

NUMERICAL ANALYSIS OF COUPLING FOR A KINETIC EQUATION

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Abstract. In this paper we introduce a coupled systems of kinetic equations for the linearized Carleman model. We then study the existence theory and the asymptotic behaviour of the resulting coupled problem. In order to solve the coupled problem we propose to use the time marching algorithm. We then develop a convergence theory for the resulting algorithm. Numerical results confirming the theory are then presented.

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1. INTRODUCTION

The coupling of kinetic equations and their hydrodynamic limits was introduced and studied in [3, 12] see also [4, 5, 8–11, 13]. This approach was introduced in order to solve several difficulties that occur at the interface between fluid mechanics and kinetic theory. Because of the practical importance of these methodologies, the establishment of their mathematical foundations is of crucial importance. The mathematical theory of such coupling started in [6, 7, 12], where the coupling of two models of hydrodynamical type is considered. In [15], the author provided an analysis of the coupling of two models of kinetic type. In this paper we shall further study the coupling of kinetic equations for the linearized Carleman model. In particular, we shall study the existence theory and the asymptotic behaviour of the resulting coupled systems. To solve the coupled problem, we propose to use the time marching algorithm also introduced in [6, 7, 12–14]. We shall then establish the convergence theory for the resulting algorithm. Finally, we provide numerical results confirming the above mentioned mathematical results.

We consider in this paper the following linearized Carleman system [1].

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = a(v - u) \quad \text{on }]0, 1[, \quad (1)$$

$$\frac{\partial v}{\partial t} - \frac{\partial v}{\partial x} = a(u - v) \quad \text{on }]0, 1[, \quad (2)$$

$$u(0, \cdot) = u_0, \quad v(0, \cdot) = v_0, \quad (3)$$

$$u(t, 0) = g(t), \quad v(t, 1) = h(t), \quad (4)$$

where $t \in \mathbb{R}^+$, $x \in [0, 1]$ and $u(t, x)$, $v(t, x)$ are functions of x which represent probability densities for particles moving in the positive and negative x – direction, respectively. a is a positive constant and $g(t)$ and $h(t)$ are

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two nonnegative functions. This model describes a random walk in one dimension. System (1–4) has a unique strong solution.

The steady state problem corresponding to equations (1–4) is

$$u' = a(v - u) \quad \text{on }]0, 1[, \quad (5)$$

$$-v' = a(u - v) \quad \text{on }]0, 1[, \quad (6)$$

$$u(0) = g, \quad v(1) = h. \quad (7)$$

The coupled strategy of [6, 7, 12–14] applied to system (1–4) leads to the following coupled systems

$$\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial x} = a(v_1 - u_1) \quad \text{on }]0, h_1[, \quad (8)$$

$$\frac{\partial v_1}{\partial t} - \frac{\partial v_1}{\partial x} = a(u_1 - v_1) \quad \text{on }]0, h_1[, \quad (9)$$

$$u_1(0, \cdot) = u_{10}, \quad v_1(0, \cdot) = v_{10}, \quad (10)$$

$$u_1(t, 0) = g(t), \quad v_1(t, h_1) = v_2(t, h_1), \quad (11)$$

and

$$\frac{\partial u_2}{\partial t} + \frac{\partial u_2}{\partial x} = a(v_2 - u_2) \quad \text{on }]h_1, 1[, \quad (12)$$

$$\frac{\partial v_2}{\partial t} - \frac{\partial v_2}{\partial x} = a(u_2 - v_2) \quad \text{on }]h_1, 1[, \quad (13)$$

$$u_2(0, \cdot) = u_{20}, \quad v_2(0, \cdot) = v_{20}, \quad (14)$$

$$u_2(t, h_1) = u_1(t, h_1), \quad v_2(t, 1) = h(t), \quad (15)$$

where $0 < h_1 < 1$. Notice that problems (8–11, 12–15) are only coupled by their boundary conditions. They can be solved by two independent solution techniques.

In Section 2 and 3, we shall state and prove results about the existence theory and asymptotic behaviour of the coupled systems. In Section 4, we shall study the convergence properties of the time marching algorithm applied to the coupled problem. Finally, in Section 5, we present a numerical study of the resulting algorithm.

2. EXISTENCE THEORY

In this section, we shall study the existence of a solution for the coupled problem introduced in the previous section. We shall work in the Hilbert space

$$X = (L^2[0, h_1])^2 \times (L^2[h_1, 1])^2,$$

with the following norm

$$\|(w_1, w_2, w_3, w_4)\| = (\|w_1\|_{L^2[0, h_1]}^2 + \|w_2\|_{L^2[0, h_1]}^2 + \|w_3\|_{L^2[h_1, 1]}^2 + \|w_4\|_{L^2[h_1, 1]}^2)^{\frac{1}{2}}.$$

The main result of this section is the following

Theorem 2.1. *Assume that $(u_{10}, v_{10}, u_{20}, v_{20}) \in X$, then the coupled problem (8–15) has a unique strong solution (u_1, v_1, u_2, v_2) .*

We shall give the proof of this theorem for the homogeneous boundary conditions: $g(t) = 0$ and $h(t) = 0$. By a standard argument the proof in the nonhomogeneous case can be reduced to the homogeneous case.

We introduce an operator A on X as follows

$$A(w_1, w_2, w_3, w_4) = \begin{pmatrix} w'_1 + a(w_1 - w_2) \\ -w'_2 + a(w_2 - w_1) \\ w'_3 + a(w_3 - w_4) \\ -w'_4 + a(w_4 - w_3) \end{pmatrix}, \tag{16}$$

$$D(A) = \left\{ \begin{array}{l} (w_1, w_2, w_3, w_4) \in X | w'_1, w'_2 \in L^2[0, h_1], w'_3, \text{ and } w'_4 \in L^2[h_1, 1] \\ w_1(0) = 0, w_4(1) = 0, (w_1 - w_3)(h_1) = 0 \text{ and } (w_2 - w_4)(h_1) = 0 \end{array} \right\}.$$

It is clear that $D(A)$ is dense in X . We shall first, prove that the coupled problem has a unique solution in $D(A)$. This will be a consequence of the following theorem which establishes that the operator $-A$ is the infinitesimal generator of a contraction semigroup of class \mathcal{C}^0 .

Theorem 2.2. *The operator $-A$ is the infinitesimal generator of a contraction semigroup of class $\mathcal{C}^0 : \{G(t); t \geq 0\}$.*

Remark 1. As a consequence of Theorem (2.2) the existence theory is established.

Proof. To prove Theorem (2.2), we shall use the following result of Hille-Phillips [2].

Theorem 2.3. *Let T be an unbounded operator with domain $D(T)$ dense in the Hilbert space X . Then T is the infinitesimal generator of a contraction semigroup of class \mathcal{C}^0 if and only if*

- (i) T is dissipative;
- (ii) the range of $D(T)$ by $I - T$ is equal to X .

Let $T = -A$, where A is the unbounded operator defined in (16). Let w be an element of $D(A)$. T satisfies

$$\begin{aligned} (Tw, w) &= -(Aw, w) \\ &= \int_0^{h_1} (-w'_1 + a(w_2 - w_1))w_1 + \int_0^{h_1} (w'_2 + a(w_1 - w_2))w_2 + \int_{h_1}^1 (-w'_3 + a(w_4 - w_3))w_3 \\ &\quad + \int_{h_1}^1 (w'_4 + a(w_3 - w_4))w_4 \\ &= -\frac{1}{2}(w_2^2(0) + w_3^2(1)) - a \int_0^{h_1} (w_1 - w_2)^2 - a \int_{h_1}^1 (w_3 - w_4)^2. \end{aligned} \tag{17}$$

To obtain the last equality, we have used the boundary conditions. Since $a > 0$, we have $(Tu, u) \leq 0$ and $-A$ is dissipative.

Next, we show that the range of $D(T)$ by $I - T$ is equal to X . Let $f \in X$ and consider the problem of finding $u \in D(T)$ such that

$$(I - T)w = f, \tag{18}$$

which corresponds to finding $w \in D(T)$ such that

$$w'_1 + (a + 1)w_1 - aw_2 = f_1, \tag{19}$$

$$-w'_2 + (a + 1)w_2 - aw_1 = f_2, \tag{20}$$

$$w'_3 + (a + 1)w_3 - aw_4 = f_3, \tag{21}$$

$$-w'_4 + (a + 1)w_4 - aw_3 = f_4. \tag{22}$$

By a density argument we may assume that $f_1, f_2, f_3,$ and f_4 are continuous. By elementary methods we obtain the general solution of system (19–20). Similarly, we obtain the general solution of system (21–22). By using the boundary conditions we conclude that problem (18) has a unique solution. The use of Theorem 2.3 concludes the proof of the theorem.

3. ASYMPTOTIC ANALYSIS OF THE COUPLED SYSTEMS

The main result of this section is

Theorem 3.1. *Assume that $u_{10}, v_{10} \in L^2[0, h_1]$ and $u_{20}, v_{20} \in L^2[h_1, 1]$, then the solution of the coupled problem (8–15) converges as t tends to $+\infty$ to the solution of the steady problem (5–7).*

Proof. As in the previous section, without loss of generality, we may assume that $g = g(t) = 0$ and $h = h(t) = 0$. Let (u_s, v_s) denote the solution of the steady problem (5–7). Let \bar{u} and \bar{v} be defined as follows

$$\begin{aligned} \bar{u} &= u_1 - u_s \text{ and } \bar{v} = v_1 - v_s \text{ on }]0, h_1[, \\ \bar{u} &= u_2 - u_s \text{ and } \bar{v} = v_2 - v_s \text{ on }]h_1, 1[, \end{aligned}$$

where (u_1, v_1, u_2, v_2) is the solution of the coupled problem (8–15). Then we have

$$\frac{\partial \bar{u}_1}{\partial t} + \frac{\partial \bar{u}_1}{\partial x} = a(\bar{v}_1 - \bar{u}_1) \text{ on }]0, h_1[, \tag{23}$$

$$\frac{\partial \bar{v}_1}{\partial t} - \frac{\partial \bar{v}_1}{\partial x} = a(\bar{u}_1 - \bar{v}_1) \text{ on }]0, h_1[, \tag{24}$$

$$\bar{u}_1(0, \cdot) = u_{10}, \quad \bar{v}_1(0, \cdot) = v_{10}, \tag{25}$$

$$\bar{u}_1(t, 0) = 0, \quad \bar{v}_1(t, h_1) = \bar{v}_2(t, h_1), \tag{26}$$

and

$$\frac{\partial \bar{u}_2}{\partial t} + \frac{\partial \bar{u}_2}{\partial x} = a(\bar{v}_2 - \bar{u}_2) \text{ on }]h_1, 1[, \tag{27}$$

$$\frac{\partial \bar{v}_2}{\partial t} - \frac{\partial \bar{v}_2}{\partial x} = a(\bar{u}_2 - \bar{v}_2) \text{ on }]h_1, 1[, \tag{28}$$

$$\bar{u}_2(0, \cdot) = u_{20}, \quad \bar{v}_2(0, \cdot) = v_{20}, \tag{29}$$

$$\bar{u}_2(t, h_1) = \bar{u}_1(t, h_1), \quad \bar{v}_2(t, 1) = 0. \tag{30}$$

Let $w = (w_1, w_2, w_3, w_4) = (\bar{u}_1, \bar{v}_1, \bar{u}_2, \bar{v}_2)$. Let φ_1 and φ_2 be two positive functions independent of t to be precised later. Multiplying equations (23, 24) and (27, 28) respectively by $\varphi_1 w_1, \varphi_2 w_2$ and $\varphi_1 w_3, \varphi_2 w_4$, integrating over $[0, h_1]$ respectively $[h_1, 1]$, and using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \frac{d}{dt} \int_0^{h_1} (\varphi_1 w_1^2 + \varphi_2 w_2^2) + \int_0^{h_1} (-\varphi_{1x} + a(\varphi_1 - \varphi_2)) w_1^2 + \int_0^{h_1} (\varphi_{2x} + a(\varphi_2 - \varphi_1)) w_2^2 \\ + \varphi_1(h_1) w_1^2(h_1) - \varphi_1(0) w_1^2(0) - \varphi_2(h_1) w_2^2(h_1) + \varphi_2(0) w_2^2(0) \leq 0 \end{aligned} \tag{31}$$

and

$$\begin{aligned} \frac{d}{dt} \int_{h_1}^1 (\varphi_1 w_3^2 + \varphi_2 w_4^2) + \int_{h_1}^1 (-\varphi_{1x} + a(\varphi_1 - \varphi_2)) w_3^2 + \int_{h_1}^1 (\varphi_{2x} + a(\varphi_2 - \varphi_1)) w_4^2 \\ + \varphi_1(1) w_3^2(1) - \varphi_1(h_1) w_3^2(h_1) - \varphi_2(1) w_4^2(1) + \varphi_2(h_1) w_4^2(h_1) \leq 0. \end{aligned} \tag{32}$$

Assuming that $\varphi_1(h_1) = \varphi_2(h_1)$, and using the boundary conditions, we obtain

$$\begin{aligned} \frac{d}{dt} \int_0^{h_1} (\varphi_1 w_1^2 + \varphi_2 w_2^2) + \int_0^{h_1} (-\varphi_{1x} + a(\varphi_1 - \varphi_2)) w_1^2 + \int_0^{h_1} (\varphi_{2x} + a(\varphi_2 - \varphi_1)) w_2^2 \\ + \frac{d}{dt} \int_{h_1}^1 (\varphi_1 w_3^2 + \varphi_2 w_4^2) + \int_{h_1}^1 (-\varphi_{1x} + a(\varphi_1 - \varphi_2)) w_3^2 + \int_{h_1}^1 (\varphi_{2x} + a(\varphi_2 - \varphi_1)) w_4^2 \leq 0. \end{aligned} \quad (33)$$

Now consider the following system of equations

$$-\varphi_{1x} + a\varphi_1 - a\varphi_2 = k \text{ on }]0, h_1[\text{ (resp. on }]h_1, 1[), \quad (34)$$

$$\varphi_{2x} + a\varphi_2 - a\varphi_1 = k \text{ on }]0, h_1[\text{ (resp. on }]h_1, 1[), \quad (35)$$

$$\varphi_1(h_1) = \varphi_2(h_1), \quad (36)$$

where k is a positive constant. The solution of this system is given by

$$\begin{aligned} \varphi_1(x) &= \varphi_1(h_1) + (h_1 - x)k - ak(h_1 - x)^2, \\ \varphi_2(x) &= \varphi_1(h_1) + (x - h_1)k - ak(h_1 - x)^2. \end{aligned}$$

Choosing for example $\varphi_1(h_1) = 2ah_1^2k + 2h_1k + (3 + a)k$, we obtain

$$k < \varphi_1(x) \leq K \quad \forall x \in [0, h_1] \text{ (resp. on } [h_1, 1]), \quad (37)$$

$$k < \varphi_2(x) \leq K \quad \forall x \in [0, h_1] \text{ (resp. on } [h_1, 1]), \quad (38)$$

where K is a positive constant independent of h_1 .

Combining (33, 34, 35), we then obtain

$$\frac{d}{dt} \left[\int_0^{h_1} (\varphi_1 w_1^2 + \varphi_2 w_2^2) + \int_{h_1}^1 (\varphi_1 w_3^2 + \varphi_2 w_4^2) \right] + \int_0^{h_1} (f_1 w_1^2 + f_2 w_2^2) + \int_{h_1}^1 (f_1 w_3^2 + f_2 w_4^2) \leq 0. \quad (39)$$

Using the properties of φ_1 and φ_2 we finally obtain

$$\int_0^{h_1} (w_1^2 + w_2^2) + \int_{h_1}^1 (w_3^2 + w_4^2) \leq ce^{-c_0 t} \left[\int_0^{h_1} (w_{10}^2 + w_{20}^2) + \int_{h_1}^1 (w_{30}^2 + w_{40}^2) \right]. \quad (40)$$

And this concludes the proof of Theorem 3.1.

4. NUMERICAL ANALYSIS OF THE COUPLING

To solve the coupled problem introduced in previous sections, we propose to use the time marching algorithm of [6, 7, 12–14]. In this section we shall study the convergence properties of the algorithm.

$$\frac{u_1^{n+1} - u_1^n}{\Delta t} + \frac{du_1^{n+1}}{dx} = a(v_1^{n+1} - u_1^{n+1}) \quad \text{on }]0, h_1[, \quad (41)$$

$$\frac{v_1^{n+1} - v_1^n}{\Delta t} - \frac{dv_1^{n+1}}{dx} = a(u_1^{n+1} - v_1^{n+1}) \quad \text{on }]0, h_1[, \quad (42)$$

$$u_1^{n+1}(0) = 0, \quad v_1^{n+1}(h_1) = v_2^{n+1}(h_1), \quad (43)$$

$$\frac{u_2^{n+1} - u_2^n}{\Delta t} + \frac{du_2^{n+1}}{dx} = a(v_2^{n+1} - u_2^{n+1}) \quad \text{on }]h_1, 1[, \quad (44)$$

$$\frac{v_2^{n+1} - v_2^n}{\Delta t} - \frac{dv_2^{n+1}}{dx} = a(u_2^{n+1} - v_2^{n+1}) \quad \text{on }]h_1, 1[, \quad (45)$$

$$u_2^{n+1}(h_1) = u_1^n(h_1), \quad v_2^{n+1}(1) = 0, \quad (46)$$

and the initial conditions

$$u_1^0 = u_{10}, \quad v_1^0 = v_{10}, \quad u_2^0 = u_{20}, \quad v_2^0 = v_{20}. \quad (47)$$

Here, without loss of generality, we have assumed that $g = g(t) = 0$ and $h = h(t) = 0$.

The convergence of the algorithm (41–47) is stated in the following theorem.

Theorem 4.1. *The algorithm (41–47) converges as n tends to ∞ .*

The proof of this theorem will be given in Section 4.2.

4.1. The time independent case

If we introduce the coupled systems directly for the steady problem and we apply the time marching algorithm, we obtain

$$\frac{du_1^{n+1}}{dx} = a(v_1^{n+1} - u_1^{n+1}) \quad \text{on }]0, h_1[, \quad (48)$$

$$-\frac{dv_1^{n+1}}{dx} = a(u_1^{n+1} - v_1^{n+1}) \quad \text{on }]0, h_1[, \quad (49)$$

$$u_1^{n+1}(0) = 0, \quad v_1^{n+1}(h_1) = v_2^{n+1}(h_1), \quad (50)$$

$$\frac{du_2^{n+1}}{dx} = a(v_2^{n+1} - u_2^{n+1}) \quad \text{on }]h_1, 1[, \quad (51)$$

$$-\frac{dv_2^{n+1}}{dx} = a(u_2^{n+1} - v_2^{n+1}) \quad \text{on }]h_1, 1[, \quad (52)$$

$$u_2^{n+1}(h_1) = u_1^n(h_1), \quad v_2^{n+1}(1) = 0, \quad (53)$$

$$u_1^0 = u_{10}, \quad v_1^0 = v_{10}, \quad u_2^0 = u_{20}, \quad v_2^0 = v_{20}. \quad (54)$$

The following theorem states the convergence of the algorithm (48–54).

Theorem 4.2. *The algorithm (48–54) converges for any $0 < h_1 < 1$.*

Proof. We first set

$$u_2^{(n)}(h_1) = \gamma^{(n)}, \quad v_1^{(n)}(h_1) = \delta^{(n)}. \quad (55)$$

The algorithm (48–54) converges if $\gamma^{(n)} \rightarrow 0$ and $\delta^{(n)} \rightarrow 0$ as $n \rightarrow \infty$. The solution of problem (48–50) is given by

$$u_1^{(n+1)}(x) = \frac{v_2^{(n+1)}(h_1)}{1 + ah_1} ax, \quad v_1^{(n+1)}(x) = \frac{v_2^{(n+1)}(h_1)}{1 + ah_1} (1 + ax). \quad (56)$$

The solution of problem (51–53) is given by

$$u_2^{(n+1)}(x) = \frac{u_1^{(n)}(h_1)}{1 + a(1 - h_1)} (1 + a(1 - x)), \quad v_2^{(n+1)}(x) = \frac{u_1^{(n)}(h_1)}{1 + a(1 - h_1)} a(1 - x). \quad (57)$$

The solution of the steady problem (5–6) with boundary conditions $u(0) = 0$ and $v(1) = 0$ is $u = 0$ and $v = 0$.

Using the coupling boundary conditions, we obtain

$$\begin{aligned} \gamma^{(1)} &= u_2^{(1)}(h_1) = u_1^{(0)}(h_1) \\ \delta^{(1)} &= v_1^{(1)}(h_1) = v_2^{(1)}(h_1) \\ &= \frac{a(1 - h_1)}{1 + a(1 - h_1)} \gamma^{(1)}. \end{aligned}$$

On the other hand using the coupling boundary conditions, we obtain

$$\begin{aligned} \gamma^{(2)} &= u_2^{(2)}(h_1) = u_1^{(1)}(h_1) \\ &= \frac{ah_1}{1 + ah_1} \frac{a(1 - h_1)}{1 + a(1 - h_1)} \gamma^{(1)}, \\ \delta^{(2)} &= v_1^{(2)}(h_1) = v_2^{(2)}(h_1) \\ &= \frac{a(1 - h_1)}{1 + a(1 - h_1)} \frac{ah_1}{1 + ah_1} \delta^{(1)}. \end{aligned}$$

Hence, the iterative process is completely determined

$$\begin{aligned} \gamma^{(n+1)} &= \frac{ah_1}{1 + ah_1} \frac{a(1 - h_1)}{1 + a(1 - h_1)} \gamma^{(n)}, \\ \delta^{(n+1)} &= \frac{a(1 - h_1)}{1 + a(1 - h_1)} \frac{ah_1}{1 + ah_1} \delta^{(n)}. \end{aligned}$$

From the last iterative process, we clearly deduce the convergence of the algorithm. And this concludes the proof of Theorem 4.2.

4.2. Analysis of the general algorithm

We shall now establish the convergence properties of the algorithm (41–47). More precisely, we shall give the proof of Theorem 4.1.

Proof of Theorem 4.1. Using the notation $u_1^{n+1} = u_1$, $v_1^{n+1} = v_1$, $u_1^n = f_1$, $v_1^n = g_1$, $u_2^{n+1} = u_2$, $v_2^{n+1} = v_2$, $u_2^n = f_2$, $v_2^n = g_2$, the algorithm (41–47) becomes

$$\begin{cases} bu_1 + u_1' = av_1 + cf_1 & \text{on }]0, h_1[, \\ bv_1 - v_1' = au_1 + cg_1 & \text{on }]0, h_1[, \\ u_1(0) = 0, \quad v_1(h_1) = v_2(h_1), \end{cases} \tag{58}$$

$$\begin{cases} bu_2 + u_2' = av_2 + cf_2 & \text{on }]h_1, 1[, \\ bv_2 - v_2' = au_2 + cg_2 & \text{on }]h_1, 1[, \\ u_2(h_1) = f_1(h_1), \quad v_2(1) = 0, \end{cases} \tag{59}$$

with the initial conditions (47). Here, we have used the notation $b = a + 1/\Delta t$ and $c = 1/\Delta t$.

Let $\varphi_1, \varphi_2, \varphi_3$ and φ_4 be four positive functions bounded below and above to be precised later. Multiplying equations (58, 59) respectively by $\varphi_1, \varphi_2, \varphi_3$ and φ_4 , integrating respectively over $]0, h_1]$ and $[h_1, 1]$ and using Cauchy-Schwarz inequality we obtain

$$\int_0^{h_1} \left[\frac{a+c}{2}\varphi_1 - \frac{1}{2}\varphi_1' - \frac{a}{2}\varphi_2 \right] u_1^2 + \int_0^{h_1} \left[\frac{a+c}{2}\varphi_2 + \frac{1}{2}\varphi_2' - \frac{a}{2}\varphi_1 \right] v_1^2 + \frac{1}{2}(\varphi_1 u_1^2 - \varphi_2 v_1^2)'_0^{h_1} \leq \frac{c}{2} \int_0^{h_1} (\varphi_1 f_1^2 + \varphi_2 g_1^2) \tag{60}$$

and

$$\int_{h_1}^1 \left[\frac{a+c}{2}\varphi_3 - \frac{1}{2}\varphi_3' - \frac{a}{2}\varphi_4 \right] u_2^2 + \int_{h_1}^1 \left[\frac{a+c}{2}\varphi_4 + \frac{1}{2}\varphi_4' - \frac{a}{2}\varphi_3 \right] v_2^2 + \frac{1}{2}(\varphi_3 u_2^2 - \varphi_4 v_2^2)'_{h_1}^1 \leq \frac{c}{2} \int_{h_1}^1 (\varphi_3 f_2^2 + \varphi_4 g_2^2). \tag{61}$$

Using the boundary conditions and combining (60, 61), we obtain

$$\begin{aligned} & \int_0^{h_1} \left[\frac{a+c}{2}\varphi_1 - \frac{1}{2}\varphi_1' - \frac{a}{2}\varphi_2 \right] u_1^2 + \int_0^{h_1} \left[\frac{a+c}{2}\varphi_2 + \frac{1}{2}\varphi_2' - \frac{a}{2}\varphi_1 \right] v_1^2 + \int_{h_1}^1 \left[\frac{a+c}{2}\varphi_3 - \frac{1}{2}\varphi_3' - \frac{a}{2}\varphi_4 \right] u_2^2 + \int_{h_1}^1 \left[\frac{a+c}{2}\varphi_4 \right. \\ & \quad \left. + \frac{1}{2}\varphi_4' - \frac{a}{2}\varphi_3 \right] v_2^2 + \frac{1}{2}\varphi_1(h_1)u_1^2(h_1) + \frac{1}{2}[\varphi_4(h_1)v_2^2(h_1) - \varphi_2(h_1)v_1^2(h_1)] + \frac{1}{2}\varphi_2(0)v_1^2(0) + \frac{1}{2}\varphi_3(1)u_2^2(1) \\ & \leq \frac{c}{2} \int_0^{h_1} (\varphi_1 f_1^2 + \varphi_2 g_1^2) + \frac{c}{2} \int_{h_1}^1 (\varphi_3 f_2^2 + \varphi_4 g_2^2) + \frac{1}{2}\varphi_3(h_1)f_1^2(h_1). \end{aligned} \tag{62}$$

Now our objective is to find γ_1 and γ_2 both larger than $c/2$ such that the solutions φ_1 and φ_2 of the following system

$$\frac{a+c}{2}\varphi_1 - \frac{1}{2}\varphi_1' - \frac{a}{2}\varphi_2 = \gamma_1\varphi_1 \text{ on }]0, h_1[, \tag{63}$$

$$\frac{a+c}{2}\varphi_2 + \frac{1}{2}\varphi_2' - \frac{a}{2}\varphi_1 = \gamma_2\varphi_2 \text{ on }]0, h_1[, \tag{64}$$

with the boundary conditions $\varphi_1(h_1)$ and $\varphi_2(h_1)$ fixed and positive to be precised later, are both positive bounded below and above by positive constants which are independent of both c and h_1 . By choosing $\gamma_1 = \gamma_2 = (c + \epsilon)/2$ and setting $\lambda = a - \epsilon$ where ϵ is a small positive parameter, we find

$$\varphi_1 = \frac{1}{a}e^{-\lambda x}v', \quad \varphi_2 = e^{-\lambda x}v \tag{65}$$

$$v = e^{\lambda x}(c_1 \cos \sqrt{\epsilon(\lambda+a)}x + c_2 \sin \sqrt{\epsilon(\lambda+a)}x), \tag{66}$$

with c_1 and c_2 given by

$$c_1 = \frac{ae^{\alpha_2 h_1} \varphi_1(h_1) - \frac{b_2}{a_2} e^{\alpha_2 h_1} \varphi_2(h_1)}{b_1 - \frac{a_1}{a_2} b_2},$$

$$c_2 = \frac{1}{a_2} e^{\alpha_2 h_1} \varphi_2(h_1) - \frac{a_1}{a_2} \frac{ae^{\alpha_2 h_1} \varphi_1(h_1) - \frac{b_2}{a_2} e^{\alpha_2 h_1} \varphi_2(h_1)}{b_1 - \frac{a_1}{a_2} b_2}.$$

Here, a_1, a_2, b_1 and b_2 are given by

$$a_1 = e^{\lambda h_1} \cos \sqrt{\epsilon(\lambda + a)} h_1, \quad a_2 = e^{\lambda h_1} \sin \sqrt{\epsilon(\lambda + a)} h_1$$

$$b_1 = \lambda a_1 - \sqrt{\epsilon(\lambda + a)} a_2$$

$$b_2 = \lambda a_2 + \sqrt{\epsilon(\lambda + a)} a_1.$$

Assuming that the boundary conditions satisfy

$$\varphi_1(h_1) = \left(1 + \frac{\delta}{a}\right) \varphi_2(h_1) \tag{67}$$

with δ a small positive parameter, we find that c_1, c_2, φ_1 and φ_2 are positive. Moreover, they satisfy the following inequalities

$$C_1 \varphi_2(h_1) \leq \varphi_1(x) \leq C_2 \varphi_2(h_1)$$

$$C_3 \varphi_2(h_1) \leq \varphi_2(x) \leq C_4 \varphi_2(h_1)$$

where C_1, C_2, C_3 and C_4 are positive constants. Using (67) and with an appropriate choice for $\varphi_2(h_1)$, we conclude that for ϵ and δ small φ_1 and φ_2 are bounded below and above by positive constants which are independent of c and h_1 .

We shall now find γ_3 and γ_4 both larger than $c/2$ such that the solutions φ_3 and φ_4 of the following system

$$\frac{a+c}{2} \varphi_3 - \frac{1}{2} \varphi_3' - \frac{a}{2} \varphi_4 = \gamma_3 \varphi_3 \quad \text{on }]h_1, 1[, \tag{68}$$

$$\frac{a+c}{2} \varphi_4 + \frac{1}{2} \varphi_4' - \frac{a}{2} \varphi_3 = \gamma_4 \varphi_4 \quad \text{on }]h_1, 1[, \tag{69}$$

$$\varphi_3(h_1) = \left(1 - \frac{\delta}{a}\right) \varphi_1(h_1), \quad \varphi_4(h_1) = \varphi_2(h_1), \tag{70}$$

are positive bounded below and above by positive constants which are independent of both c and h_1 . We shall choose $\gamma_3 = \gamma_4 = (c + \epsilon)/2$ where ϵ is a small positive parameter. Setting $\lambda = a - \epsilon$, we find

$$\varphi_3 = \frac{1}{a} e^{-\lambda x} v', \quad \varphi_4 = e^{-\lambda x} v \tag{71}$$

where v has similar form as in (66).

Proceeding as for φ_1 and φ_2 and using the boundary conditions (70), we find that for ϵ and δ small φ_3 and φ_4 are both positive bounded below and above by positive constants which are independent of both c and h_1 . In fact, we may assume that ϵ and δ introduced here are the same as those introduced for the construction of φ_1 and φ_2 . We then have $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4$.

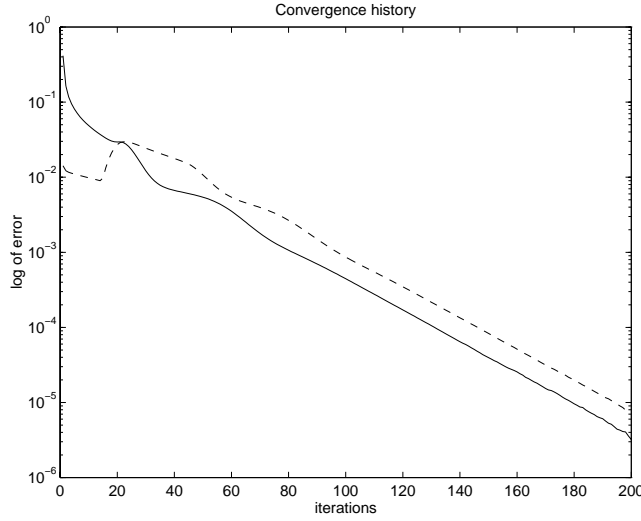


FIGURE 1. L_∞ norm of the error on $[0, h_1]$ (—) and $[h_1, 1]$ (- -).

Using (63, 64, 68, 69), (62) becomes

$$\begin{aligned} &\gamma_1 \int_0^{h_1} (\varphi_1 u_1^2 + \varphi_2 v_1^2) + \gamma_1 \int_{h_1}^1 (\varphi_3 u_2^2 + \varphi_4 v_2^2) + \frac{1}{2} \varphi_1(h_1) u_1^2(h_1) + \frac{1}{2} [\varphi_4(h_1) v_2^2(h_1) - \varphi_2(h_1) v_1^2(h_1)] \\ &\quad + \frac{1}{2} \varphi_2(0) v_1^2(0) + \frac{1}{2} \varphi_3(1) u_2^2(1) \leq \frac{c}{2} \int_0^{h_1} (\varphi_1 f_1^2 + \varphi_2 g_1^2) + \frac{c}{2} \int_{h_1}^1 (\varphi_3 f_2^2 + \varphi_4 g_2^2) + \frac{1}{2} \varphi_3(h_1) f_1^2(h_1). \end{aligned} \quad (72)$$

Using the boundary conditions in (58, 70) and the positivity of φ_2 and φ_3 , we obtain

$$\begin{aligned} &\gamma_1 \int_0^{h_1} (\varphi_1 u_1^2 + \varphi_2 v_1^2) + \gamma_1 \int_{h_1}^1 (\varphi_3 u_2^2 + \varphi_4 v_2^2) + \frac{1}{2} \varphi_1(h_1) u_1^2(h_1) \\ &\quad \leq \frac{c}{2} \int_0^{h_1} (\varphi_1 f_1^2 + \varphi_2 g_1^2) + \frac{c}{2} \int_{h_1}^1 (\varphi_3 f_2^2 + \varphi_4 g_2^2) + \frac{1}{2} (1 - \frac{\delta}{a}) \varphi_1(h_1) f_1^2(h_1). \end{aligned} \quad (73)$$

Because $\gamma_1 > c/2$ and the fact that δ is small, the operator in (73) is contractant and therefore

$$\int_0^{h_1} (\varphi_1 u_1^2 + \varphi_2 v_1^2) + \int_{h_1}^1 (\varphi_3 u_2^2 + \varphi_4 v_2^2) + \frac{1}{2\gamma_1} \varphi_1(h_1) u_1^2(h_1),$$

converges to 0. Because of our special construction of the functions $\varphi_1, \varphi_2, \varphi_3$ and φ_4 , we conclude that

$$\int_0^{h_1} (u_1^2 + v_1^2) + \int_{h_1}^1 (u_2^2 + v_2^2) + \frac{1}{2\gamma_1} u_1^2(h_1),$$

converges to 0. And this concludes the proof of Theorem 4.1.

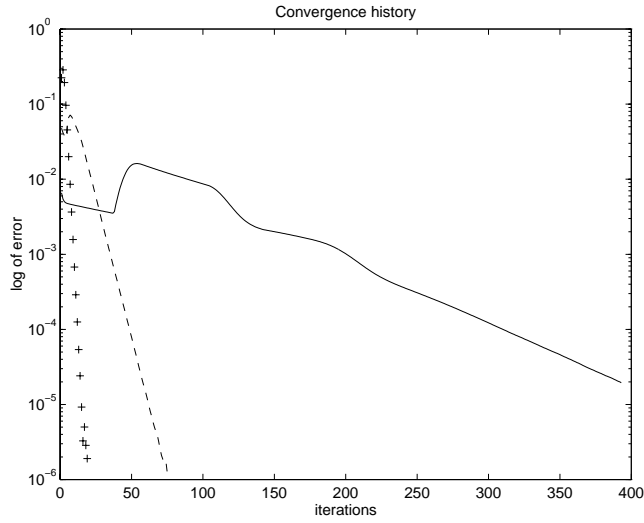


FIGURE 2. Convergence history curves for $\Delta t = 10^{-2}$ (—), $\Delta t = 10^{-1}$ (- -) and $\Delta t = 1$ (++).

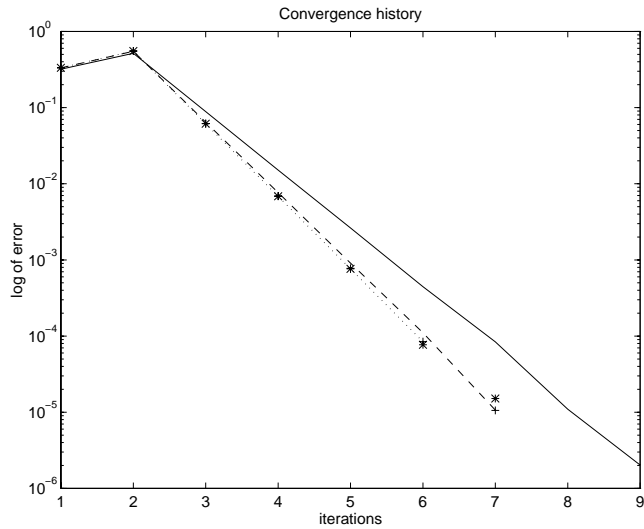


FIGURE 3. Convergence history curves for $\Delta t = 10$ (—), $\Delta t = 10^2$ (- -), $\Delta t = 10^3$ (++), $\Delta t = 10^4$ (**), and $\Delta t = 10^5$ (···).

5. NUMERICAL STUDY OF THE COUPLING

In this section we shall study numerically the convergence properties of the algorithm (41–47). This algorithm corresponds to the following systems

$$\left\{ \begin{array}{l} \frac{u_1^{n+1} - u_1^n}{\Delta t} + \frac{du_1^{n+1}}{dx} = a(v_1^{n+1} - u_1^{n+1}), \\ \frac{v_1^{n+1} - v_1^n}{\Delta t} - \frac{dv_1^{n+1}}{dx} = a(u_1^{n+1} - v_1^{n+1}), \\ u_1^{n+1}(0) = g, \quad v_1^{n+1}(h_1) = v_2^{n+1}(h_1), \end{array} \right. \quad \left\{ \begin{array}{l} \frac{u_2^{n+1} - u_2^n}{\Delta t} + \frac{du_2^{n+1}}{dx} = a(v_2^{n+1} - u_2^{n+1}), \\ \frac{v_2^{n+1} - v_2^n}{\Delta t} - \frac{dv_2^{n+1}}{dx} = a(u_2^{n+1} - v_2^{n+1}), \\ u_2^{n+1}(h_1) = u_1^n(h_1), \quad v_2^{n+1}(1) = h, \end{array} \right.$$

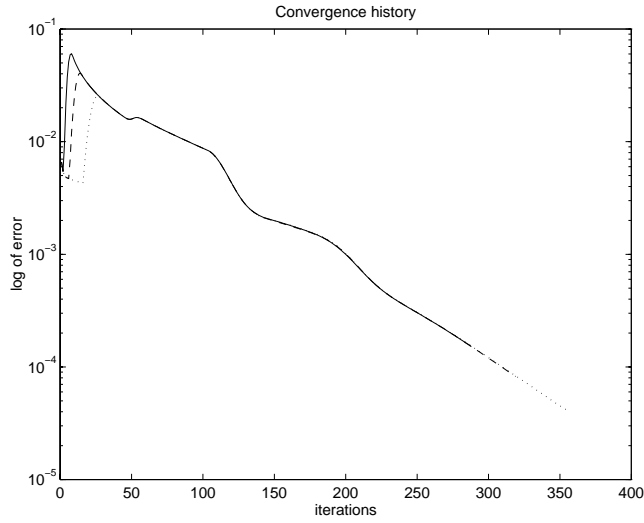


FIGURE 4. Convergence history curves for $h_1 = 0.0625$ (—), $h_1 = 0.125$ (- -) and $h_1 = 0.25$ (\cdots).

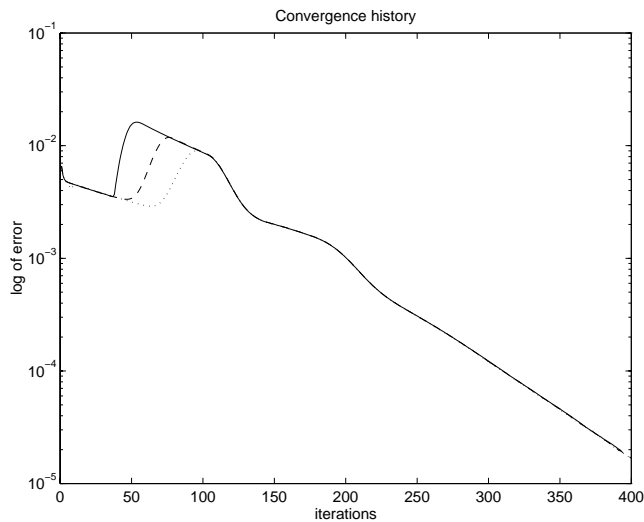


FIGURE 5. Convergence history curves for $h_1 = 0.5$ (—), $h_1 = 0.75$ (- -) and $h_1 = 0.95$ (\cdots).

and the initial conditions

$$u_1^0 = u_{10}, \quad v_1^0 = v_{10}, \quad u_2^0 = u_{20}, \quad v_2^0 = v_{20}.$$

For the discretization in space, we consider an equally spaced subdivision of $[0, h_1]$ into subintervals $[x_i, x_{i+1}]$, $x_0 = 0$, $x_i = x_{i-1} + \Delta x_i$, $i = 1, \dots, n_1$, where n_1 is the number of subdivisions and $\Delta x_i = h_1/n_1$. We also subdivide $[h_1, 1]$ into equally spaced subdivision $[y_i, y_{i+1}]$, $y_0 = h_1$, $y_i = y_{i-1} + \Delta y_i$, $i = 1, \dots, n_2$, where n_2 is the number of subdivisions and $\Delta y_i = (1 - h_1)/n_2$.

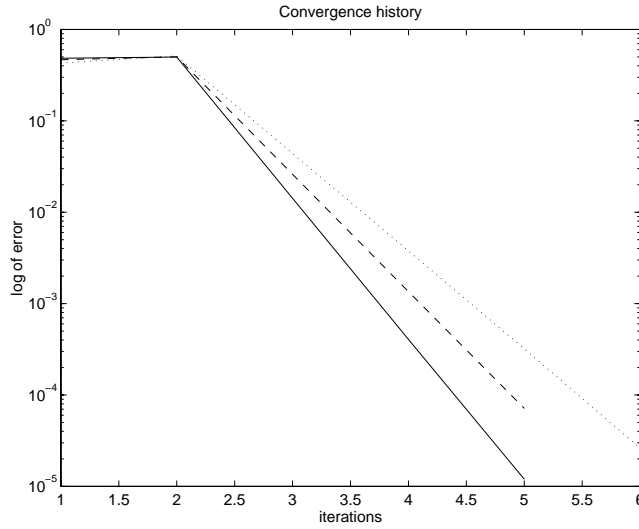


FIGURE 6. Convergence history curves for $h_1 = 0.0625$ (—), $h_1 = 0.125$ (---) and $h_1 = 0.25$ (⋯).

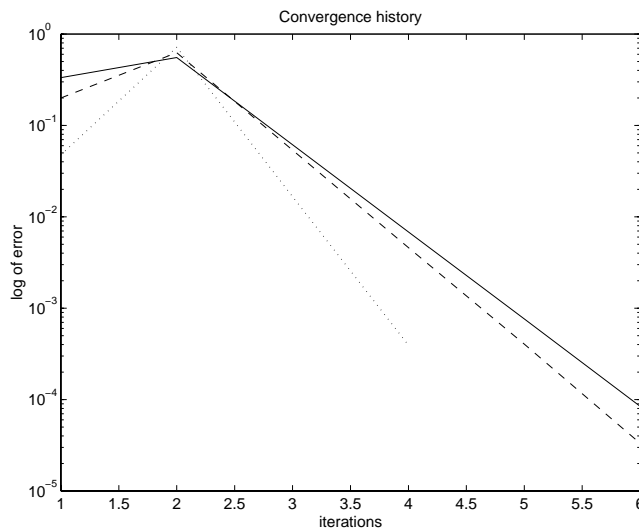


FIGURE 7. Convergence history curves for $h_1 = 0.5$ (—), $h_1 = 0.75$ (---) and $h_1 = 0.95$ (⋯).

We then use an upwinded difference method to approximate the space derivatives appearing in the algorithm. For example, for the first derivative appearing in the first equation in the algorithm, we use the approximation

$$\frac{du_1^{n+1}}{dx}(x_i) \approx \frac{u_1^{n+1}(x_i) - u_1^{n+1}(x_{i-1})}{\Delta x_i}.$$

The other derivatives in the coupled problem are approximated similarly. The two problems are coupled only through their boundary conditions. The full discretization of each of these two problems leads to an algebraic system that we solve by the incomplete factorization method.

To test the convergence properties of the algorithm, we take $a = 1$, $h_1 = 1/2$, $g = 1/2$, and $h = 1$. We initialize the coupling algorithm by 0 on the whole interval $[0, 1]$. In Figure. 1, we show the convergence history of the algorithm for $\Delta t = 0.025$. In this figure, we present the L_∞ norm of the errors $\|u_1^n - u\|_{L^\infty[0, h_1]}$ and $\|u_2^n - u\|_{L^\infty[h_1, 1]}$ versus the iterations. The exact solution of the steady problem is given by

$$u(x) = u(0) - \frac{u(0) - u(1)}{1 + a}ax, \quad v(x) = u(x) - \frac{u(0) - u(1)}{1 + a}.$$

The plotting (not shown here) of the converged solution u computed by the coupling algorithm and the exact solution u of the steady Carleman equations shows no difference between these two solutions.

These numerical results clearly show that the algorithm converges and it converges to the solution of the steady Carleman equations.

In Figures 2 and 3 we show the convergence history ($\|u_1^n - u\|_{L^\infty[0, h_1]}$) of the algorithm for a fixed $h_1 = 0.5$ and various values of Δt . In Figures 4 and 5 we show the convergence history ($\|u_1^n - u\|_{L^\infty[0, h_1]}$) of the algorithm for a fixed $\Delta t = 10^{-2}$ and various values of h_1 . In Figures 6 and 7, we show the convergence history ($\|u_1^n - u\|_{L^\infty[0, h_1]}$) of the algorithm for a fixed $\Delta t = 10^5$ and various values of h_1 . These results show that the algorithm converges in accordance with the theory established in Section 4.

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