STABILIZATION METHODS OF BUBBLE TYPE FOR THE \( Q_1/Q_1 \)-ELEMENT APPLIED TO THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS

Petr Knobloch\(^1\) and Lutz Tobiska\(^2\)

Abstract. In this paper, a general technique is developed to enlarge the velocity space \( V^1_h \) of the unstable \( Q_1/Q_1 \)-element by adding spaces \( V^2_h \) such that for the extended pair the Babuška–Brezzi condition is satisfied. Examples of stable elements which can be derived in such a way imply the stability of the well-known \( Q_2/Q_1 \)-element and the \( 4Q_1/Q_1 \)-element. However, our new elements are much more cheaper. In particular, we shall see that more than half of the additional degrees of freedom when switching from the \( Q_1 \) to the \( Q_2 \) and \( 4Q_1 \), respectively, element are not necessary to stabilize the \( Q_1/Q_1 \)-element. Moreover, by using the technique of reduced discretizations and eliminating the additional degrees of freedom we show the relationship between enlarging the velocity space and stabilized methods. This relationship has been established for triangular elements but was not known for quadrilateral elements. As a result we derive new stabilized methods for the Stokes and Navier–Stokes equations. Finally, we show how the Brezzi–Pitkäranta stabilization and the SUPG method for the incompressible Navier–Stokes equations can be recovered as special cases of the general approach. In contrast to earlier papers we do not restrict ourselves to linearized versions of the Navier–Stokes equations but deal with the full nonlinear case.

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1. INTRODUCTION

In this paper we introduce a general class of stable finite element spaces suitable for a numerical solution of the Stokes equations

\[-\nu \Delta \mathbf{u} + \nabla p = f, \quad \text{div} \, \mathbf{u} = 0 \quad \text{in} \; \Omega, \quad \mathbf{u} = 0 \quad \text{on} \; \partial \Omega, (1)\]

the Navier–Stokes equations

\[-\nu \Delta \mathbf{u} + (\nabla \mathbf{u}) \mathbf{u} + \nabla p = f, \quad \text{div} \, \mathbf{u} = 0 \quad \text{in} \; \Omega, \quad \mathbf{u} = 0 \quad \text{on} \; \partial \Omega (2)\]

or other problems describing incompressible materials. In the equations (1) and (2), \( \mathbf{u} \) is the velocity and \( p \) is the pressure in a linear viscous fluid contained in a bounded domain \( \Omega \subset \mathbb{R}^2 \) with a polygonal boundary \( \partial \Omega \).

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\(^1\) Institute of Numerical Mathematics, Faculty of Mathematics and Physics, Charles University, Malostranské náměstí 25, 118 00 Praha 1, Czech Republic. e-mail: knobloch@karlin.mff.cuni.cz

\(^2\) Institute of Analysis and Numerics, Otto von Guericke University, PF 4120, 39016 Magdeburg, Germany. e-mail: tobiska@mathematik.uni-magdeburg.de

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The parameter \( \nu > 0 \) is the kinematic viscosity and \( f \) is an external body force, e.g. the gravity. Denoting

\[
\begin{align*}
    a(u, v) &= \int_{\Omega} \nabla u \cdot \nabla v \, dx, \\
    n(u, w, v) &= \int_{\Omega} v \cdot (\nabla w) u \, dx, \\
    b(v, p) &= -\int_{\Omega} \rho \operatorname{div} v \, dx,
\end{align*}
\]

the usual weak formulation of (1) reads: Given \( \nu > 0 \) and \( f \in H^{-1}(\Omega)^2 \), find \( u \in H_0^1(\Omega)^2 \) and \( p \in L_0^2(\Omega) \) such that

\[
\nu a(u, v) + b(v, p) - b(u, q) = \langle f, v \rangle \quad \forall \, v \in H_0^1(\Omega)^2, \, q \in L_0^2(\Omega),
\]

(3)

where \( L_0^2(\Omega) \) consists of \( L^2(\Omega) \) functions having zero mean value on \( \Omega \). It can be shown that this problem has a unique solution (cf. [14], p. 80, Theorem 5.1). The weak formulation of (2) is given by

\[
\nu a(u, v) + n(u, u, v) + b(v, p) - b(u, q) = \langle f, v \rangle \quad \forall \, v \in H_0^1(\Omega)^2, \, q \in L_0^2(\Omega).
\]

(4)

The problem (4) has a solution which is unique if \( \nu \) is sufficiently large and/or \( f \) is sufficiently small (cf. [14], pp. 291 and 292).

A standard Galerkin finite element discretization of (3) reads: Find \( u_h \in V_h \) and \( p_h \in Q_h \) satisfying

\[
\nu a(u_h, v_h) + b(v_h, p_h) - b(u_h, q_h) = \langle f, v_h \rangle \quad \forall \, v_h \in V_h, \, q_h \in Q_h,
\]

(5)

where \( V_h \subset H_0^1(\Omega)^2 \) and \( Q_h \subset L_0^2(\Omega) \) are some finite element spaces defined using a triangulation \( T_h \) of \( \Omega \). In this paper, we shall consider only triangulations consisting of quadrilaterals \( T \) (cf. Sect. 2) and we shall use the spaces

\[
\begin{align*}
    V_h &= \{ v \in H_0^1(\Omega)^2; \, v \circ F_T \in Q_1(\hat{T})^2 \} \quad \forall \, T \in T_h, \\
    Q_h &= \{ q \in H^1(\Omega) \cap L^2(\Omega); \, q \circ F_T \in Q_1(\hat{T}) \} \quad \forall \, T \in T_h
\end{align*}
\]

for approximating the velocity and the pressure, respectively. Here, \( Q_1(\hat{T}) \) is the space of bilinear functions defined on the reference square \( \hat{T} \) and \( F_T \in Q_1(\hat{T})^2 \) is a one-to-one mapping which maps \( \hat{T} \) onto \( T \). It is well known that this pair of spaces does not satisfy the Babuška–Brezzi condition

\[
\exists \, \beta > 0 : \sup_{v_h \in V_h(\Omega)} \frac{b(v_h, q_h)}{\|v_h\|_{1, \Omega}} \geq \beta \|q_h\|_{0, \Omega} \quad \forall \, q_h \in Q_h, \, h > 0,
\]

(6)

which often causes that the problem (5) with \( V_h = V_1^h \) is not solvable or that its solution contains spurious oscillations. One way to suppress these oscillations and to assure the solvability is to add some extra terms to the discretization (5) (cf. e.g. [7, 9, 15, 19]). Another way is to enlarge the space \( V^h \) by a space \( V^2_h \) so that the Babuška–Brezzi condition is satisfied. Here we shall first consider the second possibility and construct a general class of spaces \( V^2_h \) assuring the fulfilment of the Babuška–Brezzi condition. Then we shall show that, for suitable spaces \( V^2_h \), the \( V^1_h \)-component of \( u_h \) and the function \( p_h \) are solutions of the stabilized methods of [9, 15].

In case of the mini element [1], which is defined by enriching continuous piecewise linear functions by cubic bubble functions, the close relation to the stabilized methods of [9, 15] was already discussed in [3, 18]. Similar results for a convection–diffusion equation were obtained in [6]. In an abstract framework, the equivalence between Galerkin methods with bubble functions and stabilized methods was investigated for linear problems in [2]. For the linearized incompressible Navier–Stokes equations, the relation between a Galerkin method with
the mini element and the streamline upwind Petrov–Galerkin method (SUPG) was studied in [12]. In [20], this relation was investigated for residual-free bubbles and it was shown for the triangular \( P_1 / P_1 \)-element that also the correct stabilization parameters in both the diffusion-dominated and the convection-dominated regimes can be recovered. However, generally, e.g. for the \( Q_1 / Q_1 \)-element considered here, a stabilization using residual-free bubbles is not equivalent to the SUPG method (cf. [8]). Finally, it was also shown that bubble functions can help to design new stabilized methods (cf. e.g. [11,13]).

There is a lot of further papers devoted to investigations of discretizations stabilized using bubble functions, but the most of them are restricted to triangular elements and to linear problems. In this paper, we deal with quadrilateral elements and, in addition, we consider more general functions than bubble functions. Apart from investigating the relations to some well-known stabilized methods, we shall also derive, eliminating a suitable quadrilateral elements and, in addition, we consider more general functions than bubble functions. Apart from the Babuška–Brezzi condition for the spaces \( V \)' that, for a modified discretization of the incompressible Navier–Stokes equations and a suitable choice of the parameter \( h \), the \( V \)-function orthogonal to the edges are used (see [5,10]).

We shall give explicit examples of spaces \( V^2_h \) to satisfy the Babuška–Brezi condition will be defined in a general way as

\[
V^2_h = \text{span}\{ \varphi_h^i t_h^i \}_{i=1}^{N_h},
\]

where \( \varphi_h^i \in H^1(\Omega) \) and \( t_h^i \in \mathbb{R}^2 \) are some suitable functions and vectors, respectively. The proof of the Babuška–Brezzi condition for the spaces \( V_h = V^1_h \oplus V^2_h \) and \( Q_h \), which uses some ideas of [4] and a modification of the Verfürth trick [22], requires that the functions \( \varphi_h^i \) have localized supports and that, for any \( \varphi_h^i \), there exists a point \( A^i_h \in \Omega \) such that

\[
\int_{\Omega} \frac{\partial \varphi_h^i}{\partial t_h^i} \, dx = \frac{\partial \varphi_h^i}{\partial t_h^i} (A_h^i) \int_{\Omega} \varphi_h^i \, dx \quad \forall \, q_h \in Q_h, \quad h \geq 1, \ldots, N_h, (7)
\]

\[
|q_h|_{1,T}^2 \leq C h^2 \sum_{A^i_h \in T} \left| \frac{\partial \varphi_h^i}{\partial t_h^i} (A^i_h) \right|^2 \forall \, q_h \in Q_h, T \in T_h. (8)
\]

We shall give explicit examples of spaces \( V^2_h \) such that (7) and (8) are fulfilled.

If \( A^i_h \) lies on an edge \( E \) of the triangulation \( T_h \), the corresponding function \( \varphi_h^i \) can be associated with \( E \) and we require that \( t_h^i \) is tangent to \( E \). In other words, vector functions associated with edges used to stabilize the \( Q_1 / Q_1 \)-element are tangent to the edges. This is not the case for a stabilization of finite elements with discontinuous pressure like the quadrilateral \( Q_1 / P_0 \)-element or the triangular \( P_1 / P_0 \)-element, where vector functions orthogonal to the edges are used (see [5,10]).

The plan of the paper is as follows. In Section 2, we introduce some notations and summarize the assumptions on the triangulations and the functions \( \varphi_h^i \) needed for proving the Babuška–Brezzi condition in Section 3. In Section 4, we give some examples of the functions \( \varphi_h^i \) and construct proper subspaces of the stable \( Q_2 / Q_1 \)-element and the stable \( 4Q_1 / Q_1 \)-element which satisfy the Babuška–Brezzi condition. Further, in this section, we also recover the stability of the \( Q_1 \)-bubble/\( Q_1 \)-element by Mons and Rogé [17]. We investigate discretizations obtained from (5) by eliminating the \( V^2_h \)-component of \( u_h \) in Section 5 and discuss the general framework between this technique and stabilized schemes. Particularly, we derive a new type of stabilization in Section 6 and show the equivalence to the stabilized methods of [9,15] in Sections 7 and 8. Finally, in Section 9, we show that, for a modified discretization of the incompressible Navier–Stokes equations and a suitable choice of the space \( V^2_h \), the \( V^1_h \)-component of \( u_h \) and the function \( p_h \) are solutions of the SUPG method analyzed in [12,21].

2. Assumptions and notations

We assume that we are given a family \( \{ T_h \} \) of triangulations of the domain \( \Omega \) parametrized by a positive parameter \( h \rightarrow 0 \) and having the following properties. Each triangulation \( T_h \) consists of a finite number of closed
convex quadrilaterals \( T \) (which will be often called elements in the following) such that \( h_T \equiv \text{diam}(T) \leq h \), \( T = \bigcup_{T \in T_h} T \) and any two different elements \( T_1, T_2 \in T_h \) are either disjoint or possess either a common vertex or a common edge. In order to prevent the elements from degenerating when \( h \) tends to zero, we assume that any triangle \( \bar{T} \), the vertices of which are three vertices of an element \( T \in T_h \), satisfies

\[
\frac{h_T}{\vartheta_T} \leq C_1,
\]

where

\[
h_T = \text{diam}(\bar{T}) \equiv \sup_{x, y \in \bar{T}} |x - y|, \quad \vartheta_T = \sup_{B \subset \bar{T}} \text{diam}(B)
\]

and the constant \( C_1 \) is independent of \( h \).

We introduce a reference Cartesian coordinate system with axes \( \hat{x}_1, \hat{x}_2 \) and we define a reference element \( \hat{T} = [0, 1]^2 \). For any \( T \in T_h \), we denote by \( F_T = (F_{T_1}, F_{T_2}) \) a fixed one-to-one mapping \( F_T \in Q_1(\hat{T})^2 \) which maps \( \hat{T} \) onto \( T \). Such a mapping always exists and the assumption (9) guarantees that

\[
|F_T|_{1, \infty, \hat{T}} \leq C h_T, \quad |F_T^{-1}|_{1, \infty, T} \leq C h_T^{-1} \quad \forall \ T \in T_h,
\]

where the constant \( C \) depends only on \( C_1 \). Thus, we have

\[
C h_T \|v \circ F_T\|_{0, \hat{T}} \leq \|v\|_{0, T} \leq \tilde{C} h_T \|v \circ F_T\|_{0, \hat{T}} \quad \forall \ v \in L^2(T), \ T \in T_h,
\]

\[
C \|v \circ F_T\|_{1, \hat{T}} \leq \|v\|_{1, T} \leq \tilde{C} \|v \circ F_T\|_{1, \hat{T}} \quad \forall \ v \in H^1(T), \ T \in T_h.
\]

We shall use the notation \( J_T(\hat{x}) = D F_T / D \hat{x}(\hat{x}) \) for the Jacobian matrix of \( F_T \).

In the following, we formulate general assumptions which are essential for the construction of the supplementary space \( V_h^2 \). Later, in Sections 4, 6, 7 and 9, we shall show how these assumptions can be satisfied in special cases.

We suppose that we are given functions \( \{ \tilde{\varphi}^\alpha \}_{\alpha \in \mathcal{P}} \subset H^1(\hat{T}) \) (where \( \mathcal{P} \) is some parameter set which is usually finite) such that, for any \( \alpha \in \mathcal{P} \), the function \( \tilde{\varphi}^\alpha \) vanishes on at least three edges of \( \hat{T} \) and there exists a point \( \hat{\alpha} \in \hat{T} \) different from the vertices of \( \hat{T} \) satisfying

\[
\int_{\hat{T}} \tilde{q} \tilde{\varphi}^\alpha \, d\hat{x} = \tilde{q}(\hat{\alpha}) \int_{\hat{T}} \tilde{\varphi}^\alpha \, d\hat{x} \quad \forall \ \tilde{q} \in Q_1(\hat{T}).
\]

(13)

Further, for any \( \alpha \in \mathcal{P} \), we introduce a unit vector \( \hat{\alpha} = (\hat{\alpha}_1, \hat{\alpha}_2) \) and we denote \( \hat{\alpha}^\beta = (\hat{\alpha}_1^\beta, \hat{\alpha}_2^\beta) \). If \( \hat{\alpha} \in \partial \hat{T} \), we require that \( \hat{\alpha}^\beta \) coincides with the direction of the edge of \( \hat{T} \) containing \( \hat{\alpha} \). We admit \( \tilde{\varphi}^\alpha = \tilde{\varphi}^\beta \) for \( \alpha \neq \beta \) in order to be able to use the same function \( \tilde{\varphi}^\alpha \) with two different directions \( \hat{\alpha} \). For formal reasons, we also admit \( \tilde{\varphi}^\alpha = 0 \). In this case, (13) is automatically satisfied for any point \( \hat{\alpha} \).

Now, using the mappings \( F_T \), we transform the functions \( \tilde{\varphi}^\alpha \) onto elements \( T \) of a triangulation \( T_h \) and introduce finite element functions \( \varphi^i_h \in H^1_0(\Omega) \setminus \{0\} \), \( i = 1, \ldots, N_h \), having their supports always in one or two elements. Precisely, we assume that, for any \( i \in \{1, \ldots, N_h\} \), either

\[
\exists \ \ T \in T_h, \ \alpha \in \mathcal{P} : \quad F_T(\hat{\alpha}) \in \partial \Omega \cup \text{int} T, \quad \varphi^i_h|_T = \tilde{\varphi}^\alpha \circ F_T^{-1}, \quad \varphi^i_h|_{\Omega \setminus T} = 0
\]

(14)

or

\[
\exists \ T, T' \in T_h, \ \alpha, \alpha' \in \mathcal{P} : \quad T \cap T' = \{ \text{edge} \}, \quad F_T(\hat{\alpha}) = F_{T'}(\hat{\alpha}'), \quad \varphi^i_h|_T = \tilde{\varphi}^\alpha \circ F_T^{-1}, \quad \varphi^i_h|_{T'} = \tilde{\varphi}^{\alpha'} \circ F_{T'}^{-1}, \quad \varphi^i_h|_{\Omega \setminus (T \cup T')} = 0.
\]

(15)
The assumption \( F_T(\widehat{A}^\alpha) = F_T(\widehat{A}^\alpha') \) in (15) implies that \( \widehat{A}^\alpha, \widehat{A}^\alpha' \in \partial T \). Note that the function \( \varphi_h^i \) defined by (15) may vanish on one of the elements \( T, T' \). In both cases (14) and (15), we set

\[
A_h^i = F_T(\widehat{A}^\alpha), \quad t_h^i = J_T(\widehat{A}^\alpha) \widehat{t}^\alpha / |J_T(\widehat{A}^\alpha) \widehat{t}^\alpha |
\]

and we denote

\[
\widehat{A}_h^i = \widehat{A}^\alpha, \quad \widehat{t}_h^i = \widehat{t}^\alpha, \quad \widehat{n}_h^i = \widehat{n}^\alpha
\]

(and \( \widehat{A}_h^j = \widehat{A}^\alpha', \quad \widehat{t}_h^j = \widehat{t}^\alpha', \quad \widehat{n}_h^j = \widehat{n}^\alpha' \)).

In the case of (15), we then also have \( A_h^i = F_T(\widehat{A}^\alpha') \) and

\[
t_h^i = J_T(\widehat{A}^\alpha') \widehat{t}^\alpha' / |J_T(\widehat{A}^\alpha') \widehat{t}^\alpha'| \quad \text{or} \quad t_h^i = -J_T(\widehat{A}^\alpha') \widehat{t}^\alpha' / |J_T(\widehat{A}^\alpha') \widehat{t}^\alpha'|.
\]

We suppose that the functions \( \{ \varphi_h^i t_h^i \}_{i=1}^{N_h} \) are linearly independent and that

\[
\text{card}\{ i \in \{1, \ldots, N_h\}; \ A_h^i \in T \} \leq C_2 \quad \forall \ T \in \mathcal{T}_h,
\]

where the constant \( C_2 \) is independent of \( h \). The support of any function \( \varphi_h^i \) is contained in the union of the elements containing the point \( A_h^i \) which will be denoted by \( P_h^i \). Thus, \( P_h^i \) consists of one or two elements. Further, we introduce the quantities

\[
\gamma_h^i = \left| \frac{\int_{\Omega} \varphi_h^i dx}{|P_h^i(\varphi_h^i)_{1,\Omega}|} \right|, \quad i = 1, \ldots, N_h, \quad \gamma_h = \min_{i=1,\ldots,N_h} \gamma_h^i
\]

which influence the magnitude of the constant in the Babuška–Brezzi condition. Defining the functions \( \varphi_h^i \) in a suitable way, the value of \( \gamma_h \) can be made arbitrarily small. However, arbitrarily large values of \( \gamma_h \) cannot be obtained. It can be shown that \( \gamma_h \leq 2C_1 \) and, if \( \mathcal{T}_h \) consists of rectangles, we even have \( \gamma_h \leq 1 \).

Finally, we introduce an assumption ensuring the validity of (8). We assume that, for any \( T \in \mathcal{T}_h \), there exist points \( A_h^i, A_h^j, A_h^k \in T \) (with \( i, j, k \in \{1, \ldots, N_h\} \)) such that the number

\[
\widehat{S}_T^{ijk} = (\widehat{n}_T^i \times \widehat{n}_T^j) \widehat{n}_T^k \cdot \widehat{A}_h^k + (\widehat{n}_T^i \times \widehat{n}_T^k) \widehat{n}_T^j \cdot \widehat{A}_h^j + (\widehat{n}_T^j \times \widehat{n}_T^k) \widehat{n}_T^i \cdot \widehat{A}_h^i
\]

(16)

satisfies

\[
|\widehat{S}_T^{ijk}| \geq C_3 > 0,
\]

(17)

where the constant \( C_3 \) is independent of \( T \) and \( h \), the vector product \( a \times b \) is defined as \( a_1 b_2 - b_1 a_2 \) and \( \widehat{n}_T^i \cdot \widehat{A}_h^i \) is defined as \( \widehat{n}_T^i \cdot (\widehat{A}_h^i - 0) \).

**Remark 1.** Using the identity \( (\widehat{n}_T^i \times \widehat{n}_T^j) \widehat{n}_T^k + (\widehat{n}_T^i \times \widehat{n}_T^k) \widehat{n}_T^j + (\widehat{n}_T^j \times \widehat{n}_T^k) \widehat{n}_T^i = 0 \), we have

\[
\widehat{S}_T^{ijk} = (\widehat{n}_T^i \times \widehat{n}_T^j) \widehat{n}_T^k \cdot (\widehat{A}_h^k - \widehat{A}_h^j) + (\widehat{n}_T^j \times \widehat{n}_T^k) \widehat{n}_T^i \cdot (\widehat{A}_h^i - \widehat{A}_h^j).
\]

If, particularly, \( \widehat{n}_T^i \) and \( \widehat{n}_T^j \) are equal and orthogonal to \( \widehat{n}_T^k \), then

\[
|\widehat{S}_T^{ijk}| = |\widehat{n}_T^k \cdot (\widehat{A}_h^i - \widehat{A}_h^j)|,
\]

which illustrates the meaning of (17).
Remark 2. If $\mathcal{T}_h$ consists of parallelograms, it is sufficient for proving the Babuška–Brezzi condition to assume

$$\hat{n}^\alpha \cdot \int_{\hat{T}} (\hat{x} - \hat{A}^\alpha) \hat{\varphi}^\alpha(\hat{x}) \, d\hat{x} = 0$$

instead of (13) (cf. Remark 6 in Sect. 3). Functions satisfying the property (18) are easier to construct than those ones satisfying (13).

Remark 3. Let $\hat{\varphi}^\alpha$ be given by a formula which is invariant to which vertex of $\hat{T}$ is chosen as the origin of the coordinate system $\hat{x}_1, \hat{x}_2$ (with axes in the directions of edges of $\hat{T}$). Let $\{\hat{q}^i\}^4_{i=1}$ be a basis of $Q_1(\hat{T})$ consisting of bilinear functions equal to 0 in three vertices of $\hat{T}$ and equal to 1 in the remaining vertex. Then $\int_{\hat{T}} \hat{\varphi}^\alpha \hat{q}^i \, d\hat{x} = \int_{\hat{T}} \hat{\varphi}^\alpha \hat{q}^i \, d\hat{x}$ for $i = 2, 3, 4$ and since $\sum_{i=1}^4 \hat{q}^i = 1$, we infer that $\int_{\hat{T}} \hat{\varphi}^\alpha \hat{q}^i \, d\hat{x} = \frac{1}{4} \int_{\hat{T}} \hat{\varphi}^\alpha \, d\hat{x}, \ i = 1, \ldots, 4$. Any $\hat{q} \in Q_1(\hat{T})$ can be written as $\hat{q} = \sum_{i=1}^4 \alpha^i \hat{q}^i$ and hence $\int_{\hat{T}} \hat{\varphi}^\alpha \hat{q} \, d\hat{x} = \frac{1}{4} \sum_{i=1}^4 \alpha^i \int_{\hat{T}} \hat{\varphi}^\alpha \, d\hat{x} = \hat{q}(\hat{C}_T) \int_{\hat{T}} \hat{\varphi}^\alpha \, d\hat{x}$, where $\hat{C}_T = (\frac{1}{2}, \frac{1}{2})$ is the barycentre of $\hat{T}$. Thus, (13) holds with $\hat{A}^\alpha = \hat{C}_T$. An example of such an invariant function $\hat{\varphi}^\alpha$ is the biquadratic function

$$\hat{\varphi}^\alpha(\hat{x}) = \hat{x}_1 (1 - \hat{x}_1) \hat{x}_2 (1 - \hat{x}_2).$$

Remark 4. It is not necessary to construct invariant functions $\hat{\varphi}^\alpha$ to satisfy (13). An example of a non-invariant function satisfying the relation (13) is the biquadratic function

$$\hat{\varphi}^\alpha(\hat{x}) = \hat{x}_1 (1 - \hat{x}_1) (1 - \hat{x}_2) (1/2 - \hat{x}_2),$$

for which $\hat{A}^\alpha = (1/2, 0)$.

Remark 5. If $A^i_h$ lies on an edge of some element of the triangulation $\mathcal{T}_h$, then $t^i_h$ is a unit vector in the direction of this edge. Therefore, the derivative $\frac{\partial q_h}{\partial t^i_h}(A^i_h)$ is well defined for any $q_h \in Q_h$ and any $i \in \{1, \ldots, N_h\}$. That is essential for our proceeding in the following section.

3. PROOF OF THE BABUŠKA–BREZZI CONDITION

In this section, we prove that, under the assumptions made in Section 2, the spaces $V_h \equiv V^1_h \oplus V^2_h$ and $Q_h$ satisfy the Babuška–Brezzi condition with a constant proportional to $\gamma_h$. First, in Lemmas 1 and 2, we prove the validity of (7) and (8). Then, in Lemma 3, we establish a Babuška–Brezzi condition with a ‘wrong’ norm of $q_h$ and, finally, in Theorem 1, we prove the desired Babuška–Brezzi condition applying the modified Verfürth trick.

Lemma 1. We have

$$\int_{\Omega} \frac{\partial q_h}{\partial t^i_h}(A^i_h) \int_{\Omega} \varphi^\alpha_h \, dx \quad \forall \ q_h \in Q_h, \ i \in \{1, \ldots, N_h\}.$$  \hspace{1cm} (19)

Proof. Consider any $T \subset P_h^i$ and $q_h \in Q_h$ and set $\hat{q}_T = q_h \circ F_T$. Since

$$(\nabla q_h)(F_T(\hat{x})) = J_T(\hat{x})^{-T} \hat{\nabla} \hat{q}_T(\hat{x}) \quad \forall \ \hat{x} \in \hat{T},$$ \hspace{1cm} (20)

where $\hat{\nabla} = (\partial / \partial \hat{x}_1, \partial / \partial \hat{x}_2)^T$, we have

$$\int_T \varphi^\alpha_h t^i_h \cdot \nabla q_h \, dx = \int_{\hat{T}} \hat{\varphi}^\alpha \hat{t}^i_h \cdot J_T^{-T} \hat{\nabla} \hat{q}_T \cdot | \det J_T | \, d\hat{x}. \hspace{1cm} (21)$$
It is easy to verify that

\[ J_T^{-1} = \frac{1}{\det J_T} \begin{pmatrix} \frac{\partial F_T}{\partial x_2} & \frac{\partial F_T}{\partial x_1} \\ -\frac{\partial F_T}{\partial x_2} & \frac{\partial F_T}{\partial x_1} \end{pmatrix} . \]

Since the \( \hat{x}_1 \)-derivative of a function from \( Q_1(\hat{T}) \) is a linear function of \( \hat{x}_2 \) which does not depend on \( \hat{x}_1 \) (and similarly for the \( \hat{x}_2 \)-derivative), we infer that (det \( J_T \)) \( J_T^{-1} \hat{\nabla} \hat{q}_T \in Q_1(\hat{T}) \). Using the fact that det \( J_T \neq 0 \) on \( \hat{T} \), it follows from (21) and (13) that

\[ \int_{\hat{T}} \varphi_h \cdot \nabla q_h \, dx = t_h \cdot (J_T^{-1} \hat{\nabla} \hat{q}_T \mid \text{det} J_T)((\tilde{A}^\alpha)) \int_{\hat{T}} \varphi^\alpha \, d\hat{x}. \]

Applying (13) with \( \hat{q} = \mid \text{det} J_T \mid \) and using (20), we get

\[ \int_{\hat{T}} \frac{\partial q_h}{\partial t_h} \varphi_h \, dx = t_h \cdot (\nabla q_h)(A^t_h) \int_{\hat{T}} \varphi^\alpha \mid \text{det} J_T \mid d\hat{x} = \frac{\partial q_h}{\partial t_h}(A^t_h) \int_{\hat{T}} \varphi_h \, dx. \]

\[ \square \]

**Remark 6.** If \( T_h \) consists of parallelograms, then \( J_T = \text{const.} \) and it follows from (21) that

\[ \int_{\hat{T}} \varphi_h \cdot t_h \cdot \nabla q_h \, dx = \int_{\hat{T}} \varphi^\alpha \cdot \hat{\nabla} \hat{q}_T \, d\hat{x} \mid \text{det} J_T \mid \mid J_T \hat{t}^\alpha \mid , \]

where we assume that \( t_h = J_T \hat{t}^\alpha / \mid J_T \hat{t}^\alpha \mid \) (if \( t_h = -J_T \hat{t}^\alpha / \mid J_T \hat{t}^\alpha \mid \), we can proceed analogously). Denoting \( \hat{q}_T(\hat{x}) = \xi_0 + \xi_1 \hat{x}_1 + \xi_2 \hat{x}_2 + \xi_3 \hat{x}_1 \hat{x}_2 \) and \( \hat{A}^\alpha = (\hat{a}_1, \hat{a}_2) \), we have for \( \hat{x} \in \hat{T} \)

\[ \frac{\partial q_T}{\partial \hat{x}_1}(\hat{x}) = \frac{\partial q_T}{\partial \hat{x}_1}(\hat{A}^\alpha) + \xi_3(\hat{x}_2 - \hat{a}_2), \quad \frac{\partial q_T}{\partial \hat{x}_2}(\hat{x}) = \frac{\partial q_T}{\partial \hat{x}_2}(\hat{A}^\alpha) + \xi_3(\hat{x}_1 - \hat{a}_1). \]

Hence

\[ \hat{t}^\alpha \cdot \hat{\nabla} \hat{q}_T(\hat{x}) = \hat{t}^\alpha \cdot \hat{\nabla} \hat{q}_T(\hat{A}^\alpha) + \xi_3 \hat{n}^\alpha \cdot (\hat{x} - \hat{A}^\alpha) \]

and we see that, for proving (19), it suffices to assume (18) instead of (13).

**Lemma 2.** There exists a constant \( C_4 \) independent of \( h \) such that, for any \( T \in T_h \), we have

\[ |q_h|_{1,T}^2 \leq C_4 \cdot h_T^2 \sum_{i=1}^{N_T} \left| \frac{\partial q_h}{\partial A_i^t}(A^t_h) \right|^2 \quad \forall q_h \in \text{Q}_h. \]  

(22)

**Proof.** Let \( \hat{A}^t \in \hat{T} \), \( \hat{n}^i \in \mathbb{R}^2 \), \( |\hat{n}^i| = 1 \), \( i = 1, 2, 3 \), be points and vectors satisfying (17) and let again \( \hat{t}^i = (\hat{n}^i_2, \hat{n}^i_1) \), \( i = 1, 2, 3 \). Let us set for \( \hat{x} \in \hat{T} \) and \( i = 1, 2, 3 \)

\[ \hat{q}^i(\hat{x}) = \{ \hat{n}^i+1 (\hat{n}^{i-1} \cdot \hat{A}^{i-1}) - \hat{n}^{i-1} (\hat{n}^{i+1} \cdot \hat{A}^{i+1}) \} \cdot (\hat{x}_1, \hat{x}_2) + (\hat{n}^{i+1} \times \hat{n}^{i-1}) \hat{x}_1 \hat{x}_2 . \]
Lemma 3. We have

\[ \sup_{v_h \in V_h \setminus \{0\}} \frac{b(v_h, q_h)}{|v_h|_{1, \Omega}} \geq \frac{\gamma_h}{2 C_1 \sqrt{C_2 C_4}} \sqrt{\sum_{T \in T_h} h_T^2 |q_h|_{1, T}^2}, \quad \forall q_h \in Q_h. \]
Proof. Consider any \( v_h \in V_h^2 \). Then \( v_h = \sum_{i=1}^{N_h} \alpha^i \varphi_h^i \) for some real numbers \( \alpha^i \). We can assume that \( \int_{\Omega} \varphi_h^i \, dx \neq 0 \) for \( i = 1, \ldots, N_h \) since otherwise \( \gamma_h = 0 \) and (25) holds. Thus, we have because of the definitions of \( C_2 \) and \( \gamma_h \)

\[
|v_h|_{1,\Omega}^2 = \sum_{T \in \mathcal{T}_h} |v_h|_{1,T}^2 \leq C_2 \sum_{T \in \mathcal{T}_h} \sum_{i=1}^{N_h} |\alpha^i|^2 |\varphi_h^i|_{1,T}^2 \leq \frac{C_2}{\gamma_h} \sum_{i=1}^{N_h} |\alpha^i|^2 \frac{\int_{\Omega} \varphi_h^i \, dx}{|P_h|^2}.
\]

(26)

Applying Lemma 1, we obtain for any \( q_h \in Q_h \)

\[
b(v_h, q_h) = \int_{\Omega} v_h \cdot \nabla q_h \, dx = \sum_{i=1}^{N_h} \alpha^i \int_{\Omega} \frac{\partial q_h}{\partial x^i} \varphi_h^i \, dx = \sum_{i=1}^{N_h} \alpha^i \frac{\partial q_h}{\partial x^i} (A_h) \int_{\Omega} \varphi_h^i \, dx.
\]

Choosing

\[
\alpha^i = \frac{|P_h|^2}{\int_{\Omega} \varphi_h^i \, dx} \frac{\partial q_h}{\partial x^i} (A_h),
\]

it follows that

\[
b(v_h, q_h) = \sum_{i=1}^{N_h} |\alpha^i|^2 \frac{\int_{\Omega} \varphi_h^i \, dx}{|P_h|^2} \leq \left( \sum_{i=1}^{N_h} |\alpha^i|^2 \frac{\int_{\Omega} \varphi_h^i \, dx}{|P_h|^2} \right)^\frac{1}{2} \left( \sum_{i=1}^{N_h} \frac{|P_h|^2}{|\partial q_h/\partial x^i (A_h)|^2} \right)^\frac{1}{2},
\]

which implies (25) owing to (26), (9) and (22). \( \square \)

**Theorem 1.** There exists a constant \( C_5 > 0 \) independent of \( h \) such that the spaces \( V_h = V_h^1 \oplus V_h^2 \) and \( Q_h \) satisfy the Babuška–Brezzi condition

\[
\sup_{v_h \in V_h \setminus \{0\}} \frac{b(v_h, q_h)}{|v_h|_{1,\Omega}^2} \geq C_5 \gamma_h \|q_h\|_{0,\Omega} \quad \forall \; q_h \in Q_h.
\]

(27)

**Proof.** Applying the modified Verfürth trick presented in [7], pp. 255–256, we obtain

\[
\sup_{v_h \in V_h \setminus \{0\}} \frac{b(v_h, q_h)}{|v_h|_{1,\Omega}^2} \geq C \|q_h\|_{0,\Omega} - \sqrt{\sum_{T \in \mathcal{T}_h} h_T^2 |q_h|_{1,T}^2} \quad \forall \; q_h \in Q_h
\]

(28)

with a constant \( C > 0 \) independent of \( h \) and the theorem follows from Lemma 3 and the bound \( \gamma_h \leq 2 C_1 \). For completeness, we recall the main arguments leading to (28). Since the spaces \( H_0^1(\Omega) \) and \( L_0^2(\Omega) \) satisfy the inf–sup condition (6), we have

\[
\beta \|q_h\|_{0,\Omega} \leq \sup_{v \in H_0^1(\Omega) \setminus \{0\}} \frac{b(v, q_h)}{|v|_{1,\Omega}} \quad \forall \; q_h \in Q_h.
\]

(29)

Using an operator \( \pi_h : H_0^1(\Omega)^2 \to V_h \) satisfying

\[
|\pi_h v|_{1,\Omega} \leq \tilde{C} |v|_{1,\Omega}, \quad \sqrt{\sum_{T \in \mathcal{T}_h} h_T^{-2} \|v - \pi_h v\|_{0,T}^2} \leq \tilde{C} |v|_{1,\Omega} \quad \forall \; v \in H_0^1(\Omega)^2,
\]
where \( \tilde{C} \) is independent of \( h \), we get for any \( v \in H^1_0(\Omega)^2 \) and \( q_h \in Q_h 
abla v q_h \ dx \leq \tilde{C} |v|_{1,\Omega} \sqrt{\sum_{T \in T_h} h_T^2 |q_h|^2_{1,T}} , 
abla \),

\[
b(\pi_h v, q_h) \leq \tilde{C} |v|_{1,\Omega} \frac{|b(\pi_h v, q_h)|}{|\pi_h v|_{1,\Omega}} \leq \tilde{C} |v|_{1,\Omega} \sup_{v_h \in V_h(\Omega)} \frac{b(v_h, q_h)}{|v_h|_{1,\Omega}} .
\]

Substituting the sum of these two relations into (29), we derive (28).

\[\square\]

Remark 7. Let \( \int_T \varphi^\alpha \, d\tilde{x} \geq 0 \) for any \( \alpha \in \mathcal{P} \). Consider any \( T \in T_h \) and \( \alpha \in \{1, \ldots, N_h\} \) and let \( \varphi^\alpha_h |_T = \varphi^\alpha \circ F_T^{-1} \) for some \( \alpha \in \mathcal{P} \). Then, according to (13),

\[
\int_T \varphi^\alpha_h \, dx = \int_T \varphi^\alpha | \det J_T | \, d\tilde{x} = | \det J_T(\tilde{\varphi}^\alpha) | \int_T \varphi^\alpha \, d\tilde{x}
\]

and hence, in view of (10) and (9),

\[
\int_\Omega \varphi^\alpha_h \, dx \geq C |P^\alpha_h| \inf_{\alpha \in \mathcal{P}, \varphi^\alpha \neq 0} \int_T \varphi^\alpha \, d\tilde{x} , \quad i = 1, \ldots, N_h ,
\]

where \( C > 0 \) is a constant independent of \( h \). Finally, applying (12), we get

\[
\gamma_h \geq C \frac{\inf_{\alpha \in \mathcal{P}, \varphi^\alpha \neq 0} \int_T \varphi^\alpha \, d\tilde{x}}{\sup_{\alpha \in \mathcal{P}} |\varphi^\alpha|_{1,T}} ,
\]

where \( C > 0 \) is again independent of \( h \). Thus, if \( \mathcal{P} \) is finite and \( \int_T \varphi^\alpha \, d\tilde{x} > 0 \) for any \( \varphi^\alpha \neq 0 \), then \( \gamma_h \geq C > 0 \) with \( C \) independent of \( h \).

4. Examples of stable elements implying the stability of known elements

In this section, we derive several explicit examples of supplementary spaces \( V_h^2 \) such that the pair \( V_h = V_h^1 \oplus V_h^2 \), \( Q_h \) satisfies the Babuška–Brezzi condition. All examples which will be discussed in this section can be divided into three classes. The first class is characterized by the fact that \( V_h^1 \oplus V_h^2 \) is a proper subspace of the space

\[
V_h^{Q2} = \{ v \in H^1_0(\Omega)^2; v \circ F_T \in Q_2(\hat{T})^2 \ \forall \ T \in T_h \} ,
\]

where \( Q_2(\hat{T}) \) is the space of biquadratic functions defined on \( \hat{T} \). In this way, we get an alternative proof for the stability of the \( Q_2/Q_1 \)-element. The second class will be constructed such that \( V_h^1 \oplus V_h^2 \) is a proper subspace of the space

\[
V_h^{Q1} = \{ v \in H^1_0(\Omega)^2; v \circ F_T \in Q_1(\hat{T})^2 \ \forall \ T \in T_{h/2} \} ,
\]

where \( T_{h/2} \) is a triangulation obtained from \( T_h \) by dividing each quadrilateral \( T \in T_h \) into four quadrilaterals connecting the midpoints of opposite edges of \( T \). This also represents a new proof for the stability of the \( 4Q_1/Q_1 \)-element. The basic feature of the last class is that the space \( V_h^2 \) consists of bubble functions, \( i.e., \), any function from \( V_h^2 \) vanishes on all edges of the triangulation \( T_h \). Here we recover the stability of the \( Q_1 \)-bubble/\( Q_1 \)-element [17] by our general approach.
We start with the first class and introduce functions $\tilde{\varphi}^\alpha$ and points $\tilde{A}^\alpha$ satisfying the relation (13). As we have already seen in Remark 4, the biquadratic function

$$\tilde{\varphi}^1(\tilde{x}) = \tilde{x}_1 (1 - \tilde{x}_1) (1 - \tilde{x}_2) (1/2 - \tilde{x}_2)$$

satisfies (13) with $\tilde{A}^1 = (1/2, 0)$. Analogously we define the biquadratic functions $\tilde{\varphi}^2$, $\tilde{\varphi}^3$ and $\tilde{\varphi}^4$ satisfying (13) with $\tilde{A}^\alpha$ equal to $(1, 1/2)$, $(1/2, 1)$ and $(0, 1/2)$, respectively. Further, we define the functions

$$\tilde{\varphi}^5(\tilde{x}) = \tilde{\varphi}^6(\tilde{x}) = \tilde{x}_1 (1 - \tilde{x}_1) \tilde{x}_2 (1 - \tilde{x}_2)$$

which satisfy (13) with $\tilde{A}^5 = \tilde{A}^6 = (1/2, 1/2)$ (cf. Remark 3). Finally, we introduce the vectors $\tilde{t}^1 = \tilde{t}^3 = \tilde{t}^5 = (1, 0)$ and $\tilde{t}^2 = \tilde{t}^4 = \tilde{t}^6 = (0, 1)$. Now, we follow the lines of Section 2 and construct the functions $\varphi^\alpha$, $\psi^\alpha$ and vectors $t^\alpha_i$ generating the space $V_h^\alpha$. Each $\varphi^\alpha$ is constructed using either the functions $\varphi^1$, $\varphi^2$, $\varphi^3$ or the functions $\varphi^5$, $\varphi^6$. In the former case, the point $A^\alpha_i$ is the midpoint of an inner edge of $T_h$, $t^\alpha_i$ is a unit vector in the direction of this edge and $\text{supp} \varphi^\alpha$ consists of the two elements containing $A^\alpha_i$. In the latter case, the point $A^\alpha_i \equiv F_T(1/2, 1/2)$ lies in the interior of an element $T$, $t^\alpha_i$ is a unit vector in the direction determined by the midpoints of two opposite edges of $T$ and $\text{supp} \varphi^\alpha = T$.

The last assumption for satisfying the Babuška–Brezzi condition is the relation (17). According to Remark 1, this relation is satisfied if, for any element $T$, there exist three points $A^\alpha_i$, $A^\alpha_j$, $A^\alpha_k$ in $T$ such that the corresponding vectors $\tilde{t}^\alpha_i$, $\tilde{t}^\alpha_j$ and $\tilde{t}^\alpha_k$ are not equal. For example, it is sufficient if, for a given element $T$, there exist three functions $\varphi^\alpha_i$ with $A^\alpha_i \in \partial T$ or if there exist one function $\varphi^\alpha_i$ with $A^\alpha_i \in \partial T$ and two functions $\varphi^\alpha_j$ with $A^\alpha_j \in \text{int} T$. Thus, using the above defined functions $\varphi^\alpha_i$ and vectors $t^\alpha_i$, we are able to define various spaces $V^\alpha_h$ such that the spaces $V^\alpha_h = V^\alpha_1 \oplus V^\alpha_2$ and $Q^\alpha_h$ satisfy the Babuška–Brezzi condition (27). Moreover, in all these cases, the Babuška–Brezzi condition holds uniformly with respect to $h$ (cf. Remark 7). As a simple consequence, we also see that, owing to

$$V_h = V^\alpha_1 \oplus V^\alpha_2 \subset V^{Q_1}_h,$$

the Babuška–Brezzi condition is also satisfied for the spaces $V^{Q_1}_h$, $Q^\alpha_h$. The remarkable aspect of the new class of elements described above is that the $Q_2/Q_1$–element remains stable if more than one half of the basis functions from the velocity space are dropped. Particularly, the functions $\varphi^\alpha_i$ defined using $\varphi^5$ and $\varphi^6$ are needed for the validity of the Babuška–Brezzi condition only on those elements which have two or three edges on $\partial \Omega$.

We now derive the second class of elements implying the stability of the $4Q_1/Q_1$–element. The construction of $V^\alpha_2$ is similar to the class above, however, we have to use piecewise bilinear functions instead of biquadratic functions $\tilde{\varphi}^\alpha$. First, we introduce the functions

$$\varphi(\tilde{x})_1 = \begin{cases} \tilde{x}_1 & \text{for } \tilde{x}_1 \in [0, \frac{1}{2}], \\ 1 - \tilde{x}_1 & \text{for } \tilde{x}_1 \in [\frac{1}{2}, 1], \end{cases} \quad \varphi(\tilde{x})_2 = \begin{cases} 3 - 7 \tilde{x}_2 & \text{for } \tilde{x}_2 \in [0, \frac{1}{2}], \\ \tilde{x}_2 - 1 & \text{for } \tilde{x}_2 \in [\frac{1}{2}, 1]. \end{cases}$$

Setting $\tilde{\varphi}^1(\tilde{x}) = \varphi(\tilde{x})_1 \tilde{\varphi}(\tilde{x})_2$, we obtain a function which is piecewise bilinear with respect to a subdivision of $\tilde{T}$ into four equal squares and which satisfies (13) with $\tilde{A}^1 = (\frac{1}{2}, 0)$. Analogously we define the functions $\tilde{\varphi}^2$, $\tilde{\varphi}^3$ and $\tilde{\varphi}^4$ satisfying (13) with the same points as in the biquadratic case. The functions $\tilde{\varphi}^5$ and $\tilde{\varphi}^6$ are now piecewise bilinear functions which vanish on the boundary of $\tilde{T}$ and are equal 1 in the point $(1/2, 1/2)$. According to Remark 3, the functions $\tilde{\varphi}^5$, $\tilde{\varphi}^6$ satisfy (13) with $\tilde{A}^5 = \tilde{A}^6 = (1/2, 1/2)$. Now we can proceed in the same way as in the biquadratic case and construct various spaces $V^\alpha_h$ which guarantee the fulfillment of the Babuška–Brezzi condition. The assumption (17) can be satisfied as in the case of the first class. In all possible cases, we have $V^\alpha_1 \oplus V^\alpha_2 \subset V^{Q_1}_h$ and hence we particularly infer that the $4Q_1/Q_1$–element satisfies
the Babuška–Brezzi condition with a constant independent of \( h \). Again, the \( 4 \mathcal{Q}_1/\mathcal{Q}_1 \)-element remains stable if more than one half of the basis functions from the velocity space are dropped.

As an example of the third class mentioned at the beginning of this section, we shall investigate the \( \mathcal{Q}_1 \)-bubble/\( \mathcal{Q}_1 \)-element by Mons and Rogé [17]. To describe the space \( \mathcal{V}_h^2 \), we divide the reference element \( \tilde{T} \) into the triangles \( \tilde{T}_1, \tilde{T}_2 \) having the vertices \((0,0), (1,0), (0, 1) \) and \((1,0), (1,1), (0,1) \), respectively. Denoting

\[
\tilde{\psi}^1(\tilde{x}) = \begin{cases} 
\tilde{x}_1 \tilde{x}_2 (1 - \tilde{x}_1 - \tilde{x}_2) & \text{for } \tilde{x} \in \tilde{T}_1, \\
0 & \text{for } \tilde{x} \in \tilde{T}_2,
\end{cases}
\]

\[
\tilde{\psi}^2(\tilde{x}) = \begin{cases} 
0 & \text{for } \tilde{x} \in \tilde{T}_1, \\
(1 - \tilde{x}_1)(1 - \tilde{x}_2)(\tilde{x}_1 + \tilde{x}_2 - 1) & \text{for } \tilde{x} \in \tilde{T}_2,
\end{cases}
\]

we have

\[
\mathcal{V}_h^2 = \{ v \in H_0^1(\Omega)^2; v \circ F_T \in [\text{span}\{\tilde{\psi}^1, \tilde{\psi}^2]\}^2 \ \forall \ T \in \mathcal{T}_h \}.
\]

We want to show that the stability of the \( \mathcal{Q}_1 \)-bubble/\( \mathcal{Q}_1 \)-element follows from our general theory. For this, we cannot use the functions \( \tilde{\psi}^1, \tilde{\psi}^2 \) since they do not satisfy (13) for any points \( \tilde{A}_1, \tilde{A}_2 \). Therefore, we introduce new basis functions

\[
\tilde{\varphi}^1 = \mu \tilde{\psi}^1 + \tilde{\psi}^2, \quad \tilde{\varphi}^2 = \tilde{\psi}^1 + \mu \tilde{\psi}^2, \quad \mu = \frac{5 + \sqrt{21}}{2},
\]

for which (13) holds with

\[
\tilde{A}_1 = \left( \frac{21 - \sqrt{21}}{42}, \frac{21 - \sqrt{21}}{42} \right), \quad \tilde{A}_2 = \left( \frac{21 + \sqrt{21}}{42}, \frac{21 + \sqrt{21}}{42} \right),
\]

respectively. Further, we set \( \varphi^3 = \varphi^1, \varphi^4 = \varphi^2, \tilde{A}_3 = \tilde{A}_1, \tilde{A}_4 = \tilde{A}_2, \tilde{t}^1 = \tilde{t}^2 = (1,0), \tilde{t}^3 = \tilde{t}^4 = (0,1) \). Defining functions \( \varphi^1_h \) and vectors \( t^i_h, i = 1, \ldots, N_h, \) \( N_h = 4 \) card \( \mathcal{T}_h \), as in Section 2, we deduce that

\[
\mathcal{V}_h^2 = \text{span}\{\varphi^i_h t^i_h\}_{i=1}^{N_h}.
\]

It is easy to check that all the assumptions made in Section 2 are fulfilled and hence it follows from Theorem 1 and Remark 7 that the \( \mathcal{Q}_1 \)-bubble/\( \mathcal{Q}_1 \)-element satisfies the Babuška–Brezzi condition with a constant independent of \( h \).

5. GENERAL RELATIONSHIP BETWEEN ENLARGING THE VELOCITY SPACE AND STABILIZING THE CONTINUITY EQUATION

It is well known that a standard Galerkin finite element discretization of the Stokes or Navier–Stokes equations with the spaces \( \mathcal{V}_h^1 \) and \( \mathcal{Q}_h \) cannot be used because of failing the Babuška–Brezzi condition (6). We have already seen in Section 4 that, enlarging the velocity space by \( \mathcal{V}_h^2 \), the stability for the spaces \( \mathcal{V}_h = \mathcal{V}_h^1 \oplus \mathcal{V}_h^2, \mathcal{Q}_h \) can be achieved. An alternative way for stabilizing a Galerkin finite element discretization using \( \mathcal{V}_h^1, \mathcal{Q}_h \) consists in adding some terms to the continuity equation

\[
b(u_h, q_h) = 0 \quad \forall q_h \in \mathcal{Q}_h.
\]
Here, we shall show that this technique is in some sense equivalent to eliminating the degrees of freedom of the corresponding supplementary space $V_h^2$ from the discrete problem formulated for the spaces $V_h = V_h^1 \oplus V_h^2$, $Q_h$.

In the following, we shall confine ourselves to functions $\varphi_h^i$ and vectors $t_h^i$ satisfying

$$a(\varphi_h^i t_h^i, \varphi_h^j t_h^j) = 0 \quad \forall i \neq j, \quad i, j \in \{1, \ldots, N_h\}. \quad (30)$$

Examples of such functions and vectors will be given in the following sections.

We start with a reduced discretization of the Stokes equations given by:

Find $u_h^1 \in V_h^1$, $u_h^2 \in V_h^2$ and $p_h \in Q_h$ satisfying

$$\nu a(u_h^1, v_h^1) + b(v_h^1, p_h) = \langle f, v_h^1 \rangle \quad \forall v_h^1 \in V_h^1, \quad (31)$$

$$\nu a(u_h^2, v_h^2) + b(v_h^2, p_h) = 0 \quad \forall v_h^2 \in V_h^2, \quad (32)$$

$$b(u_h^1, q_h) + b(u_h^2, q_h) = 0 \quad \forall q_h \in Q_h. \quad (33)$$

This problem was obtained from the discretization (5) with $V_h = V_h^1 \oplus V_h^2$ by dropping the terms $\nu a(u_h^2, v_h^1)$, $\nu a(u_h^1, v_h^2)$ and $\langle f, v_h^2 \rangle$. In [16] it has been shown that, for $f \in L^2(\Omega)^2$, the solution $u_h = u_h^1 + u_h^2$, $p_h$ of (31)–(33) has asymptotically the same rate of convergence as the solution of the original problem (5). Note, in the special case when the triangulation $T_h$ consists of rectangles only and the functions from $V_h^2$ vanish on all edges of $T_h$, the terms $a(u_h^2, v_h^1)$ and $a(u_h^1, v_h^2)$ vanish identically. Owing to (30), the elimination of $u_h^2$ by means of (32) becomes simple. Indeed, using the basis representation $u_h^2 = \sum_{j=1}^{N_h} \alpha^j \varphi_h^j t_h^j$ and setting $v_h^2 = \varphi_h^j t_h^j$ in (32), we get

$$\alpha^i \nu a(\varphi_h^i t_h^i, \varphi_h^j t_h^j) + b(\varphi_h^i t_h^i, p_h) = 0.$$ 

Since $a(\varphi_h^i t_h^i, \varphi_h^j t_h^j) = |\varphi_h^i|^2_{\Omega} \neq 0$, we can eliminate $u_h^2$ from (31)–(33). Lemma 1 implies that

$$b(\varphi_h^i t_h^i, q_h) = \frac{\partial q_h}{\partial t_h} (A_h^i) \int_{\Omega} \varphi_h^i \, dx \quad \forall q_h \in Q_h, \quad i \in \{1, \ldots, N_h\}$$

and hence we obtain a stabilized $Q_1$/$Q_1$-discretization of the Stokes equations in the form:

Find $u_h^1 \in V_h^1$ and $p_h \in Q_h$ satisfying

$$\nu a(u_h^1, v_h^1) + b(v_h^1, p_h) = \langle f, v_h^1 \rangle \quad \forall v_h^1 \in V_h^1, \quad (34)$$

$$b(u_h^1, q_h) - c_h(p_h, q_h) = 0 \quad \forall q_h \in Q_h, \quad (35)$$

where the stabilizing term is given by

$$c_h(p_h, q_h) = \sum_{i=1}^{N_h} \frac{\partial q_h}{\partial t_h} (A_h^i) \frac{\partial \varphi_h^i}{\partial t_h} (A_h^i) \left| \int_{\Omega} \varphi_h^i \, dx \right|^2_{\Omega}. \quad (36)$$

It is easy to show (cf. [16]) that $u_h^1$ converges to the solution $u$ of (3) with the same rate as $u_h = u_h^1 + u_h^2$. That means that any stabilized discretization of the Stokes equations having the form (34),(35) with $c_h(p_h, q_h)$ given by (36) possesses optimal approximation properties.
Similarly as for the Stokes equations we can also proceed for the Navier–Stokes equations. We start with the
reduced discretization:

Find \( u_h^1 \in V_h^1, u_h^2 \in V_h^2 \) and \( p_h \in Q_h \) satisfying

\[
\begin{align*}
v a(u_h^1, v_h^1) + n(u_h^1, u_h^1, v_h^1) + b(v_h^1, p_h) &= \langle f, v_h^1 \rangle \quad \forall v_h^1 \in V_h^1, \\
v a(u_h^2, v_h^2) + b(v_h^2, p_h) &= 0 \quad \forall v_h^2 \in V_h^2, \\
b(u_h^1, q_h) + b(u_h^2, q_h) &= 0 \quad \forall q_h \in Q_h,
\end{align*}
\]

the solution of which has asymptotically, for \( f \in L^2(\Omega)^2 \), the same convergence rate as the solution of the
standard Galerkin finite element discretization of the Navier–Stokes equations (cf. [16]). Assuming (30) and
eliminating \( u_h^2 \) from (37)–(39), we arrive at the stabilized discretization:

Find \( u_h^1 \in V_h^1 \) and \( p_h \in Q_h \) satisfying

\[
\begin{align*}
v a(u_h^1, v_h^1) + n(u_h^1, u_h^1, v_h^1) + b(v_h^1, p_h) &= \langle f, v_h^1 \rangle \quad \forall v_h^1 \in V_h^1, \\
b(u_h^1, q_h) - c_h(p_h, q_h) &= 0 \quad \forall q_h \in Q_h,
\end{align*}
\]

where the stabilizing term \( c_h(p_h, q_h) \) is given by (36). Again, it follows that this stabilized discretization has
optimal approximation properties.

**Remark 8.** Since the matrix \( \{a(\varphi_h^i, t_h^j, \varphi_h^j, t_h^i)\}_{i,j=1}^{N_h} \) is regular, the assumption (30) is not necessary for trans-
forming the problem (31)–(33) (resp. (37), (39)) into the form (34)–(35) (resp. (40), (41)) with some stabilizing
term \( c_h(p_h, q_h) \). However, the stabilizing term then generally cannot be written in a compact form like (36).

### 6. NEW STABILIZATION TERMS

In this section, we discuss some choices of \( \varphi_h^j \) and \( t_h^i \) satisfying (30) and leading to new stabilization terms.
A sufficient condition for (30) is

\[
\text{int} \supp \varphi_h^i \cap \text{int} \supp \varphi_h^j = \emptyset \quad \text{or} \quad t_h^i \cdot t_h^j = 0 \quad \forall i \neq j, \; i, j \in \{1, \ldots, N_h\}.
\]

The functions \( \varphi_h^i \) introduced in Section 4 do not satisfy this condition but we can easily modify them so that
(42) holds. We define the sets \( \tilde{\Sigma}^1, \ldots, \tilde{\Sigma}^5 \subset \tilde{T} \) as depicted in Figure 1 (the set \( \tilde{\Sigma}^5 \) is a square with the vertices
\((1/4, 1/4), (3/4, 1/4), (3/4, 3/4), (1/4, 3/4)\)) and we transform the biquadratic functions \( \varphi^1, \ldots, \varphi^5 \) from Section 4 onto \( \tilde{\Sigma}^1, \ldots, \tilde{\Sigma}^5 \), respectively. For simplicity, we denote the transformed functions again \( \tilde{\varphi}^1, \ldots, \tilde{\varphi}^5 \) and we set \( \tilde{\varphi} = \tilde{\varphi}^5 \). We remark that the sets \( \tilde{\Sigma}^1 \) could be defined in many other ways (it is only important that
their interiors are disjoint) and we could also use various other functions \( \tilde{\varphi} \) (e.g., we could transform the
piecewise bilinear functions from Section 4 onto the sets \( \tilde{\Sigma}^i \)). However, to fix ideas, we shall now consider only

\[
\begin{array}{c|c|c|c|c}
\tilde{T} & \tilde{\Sigma}^1 & \tilde{\Sigma}^2 & \tilde{\Sigma}^4 & \tilde{\Sigma}^5 \\
\hline
\tilde{\Sigma}^3 & & & & \\
\end{array}
\]

**Figure 1.** Supports of the functions \( \{\tilde{\varphi}^i\}_{i=1}^5 \).
the biquadratic functions defined on the sets from Figure 1. Thus, for example, the function \( \tilde{\varphi}^1 \) satisfies

\[
\tilde{\varphi}^1(\tilde{x}) = (\tilde{x}_1 - \frac{1}{4})(\frac{3}{4} - \tilde{x}_1)(\frac{1}{4} - \tilde{x}_2)(\frac{1}{8} - \tilde{x}_2) \quad \text{for } \tilde{x} \in \tilde{\Sigma}^1
\]

and vanishes in \( \tilde{T} \setminus \tilde{\Sigma}^1 \). The points \( \tilde{A}^1, \ldots, \tilde{A}^6 \) and the vectors \( \tilde{t}^1, \ldots, \tilde{t}^5 \) remain the same as in Section 4, i.e.,

\[
\tilde{A}^1 = (\frac{1}{2}, 0), \quad \tilde{A}^2 = (1, \frac{1}{2}), \quad \tilde{A}^3 = (\frac{1}{2}, 1), \quad \tilde{A}^4 = (0, \frac{1}{2}), \quad \tilde{A}^5 = \tilde{A}^6 = (\frac{1}{2}, \frac{1}{2})
\]

and

\[
\tilde{t}^1 = \tilde{t}^3 = \tilde{t}^5 = (1, 0), \quad \tilde{t}^2 = \tilde{t}^4 = \tilde{t}^6 = (0, 1).
\]

The construction of a space \( V_h^2 \) is analogous as in Section 4. Thus, using the functions \( \tilde{\varphi}^1, \ldots, \tilde{\varphi}^4 \), we construct functions \( \varphi_h^1 \) with points \( A_h^1 \) lying on inner edges and, using the functions \( \tilde{\varphi}^5 \) and \( \tilde{\varphi}^6 \), we construct functions \( \varphi_h^1 \) with \( A_h^1 \) lying in the interiors of elements. However, using the functions \( \tilde{\varphi}^1, \ldots, \tilde{\varphi}^6 \), we are not able to construct \( \varphi_h^i \) with \( A_h^1 \) lying on a boundary edge. Therefore, for \( i = 1, \ldots, 4 \), we further introduce functions \( \tilde{\varphi}^{*} \in H_0^1() \) with \( \text{supp } \tilde{\varphi}^\ast \subset \tilde{\Sigma}^1 \) satisfying (13) with \( \tilde{\varphi} \). The function \( \tilde{\varphi}^{1*} \) is given in \( \tilde{\Sigma}^1 \) by

\[
\tilde{\varphi}^{1*}(\tilde{x}) = (\tilde{x}_1 - \frac{1}{4})(\frac{3}{4} - \tilde{x}_1)(\frac{1}{4} - \tilde{x}_2)(\frac{3}{20} - \tilde{x}_2)
\]

and the functions \( \tilde{\varphi}^{2*}, \tilde{\varphi}^{3*}, \tilde{\varphi}^{4*} \) are defined analogously. Now, for any edge lying on \( \partial \Omega \), we introduce a function \( \varphi_h^{1} \) with \( A_h^1 \) belonging to this edge. For clarity, we shall not use the functions \( \tilde{\varphi}^{1*}, \ldots, \tilde{\varphi}^{4*} \) for constructing functions \( \varphi_h^{1} \) with \( A_h^1 \in \Omega \). Now, if we choose a subset of the functions \( \{\varphi_h^1\} \) so that the assumption (17) is satisfied and define the space \( V_h^2 \) as the linear hull of this subset, then the Babuška–Brezzi condition (27) will hold. The question which functions \( \varphi_h^1 \) should be chosen in order to have (17) was already discussed in Section 4.

We recall that, for example, the assumption (17) is satisfied if, for any element \( T \), we have three functions \( \varphi_h^{i} \) with \( A_h^i \) on edges of \( T \). As we know, it is also sufficient if, for a given element, we have two functions \( \varphi_h^{i} \) with \( A_h^i \in \text{int} \ T \) and one function \( \varphi_h^{i} \) with \( A_h^i \) on an edge of \( T \). However, there is an important difference between these two possibilities: A set of functions \( \varphi_h^{i} \) with \( A_h^i \) on edges of the triangulation satisfies the condition (42) (the supports of these \( \varphi_h^{i} \) are disjoint) whereas if the points \( A_h^i \) also lie in the interiors of elements, (42) generally does not hold. The reason is that functions \( \varphi_h^{i} \) defined by \( \varphi_h|_T = \tilde{\varphi}^i \circ F^{-1}_T \) for some element \( T \) have the same supports and the corresponding vectors \( t_h^i \), \( t_h^i \) are not orthogonal unless \( T \) is a rectangle.

We denote by \( E_h \) the set of all edges \( E \) of the triangulation \( T_h \), by \( C_E \) the midpoint of each edge \( E \) and by \( t_E \) a unit vector in the direction of \( E \). Further, for each edge \( E \), we have a function \( \varphi_h^i \) with \( A_h^i = C_E \) and we denote \( \varphi_E = \varphi_h^i \). Finally, for any \( T \in T_h \), we set \( \varphi_T = \tilde{\varphi}^6 \circ F^{-1}_T \) and we denote by \( C_T \) the barycentre of \( T \). As we know from the previous paragraph, the space \( V_h^2 = \text{span}\{\varphi_E t_E\} \) guarantees the validity of (27) with \( \gamma_h \geq C > 0 \). For this choice of \( V_h^2 \), the stabilizing term (36) can be rewritten into

\[
c_h(p_h, q_h) = \sum_{E \in E_h} \frac{\delta_E h_E}{\nu} \frac{\partial p_h}{\partial t_E}(C_E) \frac{\partial q_h}{\partial t_E}(C_E),
\]

where \( h_E \) denotes the length of the edge \( E \) and the parameter

\[
\delta_E = \frac{\|f_{\Omega} \varphi_E \partial x\|}{h_E^3 |\varphi_E|^2 \Omega}
\]

is bounded from below and from above by positive constants independent of \( h \) (cf. Remark 7).
supports of the functions \( \{ \tilde{\phi}^i \}_{i=1}^4 \) and directions of the vectors \( \{ \tilde{t}^i \}_{i=1}^4 \).

If we also use the functions \( \varphi_T \) and the triangulation \( T_h \) consists of rectangles, then (36) can be rewritten into

\[
c_h(p_h, q_h) = \sum_{E \in \mathcal{E}_h} \frac{\delta_E h^4}{\nu} \frac{\partial p_h}{\partial t_E} (C_E) \frac{\partial q_h}{\partial t_E} (C_E) + \sum_{T \in \mathcal{T}_h} \frac{\delta_T h^4}{\nu} \nabla p_h (C_T) \cdot \nabla q_h (C_T),
\]

where \( \delta_E \) is the same as above and

\[
\delta_T = \frac{\left| \int_{\Omega} \varphi_T \, dx \right|^2}{h^4_T |\varphi_T|^1_{1, \Omega}}.
\]

Again, \( \delta_T \) is bounded from below and from above by positive constants independent of \( h \). We recall that, in the sums of (43) and (44), it is sufficient to consider only those terms which assure that the assumption (17) is satisfied. For instance, if some \( T \in \mathcal{T}_h \) is present in the second sum of (44), we need only one edge \( E \subset T \) in the first sum of (44). Note also that the number of entries in each row of the matrix corresponding to (43) is equal to one plus the number of edges containing the vertex associated with the given row. Thus, for a uniform triangulation, the matrix corresponding to (43) has only five entries in each row like the usual five point star for the discretization of the Laplacian. The matrix corresponding to (44) has typically nine entries per row.

We have seen in Section 5 that the stabilized finite element discretizations (34), (35) and (40), (41) of the Stokes and Navier–Stokes equations, respectively, possess optimal approximation properties provided that the stabilizing term \( c_h(p_h, q_h) \) can be written in the form (36). The results of this section show that, particularly, the mentioned discretizations have optimal approximation properties if \( c_h(p_h, q_h) \) is defined by (43) or (44) with some suitable parameters \( \delta_E \) and \( \delta_T \), which is a new result.

7. Recovering of the Brezzi–Pitkärranta stabilization

Eliminating the space \( V_h^2 \) from a discretization, we can not only derive new stabilized discretizations as in the previous section, but we can also obtain some existing ones. That often provides a deeper insight into their behaviour. Here, we show that, choosing the functions \( \varphi^i \) and vectors \( t^i \) in (36) in a suitable way, we can recover a stabilization introduced and studied by Brezzi and Pitkärranta [9]. For this, we introduce functions \( \tilde{\varphi}^1, \ldots, \tilde{\varphi}^4 \in H_0^1(\tilde{T}) \) having their supports in the sets \( \tilde{\Sigma}^1, \ldots, \tilde{\Sigma}^4 \) depicted in Figure 2. The sets \( \tilde{\Sigma}^i \) are squares with side length \( \frac{3+\sqrt{3}}{6} \) and barycentres in the points \( \tilde{A}^1 = (\frac{1}{2}, \frac{3-\sqrt{3}}{6}), \tilde{A}^2 = (\frac{3+\sqrt{3}}{6}, \frac{1}{2}), \tilde{A}^3 = (\frac{1}{2}, \frac{3+\sqrt{3}}{6}) \) and
\( \hat{A}^i = (\frac{\partial \varphi_i}{\partial x_j}, \frac{\partial \varphi_i}{\partial x_k}) \), respectively. The choice of the points \( \hat{A}^i \) assures that

\[
\int_T \hat{g} \, d\tilde{x} = \frac{1}{2} [\hat{g}(\hat{A}^2) + \hat{g}(\hat{A}^4)] \quad \text{for} \quad \hat{g}(\tilde{x}) = (\xi_0 + \xi_1 \tilde{x}_1) \tilde{q}(\tilde{x}),
\]

(45)

\[
\int_T \hat{h} \, d\tilde{x} = \frac{1}{2} [\hat{h}(\hat{A}^1) + \hat{h}(\hat{A}^3)] \quad \text{for} \quad \hat{h}(\tilde{x}) = (\xi_0 + \xi_1 \tilde{x}_2) \tilde{q}(\tilde{x}),
\]

(46)

where \( \tilde{q} \in Q_1(\hat{T}) \) and \( \xi_0, \xi_1 \) are arbitrary real numbers. Each function \( \hat{\varphi}^i \) is biquadratic in \( \hat{S}^i \), vanishes on the boundary of \( \bar{S}^i \) and is equal to 1 in \( \hat{A}^i \). The corresponding vectors \( \hat{t}^i \) are depicted in Figure 2 and are defined by \( \hat{t}^1 = \hat{t}^3 = (1, 0) \) and \( \hat{t}^2 = \hat{t}^4 = (0, 1) \). Now, for any \( T \in T_h \) and \( i \in \{1, \ldots, 4\} \), we set

\[
\varphi^i_T = \hat{\varphi}^i \circ F_T^{-1}, \quad \varphi^i_T|_{\partial T} = 0, \quad A_T^i = F_T(\hat{A}^i), \quad t_T^i = J_T(\hat{A}^i)^T/|J_T(\hat{A}^i)|.
\]

The functions \( \varphi^i_T \) and vectors \( t_T^i \) satisfy all the assumptions made in Section 2 and the space \( V_h^2 = \text{span}\{\varphi^j_T t_T^j\}_{T \in T_h, i=1, \ldots, 4} \) guarantees the validity of the Babuska–Brezzi condition for the spaces \( V_h^1 \oplus V_h^2 \) and \( Q_h \). The condition (42) is clearly satisfied and therefore, the discretizations (34), (35) and (40), (41) are stable for the stabilizing term

\[
c_h(p_h, q_h) = \sum_{T \in T_h} \sum_{i=1}^4 \frac{\partial p_h}{\partial t_T^i} (A_T^i) \frac{\partial q_h}{\partial t_T^i} (A_T^i) \left| \int_{\Omega} \varphi^i_T \, d\tilde{x} \right|^2.
\]

Denoting \( \hat{p}_T = p_h \circ F_T, \hat{q}_T = q_h \circ F_T \) for any \( T \in T_h \), it follows using (13) and (20) that

\[
c_h(p_h, q_h) = \sum_{T \in T_h} \sum_{i=1}^4 \frac{\partial \hat{p}_T}{\partial \hat{t}^i_T} (A_T^i) \frac{\partial \hat{q}_T}{\partial \hat{t}^i_T} (A_T^i) \left| \det J_T(\hat{A}^i) \right|^2 \frac{\left| \int_{\hat{T}} \varphi^i_T \, d\hat{\tilde{x}} \right|^2}{|J_T(\hat{A}^i)|^2 |\hat{T}|^2}.
\]

If the triangulation \( T_h \) consists of parallelograms, then \( J_T \) is constant for any \( T \in T_h \) and we obtain

\[
c_h(p_h, q_h) = \sum_{T \in T_h} \left[ \left( \frac{\partial \hat{p}_T}{\partial \hat{x}_1} (A_T^1) + \frac{\partial \hat{p}_T}{\partial \hat{x}_3} (A_T^3) \right) \frac{1}{|J_T \hat{T}|^2} + \left( \frac{\partial \hat{q}_T}{\partial \hat{x}_1} (A_T^1) + \frac{\partial \hat{q}_T}{\partial \hat{x}_3} (A_T^3) \right) \frac{1}{|J_T \hat{T}|^2} \right] \frac{\left| \det J_T \right| \left| \int_{\hat{T}} \varphi^1_T \, d\hat{\tilde{x}} \right|^2}{|\hat{T}|^2}.
\]

Since \( \partial \hat{p}_T/\partial \hat{x}_k \) is a linear function independent of \( \hat{x}_k, k = 1, 2 \), we infer applying (45) and (46) that the terms in the square brackets are equal to

\[
2 \int_T \frac{\partial \hat{p}_T}{\partial x_1} \frac{\partial \hat{q}_T}{\partial x_1} \frac{1}{|J_T \hat{T}|^2} + \frac{\partial \hat{p}_T}{\partial x_3} \frac{\partial \hat{q}_T}{\partial x_3} \frac{1}{|J_T \hat{T}|^2} \, d\tilde{x}.
\]

Applying (20), we obtain

\[
c_h(p_h, q_h) = \sum_{T \in T_h} \frac{\gamma_T h_T^2}{\nu} \int_T \frac{\partial p_h}{\partial t_T^1} \frac{\partial q_h}{\partial t_T^1} + \frac{\partial p_h}{\partial t_T^3} \frac{\partial q_h}{\partial t_T^3} \, dx,
\]

where the parameter

\[
\gamma_T = \frac{2 \left| \int_{\Omega} \varphi^1_T \, dx \right|^2}{h_T^2 |T| |\varphi^1_T|_{H^1}^2}.
\]
is bounded from below and from above by positive constants independent of \( h \). The vectors \( t_T^1, t_T^2 \) are unit vectors in the directions of the edges of the parallelogram \( T \). Thus, if the triangulation \( T_h \) consists of rectangles, we obtain

\[
ch(p_h, q_h) = \sum_{T \in T_h} \frac{\gamma_T h_T^2}{\nu} \int_T \nabla p_h \cdot \nabla q_h \, dx,
\]

which is the stabilization introduced by Brezzi and Pitkäranta [9] for stabilizing a discretization of the Stokes equations.

8. Consistent stabilized discretizations of the Stokes equations

A drawback of the stabilizations discussed up to now is that they are not consistent. First of all, the consistency error comes from the dropped right-hand side in (32), resp. in (38). Let us consider the Stokes equation (32) in the reduced discretization (31)–(33) by

\[
\nu a(u_h^2, v_h^2) + b(v_h^2, p_h) = (f, v_h^2) \quad \forall \ v_h^2 \in V_h^2.
\] (47)

The resulting discrete problem (31), (47), (33) has a solution which converges to the solution of (3) with the same rate as the solution of (31)–(33) (cf. [16]). If the triangulation \( T_h \) consists of rectangles, the equation (47) can be written as \( \nu a(u_h^2, v_h^2) = r_h(u_h^1, p_h, v_h^2) \) with

\[
r_h(w, q, v) = (f, v) - b(v, q) + \nu \sum_{T \in T_h} \int_T v \cdot \Delta w \, dx.
\]

A solution of (3) with \( u \in H^2(\Omega)^2 \) satisfies \( r_h(u, p, v) = 0 \) for any \( v \in H^1_0(\Omega)^2 \) and hence \( u_h^1 = u \), \( p_h = p \) solves the new discrete problem (31), (47), (33). In this sense, the new discrete problem is consistent. If the elements of \( T_h \) are not rectangular, the discretization is not consistent any more, but if they are nearly rectangular, we can hope that the consistency error is small.

Similarly as at the beginning of this section, we can eliminate \( u_h^2 \) from the discretization (31), (47), (33) and obtain a stabilized \( Q_1/Q_1 \)-discretization of the Stokes equations. This discretization now reads:

Find \( u_h^1 \in V_h^1 \) and \( p_h \in Q_h \) satisfying

\[
\nu a(u_h^1, v_h^1) + b(v_h^1, p_h) - c_h(p_h, q_h) = (f, v_h^1) \quad \forall \ v_h^1 \in V_h^1, \quad \forall \ q_h \in Q_h,
\]

where \( c_h(p_h, q_h) \) is defined by (36) and

\[
l_h(q_h) = -\sum_{i=1}^{N_h} \frac{\partial q_h}{\partial t_h} (A_h^i) (f, \phi_h^i t_h) \int_{\Omega} \frac{\phi_h^i \, dx}{\nu |\phi_h^i|_{1, \Omega}^2}.
\]

The particular formulas for \( l_h(q_h) \) corresponding to (43) or (44) can be introduced in a straightforward way. We only derive a formula for \( l_h(q_h) \) in case of the functions \( \phi_T \).

Let \( T_h \) consists of parallelograms and let

\[
f \in \{ v \in L^2(\Omega)^2; v \circ F_T \in Q_1(T)^2 \quad \forall \ T \in T_h \}.
\]
Denoting \( \hat{\mathbf{f}}_T = \mathbf{f} \circ F_T \) and \( \hat{\mathbf{q}}_T = \mathbf{q}_h \circ F_T \) for any \( T \in T_h \), we infer using \((13)\) and \((20)\) that

\[
l_h(q_h) = - \sum_{T \in T_h} \sum_{i=1}^4 \gamma_T h_T^2 h_T \frac{\partial \hat{\mathbf{q}}_T}{\partial t} (\hat{A}^T) \hat{f}_T(\hat{A}^T) \cdot J_T \mathbf{e}_i \left( \frac{|J_T|}{|J_T^T|^2} \right).
\]

Applying \((45)\), \((46)\) and \((20)\), we obtain

\[
l_h(q_h) = - \sum_{T \in T_h} \gamma_T h_T^2 h_T \int_T \frac{\partial q_h}{\partial t} \mathbf{f} \cdot \mathbf{v}_h^1 + \frac{\partial q_h}{\partial t} \mathbf{f} \cdot \mathbf{v}_h^2 \, dx
\]

and hence, if the triangulation \( T_h \) consists of rectangles, the stabilized continuity equation reads

\[
b(u_h^1, q_h) + \sum_{T \in T_h} \gamma_T h_T^2 h_T \int_T (f - \nabla p_h) \cdot \nabla q_h \, dx = 0. 
\]

This stabilization is identical with the Petrov–Galerkin formulation of the Stokes equations introduced in \([15]\). Increasing the number of the bubble functions, we can derive this equation also for \( f \) being generated by higher degree polynomials defined on the reference element. That will be also seen in the next section.

9. Recovering of the SUPG Method

The aim of this section is to show that, eliminating a sufficiently rich space \( V^2 \) from a modified reduced discretization of the Navier–Stokes equations, we can obtain the streamline upwind/Petrov–Galerkin (SUPG) method of \([12]\) analyzed for arbitrary combinations of approximation spaces for the velocity and pressure in \([21]\). This equivalence will be established without linearizing the convective term, unlike other papers investigating the relationship between Galerkin methods with bubble functions and the SUPG method.

We confine ourselves to triangulations consisting of rectangles and, similarly as in Section 5, we again start with a reduced discretization of the Navier–Stokes equations. In contrast with \((37)-(39)\), we now drop only the terms \( \nu a(u_h^2, v_h^1) \), \( \nu a(u_h^1, v_h^2) \), \( n(u_h^1, u_h^2, v_h^2) \) and \( n(u_h^1, u_h^1, v_h) \). Then we replace the term \( n(u_h^1, u_h^2, v_h^2) \) by the term \(-n(u_h^1, v_h^1, u_h^1)\). The last modification is motivated by the fact that

\[
n(u, w, v) = -n(u, v, w) - \int_\Omega (v \cdot w) \text{div} \mathbf{u} \, dx \quad \forall \, u, v, w \in H^1_0(\Omega)^2.
\]

Thus, we consider the following modified discretization of the Navier–Stokes equations:

Find \( u_h^1 \in V^1_h, u_h^2 \in V^2_h \) and \( p_h \in Q_h \) satisfying

\[
\nu a(u_h^1, v_h^1) + n(u_h^1, u_h^1, v_h^2) - n(u_h^1, v_h^1, u_h^1) + b(v_h^1, p_h) = \langle f, v_h^1 \rangle, 
\]

\[
\nu a(u_h^2, v_h^2) + n(u_h^1, u_h^1, v_h^2) + b(v_h^2, p_h) = \langle f, v_h^2 \rangle,
\]

\[
b(u_h^1, q_h) + b(u_h^2, q_h) = 0
\]

for any \( v_h^1 \in V^1_h, v_h^2 \in V^2_h \) and \( q_h \in Q_h \). Using the techniques of \([16]\), it is possible to prove the same convergence results for \((49)-(51)\) as we have for the standard Galerkin finite element discretization of the Navier–Stokes equations. Since \( \Delta(u_h^1|T) = 0 \) for any \( T \in T_h \) and \( u_h^2 \) can be considered as a stabilization device only, the above discrete problem for \( u_h^1, p_h \) is consistent (cf. the previous section). The term \(-n(u_h^1, v_h^1, u_h^1)\) introduces an influence of the space \( V^2_h \) into the momentum balance \((49)\). We shall see that this influence corresponds to a stabilization of the convective term \( n(u_h^1, u_h^1, v_h^2) \) analogous to the SUPG effect.

In view of the presence of the convective terms in \((49)\) and \((50)\), we shall need a more accurate integration formula than \((45)\) and \((46)\). Therefore, we have to increase the number of the points \( \hat{A}^T \) and hence of the bubble
functions $\tilde{\varphi}^i$. We introduce 16 points $\tilde{A}^i \in \tilde{T}$, $i = 1, \ldots, 16$, depicted in Figure 3 whose coordinates are all possible combinations of the values $\frac{1}{2}(1 + \xi), \frac{1}{2}(1 - \xi), \frac{1}{2}(1 + \eta), \frac{1}{2}(1 - \eta)$, where

$$\xi = \sqrt{\frac{1}{3} \left(1 + \frac{2}{\sqrt{3}}\right)}, \quad \eta = \sqrt{\frac{1}{3} \left(1 - \frac{2}{\sqrt{3}}\right)}.$$  

Then

$$\int_{\tilde{T}} \tilde{\varphi} \, d\tilde{x} = \frac{1}{16} \sum_{i=1}^{16} \tilde{\varphi}(\tilde{A}^i) \quad \forall \tilde{\varphi} \in Q_5(\tilde{T}),$$

where $Q_5(\tilde{T})$ is the space of polynomials of degrees less than or equal to 5 in each variable. To define the functions $\tilde{\varphi}^i$ corresponding to the points $\tilde{A}^i$, we first prove the following lemma.

**Lemma 4.** Let $\tilde{A} \in \tilde{T}$ and $\hat{\psi} \in H^1_0(\tilde{T}) \setminus \{0\}$ with $\hat{\psi} \geq 0$ on $\tilde{T}$ be given. Then there exists $\hat{\varphi} \in Q_2(\tilde{T})$ such that the function $\hat{\varphi} = \hat{p} \hat{\psi}$ satisfies $\hat{\varphi} \in H^1_0(\tilde{T}) \setminus L^2_0(\tilde{T})$ and

$$\int_{\tilde{T}} \hat{\varphi} \hat{\psi} \, d\tilde{x} = \hat{\varphi}(\hat{\psi}) \int_{\tilde{T}} \hat{\varphi} \, d\tilde{x} \quad \forall \hat{\varphi} \in Q_2(\tilde{T}).$$

**Proof.** Let us denote $((\hat{\psi}, \hat{\varphi})) = \int_{\tilde{T}} \hat{\varphi} \hat{\psi} \, d\tilde{x}$ and $\hat{M} = \{\hat{\varphi} \in Q_2(\tilde{T}); \hat{\varphi}(\hat{\psi}) = 0\}$. Then $(\cdot, \cdot)$ is an inner product on $Q_2(\tilde{T})$ and $\hat{M}$ is a linear subspace of $Q_2(\tilde{T})$ with dim $\hat{M} = \dim Q_2(\tilde{T}) - 1$. Thus, the orthogonal complement $\hat{M}^\perp$ of $\hat{M}$ in $Q_2(\tilde{T})$ with respect to $(\cdot, \cdot)$ is a one-dimensional space and hence there exists $\hat{p} \in \hat{M}^\perp \setminus \{0\}$ such that $((\hat{p}, \hat{\varphi})) = 0$ $\forall \hat{\varphi} \in \hat{M}$. That means that the function $\hat{\varphi} = \hat{p} \hat{\psi}$ satisfies $\int_{\tilde{T}} \hat{\varphi} \hat{\psi} \, d\tilde{x} = 0$ $\forall \hat{\varphi} \in \hat{M}$. Since $\hat{\varphi} - \hat{\varphi}(\hat{A}) \in \hat{M}$ for any $\hat{\varphi} \in Q_2(\tilde{T})$, (53) holds. Let us assume that $\hat{\varphi} \in L^2_0(\tilde{T})$. Then $\int_{\tilde{T}} \hat{\varphi} \hat{\psi} \, d\tilde{x} = 0$ $\forall \hat{\varphi} \in Q_2(\tilde{T})$, i.e., $(\hat{\psi}, \hat{\varphi}) = 0$ $\forall \hat{\varphi} \in Q_2(\tilde{T})$, which is in contradiction with $\hat{p} \neq 0$. 

For each point $\tilde{A}^i$, $i = 1, \ldots, 16$, we define a square with a side length 0.1 and a barycentre in $\tilde{A}^i$ (cf. Fig. 3). Transforming the function $\hat{\varphi}$ from Lemma 4 (for $\hat{A} = (1/2, 1/2)$ and some fixed function $\hat{\psi}$) onto the squares around the points $\tilde{A}^i$, we obtain functions $\tilde{\varphi}^i \in H^1_0(\tilde{T}), i = 1, \ldots, 16$, satisfying

$$\int_{\tilde{T}} \tilde{\varphi} \tilde{\varphi}' \, d\tilde{x} = \tilde{\varphi}(\tilde{A}^i) \int_{\tilde{T}} \tilde{\varphi}' \, d\tilde{x} \quad \forall \tilde{\varphi} \in Q_2(\tilde{T}), \; i \in \{1, \ldots, 16\}.$$
Further, we denote \( \varphi^i_{16} = \tilde{\varphi}^i, \tilde{A}^i_{16} = \tilde{A}^i, \tilde{t}^i = (1,0) \) and \( \tilde{t}^{i+16} = (0,1), \) \( i = 1, \ldots, 16. \) Finally, we again set for any \( T \in \mathcal{T}_h \) and \( i \in \{1, \ldots, 32\} \)
\[
\varphi^i_T|_T = \tilde{\varphi}^i \circ F_T^{-1}, \quad \varphi^i_T|_{\Omega \setminus T} = 0, \quad \tilde{A}^i_T = F_T(A^i), \quad \tilde{t}^i_T = J_T \tilde{t}^i / |J_T \tilde{t}^i|.
\]
(Note that \( J_T = \text{const. for each } T \in \mathcal{T}_h. \)) Since \( \mathcal{T}_h \) consists of rectangles, we have
\[
\int_T q \varphi^i_T \, dx = q(A^i_T) \int_T \varphi^i_T \, dx \quad \forall q \in Q_2(T), \ T \in \mathcal{T}_h, \ i \in \{1, \ldots, 32\}.
\]
(54)
The space \( \mathcal{V}^2_h = \text{span}\{\varphi^i_T, \tilde{t}^i_T|_{T \in \mathcal{T}_h, i = 1, \ldots, 32}\} \) guarantees the validity of the Babuška–Brezzi condition (27) for the spaces \( \mathcal{V}^1_h \oplus \mathcal{V}^2_h \) and \( \mathcal{Q}_h \) with \( \gamma_h \geq C > 0. \) Of course, much less bubble functions would be enough to get a stable element (it would be sufficient to have three functions \( \varphi^i_T \) in each element \( T \) ) but the supplementary space \( \mathcal{V}^2_h = \text{span}\{\varphi^i_T, \tilde{t}^i_T|_{T \in \mathcal{T}_h, i = 1, \ldots, 32}\} \) enables us to show a relationship to the SUPG method.

Now, let us eliminate the function
\[
u^i_T = \sum_{T \in \mathcal{T}_h} \sum_{i=1}^{32} \alpha^i_T \varphi^i_T \tilde{t}^i_T
\]
from the discrete problem (49)–(51). Since the basis functions of \( \mathcal{V}^2_h \) are orthogonal with respect to the bilinear form \( a(\cdot, \cdot), \) the elimination of \( \nu^i_T \) is again easy. We shall assume that
\[
f \in \{v \in L^2(\Omega)^2; v|_T \in Q_2(T)^2 \ \forall \ T \in \mathcal{T}_h\}
\]
and we shall employ that the spaces \( \mathcal{V}^1_h \) and \( \mathcal{Q}_h \) now consist of piecewise bilinear functions and that \( \tilde{t}^i_T \cdot \tilde{t}^{i+16}_T = 0 \) for any \( T \in \mathcal{T}_h \) and \( i \in \{1, \ldots, 16\}. \) Applying (54) and the fact that
\[
(f - (\nabla u^1_h)u^1_h - \nabla p_h)|_T \in Q_2(T)^2 \ \forall \ T \in \mathcal{T}_h,
\]
we obtain from (50)
\[
\alpha^i_T \nu|_{\mathcal{V}^1_T|_T, \Omega} = \tilde{t}^i_T \cdot (f - (\nabla u^1_h)u^1_h - \nabla p_h)(A^i_T) \int_{\Omega} \varphi^i_T \, dx.
\]
Using (54) and the fact that each function \( \varphi^i_T \) is only a shifted function \( \varphi^i_T, \) we further infer that
\[
n(u^1_h, v^1_h, u^2_h) = \sum_{T \in \mathcal{T}_h} \sum_{i=1}^{16} \left( (f - (\nabla u^1_h)u^1_h - \nabla p_h) \cdot (\nabla v^1_h)u^1_h \right) (A^j_T) \left| \int_{\Omega} \varphi^i_T \, dx \right|^2.
\]
Since the quadrature rule (52) is exact for polynomials from \( Q_3(T), \) we finally get
\[
n(u^1_h, v^1_h, u^2_h) = \sum_{T \in \mathcal{T}_h} \sum_{i=1}^{16} \frac{\gamma_T h^2}{\nu} \int_T (f - (\nabla u^1_h)u^1_h - \nabla p_h) \cdot (\nabla v^1_h)u^1_h \, dx,
\]
where
\[
\gamma_T = \frac{16 \left| \int_{\Omega} \varphi^1_T \, dx \right|^2}{h^2 |T| \left| \varphi^1_T \right|^2_1, \Omega}
\]
is again bounded from below and from above by positive constants independent of $h$. Analogously we obtain

$$b(u^i_h, q_h) = \sum_{T \in \mathcal{T}_h} \frac{\gamma_T h_T^2}{\nu} \int_T (f - (\nabla u^i_h) u^i_h - \nabla p_h) \cdot \nabla q_h \, dx.$$ 

Thus, the discrete problem (49)–(51) can be equivalently written in the form:

Find $u^i_h \in V^1_h$ and $p_h \in Q_h$ satisfying

$$\nu a(u^i_h, v^i_h) + n(u^i_h, u^i_h, v^i_h) + b(v^i_h, p_h) - b(u^i_h, q_h)$$

$$= (f, v^i_h) + \sum_{T \in \mathcal{T}_h} \frac{\gamma_T h_T^2}{\nu} \int_T (f - (\nabla u^i_h) u^i_h - \nabla p_h) \cdot ((\nabla v^i_h) u^i_h + \nabla q_h) \, dx \quad (55)$$

for any $v^i_h \in V^1_h$ and $q_h \in Q_h$. This form of the discrete problem (49), (51) is identical with the SUPG method of [12, 21] in the diffusion–dominated case, i.e., for small values of the element Reynolds number $Re_T = |u^i_h|_{1, \infty, T} h_T / \nu$. The stabilized continuity equation now reads

$$b(u^i_h, q_h) + \sum_{T \in \mathcal{T}_h} \frac{\gamma_T h_T^2}{\nu} \int_T (f - (\nabla u^i_h) u^i_h - \nabla p_h) \cdot \nabla q_h \, dx = 0,$$

which is a generalization of (48) to the nonlinear case.

In the convection–dominated case, i.e., for large values of $Re_T$, the factor in front of the integral in (55) is usually chosen proportional to $h_T / |u^i_h|_{1, \infty, T}$. Thus, for large values of $Re_T$, we should have $\gamma_T \sim 1 / Re_T$. That can be always fulfilled since

$$\gamma_T \leq \frac{1}{625 C^2 [\text{diam(supp } \hat{\psi}]^4,$$

where $\hat{\psi}$ is the function from Lemma 4 and $C$ is the same constant as in (12). For each element $T$, we can use another function $\hat{\psi}$ and obtain the correct value of $\gamma_T$ (the parameter set $\mathcal{P}$ introduced in Section 2 is then generally infinite). However, if $\gamma_T \sim 1 / Re_T$, then the parameter $\gamma_h$ from the Băuśka–Brezzi condition (27) behaves like

$$\gamma_h \sim \frac{1}{\max_{T \in \mathcal{T}_h} \sqrt{Re_T}},$$

which means that, for large values of $Re_T$, the SUPG method is equivalent to the problem (49)–(51) with spaces $V_h = V^1_h \oplus V^2_h$ and $Q_h$ satisfying the Băuśka–Brezzi condition (27) with a small parameter $\gamma_h$. Although this dependency has been not focussed in [20], a careful inspection shows that also for the $P_1 / P_1$–element enlarged by residual–free bubbles the condition in the Băuśka–Brezzi condition behaves in the convection–dominated case like $O(1 / \sqrt{Re})$. Therefore, it seems to be more convenient to stabilize the continuity equation and the convective term separately with different parameters $\gamma_T$. This discretization corresponds to a modified discretization of the Navier–Stokes equations with a velocity space $V_h = V^1_h \oplus V^2_h \oplus V^3_h$, where the supplementary space $V^3_h$ guarantees the fulfillment of the Băuśka–Brezzi condition and $V^3_h$ gives additional stability in the convection–dominated case (like in the SUPG approach).

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