_e)\|. \end{aligned}$$

Together with (4.17) we conclude (4.11) in the case $\gamma_3 = 0$. □

5. The operator L_h: A modification of Z_h by choosing large sides with a projection property

In contrast to Section 4 we will now employ large edges/faces and investigate the resulting operator L_h. We still assume that e is an element of tensor product type, see Definition 1 in Section 2. The notation is used as follows: We keep Properties (P1, P2, P3) from Section 4 and simply turn the relation (4.1):

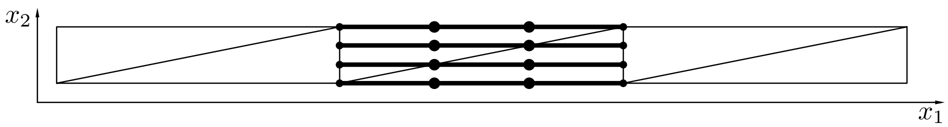
$$h_j \ge h_d \quad \text{in } S_e \quad (j = 1, \dots, d). \tag{5.1}$$

But due to the conclusions of Example 1 in Section 3, we do not have so much freedom for the choice of the σ_i as in the case of S_h. We must assume the following projection property (P4), compare also Figure 7.

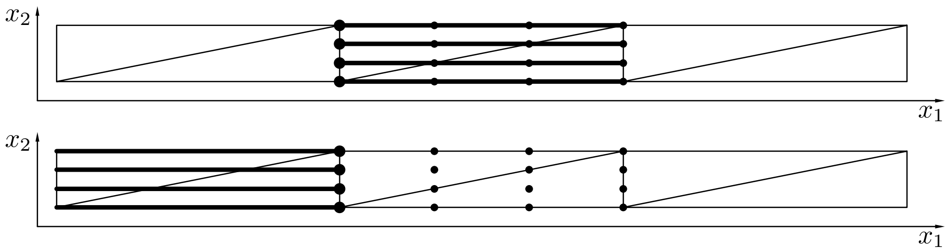
(P4) If the projections of any two points X_i and X_j on the x_1-axis/x_1, x_2-plane coincide then so do the projections of σ_i and σ_j.

We can prove the results of Theorem 6 for this case as well. Moreover, these results extend to the case $m = \ell$. But in contrast to the needle elements of Section 4 the three-dimensional elements are now flat, $h_1 \sim h_2 \gtrsim h_3$. The idea for this choice of σ_i was found in Chapter 5 of [15] where the special case of rectangular and brick elements was considered for $k = 1$, $p = q = 2$. We extend this theory to more element types and to general $k \in \mathbb{N}$, $p, q \in [1, \infty]$. Our proof differs from that in [15].

We start as in Section 4 with the separate consideration of the stability of first derivatives of $\mathrm{L}_h u$. This time the derivative in x_1-direction is the simpler one.



(a) Points where σ_i is uniquely determined.



(b) Two choices for σ_i for points on vertical mesh lines.

FIGURE 7. Choice of σ_i in dependence of X_i in the case of operator L_h.

Lemma 8. *Consider an element e of a mesh of tensor product type and assume that* (5.1) *is valid. Then the estimate of type* $(1,1)$

$$\left\| \frac{\partial}{\partial x_n} \mathrm{L}_h u; L^q(e) \right\| \lesssim (\operatorname{meas} e)^{1/q-1/p} |u; W^{1,p}(S_e)|, \quad n = 1, \dots, d \tag{5.2}$$

holds for $u \in W^{1,p}(S_e)$ and all $p, q \in [1, \infty]$.

Proof. For $n = 1, \dots, d-1$ the proof can be carried out with the same arguments as the proof of Lemma 4. The only difference is that the role of x_d and h_d is now played by x_n and h_n.

For the case $n = d$ we will reformulate $\mathrm{L}_h u$. For this consider first a one-dimensional situation, that means a single finite element formed by an interval (ξ, η). Let ϕ_i, $i = 0, \dots, k$, be the nodal basis functions in (ξ, η). We change now to a new basis

$$\chi_i = \sum_{j=0}^{i} \phi_j, \quad i = 0, \dots, k.$$

Consequently,

$$\sum_{i=0}^{k} a_i \phi_i = \sum_{i=0}^{k-1} (a_i - a_{i+1}) \chi_i + a_k,$$

where we also used that $\sum_{i=0}^{k} \phi_i = 1$. Note further that

$$\|\chi_i; L^\infty(\xi, \eta)\| \lesssim 1, \quad \|\chi_i'; L^\infty(\xi, \eta)\| \lesssim |\eta - \xi|^{-1}. \tag{5.3}$$

We use this kind of a new basis in the case of a rectangular element $e = (\xi_1, \eta_1) \times (\xi_2, \eta_2)$. The nodal basis functions are (for simplicity with a double index)

$$\varphi_{i,j}(x_1, x_2) = \phi^i(x_1) \phi_j(x_2), \quad i, j = 0, \dots, k, \tag{5.4}$$

where ϕ^i and ϕ_j are the nodal basis functions with respect to (ξ_1, η_1) and (ξ_2, η_2), respectively. Thus

$$\begin{aligned} \mathrm{L}_h u &= \sum_{i=0}^{k}\sum_{j=0}^{k} a_{i,j}\phi^i(x_1)\phi_j(x_2) \\ &= \sum_{i=0}^{k}\phi^i(x_1)\left(\sum_{j=0}^{k-1}(a_{i,j}-a_{i,j+1})\chi_j(x_2)+a_{i,k}\right), \\ \frac{\partial}{\partial x_2}\mathrm{L}_h u &= \sum_{i=0}^{k}\phi^i(x_1)\sum_{j=0}^{k-1}(a_{i,j}-a_{i,j+1})\chi_j'(x_2). \end{aligned} \tag{5.5}$$

Because of Property (P4) the subdomains $\sigma_{i,j}$ belonging to the node (i,j) depend only on i. We can write

$$\begin{aligned} a_{i,j} &= \int_{\sigma_{i,j}}\psi_i(x_1)u(x_1,y_j)\,\mathrm{d}x_1, \\ a_{i,j}-a_{i,j+1} &= -\int_{\sigma_{i,j}}\psi_i(x_1)\int_{y_j}^{y_j+1}\frac{\partial u}{\partial x_2}(x_1,y)\,\mathrm{d}y\mathrm{d}x_1, \\ \sum_{j=0}^{k-1}|a_{i,j}-a_{i,j+1}| &\le \int_{S_e}\left|\psi_i\frac{\partial u}{\partial x_2}\right|, \end{aligned} \tag{5.6}$$

where y_j is the value of the x_2-coordinate of points $X_{i,j}$. The proof of (5.2) is now standard:

$$\begin{aligned} \left\|\frac{\partial}{\partial x_d}\mathrm{L}_h u; L^q(e)\right\| &\lesssim \sum_{i=0}^{k}\sum_{j=0}^{k-1}|a_{i,j}-a_{i,j+1}|\cdot\|\phi^i(x_1)\chi_j'(x_2); L^q(e)\| \\ &\lesssim h_2^{-1}(\operatorname{meas} e)^{1/q}\sum_{i=0}^{k}\int_{S_e}\left|\psi_i\frac{\partial u}{\partial x_2}\right| \\ &\lesssim h_2^{-1}(\operatorname{meas} e)^{1/q+1-1/p}\sum_{i=0}^{k}(\operatorname{meas}\sigma_i)^{-1}\left\|\frac{\partial u}{\partial x_2}; L^p(S_e)\right\|. \end{aligned}$$

For pentahedral and hexahedral elements the proof is similar. We only replace (5.4) by

$$\varphi_{i,j}(x_1,x_2,x_3) = \phi^i(x_1,x_2)\phi_j(x_3), \quad i=0,\dots,K,\ j=0,\dots,k,$$

with appropriate basis functions $\phi^i(x_1,x_2)$ and

$$K=(k+1)^2-1 \quad \text{for hexahedra,} \quad K=\binom{k+2}{2}-1 \quad \text{for pentahedra.} \tag{5.7}$$

In the case of simplicial elements we have to modify these considerations slightly. We will explain it in the two-dimensional case. Consider an element e with nodes $X_{i,j}$,

$$\begin{aligned} e &= \left\{(x_1,x_2): \xi_1\le x_1\le\eta_1,\ \xi_2\le x_2\le\eta_2-(x_1-\xi_1)\frac{\eta_2-\xi_2}{\eta_1-\xi_1}\right\}, \\ X_{i,j} &= \left(\xi_1+\frac{i}{k}(\eta_1-\xi_1), \xi_2+\frac{j}{k}(\eta_2-\xi_2)\right), \end{aligned}$$

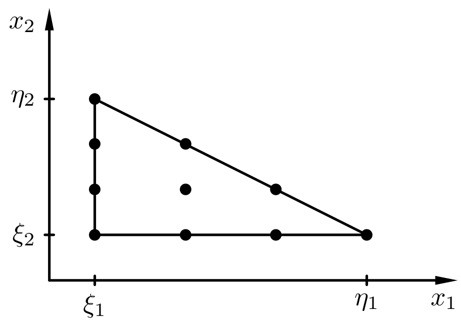


FIGURE 8. Illustration of the case of a triangle.

and nodal basis functions $\varphi_{i,j}$, $i = 0, \ldots, k$, $j = 0, \ldots, k-i$, as illustrated in Figure 8. The new basis functions are

$$\chi_{i,j} = \sum_{s=0}^{j} \varphi_{i,s}, \quad i = 0, \ldots, k, \ j = 0, \ldots, k-i.$$

We get

$$\begin{aligned} \mathrm{L}_h u &= \sum_{i=0}^{k} \sum_{j=0}^{k-i} a_{i,j}\varphi_{i,j} = \sum_{i=0}^{k} \left(\sum_{j=0}^{k-i-1} (a_{i,j} - a_{i,j+1})\chi_{i,j} + a_{i,k-i}\chi_{i,k-i} \right), \\ \left\| \frac{\partial \mathrm{L}_h u}{\partial x_2}; L^q(e) \right\| &\lesssim \sum_{i=0}^{k} \left(\sum_{j=0}^{k-i-1} |a_{i,j} - a_{i,j+1}| \left\| \frac{\partial \chi_{i,j}}{\partial x_2}; L^q(e) \right\| + |a_{i,k-i}| \left\| \frac{\partial \chi_{i,k-i}}{\partial x_2}; L^q(e) \right\| \right). \end{aligned}$$

To conclude (5.2) with the same arguments as above it remains to show that

$$\frac{\partial \chi_{i,k-i}}{\partial x_2} = 0 \quad \text{for all } i = 0, \ldots, k. \tag{5.8}$$

For this we observe that $\chi_{i,k-i}$ is uniquely determined by

$$\chi_{i,k-i}(X_{s,j}) = \begin{cases} 1 & \text{for } s = i, \ j = 0, \ldots, k-i, \\ 0 & \text{else.} \end{cases}$$

Thus $\chi_{i,k-i} = \phi^i(x_1)$ with ϕ^i in the sense of (5.4), and (5.8) is proved.

The proof for tetrahedral elements is analogous. □

Theorem 9. *Assume that* (5.1) *is valid. On anisotropic meshes of tensor-product type the modified Scott-Zhang operator* L_h *satisfies the following estimates:*

$$|\mathrm{L}_h u; W^{m,q}(e)| \lesssim (\operatorname{meas} e)^{1/q-1/p} |u; W^{m,p}(S_e)|, \tag{5.9}$$

$$|u - \mathrm{L}_h u; W^{m,q}(e)| \lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|=\ell-m} h^\alpha |D^\alpha u; W^{m,p}(S_e)|, \tag{5.10}$$

$0 \le m \le \ell$, $1 \le \ell \le k+1$, *provided that* $u \in W^{\ell,p}(S_e)$*. For* (5.10) *the numbers* $p, q \in [1, \infty]$ *must be such that* $W^{\ell,p}(e) \hookrightarrow W^{m,q}(e)$.

Proof. Estimate (5.10) follows from (5.9) *via* Lemma 1 as it was done for S_h in the proof of Theorem 6. So the main point is to prove (5.9). For $m = 0$, this can be done as in the proof of (3.9). The case $m = 1$ is treated in Lemma 8.

Let $m \geq 2$. Consider a multi-index γ with $|\gamma| = m$ and define $m_2 := \gamma_d$, $m_1 := m - m_2$. In the proof of Lemma 8, we made for the case $m_2 = 1$ a transformation of the nodal basis $\varphi_{i,j}$ to a basis $\chi_{i,j}$ in order to obtain differences of first order:

$$\frac{\partial}{\partial x_d} \sum_{i=0}^{K} \sum_{j=0}^{k} a_{i,j} \varphi_{i,j} = \frac{\partial}{\partial x_d} \sum_{i=0}^{K} \sum_{j=0}^{k-1} (a_{i,j} - a_{i,j+1}) \chi_{i,j}.$$

This process is repeated until differences of order m_2 are created: For simplicity consider again the one-dimensional situation. We define recursively coefficients $a_i^{(n)}$ and functions $\chi_i^{(n)}$, $i = 0, \ldots, k-n$, $n = 0, \ldots, m_2$, by

$$a_i^0 := a_i, \quad a_i^{(n+1)} := a_i^{(n)} - a_{i+1}^{(n)}, \quad i = 0, \ldots, k-n,$$
$$\chi_i^0 := \varphi_i, \quad \chi_i^{(n+1)} := \sum_{s=0}^{i} \chi_s^{(n)}, \qquad i = 0, \ldots, k,$$

and obtain

$$\frac{\partial^{m_2}}{\partial x^{m_2}} \sum_{i=0}^{k} a_i \varphi_i = \frac{\partial^{m_2}}{\partial x^{m_2}} \sum_{i=0}^{k-m_2} a_i^{(m_2)} \chi_i^{(m_2)}. \tag{5.11}$$

We get this by induction in analogy to the proof of Lemma 8. The only point is to prove that

$$\frac{\partial^{n+1}}{\partial x^{n+1}} \chi_{k-n}^{(n+1)} = 0 \quad \text{for } n = 0, \ldots, m_2 - 1.$$

This can be shown for any fixed n *via* $\chi_i^{(n+1)} = \sum_{s=0}^{i} \binom{i-s+n}{n} \chi_s^{(0)}$ (proof by induction) which yields $\chi_k^{(n+1)} = \sum_{s=0}^{k} \binom{k-s+n}{n} \varphi_s$, $\chi_k^{(n+1)}(X_r) = \binom{k-r+n}{n}$, $r = 0, \ldots, k$, $\chi_k^{(n+1)} \in \mathcal{P}_n^1$. From $\chi_i^{(n+1)} = \chi_{i+1}^{(n+1)} - \chi_{i+1}^{(n)}$ this gives by induction $\chi_i^{(n+1)} \in \mathcal{P}_n^1$ for $i = k, k-1, \ldots, k-n$. Thus

$$\frac{\partial^{n+1}}{\partial x^{n+1}} \chi_i^{(n+1)} = 0 \text{ for } i = k-n, \ldots, k.$$

Consider now rectangular elements ($d = 2$) and transfer this basis transformation to the x_2-direction. We derive (again by induction) from (5.11)

$$\frac{\partial^{m_2}}{\partial x_d^{m_2}} \sum_{i=0}^{k} \sum_{j=0}^{k} a_{i,j} \varphi_{i,j} = \frac{\partial^{m_2}}{\partial x_d^{m_2}} \sum_{i=0}^{k} \sum_{j=0}^{k-m_2} a_{i,j}^{(m_2)} \chi_{i,j}^{(m_2)}. \tag{5.12}$$

The so created differences $a_{i,j}^{(n+1)} = a_{i,j}^{(n)} - a_{i,j+1}^{(n)}$ are used now to establish an integral representation; compare (5.6):

$$a_{i,j}^{(1)} = - \int_{\sigma_{i,j}} \psi_i(x_1) \int_0^{\delta} \frac{\partial u}{\partial x_d}(x_1, y_j + \eta_1) \, d\eta_1 dx_1,$$

$\delta = y_{j+1} - y_j$ is assumed to be independent of j. We continue recursively and obtain

$$\begin{aligned} a_{i,j}^{(2)} &= -\int_{\sigma_{i,j}} \psi_i(x_1) \left[\int_0^\delta \frac{\partial u}{\partial x_d}(x_1, y_j + \eta_1)\,\mathrm{d}\eta_1 - \int_0^\delta \frac{\partial u}{\partial x_d}(x_1, y_{j+1} + \eta_1)\,\mathrm{d}\eta_1 \right] \mathrm{d}x_1 \\ &= (-1)^2 \int_{\sigma_{i,j}} \psi_i(x_1) \int_0^\delta \int_0^\delta \frac{\partial^2 u}{\partial x_d^2}(x_1, y_j + \eta_1 + \eta_2)\,\mathrm{d}\eta_1 \mathrm{d}\eta_2 \mathrm{d}x_1, \\ a_{i,j}^{(n)} &= (-1)^n \int_{\sigma_{i,j}} \psi_i(x_1) \underbrace{\int_0^\delta \cdots \int_0^\delta}_{n \text{ times}} \frac{\partial^n u}{\partial x_d^n}(x_1, y_j + \eta_1 + \cdots + \eta_n)\,\mathrm{d}\eta_1 \cdots \mathrm{d}\eta_n \mathrm{d}x_1. \end{aligned}$$

Using (3.11) and $\delta \sim h_2$ we obtain

$$|a_{i,j}^{(n)}| \lesssim (\operatorname{meas} \sigma_{i,j})^{-1} h_d^{n-1} \left\| \frac{\partial^n u}{\partial x_d^n}; L^1(S_e) \right\|.$$

Replace now $\operatorname{meas} \sigma_{i,j}$ by $\operatorname{meas} \sigma := \min_{i,j} \operatorname{meas} \sigma_{i,j}$ and u by $u - w$, $w \in \mathcal{P}_{m-1}^2$ arbitrary. Together with (5.12) we conclude that

$$\begin{aligned} \|D^\gamma \mathrm{L}_h u; L^q(e)\| &= \|D^\gamma \mathrm{L}_h (u - w); L^q(e)\| \\ &\lesssim \sum_{i=0}^{k} \sum_{j=0}^{k-m_2} |a_{i,j}^{(m_2)}| \|D^\gamma \chi_{i,j}^{(m_2)}; L^q(e)\| \\ &\lesssim h^{-\gamma} (\operatorname{meas} e)^{1/q} \sum_{i=0}^{k} \sum_{j=0}^{k-m_2} |a_{i,j}^{(m_2)}| \\ &\lesssim h^{-\gamma} (\operatorname{meas} e)^{1/q} (\operatorname{meas} \sigma)^{-1} h_d^{m_2 - 1} \left\| \frac{\partial^{m_2}}{\partial x_d^{m_2}}(u - w); L^1(S_e) \right\| \\ &\lesssim h^{-\gamma} h_d^{m_2} (\operatorname{meas} e)^{1/q - 1/p} \left\| \frac{\partial^{m_2}}{\partial x_d^{m_2}}(u - w); L^p(S_e) \right\| \qquad (5.13) \\ &\lesssim h_1^{-m_1} (\operatorname{meas} e)^{1/q - 1/p} \sum_{|\alpha| \le m - m_2} h^\alpha \left\| D^\alpha \frac{\partial^{m_2}}{\partial x_d^{m_2}}(u - w); L^p(S_e) \right\|. \end{aligned}$$

In order to derive (5.13) we have used that $h_d \operatorname{meas} \sigma \sim \operatorname{meas} e$. *Via* Corollary 2, (5.1), and $m = m_1 + m_2$ we obtain

$$\begin{aligned} \|D^\gamma \mathrm{L}_h u; L^q(e)\| &\lesssim h_1^{-m_1} (\operatorname{meas} e)^{1/q - 1/p} \sum_{|\alpha| = m - m_2} h^\alpha \left\| D^\alpha \frac{\partial^{m_2} u}{\partial x_d^{m_2}}; L^p(S_e) \right\| \\ &\le (\operatorname{meas} e)^{1/q - 1/p} \sum_{|\alpha| = m - m_2} \left\| D^\alpha \frac{\partial^{m_2} u}{\partial x_d^{m_2}}; L^p(S_e) \right\| \\ &\le (\operatorname{meas} e)^{1/q - 1/p} |u; W^{m,p}(S_e)| \end{aligned}$$

and (5.9) is proved for rectangular elements. The proof for all other types of elements is similar using the ideas explained in the proof of Lemma 8. ☐

6. The operator E_h: Choosing long edges in the three-dimensional case

6.1. Stability and approximation in classical Sobolev spaces

In Sections 4 and 5 we assumed $h_1 \sim h_2$ in the three-dimensional case. We will now investigate the general three-dimensional situation of independent mesh sizes h_1, h_2, and h_3. But for simplicity, we treat only *tensor product meshes* in the sense of Definition 2 in Section 2. In order to obtain in Subsection 6.2 a notation which is compatible with that in Subsection 4.2 we let

$$h_1 \le h_2 \le h_3. \tag{6.1}$$

The investigation of the operators S_h and L_h was based on taking σ_i as isotropic faces, that means that h_2 is of the same order as h_1 or h_3. In [15] it was suggested to overcome this restriction by taking *one-dimensional* σ_i but this was not elaborated thoroughly. We will now investigate which estimates can be obtained in this case. We assume the following properties which are analogous to the ones in Section 5.

- (P1′) σ_i is parallel to the x_3-axis.
- (P2) $X_i \in \overline{\sigma_i}$.
- (P3′) There exists an edge ς of some element e such that the projection of ς on the x_3-axis is identical with the projection of σ_i.
- (P4′) If the projections of any two points X_i and X_j on the x_3-axis coincide then so do the projections of σ_i and σ_j.

The corresponding operator is denoted by $\mathrm{E}_h : W^{\ell,p}(\Omega) \to V_h$. Note that it is defined only for $u \in W^{\ell,p}(\Omega)$ with

$$\ell \ge 2 \quad \text{for } p = 1, \qquad \ell > \frac{2}{p} \quad \text{otherwise}, \tag{6.2}$$

to guarantee that $u|_{\sigma_i} \in L^1(\sigma_i)$. Condition (6.2) can be reformulated to

$$\ell \ge 2,\ p \in [1,\infty] \quad \text{or} \quad \ell = 1,\ p \in (2,\infty]. \tag{6.3}$$

Theorem 10. *Consider an element e of a tensor product mesh and assume that* (6.1) *and* (2.4) *are fulfilled. Then the operator E_h satisfies for all $q \in [1,\infty]$ the following estimates:*

$$|\mathrm{E}_h u; W^{m,q}(e)| \lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|\le 1} h^\alpha |D^\alpha u; W^{m,p}(S_e)| \tag{6.4}$$

if $m \ge 1$ or $p > 2$, and

$$\|\mathrm{E}_h u; L^q(e)\| \lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|\le \ell} h^\alpha \|D^\alpha u; L^p(S_e)\| \tag{6.5}$$

with ℓ and p satisfying (6.3). The approximation error estimate

$$|u - \mathrm{E}_h u; W^{m,q}(e)| \lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|=\ell-m} h^\alpha |D^\alpha u; W^{m,p}(S_e)| \tag{6.6}$$

holds if $0 \le m \le \ell - 1 \le k$, *$p$ satisfies* (6.3), *q is such that $W^{\ell,p}(e) \hookrightarrow W^{m,q}(e)$, and $u \in W^{\ell,p}(S_e)$.*

We will see in the proof that for certain derivatives $D^\gamma \mathrm{E}_h u$ the stability estimate (6.4) can still be improved.

Proof. We prove the theorem for brick elements. Other element types are treated similarly, see the discussion in the proof of Lemma 8. We have to consider different cases separately.

First, let γ be a multi-index with $|\gamma| = m$ and $\gamma_1 \neq 0$, $\gamma_2 \neq 0$. We use the difference technique developed in the proof of Theorem 9 for both directions x_1 and x_2. In analogy to (5.13) we obtain for all $w \in \mathcal{P}^3_{m-1}$

$$\begin{aligned}\|D^\gamma \mathrm{E}_h u, L^q(e)\| &= \|D^\gamma \mathrm{E}_h(u-w), L^q(e)\| \\ &\lesssim h^{-\gamma} h_1^{\gamma_1} h_2^{\gamma_2} (\operatorname{meas} e)^{1/q-1/p} \left\| \frac{\partial^{\gamma_1}}{\partial x_1^{\gamma_1}} \frac{\partial^{\gamma_2}}{\partial x_2^{\gamma_2}} (u-w); L^p(S_e) \right\| \\ &\leq h_3^{-\gamma_3} (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha| \leq \gamma_3} h^\alpha |D^\alpha u; W^{\gamma_1+\gamma_2,p}(S_e)|.\end{aligned}$$

Using Corollary 2 and (6.1) we conclude

$$\begin{aligned}\|D^\gamma \mathrm{E}_h u, L^q(e)\| &\lesssim h_3^{-\gamma_3} (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha| = \gamma_3} h^\alpha |D^\alpha u; W^{\gamma_1+\gamma_2,p}(S_e)| \\ &\leq (\operatorname{meas} e)^{1/q-1/p} |u; W^{m,p}(S_e)|.\end{aligned}$$

In a second case we assume $\gamma_n \neq 0$, $n = 1$ or $n = 2$, but $\gamma_{3-n} = 0$, $\gamma_3 \neq 0$. Then we can use the difference technique only within some faces f_i $(i = 0, \dots, k)$ which are parallel to the x_n, x_3-plane. Defining $f := \bigcup_{i=0}^k f_i$ we find as above that for all $w \in \mathcal{P}^3_{m-1}$

$$\begin{aligned}\|D^\gamma \mathrm{E}_h u, L^q(e)\| &= \|D^\gamma \mathrm{E}_h(u-w), L^q(e)\| \\ &\lesssim h^{-\gamma} h_n^{\gamma_n} (\operatorname{meas} e)^{1/q} (\operatorname{meas} f)^{-1/p} \left\| \frac{\partial^{\gamma_n}}{\partial x_n^{\gamma_n}} (u-w); L^p(f) \right\|. \end{aligned} \tag{6.7}$$

Using the trace theorem $W^{\gamma_3,p}(S_e) \hookrightarrow L^p(f)$ and again Corollary 2 as well as (6.1) we obtain

$$\begin{aligned}\|D^\gamma \mathrm{E}_h u, L^q(e)\| &\lesssim h_3^{-\gamma_3} (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha| \leq \gamma_3} h^\alpha |D^\alpha (u-w); W^{\gamma_n,p}(S_e)| \\ &\lesssim h_3^{-\gamma_3} (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha| = \gamma_3} h^\alpha |D^\alpha u; W^{\gamma_n,p}(S_e)| \\ &\leq (\operatorname{meas} e)^{1/q-1/p} |u; W^{m,p}(S_e)|.\end{aligned}$$

Consider now the remaining pure derivatives. Let first be $\gamma_n = m$, $n = 1$ or $n = 2$, $\gamma_3 = 0$. Estimate (6.7) holds in this case as well. By using $p = 1$ and $w = 0$ it reads now

$$\|D^\gamma \mathrm{E}_h u, L^q(e)\| \lesssim (\operatorname{meas} e)^{1/q} (\operatorname{meas} f)^{-1} \|D^\gamma u; L^1(f)\|. \tag{6.8}$$

With the trace theorem $W^{1,p}(S_e) \hookrightarrow L^1(f)$ for all $p \in [1, \infty]$ we conclude the assertion (6.4).

Finally, for $\gamma_3 = m$, $\gamma_1 = \gamma_2 = 0$, the proof of the stability is completely analogous to the proof of Lemma 4. We have for all $w \in \mathcal{P}^3_{m-1}$

$$\|D^\gamma \mathrm{E}_h u, L^q(e)\| \lesssim h_3^{-m} (\operatorname{meas} e)^{1/q} \sum_{i \in I_e} (\operatorname{meas} \sigma_i)^{-1} \|u - w; L^1(\sigma_i)\|.$$

The trace theorem $W^{m+1,p}(S_e) \hookrightarrow L^1(\sigma_i)$ (which is the reason for the assumption $m \geq 1$ or $p > 2$) and Corollary 2 yield

$$\begin{aligned}\|D^\gamma \mathrm{E}_h u, L^q(e)\| &\lesssim h_3^{-m}(\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|\leq m} \sum_{|\beta|\leq 1} h^{\alpha+\beta} \|D^{\alpha+\beta}(u-w); L^p(S_e)\| \\ &\lesssim h_3^{-m}(\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|= m} \sum_{|\beta|\leq 1} h^{\alpha+\beta} \|D^{\alpha+\beta}u; L^p(S_e)\| \\ &\lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\beta|\leq 1} h^{\beta} |D^{\beta}u; W^{m,p}(S_e)|.\end{aligned}$$

Note that in this last case ($\gamma_3 = m$) for $m \geq 2$ and for $m = 1$, $p > 2$, it can even be proved that

$$\|D^\gamma \mathrm{E}_h u, L^q(e)\| \lesssim (\operatorname{meas} e)^{1/q-1/p} |u; W^{m,p}(S_e)|$$

because then $W^{m,p}(S_e) \hookrightarrow L^1(\sigma_i)$ holds.

Estimate (6.5) is trivial since

$$\|\mathrm{E}_h u, L^q(e)\| \lesssim (\operatorname{meas} e)^{1/q} \sum_{i\in I_e} (\operatorname{meas} \sigma_i)^{-1} \|u; L^1(\sigma_i)\|,$$

and the embedding $W^{\ell,p}(S_e) \hookrightarrow L^1(\sigma_i)$ holds just for ℓ, p satisfying (6.3).

Estimate (6.6) is concluded from (6.4, 6.5) as in the proof of Theorem 6. ☐

It is interesting to point out that the proof shows that

$$\|D^\gamma \mathrm{E}_h u, L^q(e)\| \lesssim (\operatorname{meas} e)^{1/q-1/p} |u; W^{m,p}(S_e)| \tag{6.9}$$

holds for γ with $|\gamma| = m$ if at most one of the numbers $\gamma_1, \gamma_2, \gamma_3$ vanishes. Our way of proof does not work for pure derivatives. Consider for example the case $\gamma = (1, 0, 0)$. To prove (6.9) with $p > 2$ ($\mathrm{E}_h u$ is defined only for $u \in W^{1,p}(\Omega)$ with $p > 2$.) one would have to skip the trace on f and to use a trace theorem in the form (3.13). But this leads to

$$\|D^\gamma \mathrm{E}_h u, L^q(e)\| \lesssim h_1^{-1}(\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|\leq 1} h^{\alpha} \|D^{\alpha}u; L^p(S_e)\|$$

with some diverging terms at the right-hand side. The case $\gamma = (1, 0, 0)$ would be tractable only if

$$\|D^\gamma \mathrm{E}_h u, L^q(e)\| \lesssim (\operatorname{meas} e)^{1/q-1/p} \|D^\gamma u; L^p(S_e)\|$$

was valid. It is not clear whether this estimate holds.

Remark 1. Our motivation for introducing the operator E_h was to be able to treat the general case of three independent mesh sizes $h_1 \leq h_2 \leq h_3$. Of course this includes the special case $h_1 \sim h_2$. We point out that in this case the transformation (2.4) can be generalized to (2.2, 2.3). To see that then the statement of Theorem 10 is still true consider an arbitrary element $e \in \mathcal{T}_h$ and denote its projection into the x_1, x_2-plane by ζ. Because $\mathcal{T}_h$ is of tensor product type, and because all σ_i are perpendicular to the x_1, x_2-plane, it suffices to choose S_e such that its projection to the x_1, x_2-plane is again ζ (and $\sigma_i \subset \overline{S_e}$), compare Figure 9. *Via* the transformation

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} h_1^{-1} B_e & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} =: \tilde{B} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix},$$

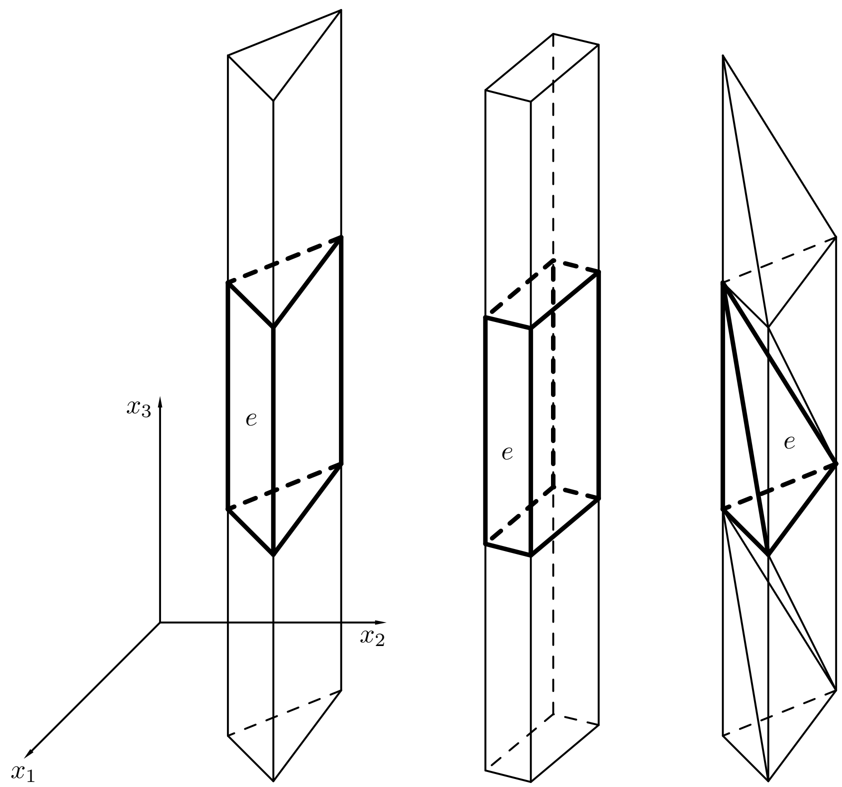


FIGURE 9. Illustration of the possible choice of a smaller S_e in the case of E_h (three element types).

B_e from (2.2), the domains e and S_e can be mapped to $\tilde{e}$ and $\tilde{S}_e = S_{\tilde{e}}$ which satisfy (locally) the assumptions made at the beginning of this section. That means that Theorem 10 holds true with respect to the coordinate system $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$. By observing that

$$\det \tilde{B} \sim 1, \quad \|\tilde{B}\| \sim 1, \quad \|\tilde{B}^{-1}\| \sim 1$$

we find that Theorem 10 extends to the meshes described above.

6.2. Stability in weighted Sobolev spaces

As in Subsection 4.2 we do not have an estimate of type $(1,1)$ for E_h. Therefore we consider a stability estimate for functions from weighted Sobolev spaces $V_\beta^{\ell,p}(S_e)$. These spaces were introduced in (4.9, 4.10). To be able to apply the transformation (2.4) to the weight we will restrict the consideration to the case $h_1 \sim h_2$. However, we can then relax (2.4) to (2.2), see Remark 1.

Lemma 11. *Consider an element e of a tensor product mesh and assume that* (6.1) *and* (2.4) *are fulfilled. Let m be an integer and β, p, q be real numbers with $0 \le m \le k$, $p, q \in [1,\infty]$, $\beta < 2 - 2/p$, $\beta \le 1$. Then for $u \in W^{m,p}(S_e) \cap V_\beta^{m+1,p}(S_e)$ the stability estimate*

$$|\mathrm{E}_h u; W^{m,q}(e)| \le (\operatorname{meas} e)^{1/q-1/p}\, h_1^{-\beta} \sum_{|\alpha|=m-1} \sum_{|t|=1} h^t \|D^{\alpha+t} v; V_\beta^{1,p}(S_e)\| \tag{6.10}$$

holds if $m \ge 1$ or $p \ge 2$.

Proof. Observe that the relations

$$\|v; L^1(S_e)\| \leq \|r^{-\beta}; L^{p'}(S_e)\| \|r^\beta v; L^p(S_e)\|, \quad (6.11)$$
$$\|r^{-\beta}; L^{p'}(S_e)\| \lesssim (\operatorname{meas} S_e)^{1-1/p} h_1^{-\beta} \quad (6.12)$$

(compare (4.15, 4.16)) lead to the embedding

$$V_\beta^{m+1,p}(S_e) \hookrightarrow V_0^{m+1,1}(S_e) \hookrightarrow W^{m+1,1}(S_e), \quad \beta < 2 - \frac{2}{p},$$

that means $u \in W^{m+1,1}(S_e)$. Therefore we can apply Theorem 10 (see also Rem. 1) with $p = 1$:

$$|\mathrm{E}_h u; W^{m,q}(e)| \lesssim (\operatorname{meas} e)^{1/q-1} \sum_{|\alpha|\leq 1} h^\alpha |D^\alpha u; W^{m,1}(S_e)|. \quad (6.13)$$

Notice further that (6.11, 6.12) lead to the estimate

$$\|v; L^1(S_e)\| \lesssim (\operatorname{meas} S_e)^{1-1/p} h_1^{-\beta} \|r^\beta v; L^p(S_e)\|, \quad \beta < 2 - \frac{2}{p}\cdot$$

So we get

$$\begin{aligned} &\sum_{|\alpha|\leq 1} \sum_{|t|=1} h^\alpha \|D^{\alpha+t} v; L^1(S_e)\| \\ &\lesssim (\operatorname{meas} S_e)^{1-1/p} h_1^{-\beta} \left(\sum_{|\alpha|=1} \sum_{|t|=1} h^\alpha \|r^\beta D^{\alpha+t} v; L^p(S_e)\| + \sum_{|t|=1} h_1 \|r^{\beta-1} D^t v; L^p(S_e)\| \right) \\ &\lesssim (\operatorname{meas} S_e)^{1-1/p} h_1^{-\beta} \sum_{|s|=1} h^s \|D^s v; V^{1,p}(S_e)\|. \end{aligned}$$

Together with (6.13) the assertion (6.10) is concluded. □

7. Application to the Poisson problem in a domain with an edge

Consider the Poisson problem with in general mixed boundary conditions in a three-dimensional polyhedral domain Ω. It is well known that the solution has in general singularities near corners and edges and near the lines where the type of the boundary condition changes. As a result, the finite element method on quasi-uniform meshes loses accuracy. The rate of convergence is smaller in comparison with that for problems with smooth solutions. To compensate this, specially adapted numerical methods have been developed. The *singular function method* which is well developed for two-dimensional problems, is used for three-dimensional problems in [14, 26]. However, *mesh refinement* techniques seem to be easier to handle. Refined *isotropic* meshes were considered in [5, 12, 25] for the finite element method and the boundary element method but this approach leads to overrefinement near edges. This overrefinement can be avoided by using *anisotropic meshes* in the neighbourhood of the edges [4, 9, 29].

In [4, 9] we considered the Dirichlet problem for the Poisson equation over a prismatic domain

$$\Omega = G \times Z \quad (7.1)$$

where $G \subset \mathbb{R}^2$ is a bounded polygonal domain and $Z := (0, z_0) \subset \mathbb{R}$ is an interval. This restriction was made there because we wanted to focus on *edge singularities*, and such domains do not introduce additional corner singularities [33,34]. The finite element meshes in [4,9] were of tensor product type, graded perpendicularly to the edge and quasi-uniform in the edge direction. Pentahedral meshes seem to be natural but in that papers the pentahedra were divided into three tetrahedra each. Pentahedral elements were used in [10], an unpublished version of the paper [9]. Note that this class of domains and the meshes exactly match the assumptions made in Section 2 for the present paper.

The estimation of the finite element error in the energy norm can be reduced to a general approximation problem due to the projection property of the finite element method. In the previous papers the interpolation error $u - \mathrm{I}_h u$ was investigated and it was shown that the family of meshes considered there is suited for the treatment of edge singularities. However, two points are still insufficient: First, the assumptions on the regularity of the right-hand side f of the Poisson equation were quite high in [4]. This drawback was partially removed in [9], but the case $f \in L^2(\Omega)$ is still not treated. This is deficient because Nitsche's method for obtaining an $L^2(\Omega)$-estimate of the finite element error is not applicable. Second, the refinement condition in [9] is slightly stronger than in [4]; this seems to be unnecessary. The aim of this section is to prove optimal estimates of the finite element error in the $W^{1,2}(\Omega)$- and the $L^2(\Omega)$-norm for $f \in L^2(\Omega)$ and the weaker refinement condition of [4]. This is now possible due to the local anisotropic estimates for the quasi-interpolation operators S_h and E_h. We point out that one essential ingredient of the proof of these optimal global error estimate is the anisotropic local estimate

$$|u - \mathrm{Q}_h u; W^{1,2}(e)| \lesssim \sum_{|\alpha|=1} h_e^\alpha |D^\alpha u; W^{1,2}(S_e)|.$$

This estimate is neither satisfied for $\mathrm{Q}_h = \mathrm{I}_h$ (see [4]) nor for $\mathrm{Q}_h = \mathrm{Z}_h$, $\mathrm{Q}_h = \mathrm{C}_h$, or $\mathrm{Q}_h = \mathrm{O}_h$, see the discussion in Section 3.

The plan of this section is the following. First we pose two model problems which differ in their boundary conditions. Then we introduce the family of finite element meshes. The global quasi-interpolation error is estimated in the $W^{1,2}(\Omega)$-seminorm. Because in general the operators do not preserve Dirichlet boundary conditions the model problems are chosen such that in one case S_h and in the other case E_h are appropriate and no modification of the operator is necessary near the boundary. The main result of this section can then be concluded, namely the finite element error estimates. Some remarks on other than the model problems complete this section.

Consider a prismatic domain Ω as described in (7.1) and denote $\Gamma_{\mathrm{B}} := \{x \in \partial\Omega : x_3 = 0 \text{ or } x_3 = z_0\}$ and $\Gamma_{\mathrm{M}} := \{x \in \partial\Omega : 0 < x_3 < z_0\} = \partial\Omega \setminus \Gamma_{\mathrm{B}}$. Then we treat the mixed boundary value problems

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_{\mathrm{B}}, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_{\mathrm{M}}, \tag{7.2}$$

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_{\mathrm{M}}, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_{\mathrm{B}}, \tag{7.3}$$

with $f \in L^2(\Omega)$. We assume that the cross-section G has only one corner with interior angle $\omega > \pi$ at the origin; thus Ω has only one "singular edge" which is part of the x_3-axis. The case of more than one singular edge introduces no additional difficulties because the edge singularities are of local nature.

Let $V_0 \subset W^{1,2}(\Omega)$ be the space of all $W^{1,2}(\Omega)$-functions which vanish at the Dirichlet part of the boundary (different for problems (7.2, 7.3)), and introduce the bilinear form $a(.,.) : V_0 \times V_0 \to \mathbb{R}$ and the linear form $(f,.) : V_0 \to \mathbb{R}$ by

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v, \quad (f,v) := \int_\Omega fv.$$

The variational form of problems (7.2, 7.3) is given by

$$\textit{Find } u \in V_0 \textit{ such that } a(u,v) = (f,v) \textit{ for all } v \in V_0. \tag{7.4}$$

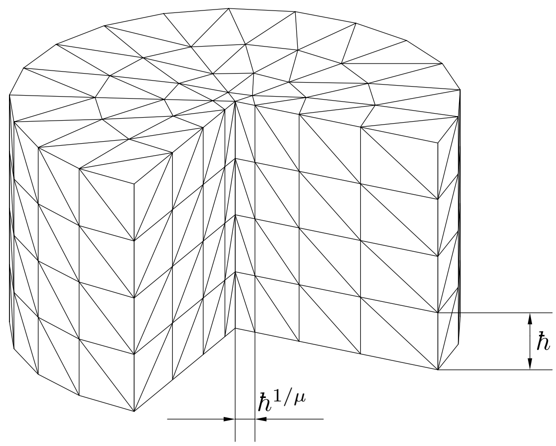


FIGURE 10. Example for an anisotropic mesh.

The existence of a unique variational solution u follows from the Lax-Milgram lemma.

The properties of the solution u can be described favourably using the weighted Sobolev spaces $V_\beta^{\ell,p}$ introduced in Subsection 4.2.

Lemma 12. *The solutions u of both problems (7.2) and (7.3) satisfy*

$$\frac{\partial u}{\partial x_i} \in V_\beta^{1,2}(\Omega), \qquad \left\| \frac{\partial u}{\partial x_i}; V_\beta^{1,2}(\Omega) \right\| \lesssim \|f; L^2(\Omega)\|, \quad i = 1, 2, \ \beta > 1 - \frac{\pi}{\omega}, \tag{7.5}$$

$$\frac{\partial u}{\partial x_3} \in V_0^{1,2}(\Omega), \qquad \left\| \frac{\partial u}{\partial x_3}; V_0^{1,2}(\Omega) \right\| \lesssim \|f; L^2(\Omega)\|. \tag{7.6}$$

Proof. The singularity of the edge at the x_3-axis can be described by (7.5, 7.6), see for example §26 and §30 in [23] or Section 2 in [9]. One can show by mirror techniques that the corners (see also [33, 34] for a different proof) and edges at the bottom and the top face do not introduce singularities. Finally, the remaining edges parallel to x_3-axis were assumed to have an opening angle smaller than π such that no singularity occurs. □

We define now a family of meshes $\mathcal{T}_h = \{e\}$ of tensor product type by introducing in G the standard mesh grading for two-dimensional corner problems, see for example [27]. Let $\{\eta\}$ be a regular isotropic triangulation of G; the elements are triangles. With $\hbar$ being the global mesh parameter, $\mu \in (0, 1]$ being the grading parameter, r_η being the distance of η to the corner,

$$r_\eta := \min_{(x_1,x_2)\in\overline{\eta}} (x_1^2 + x_2^2)^{1/2},$$

and some constant $R > 0$, we assume that the element size $h_\eta := \operatorname{diam} \eta$ satisfies

$$h_\eta \sim \begin{cases} \hbar^{1/\mu} & \text{for } r_\eta = 0, \\ \hbar r_\eta^{1-\mu} & \text{for } 0 < r_\eta \le R, \\ \hbar & \text{for } r_\eta > R. \end{cases}$$

This graded two-dimensional mesh is now extended in the third dimension using a uniform mesh size $\hbar$. In this way we obtain a pentahedral or, by dividing each pentahedron, a tetrahedral triangulation of Ω, see Figure 10 for an illustration. Note that the number of elements is of the order $\hbar^{-3}$ for the full range of μ. The notation is extended to the three-dimensional case as follows. Let r_e be the distance of an element e to the edge (x_3-axis). Then the element sizes satisfy

$$h_{1,e} \sim h_{2,e} \sim \begin{cases} \hbar^{1/\mu} & \text{for } r_e = 0, \\ \hbar r_e^{1-\mu} & \text{for } 0 < r_e \le R, \\ \hbar & \text{for } r_e > R. \end{cases} \qquad h_{3,e} \sim \hbar. \tag{7.7}$$

We introduce now the finite element space $V_{0h} := V_h \cap V_0$ where V_h is defined in Section 1. The finite element solution u_h is determined by

$$\textit{Find } u_h \in V_{0h} \textit{ such that } a(u_h, v_h) = (f, v_h) \textit{ for all } v_h \in V_{0h}. \tag{7.8}$$

Remember that V_{0h} is adapted to the Dirichlet boundary condition and therefore different for problems (7.2, 7.3).

Theorem 13. *Let u be the solution of* (7.2). *Then the estimate*

$$|u - \mathrm{S}_h u; W^{1,2}(\Omega)| \lesssim \hbar \, \|f; L^2(\Omega)\|$$

holds if $\mu < \pi/\omega$.

Proof. We reduce the estimation of the global error to the evaluation of the local errors and distinguish between the elements far from the edge M and the elements close to M.

For all elements e with $\overline{S_e} \cap M = \emptyset$ we can use Theorem 6 with $m = k = 1$ and $\ell = p = q = 2$:

$$\begin{aligned} |u - \mathrm{S}_h u; W^{1,2}(e)| &\lesssim \sum_{|\alpha|=1} h_e^\alpha |D^\alpha u; W^{1,2}(S_e)| \\ &\lesssim \sum_{i=1}^{2} h_{i,e} r_e^{-\beta} \left| \frac{\partial u}{\partial x_i}; V_\beta^{1,2}(S_e) \right| + h_{3,e} \left| \frac{\partial u}{\partial x_3}; V_0^{1,2}(S_e) \right| \end{aligned} \tag{7.9}$$

for any $\beta > 1 - \pi/\omega$. Here, we have used the fact that $r_e \lesssim \operatorname{dist}(S_e, M)$ holds, which follows from

$$r_e \le \operatorname{dist}(S_e, M) + h_{1,e'} \sim \operatorname{dist}(S_e, M) + \hbar \, [\operatorname{dist}(S_e, M)]^{1-\mu}$$

for sufficiently small $\hbar$. We apply now the assumption (7.7) and obtain for $r_e \le R$ and $\beta = 1 - \mu$ the relation $h_{i,e} r_e^{-\beta} \sim \hbar r_e^{1-\mu-\beta} = \hbar$ ($i = 1, 2$). The choice $\beta = 1 - \mu$ is admissible due to the refinement condition $\mu < \pi/\omega$. — In the case $r_e > R$ we have $h_{i,e} r_e^{-\beta} \lesssim \hbar R^{-\beta} \sim \hbar$. Combining this with (7.9) we obtain

$$|u - \mathrm{S}_h u; W^{1,2}(e)| \lesssim \hbar \sum_{i=1}^{2} \left| \frac{\partial u}{\partial x_i}; V_\beta^{1,2}(S_e) \right| + \hbar \left| \frac{\partial u}{\partial x_3}; V_0^{1,2}(S_e) \right|. \tag{7.10}$$

Consider now the elements e with $\overline{S_e} \cap M \ne \emptyset$. We use the triangle inequality and Lemma 7 with $m = k = 1$, $p = q = 2$, $\beta \in (1 - \pi/\omega, 1)$:

$$\begin{aligned} |u - \mathrm{S}_h u; W^{1,2}(e)| &\lesssim |u; W^{1,2}(e)| + |\mathrm{S}_h u; W^{1,2}(e)| \\ &\lesssim \sum_{|\alpha|=1} \|D^\alpha u, L^2(e)\| + h_{1,e}^{-\beta} \sum_{|\alpha|=1} h_e^\alpha \|D^\alpha u, V_\beta^{1,2}(S_e)\|. \end{aligned} \tag{7.11}$$

For the first term we use that $r \lesssim h_{1,e}$ in e and $1-\beta > 0$ and obtain

$$\begin{aligned} \sum_{|\alpha|=1} \|D^\alpha u, L^2(e)\| &\lesssim \sum_{i=1}^{2} h_{1,e}^{1-\beta} \left\| \frac{\partial u}{\partial x_i}; V_{\beta-1}^{0,2}(e) \right\| + h_{1,e} \left\| \frac{\partial u}{\partial x_3}; V_{-1}^{0,2}(e) \right\| \\ &\lesssim \hbar \sum_{i=1}^{2} \left\| \frac{\partial u}{\partial x_i}; V_\beta^{1,2}(e) \right\| + \hbar \left\| \frac{\partial u}{\partial x_3}; V_0^{1,2}(e) \right\|. \end{aligned} \tag{7.12}$$

We also used that $h_{1,e}^{1-\beta} \sim \hbar^{(1-\beta)/\mu} = \hbar$ for $\beta = 1-\mu$. The second term is treated with similar arguments:

$$\begin{aligned} h_{1,e}^{-\beta} \sum_{|\alpha|=1} h_e^\alpha \|D^\alpha u, V_\beta^{1,2}(S_e)\| &\lesssim \sum_{i=1}^{2} h_{1,e}^{1-\beta} \left\| \frac{\partial u}{\partial x_i}; V_\beta^{1,2}(S_e) \right\| + h_{1,e}^{-\beta} \hbar \left\| \frac{\partial u}{\partial x_3}; V_\beta^{1,2}(S_e) \right\| \\ &\lesssim \hbar \sum_{i=1}^{2} \left\| \frac{\partial u}{\partial x_i}; V_\beta^{1,2}(S_e) \right\| + \hbar \left\| \frac{\partial u}{\partial x_3}; V_0^{1,2}(S_e) \right\|. \end{aligned} \tag{7.13}$$

The last term was estimated using $r^\beta \le h_{1,e}^\beta$.

Inserting (7.12, 7.13) in (7.11) we find that (7.10) (with full norms instead of seminorms at the right-hand side) holds for elements with $\overline{S_e} \cap M \neq \emptyset$ as well. Summing up over all elements we obtain

$$|u - \mathrm{S}_h u; W^{1,2}(\Omega)| \lesssim \hbar \sum_{i=1}^{2} \left\| \frac{\partial u}{\partial x_i}; V_\beta^{1,2}(\Omega) \right\| + \hbar \left\| \frac{\partial u}{\partial x_3}; V_0^{1,2}(\Omega) \right\|,$$

$\beta = 1-\mu \in (1-\pi/\omega, 1)$. Here we used that only a finite number (independent of $\hbar$) of patches S_e overlap. By applying Lemma 12 the theorem is proved. □

Theorem 14. *Let u be the solution of* (7.3)*. Then the estimate*

$$|u - \mathrm{E}_h u; W^{1,2}(\Omega)| \lesssim \hbar \, \|f; L^2(\Omega)\|$$

holds if $\mu < \pi/\omega$.

Proof. The theorem can be proved in the same way as Theorem 7.2. Note that we used only the following properties of S_h:

$$\begin{aligned} |u - \mathrm{S}_h u; W^{1,2}(e)| &\lesssim \sum_{|\alpha|=1} h_e^\alpha |D^\alpha u; W^{1,2}(S_e)|, \\ |\mathrm{S}_h u; W^{1,2}(e)| &\lesssim h_{1,e}^{-\beta} \sum_{|\alpha|=1} \|D^\alpha u, V_\beta^{1,2}(S_e)\|. \end{aligned}$$

Both estimates hold true for E_h as well, see Theorem 10 and Lemma 11. □

Corollary 15. *Let u be the solution of (7.2) or (7.3) and let u_h be the finite element solution defined by (7.8). Assume that the mesh is refined according to $\mu < \pi/\omega$. Then the finite element error can be estimated by*

$$\begin{aligned} |u - u_h; W^{1,2}(\Omega)| &\lesssim \hbar \, \|f; L^2(\Omega)\|, \\ \|u - u_h; L^2(\Omega)\| &\lesssim \hbar^2 \, \|f; L^2(\Omega)\|. \end{aligned}$$

Proof. The first estimate follows from Theorems 13 and 14 *via* the projection property of the finite element method. Note that $\mathrm{S}_h u \in V_{0h}$ in the case of problem (7.2) and $\mathrm{E}_h u \in V_{0h}$ for (7.3). The $L^2(\Omega)$-estimate is obtained by Nitsche's method. □

By analogy one can prove for $\pi/\omega < \mu \leq 1$ that

$$\begin{aligned} |u-u_h;W^{1,2}(\Omega)| &\lesssim \hbar^{\pi/(\mu\omega)-\varepsilon}\,\|f;L^2(\Omega)\|, \\ \|u-u_h;L^2(\Omega)\| &\lesssim \hbar^{2[\pi/(\mu\omega)-\varepsilon]}\,\|f;L^2(\Omega)\|, \end{aligned}$$

for arbitrary small $\varepsilon > 0$. That means that we get for the unrefined mesh ($\mu = 1$) only an approximation order $\pi/\omega - \varepsilon$ ($W^{1,2}(\Omega)$-norm) or $2(\pi/\omega - \varepsilon)$ ($L^2(\Omega)$-norm). We conjecture that the ε can be omitted. But this needs another way of proof, for example using the theory of interpolation spaces, compare [13] for the two-dimensional case. However, one can show by an example that these estimates cannot be improved further [1]. Numerical tests support the results, see [4, 8, 10].

In the same way as above on can treat certain other boundary conditions. Conditions of third kind impose no further difficulties. Moreover, we can treat cases where Dirichlet boundary conditions are given only on a part of either Γ_{B} or Γ_{M}. In particular, if the type of the boundary condition changes at the edge M we have to substitute the expression π/ω by $\pi/2\omega$ in the whole text. Note further that for $\omega \geq \pi$ the solution is not any more contained in $W^{3/2+\varepsilon,2}(\Omega)$ which implies that the interpolation operator I_h is not applicable to u.

However, if Dirichlet boundary conditions are given on (parts of) both Γ_{B} and Γ_{M} then neither $\mathrm{S}_h u \in V_{0h}$ nor $\mathrm{E}_h u \in V_{0h}$. In such cases we have to modify S_h or E_h near the Dirichlet boundary, as it was done by Clément for C_h [19]. But we will not develop this here.

8. Summary

The starting point of our investigation was the quasi-interpolation operator Z_h introduced by Scott and Zhang [30]. We have seen in Section 3 that anisotropic estimates of type (m, ℓ) are valid for $m = 0$ but in general not for $m \geq 1$. Therefore we introduced three modifications and investigated the resulting operators S_h, L_h, and E_h. In order to summarize and to compare the different Scott-Zhang type quasi-interpolation operators we give a tabular overview. For comparison we add also the results for the nodal interpolant I_h and for the operators C_h (Clément) and O_h (Oswald).

In Table 1 we find the element types which the operator is applicable for. Note the slight difference of *tensor product type* and *tensor product* elements in three dimensions. Tensor product type corresponds to transformation (2.2, 2.3), and tensor product means the restriction to transformation (2.4). The operator I_h is widely investigated for more general elements including non-affine ones, see [2,4,7]. A comprehensive monograph is [3].

Table 2 compares the conditions for which the stability estimate

$$|\mathrm{Q}_h u; W^{m,q}(e)| \lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|\leq \ell-m} h^\alpha |D^\alpha u; W^{m,p}(S_e)| \tag{8.1}$$

holds, $\mathrm{Q}_h \in \{\mathrm{C}_h, \mathrm{O}_h, \mathrm{Z}_h, \mathrm{S}_h, \mathrm{L}_h, \mathrm{E}_h, \mathrm{I}_h\}$. In the case of S_h and E_h we additionally proved stability in weighted Sobolev spaces. The estimate

$$|\mathrm{Q}_h u; W^{m,q}(e)| \leq (\operatorname{meas} e)^{1/q-1/p} h_1^{-\beta} \sum_{|\alpha|=m-1} \sum_{|t|=1} h^t \|D^{\alpha+t} u; V_\beta^{1,p}(S_e)\|$$

holds under the conditions given in Table 3. The approximation error estimate

$$|u - \mathrm{Q}_h u; W^{m,q}(e)| \lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|=\ell-m} h^\alpha |D^\alpha u; W^{m,p}(S_e)| \tag{8.2}$$

Table 1. Treated finite elements.

	$d = 2$	$d = 3$
Z_h, C_h, O_h	tensor product meshes h_1, h_2 arbitrary	meshes of tensor product type $h_1 \sim h_2 \lesssim h_3$ or $h_1 \sim h_2 \gtrsim h_3$
		tensor product meshes h_1, h_2, h_3 independent
S_h	tensor product meshes $h_1 \lesssim h_2$	meshes of tensor product type $h_1 \sim h_2 \lesssim h_3$
L_h	tensor product meshes $h_1 \gtrsim h_2$	meshes of tensor product type $h_1 \sim h_2 \gtrsim h_3$
E_h		meshes of tensor product type $h_1 \sim h_2 \lesssim h_3$
		tensor product meshes $h_1 \lesssim h_2 \lesssim h_3$
I_h	tensor product meshes h_1, h_2 arbitrary	meshes of tensor product type $h_1 \sim h_2 \lesssim h_3$ or $h_1 \sim h_2 \gtrsim h_3$
		tensor product meshes h_1, h_2, h_3 independent
	even for more general meshes, see [2–4, 7]	even for more general meshes, see [2–4, 7]

Table 2. Conditions for the stability and error estimates.

C_h, O_h	$m = 0$, $0 \le \ell \le k+1$, $p, q \in [1, \infty]$
Z_h	$m = 0$, $1 \le \ell \le k+1$, $p, q \in [1, \infty]$
S_h	$0 \le m \le \ell - 1$, $1 \le \ell \le k+1$, $p, q \in [1, \infty]$ for $m \ge 2$ triangles and tetrahedra are excluded
L_h	$0 \le m \le \ell$, $1 \le \ell \le k+1$, $p, q \in [1, \infty]$
E_h	$1 \le m \le \ell - 1$, $1 \le \ell \le k+1$, $p, q \in [1, \infty]$
	$m = 0$, $2 \le \ell \le k+1$, $p, q \in [1, \infty]$
	$m = 0$, $\ell = 1$, $p \in (2, \infty]$, $q \in [1, \infty]$
I_h	$0 \le m \le \ell - 1$, $1 \le \ell \le k+1$, $q = p$, $p > d/\ell$ if $\ell < d$ and $m = 0$, $p > 2$ if $d = 3$ and $m = \ell - 1 > 0$
	$m = 0$, $\ell = 0$, $p = \infty$, $q \in [1, \infty]$

Table 3. Conditions for the stability in weighted Sobolev spaces.

C_h, O_h, Z_h	not treated
S_h	$0 \le m \le k$, $p, q \in [1, \infty]$, $\beta < 2 - \frac{2}{p}$, $\beta \le 1$ for $m \ge 2$ triangles and tetrahedra are excluded
L_h	not treated
E_h	$1 \le m \le k$, $p, q \in [1, \infty]$, $\beta < 2 - \frac{2}{p}$, $\beta \le 1$
	$m = 0$, $p \in (2, \infty]$, $q \in [1, \infty]$, $\beta < 2 - \frac{2}{p}$, $\beta \le 1$
I_h	not treated in this form

TABLE 4. Restrictions in the applicability of the operators.

C_h, O_h, Z_h	only $m = 0$
S_h	$m = \ell$ excluded, only $m = 0, 1$ for simplices, in 3D only needle elements
L_h	in 3D only flat elements
E_h	$m = \ell$ excluded, restrictions on p when $m = 0$, $\ell = 1$
I_h	$m = \ell$ excluded, restrictions on p when $m = 0$, $\ell < d$ or $m = \ell - 1 > 0$

holds if the conditions of Table 2 are satisfied and the parameters ℓ, p, m, q are such that the embedding $W^{\ell,p}(e) \hookrightarrow W^{m,q}(e)$ holds. The operator I_h plays an exceptional role also here, because estimate (8.2) is proved directly. The stability in the sense of (8.1) can be concluded *via* $|\mathrm{Q}_h u| \le |u| + |u - \mathrm{Q}_h u|$. Second, we mentioned in Table 2 only $q = p$ (published result), but meanwhile the estimates are derived also for general $q \in [1, \infty]$ satisfying $W^{\ell,p}(e) \hookrightarrow W^{m,q}(e)$ [3]. Finally, anisotropic interpolation error estimates are derived in [9–11] for functions from weighted Sobolev spaces with $k = 1$, $m = 0, 1$, $\ell = 2$, $q = p$. For more general results we refer also to [3].

Some shortcomings of the operators are given in Table 4. Additionally, we state that Dirichlet boundary conditions $u = g \in V_h|_{\Gamma_1}$ on Γ_1 can be satisfied on any part of $\partial\Omega$ for Z_h and I_h, on parts of the boundary which are parallel to the x_1-axis/x_1, x_2-plane for S_h and L_h, and on parts of $\partial\Omega$ which are perpendicular to the x_1, x_2-plane for E_h.

Finally, we mention that S_h and E_h have been successfully applied in the study of the Poisson problem in a domain with an edge where the singularity was treated with anisotropic mesh refinement, see Section 7. The operator L_h was applied by Becker [15] to show the stability and an approximation error estimate of the stabilized Q_1/Q_0-element pair in the context of the Stokes equation. I_h has been applied in the study of diffusion problems in domains with corners and edges [3, 4, 9–11, 29], as well as for singularly perturbed convection-diffusion-reaction problems with anisotropic refinement in boundary layers [2, 3, 6, 7, 20].

The work of the author is supported by DFG (German Research Foundation), Sonderforschungsbereich 393. The author wishes also to thank an anonymous referee for many helpful comments.

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