

## LEAST REGRET CONTROL, VIRTUAL CONTROL AND DECOMPOSITION METHODS \*

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**Abstract.** “Least regret control” consists in trying to find a control which “optimizes the situation” with the constraint of not making things too worse with respect to a known reference control, in presence of more or less significant perturbations. This notion was introduced in [7]. It is recalled on a simple example (an elliptic system, with distributed control and boundary perturbation) in Section 2. We show that the problem reduces to a standard optimal control problem for *augmented state equations*. On another hand, we have introduced in recent notes [9–12] the method of *virtual control*, aimed at the “decomposition of everything” (decomposition of the domain, of the operator, etc). An introduction to this method is presented, without *a priori* knowledge needed, in Sections 3 and 4, directly on the augmented state equations. For problems without control, or with “standard” control, numerical applications of the virtual control ideas have been given in the notes [9–12] and in the note [5]. One of the first systematic paper devoted to all kind of decomposition methods, including multicriteria, is a joint paper with A. Bensoussan and R. Temam, to whom this paper is dedicated, *cf.* [1].

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### 1. INTRODUCTION

#### 1.1. Least regret control

Let us first recall what “*least regret control*” is all about. We present it on the simplest possible example.

Let  $\Omega$  be a (bounded) open set in  $\mathbb{R}^d$  ( $d = 2, 3$  in most of the applications), with (smooth) boundary  $\partial\Omega = \Gamma$ . In  $\Omega$  we are given a second order elliptic operator

$$A\varphi = - \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial \varphi}{\partial x_j} \right) + a_0 \varphi \quad (1.1)$$

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\* *Dedicated to Roger Temam for his 60th birthday*

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where  $a_0, a_{i,j} \in L^\infty(\Omega)$  satisfy a.e.

$$\sum_{i,j=1}^d a_{ij}(x)\zeta_i\zeta_j \geq c \sum_{i=1}^d \zeta_i^2, \quad a_0 \geq c, \quad c > 0.$$

The state  $z$  of the equation is given by the solution of

$$Az = v1_{\mathcal{O}} \quad \text{in } \Omega, \tag{1.2}$$

$$\frac{\partial z}{\partial n_A} = g \quad \text{on } \Gamma, \tag{1.3}$$

where

$$\left\{ \begin{array}{l} \mathcal{O} = \text{open set contained in } \Omega, \quad v \in L^2(\mathcal{O}), \quad v = \text{control variable}, \\ 1_{\mathcal{O}} = \text{characteristic function of } \mathcal{O}, \\ \frac{\partial}{\partial n_A} = \text{conormal derivative with respect to } A, \\ g \in L^2(\Gamma), \quad g = \text{perturbation (or unknown) variable.} \end{array} \right. \tag{1.4}$$

(Condition (1.3) is taken through a weak variational formulation).

Problem (1.2)–(1.3) admits a unique solution:

$$z = z(v, g) \in H^1(\Omega). \tag{1.5}$$

We then introduce the cost function

$$J(v, g) = \frac{1}{2} \int_{\Omega} \rho(x)(z(v, g) - z_0)^2 dx + \frac{1}{2} \int_{\mathcal{O}} v^2 dx \tag{1.6}$$

where  $\rho$  is given in  $L^\infty(\Omega)$ ,  $\rho \geq 0$ , and where  $z_0$  is the “optimal” state we wish to get close to, taking into account the “cost of the control” (expressed by the term  $\frac{1}{2} \int_{\mathcal{O}} v^2 dx$  in (1.6)).

If  $g$  is known, say  $g = g_0$ ,  $J(v, g)$  does not depend on  $g$ , and the problem we wish to solve is to find

$$\inf_v J(v, g_0).$$

This is a standard problem of optimal control for distributed systems, in one of the simplest possible case. Cf. [6]. □

But here we have a perturbation on the boundary, expressed by  $g$ . We assume that, by tradition or by formal computation, one is used to apply a “nominal policy”, i.e. that one uses

$$v = v_0 \quad \text{given in } L^2(\mathcal{O}). \tag{1.7}$$

We want to choose  $v$  in the best possible way with respect to (1.6), with the natural constraint that we *do not want to deteriorate the situation with respect to the traditional policy*  $v = v_0$ . Analytically, this is expressed by the problem to find

$$\inf_v \sup_g [J(v, g) - J(v_0, g)]. \tag{1.8}$$

If there is a solution, it is called “policy without regret”. But (1.8) is too restrictive. There is *no solution* of (1.8) in general. Hence the introduction of the following (relaxed) problem

$$\inf_v \sup_g \left[ J(v, g) - J(v_0, g) - \frac{\gamma}{2} \int_{\Gamma} g^2 \, d\Gamma \right], \tag{1.9}$$

where  $\gamma$  is  $> 0$  and “small”.

Problem (1.9) admits (in the present situation) a unique solution  $v$  (see below in Sect. 2) which is called “policy with least regret”, or “least regret control”.

**Remark 1.1.** The notions of “without regret control” or “least regret control” have been introduced in [7].

The notion of regret was first introduced by [13] (a reference indicated to me by D. Gabay, after publication of the note [7]).

The notion of “least regret control” is *completely general* and immediately extends to evolution problems and non linear systems.

It has been extended in [3] to multi criteria and multi agents. □

*Our goal here is to study (1.9) with the state given by (1.2) (1.3), with the cost function given by (1.6), in the framework of Decomposition Methods.*

### 1.2. Decomposition methods

Given *any* problem involving a partial differential operator  $A$  (of *any type*) in a domain  $\Omega$ , an important question is to *decompose*  $A$  and, or, the domain  $\Omega$ , so as to “cut the problem in a large number of small and simple pieces” (with parallelism in sight).

A huge amount of work is devoted to these methods (no attempt is made for a significant bibliography). A *systematic* paper was devoted to these questions, namely [1] (a paper which seems to have been forgotten, including by his authors...). □

In a series of notes [9–12] we have introduced the technique of *virtual control* for “the decomposition of everything” (domain decomposition, decomposition of operator), applied in [5] to decomposition of the “energy space”.

*Our goal is to show here that these techniques can be applied to least regret controls problems.* □

In order not to snow the ideas under complicated technicalities we present the virtual control technique in Section 3 for the problem (1.9) in a single preliminary framework, somewhat connected with fictitious domains, further extensions being briefly indicated in Section 4.

We now proceed with (1.9) and the introduction of the *augmented state equations*. □

## 2. AUGMENTED SYSTEM

### 2.1. Preliminary computations

Let us introduce the states  $y = y(v)$  and  $\varphi(g)$  defined by

$$Ay = v1_{\mathcal{O}} \text{ in } \Omega, \quad \frac{\partial y}{\partial n_A} = 0 \text{ on } \Gamma, \tag{2.1}$$

$$A\varphi = 0 \text{ in } \Omega, \quad \frac{\partial \varphi}{\partial n_A} = g \text{ on } \Gamma, \tag{2.2}$$

so that

$$z(v, g) = y(v) + \varphi(g). \tag{2.3}$$

Moreover in order to (slightly) simplify the exposition and without restriction (in linear problems) we assume that

$$v_0 = 0. \tag{2.4}$$

Then

$$J(v, g) - J(v_0, g) = \frac{1}{2} \int_{\Gamma} \rho(y - z_0)^2 \, d\Gamma - \frac{1}{2} \int_{\Gamma} \rho z_0^2 \, d\Gamma + \int_{\Gamma} \rho y \varphi \, d\Gamma$$

(where we have written  $y$  for  $y(v)$ ,  $\varphi$  for  $\varphi(g)$ ), or

$$J(v, g) - J(v_0, g) = J(v, 0) - J(v_0, 0) + \int_{\Gamma} \rho y \varphi \, d\Gamma. \tag{2.5}$$

We then introduce the function  $\eta$  defined by

$$A^* \eta = 0 \text{ in } \Omega, \quad \frac{\partial \eta}{\partial n_{A^*}} = \rho y \text{ on } \Gamma, \tag{2.6}$$

where  $A^*$  denotes the adjoint of  $A$ .

The set  $\{y, \eta\}$  given by (2.1) (2.6) is the *augmented state*. Using (2.6) one has

$$\int_{\Gamma} \rho y \varphi \, d\Gamma = \int_{\Gamma} \eta g \, d\Gamma$$

so that

$$J(v, g) - J(v_0, g) - \frac{\gamma}{2} \int_{\Gamma} g^2 \, d\Gamma = J(v, 0) - J(v_0, 0) + \int_{\Gamma} \eta g \, d\Gamma - \frac{\gamma}{2} \int_{\Gamma} g^2 \, d\Gamma. \tag{2.7}$$

Therefore

$$\left| \begin{aligned} & \sup . \left[ J(v, g) - J(v_0, g) - \frac{\gamma}{2} \int_{\Gamma} g^2 \, d\Gamma \right] = \\ & = J(v, 0) - J(v_0, 0) + \frac{1}{2\gamma} \int_{\Gamma} \eta^2 \, d\Gamma. \end{aligned} \right. \tag{2.8}$$

We now have to minimize the expression in (2.8). Of course  $J(v_0, 0)$  is fixed. Therefore *the problem reduces to*

$$\inf_v . \left[ J(v, 0) + \frac{1}{2\gamma} \int_{\Gamma} \eta^2 \, d\Gamma \right] \tag{2.9}$$

where  $\eta = \eta(v)$ , the state  $\{y(v), \eta(v)\}$  being given by the solution of the augmented system (2.1) (2.6). □  
 A few remarks are now in order.

2.2. **Remarks**

**Remark 2.1.** Problem (2.9) is now a standard problem of optimal control for a distributed system, when the state equation is a set of two elliptic equations (2.1)–(2.6) (coupled by the boundary condition  $\frac{\partial \eta}{\partial n_{A^*}} = \rho y$ ). It is therefore “normal” that the general methods of virtual control apply to the present situation! We simply show in Section 3 how the methods of O. Pironneau and the A. already referred to can be adapted to the present situation. □

**Remark 2.2.** One can also view “least regret” as a way to “increase the robustness” of the control, subject to perturbations on the boundary. □

**Remark 2.3.** Let us assume that we do have *some* information on the perturbation  $g$ , expressed by

$$g \in G = \text{closed convex subset of } L^2(\Gamma). \tag{2.10}$$

Then of course (2.9) is replaced by

$$\inf_v \left[ J(v, 0) + \sup_{g \in G} \left( \int_{\Gamma} \eta g \, d\Gamma - \frac{\gamma}{2} \int_{\Gamma} g^2 \, d\Gamma \right) \right]. \tag{2.11}$$

Section 3 can be applied to (2.11). □

**Remark 2.4.** We can see in the above situation why the introduction of  $\gamma > 0$  is necessary (in general). Indeed if  $\gamma = 0$  (control without regret), one should have  $\eta = 0$  on  $\Gamma$ . Therefore (2.6) implies that  $\eta = 0$  in  $\Omega$ , so that  $y = 0$  on  $\Gamma$  and the problem amounts to finding

$$\inf \frac{1}{2} \int_{\Omega} v^2 \, dx \tag{2.12}$$

for all  $v$ 's such that

$$Ay = v1_{\mathcal{O}}, \quad y = 0 \text{ and } \frac{\partial y}{\partial n_A} = 0 \text{ on } \Gamma, \tag{2.13}$$

the solution being  $v = 0$ ! □

**Remark 2.5.** For non linear state equations, the decomposition (2.3) is of course not valid. One then replaces  $J(v, g)$  by

$$J(v) + \left\langle \frac{\partial J}{\partial g}(v, g), g \right\rangle$$

if the state equation is differentiable with respect to  $g$ , and which makes sense in case (2.10) with  $G$  “small”. □

We now introduce a virtual control technique.

3. VIRTUAL CONTROLS

3.1. **Embedding and virtual controls**

We embed  $\Omega$  (which can have a “complicated” boundary) into a large set  $\tilde{\Omega}$  (say a cube, or a sphere) and we introduce a new system in  $\tilde{\Omega}$ . It is an “extension to  $\tilde{\Omega}$ ” of the augmented system introduced in Section 2.1.

Let  $\omega$  and  $\omega_*$  be two open sets contained in  $\tilde{\Omega}/\bar{\Omega}$  (cf. Fig. 1).

Let  $\tilde{A}$  denote any extension of  $A$  into  $\tilde{\Omega}$ ,  $\tilde{A}$  being elliptic strictly coercive in  $\tilde{\Omega}$  (this is possible!).

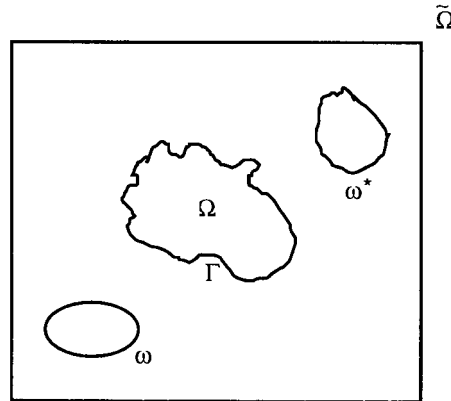


FIGURE 1

We now introduce the following system:

$$\begin{cases} \tilde{A} \tilde{y} = v1_{\mathcal{O}} + \lambda\chi, \\ \tilde{A}^* \tilde{\eta} = \mu\chi_*, \\ \tilde{y}, \tilde{\eta} \text{ subject to any "simple" boundary condition on } \partial\tilde{\Omega} \end{cases} \tag{3.1}$$

(for instance we can make the coefficients of  $\tilde{A}$  periodic in  $\tilde{\Omega}$  and take periodic boundary conditions). In (3.1)  $\chi$  (resp.  $\chi_*$ ) denotes the characteristic function of  $\omega$  (resp.  $\omega_*$ ), and  $\lambda$  and  $\mu$  are (for the time being) arbitrary functions,

$$\lambda \in L^2(\omega), \mu \in L^2(\omega_*). \tag{3.2}$$

They are the virtual control. □

**Remark 3.1.** Of course the restrictions of  $\tilde{y}, \tilde{\eta}$  to  $\Omega$ , say  $y, \eta$ , satisfy  $Ay = v1_{\mathcal{O}}, A^*\eta = 0$  in  $\Omega$ , but do not satisfy in general the boundary conditions for  $y$  and  $\eta$ ! The virtual controls have precisely to be chosen in such a way that these boundary conditions are satisfied, at least approximately. This is possible, as we now show. □

Given  $v \in L^2(\mathcal{O})$ , it is possible to choose  $\lambda \in L^2(\omega)$  and  $\mu \in L^2(\omega_*)$  in such a way that

$$\begin{cases} \left| \frac{\partial \tilde{y}}{\partial n_A} \right|_{L^2(\Gamma)} \text{ and } \left| \frac{\partial \tilde{\eta}}{\partial n_{A^*}} - \rho \tilde{y} \right|_{L^2(\Gamma)} \\ \text{are arbitrarily small.} \end{cases} \tag{3.3}$$

Indeed, let us consider the mapping

$$\lambda, \mu \rightarrow \frac{\partial \tilde{y}}{\partial n_A}, \frac{\partial \tilde{\eta}}{\partial n_{A^*}} - \rho \tilde{y} \tag{3.4}$$

from  $L^2(\omega) \times L^2(\omega_*) \rightarrow L^2(\Gamma) \times L^2(\Gamma)$ .

Its range is dense.

Indeed one shows first (by duality and a unique continuation argument) that the range of  $\lambda \rightarrow \frac{\partial \tilde{y}}{\partial n_A}$  is dense in  $L^2(\Gamma)$  ( $\tilde{y}$  depends only on  $\lambda$ , once  $v$  is fixed). Then it suffices to show that  $\mu \rightarrow \frac{\partial \tilde{\eta}}{\partial n_{A^*}}$  has a dense range in  $L^2(\Gamma)$  (which is exactly the same result than for  $\tilde{y}$ ). □

If  $\lambda$  and  $\mu$  are chosen according to (3.3), problem (2.9) is approximated (as closely as we want) by

$$\inf_v \tilde{\mathcal{J}}(v) \tag{3.5}$$

where

$$\tilde{\mathcal{J}}(v) = \frac{1}{2} \int_{\Gamma} \rho(\tilde{y} - z_0)^2 \, d\Gamma + \frac{1}{2} \int_{\mathcal{O}} v^2 \, dx + \frac{1}{2\gamma} \int_{\Gamma} (\tilde{\eta})^2 \, d\Gamma. \tag{3.6}$$

In order to proceed one *penalizes the conditions* (3.3). We introduce

$$\tilde{\mathcal{J}}(v, \lambda, \mu) = \tilde{\mathcal{J}}(v) + \frac{1}{2\varepsilon} \int_{\Gamma} \left( \frac{\partial \tilde{y}}{\partial n_A} \right)^2 \, d\Gamma + \frac{1}{2\varepsilon} \int_{\Gamma} \left( \frac{\partial \tilde{\eta}}{\partial n_{A^*}} - \rho y \right)^2 \, d\Gamma \tag{3.7}$$

where  $\varepsilon$  is fixed “small”. Then an approximation of the *least regret control problem considered here, is given by the solution of the problem.*

$$\inf_{v, \lambda, \mu} \tilde{\mathcal{J}}(v, \lambda, \mu). \tag{3.8}$$

**Remark 3.2.** Problem (3.8) certainly looks more complicated than the formulation (2.9)! But

1. the domain  $\tilde{\Omega}$  is chosen to be much simpler than  $\Omega$ ;
2. the method presented here can be thought of as an introduction to domain decomposition for least regret control problems. □

**Remark 3.3.** Let’s suppose we shall be happy with the (small) errors  $|\frac{\partial \tilde{y}}{\partial n_A}|_{L^2(\Gamma)} \leq \varepsilon_1$ ,  $|\frac{\partial \tilde{\eta}}{\partial n_{A^*}} - \rho \tilde{y}|_{L^2(\Gamma)} \leq \varepsilon_1$ . One has then to choose  $\varepsilon$  in (3.7) so that these conditions are (approximately) satisfied. This can be made more precise by transforming (3.7) by a duality argument (based on Fenchel-Rockafellar duality theorem, cf. for instance [2]), as it is used in a different situation by [4]. □

Assuming that  $\varepsilon$  has been chosen in (3.7), we now give a simple algorithm of approximation of (3.8).

### 3.2. Algorithm

We do not use in this section the state  $y, \eta$  given by (2.1) (2.6). Therefore we drop here the symbols  $\sim$  in equation (3.1) and in the functional (3.7).

The first variation of  $\mathcal{J}(v, \lambda, \mu)$  is given by

$$\begin{aligned} \delta \mathcal{J}(v, \lambda, \mu) = & \int_{\Gamma} \rho(y - z_0)\delta y + \int_{\mathcal{O}} v\delta v + \frac{1}{\gamma} \int_{\Gamma} \eta\delta\eta \\ & + \frac{1}{\varepsilon} \int_{\Gamma} \frac{\partial y}{\partial n_A} \frac{\partial \delta y}{\partial n_A} + \frac{1}{\varepsilon} \int_{\Gamma} \left( \frac{\partial \eta}{\partial n_{A^*}} - \rho y \right) \left( \frac{\partial \delta y}{\partial n_{A^*}} - \rho \delta y \right) \end{aligned} \tag{3.9}$$

where we skip the surface and volume elements  $d\Gamma$  and  $dx$ .

We introduce now the adjoint functions  $p$  and  $\pi$  defined as follows:

$$\int_{\tilde{\Omega}} p \tilde{A}\varphi = \int_{\Gamma} \rho(y - z_0)\varphi + \frac{1}{\varepsilon} \int_{\Gamma} \frac{\partial y}{\partial n_A} \frac{\partial \varphi}{\partial n_A} - \frac{1}{\varepsilon} \int_{\Gamma} \left( \frac{\partial \eta}{\partial n_{A^*}} - \rho y \right) \rho \varphi. \tag{3.10}$$

$\forall \varphi \in H^2(\tilde{\Omega})$ , satisfying the approximate boundary conditions on  $\partial\tilde{\Omega}$ , and

$$\int_{\tilde{\Omega}} \pi \tilde{A}^* \Psi = \frac{1}{\gamma} \int_{\Gamma} \eta \Psi + \frac{1}{\varepsilon} \int_{\Gamma} \left( \frac{\partial \eta}{\partial n_{A^*}} - \rho y \right) \frac{\partial \Psi}{\partial n} \quad (3.11)$$

$\forall \Psi$  enjoying the same properties than  $\varphi$  above.

We use in (3.10)–(3.11) the weak transposed solutions of the problems involved, as in [8].

We now plug  $\delta g$  for  $\varphi$  (resp.  $\delta \eta$  for  $\Psi$ ) in (3.10) (resp. (3.11)). We obtain

$$\delta \mathcal{J} = \int_{\tilde{\Omega}} \eta \tilde{A} \delta y + \int_{\tilde{\Omega}} \pi \tilde{A}^* \delta \eta + \int_{\mathcal{O}} v \delta v.$$

Using the first variations of (3.1), we finally obtain

$$\delta \mathcal{J} = \int_{\mathcal{O}} (\eta + v) \delta v + \int_{\omega} \eta \delta \lambda + \int_{\omega^*} \pi \delta \mu. \quad (3.12)$$

One can then use the simple gradient algorithm

$$\begin{cases} v^{n+1} = v^n - \rho(\eta^n + v^n) \\ \lambda^{n+1} = \lambda^n - \rho \eta^n \\ \mu^{n+1} = \mu^n - \rho \pi^n \end{cases} \quad (3.13)$$

which is convergent for  $\rho > 0$  and small enough.  $\square$

#### 4. REMARKS

**Remark 4.1.** Let us consider again the augmented state equations (2.1)–(2.6) and let us introduce a domain decomposition

$$\Omega = \Omega_1 \cup \Omega_2 \quad (4.1)$$

with overlapping, *i.e.*  $\Omega_1 \cap \Omega_2 \neq \emptyset$ .

We set

$$\Gamma_i = \partial\Omega_i \cap \partial\Omega, \quad S_i = \partial\Omega_i \cap \Omega_j, \quad j \neq i \quad (4.2)$$

so that

$$\partial\Omega_i = \Gamma_i \cup S_i.$$

we introduce two sets

$$\omega, \omega_* \subset \Omega_1 \cap \Omega_2 \quad (4.3)$$

(compare to Sect. 3). We define

$$\begin{cases} \chi_i = \text{characteristic function of } \mathcal{O} \cap \Omega_i \text{ (}\chi_i \text{ can be zero)} \\ \chi, \chi_* = \text{characteristic function of } \omega, \omega_*. \end{cases} \quad (4.4)$$



We now define  $y_i, \eta_i$  in  $\Omega_i$ , in the following fashion:

$$\left\{ \begin{array}{l} A_1 y_1 = v_1 \chi_1 + \lambda \chi \text{ in } \Omega_1, \\ \frac{\partial y_1}{\partial n_{A_1}} = 0 \text{ on } \partial \Omega_1, \\ A_1^* \eta_1 = \mu \chi_* \text{ in } \Omega_1, \\ \frac{\partial \eta_1}{\partial n_{A_1^*}} = \rho y_1 \text{ on } \Gamma_1, 0 \text{ on } S_1 \end{array} \right. \quad (4.5)$$

$$\left\{ \begin{array}{l} A_2 y_2 = v_2 \chi_2 - \lambda \chi \text{ in } \Omega_2, \\ \frac{\partial y_2}{\partial n_{A_2}} = 0 \text{ on } \partial \Omega_2, \\ A_2^* \eta_2 = -\mu \chi_* \text{ in } \Omega_2, \\ \frac{\partial \eta_2}{\partial n_{A_2^*}} = \rho y_2 \text{ on } \Gamma_2, 0 \text{ on } S_2 \end{array} \right. \quad (4.6)$$

where  $A_i =$  restriction of  $A$  to  $\Omega_i$  and where  $v_1 \chi_1 + v_2 \chi_2 = v 1_{\mathcal{O}}$ .  
 If one chooses (as it is possible) *the virtual controls*  $\lambda$  and  $\mu$  in such a way that

$$\left\{ \begin{array}{l} y_i, \eta_i \text{ are "approximately" } 0 \text{ on } S_i \text{ (in the} \\ L^2(S_i) \text{ topology)} \end{array} \right. \quad (4.7)$$

then extending  $y_i, \eta_i$  in  $\tilde{y}_i, \tilde{\eta}_i$  by 0 outside  $\Omega_i$ , one has *approximately*

$$\left\{ \begin{array}{l} A(\tilde{y}_1 + \tilde{y}_2) = v 1_{\mathcal{O}} \text{ in } \Omega, \\ A^*(\tilde{\eta}_1 + \tilde{\eta}_2) = 0 \text{ in } \Omega, \\ \frac{\partial}{\partial n_A}(\tilde{y}_1 + \tilde{y}_2) = 0 \text{ on } \partial \Omega, \\ \frac{\partial}{\partial n_{A^*}}(\tilde{\eta}_1 + \tilde{\eta}_2) = \rho(\tilde{y}_1 + \tilde{y}_2) \text{ on } \partial \Omega. \end{array} \right. \quad (4.8)$$

One can then proceed in a similar manner as in Section 3.2, that it is not necessary to make explicit. □

**Remark 4.2.** The above method readily extends to the case where

$$\Omega = \Omega_1 \cup \dots \cup \Omega_N,$$

where the covering is overlapping. □

**Remark 4.3.** One can also apply in the present situation all the various methods introduced in [9–12] and in [5], *loc.cit.* □

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