FLUIDS WITH ANISOTROPIC VISCOSITY

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Abstract. Motivated by rotating fluids, we study incompressible fluids with anisotropic viscosity. We use anisotropic spaces that enable us to prove existence theorems for less regular initial data than usual. In the case of rotating fluids, in the whole space, we prove Strichartz-type anisotropic, dispersive estimates which allow us to prove global wellposedness for fast enough rotation.

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INTRODUCTION

The aim of this paper is the study of the following system

\[
\begin{aligned}
\partial_t v + v \cdot \nabla v - \nu \Delta_h v - \nu_v \partial^2_{x_1} v &= -\nabla p \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3 \\
\text{div} v &= 0 \\
v_{i=0} &= v_0,
\end{aligned}
\]

where $\Delta_h$ denotes the horizontal Laplacian, \textit{i.e.} the operator $\partial^2_{x_1} + \partial^2_{x_2}$. The constants $\nu > 0$ and $\nu_v \geq 0$ represent respectively the horizontal and vertical viscosities. Obviously, this system is locally well posed when the initial data $v_0$ belongs to a Sobolev space $H^{s+\eta}(\mathbb{R}^3)$ for any positive real number $\eta$ simply because the energy estimates are better than the ones for Euler \textit{i.e.} $\nu = \nu_v = 0$. On the other hand, it is well known that if $\nu$ and $\nu_v$ are positive, and if $v_0$ is only in $L^2(\mathbb{R}^3)$, then a global weak solution exists (that is Leray’s theorem, see [16]). Finally, if $v_0$ belongs to the homogeneous Sobolev space $\dot{H}^{s}(\mathbb{R}^3)$, then a unique local solution exists (that is Fujita and Kato’s theorem, see [10]). Moreover, if $\|v_0\|_{\dot{H}^{s}}$ is small enough with respect to $\min\{\nu, \nu_v\}$, then the system is globally well posed. Let us note that the homogeneous space $\dot{H}^{s}(\mathbb{R}^3)$ is well adapted to the Navier–Stokes system, in the sense that it is invariant under the scaling of the equation: if $v$ is a solution of the Navier–Stokes equation, with data $v_0$, then the same goes for $v_\lambda$ defined by $v_\lambda(t,x) \overset{\text{def}}{=} \lambda v(\lambda^2 t, \lambda x)$, with

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data \( v_{0,\lambda}(x) \equiv \lambda v_0(\lambda x) \); it is easy to see that

\[ \|v_{0,\lambda}\|_{H^s(R^3)} = \|v_0\|_{H^s(R^3)}. \]

What we want to investigate here is the case when \( V \) is 0 or converges to 0. We want to prove existence and if possible uniqueness results as close as possible to the case when the two viscosities are positive and fixed.

Let us begin by saying something about the motivation of this problem. It is well known since a series of work devoted to Ekman boundary layers for rotating fluids (see for instance [13], [7] and [8]) that it makes sense to consider anisotropic viscosities and more precisely viscosities of the type

\[ -\nu \Delta_h - \beta \epsilon \partial^3_z. \]

That is obviously a particular case of the system \((NS_h)\) stated at the beginning of this introduction.

Considering the anisotropy of the problem, we shall have to use spaces of functions that take into account that anisotropy. All we shall do here is strongly related to energy estimates, so the tool will be anisotropic Sobolev spaces. Such spaces have been introduced by D. Iftimie in [15] for the study of incompressible Navier-Stokes equations in thin domains.

Let us recall the definition of those spaces. It requires an anisotropic dyadic decomposition of the Fourier space, so let us start by recalling the definition of the following operators of localization in Fourier space:

\[ \Delta^j_a \equiv \mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\hat{a}) \quad \text{for} \quad j \in \mathbb{Z}, \]

\[ \Delta^k_a \equiv \mathcal{F}^{-1}(\varphi(2^{-k}|\xi|)\hat{a}) \quad \text{for} \quad k \in \mathbb{N}, \]

\[ \Delta^{u-1}_a \equiv \mathcal{F}^{-1}(\psi(\xi)\hat{a}) \]

\[ \Delta^k_a \equiv 0 \quad \text{for} \quad k \leq -2, \]

where \( \mathcal{F}a \) and \( \hat{a} \) denote the Fourier transform of any function \( a \). The functions \( \varphi \) and \( \psi \) are smooth, compactly supported functions, with support respectively in a fixed ring of \( \mathbb{R} \) far from the origin, and in a fixed ball containing the origin and such that

\[ \forall t \in \mathbb{R} \setminus \{0\}, \sum_{j \in \mathbb{Z}} \varphi(2^{-j}t) = 1 \quad \text{and} \quad \forall t \in \mathbb{R}, \psi(t) + \sum_{j \in \mathbb{N}} \varphi(2^{-j}t) = 1. \]

We refer to [3] for a precise construction of an isotropic decomposition of the Fourier space; let us note that there exists an integer \( N_0 \) such that if \( |j - j'| \geq N_0 \), then supp \( \varphi(2^{-j'}) \cap \text{supp} \varphi(2^{-j'}) = \emptyset. \)

Let us remark that we consider a homogeneous decomposition in the horizontal variable and an inhomogeneous decomposition in the vertical variable. The associate Sobolev spaces are defined as follows.

**Definition 1.** Let \( s \) and \( s' \) be two real numbers and \( a \) a tempered distribution. Let

\[ \|a\|_{s,s'} \equiv \left( \sum_{j,k} 2^{2(j+s+k)} \|\Delta^j_a \Delta^k_z a\|^2_{L^2} \right)^{\frac{1}{2}}. \]

The space \( H^{s,s'} \) is the closure of \( \mathcal{D}(\mathbb{R}^3) \) for the above (semi-)norm.

Let us remark that, as usual when homogeneous Sobolev spaces are involved, \( H^{s,s'} \) is a Hilbert space if and only if \( s < 1 \). Spaces of this type have been used by J. Rauch and M. Reed in [18] and by M. Sablé-Tougeron in [19].

Let us state the three main theorems of this paper.
Theorem 2 (Existence). Let \( s_0 > 1/2 \) be a real number, and let \( v_0 \in H^{0,s_0} \). Then a positive time \( T \) and a solution \( v \) of \((NS_h)\) defined on \([0,T] \times \mathbb{R}^3\) exist such that

\[
v \in L^\infty([0,T]; H^{0,s_0}) \cap L^2([0,T]; H^{1,s_0}).
\]

Furthermore, there exists a constant \( c \) such that if \( \|v_0\|_{H^{0,s_0}} \) is less than \( c \nu \), then we can choose \( T = +\infty \).

Theorem 3 (Uniqueness). Let \( s_0 \) and \( s \) be two real numbers, greater respectively than \( 1/2 \) and \( 3/2 \), and let \( v_0 \) be an initial data in \( H^{0,s_0} \). Then at most one solution \( v \) of \((NS_h)\) exists in the space

\[
v \in L^\infty(\mathbb{R}^+; H^{0,s}) \cap L^2(\mathbb{R}^+; H^{1,s}).
\]

Let us denote by \( T^* \) the maximal time of existence; if \( T^* \) is finite, then

\[
\int_0^{T^*} \|v(t)\|_{H^{3/2,s_0}}^2 \, dt + \int_0^{T^*} \|v(t)\|_{H^{1,s}}^2 \, dt = +\infty.
\]

Those statements deserve a few comments. The initial data in Theorem 2 have only \( \nu \tau \) regularity in the horizontal variable, and \( H^{s+\nu} \) in the vertical variable; for positive viscosities, it is proved by D. Iftimie in [15] that the problem is well posed. Here, with that regularity (which is very close to the scaling of the equation), we obtain short time existence for large data and global existence for small data as in Fujita-Kato’s Theorem. Uniqueness requires an additional vertical derivative. That comes from the fact that the lack of vertical viscosity prevents us from gaining vertical regularity.

The third result of this paper concerns the rotating fluid equations, obtained from \((NS_h)\) by adding the Coriolis force \( \frac{1}{\varepsilon} v \times e_3 \), where \( e_3 \) is the Rossby number; we refer to [12] and [17] for a physical interpretation of that system. Note that the skew-symmetry of the operator \( L v \) implies that Theorems 2 and 3 hold for the rotating fluid equations: their proof only involves energy estimates. Moreover, the rotation induces a dispersive effect which leads to the following result.

Theorem 4 (Rotating fluids). Let \( v_0 \) be an initial data in \( H^{0,s} \) with \( s \) strictly greater than \( 3/2 \). Let \( \nu > 0 \) and \( \nu_\nu \geq 0 \) be two real numbers. Then a positive real number \( \varepsilon_0 \) exists, depending only on \( \nu \) and \( v_0 \), such that, for any \( \varepsilon \leq \varepsilon_0 \), the system

\[
\left\{
\begin{array}{l}
\partial_t v + v \cdot \nabla v - \nu \Delta_h v - \nu_\nu v \partial^2_3 v + \frac{1}{\varepsilon} v \times e_3 = -\nabla p \\
\text{div } v = 0 \\
v|_{t=0} = v_0,
\end{array}
\right.
\]

has a unique global solution in the space \( L^\infty([0,T]; H^{0,s}) \cap L^2([0,T]; H^{1,s}) \).

Let us notice that this global regularity result is different from those proved in the case of periodic boundary conditions (see for instance [1] and [11] in those cases, the global regularity is due to non resonance conditions).

The structure of the paper is the following:

- the first section is devoted to some basic properties of the spaces \( H^{s+s'} \);
- the second section, the core of the proof of Theorems 2 and 3, is devoted to the proof of an energy estimate in anisotropic Sobolev spaces for the convective term \( v \cdot \nabla \). The fact that the vector field \( v \) is divergence free will allow us to avoid bad terms containing \( \partial_t v \) which are not compensated by viscosity;
- the third section is nothing but an application of that energy estimate through classical contraction arguments, in order to prove Theorems 2 and 3;
- the fourth section consists in showing a dispersive effect for rotating fluids, as pointed out in [6], and applying it in this context in order to prove Theorem 4.
1. Algebraic properties of spaces $H^{s,s'}$

Before stating the theorem we want to prove, let us give a few properties concerning those spaces, which we shall use throughout the study. The main results are essentially contained in [18], in [19] and in [15]. They concern product rules in the spaces $H^{s,s'}$; let us start by recalling the classical product rules in isotropic Sobolev spaces: if $a$ and $b$ are two tempered distributions, then for any $d \geq 1$,

$$\forall s, t < \frac{d}{2}, \ s + t > 0, \quad \|ab\|_{\dot{H}^{s + t - \frac{d}{2}}(\mathbb{R}^d)} \leq C_{s,t} \|a\|_{\dot{H}^s(\mathbb{R}^d)} \|b\|_{\dot{H}^t(\mathbb{R}^d)}, \quad (1.1)$$

where the $\dot{H}^s$ spaces are the usual homogeneous Sobolev spaces, defined by the norm $\|u\|_{\dot{H}^s} \overset{\text{def}}{=} \| \cdot \|^s \mathcal{F}(u) \|_{L^2}$.

Furthermore, in the final section, we shall be working with bidimensional functions as well as tridimensional functions, and in particular taking the product of such functions. We shall be in need of the following result, whose proof can be found in [11] (in the case of periodic boundary conditions, but it is identical in the case of the whole space):

$$\forall s, t < 1, \ s + t > 0, \quad \|ab\|_{\dot{H}^{s + t - 1}(\mathbb{R}^3)} \leq C_{s,t} \|a\|_{\dot{H}^s(\mathbb{R}^3)} \|b\|_{\dot{H}^t(\mathbb{R}^3)}, \quad (1.2)$$

The same results hold for inhomogeneous Sobolev spaces.

Now we shall prove the following product rule in our anisotropic Sobolev spaces.

**Lemma 1.** Let $\sigma$ and $\sigma'$ be two real numbers, smaller than 1, such that $\sigma + \sigma' > 0$, and let $s_0 > 1/2$ and $s$ be such that $s + s_0 > 0$; a constant $C$ exists such that the following holds. Let $a$ and $b$ be two tempered distributions. Then

$$\|ab\|_{H^{s + \sigma' - 1,s}} \leq C(\|a\|_{H^{s_0}} \|b\|_{H^{\sigma',s}} + \|a\|_{H^{s_0}} \|b\|_{H^{\sigma',s}}).$$

**Proof of the lemma.** By definition of $H^{\sigma,s}$, we have

$$\|a\|_{H^{\sigma,s}}^2 = \sum_{j,k} 2^{2j(\sigma + ks)} \int |\Delta_j^s \Delta_k^v a(x_h, x_3)|^2 \, dx_h \, dx_3$$

$$= \sum_k 2^{2k s} \|\Delta_k^v a(\cdot, x_3)\|_{H^s(\mathbb{R}^3)}^2 \, dx_3,$$

so we can write

$$\|a\|_{H^{\sigma,s}} = \left\|2^{-ks} |\Delta_k^v a|_{L^2(\mathbb{R}^{3};H^s(\mathbb{R}^3))}\right\|_{L^2(\mathbb{Z})}. \quad (1.3)$$

Then we just have to write the paraproduct algorithm of J.-M. Bony (see [2]), in the vertical direction, which reads

$$\Delta_k^v(ab) = \Delta_k^v(T^v_a b + T^v_b a + R^v(a, b)),$$

with

$$T^v_a b \overset{\text{def}}{=} \sum_{k'} S^v_{k'} a \Delta_k^v b \quad \text{and} \quad R^v(a, b) \overset{\text{def}}{=} \sum_{k' j \in \{-1,0,1\}} \Delta_{k'}^v a \Delta_{k - j}^v b.$$

Let us state the following lemma, whose proof is obvious.

**Lemma 2.** For any $s > 1/2$ and any $\sigma \in \mathbb{R}$, a constant $C$ exists such that for any function $a$,

$$\|a\|_{L^\infty(\mathbb{R}^{3};H^s(\mathbb{R}^3))} \leq C\|a\|_{H^{s,s}}.$$
Then we have, by the bidimensional product rules recalled in (1.1),

\[ \| S_k^v a \Delta_k^v b \|_{L^2(\mathbb{R}^3; H^{s+\sigma'-1})} \leq \| S_k^v a \|_{L^\infty(\mathbb{R}^3; H^s)} \| \Delta_k^v b \|_{L^2(\mathbb{R}^3; H^s)} \leq c_k 2^{-ks} \| a \|_{H^{s,0}} \| b \|_{H^{s',0}}, \]

where \( c_k \in \ell^2(\mathbb{Z}) \), according to Lemma 2. Similarly we have

\[ \| \Delta_k^v a \Delta_k^v b \|_{L^1(\mathbb{R}^3; H^{s+\sigma'-1})} \leq \| \Delta_k^v a \|_{L^2(\mathbb{R}^3; H^s)} \| \Delta_k^v b \|_{L^2(\mathbb{R}^3; H^s)} \leq c_k^2 2^{-k(s+s_0)} \| a \|_{H^{s,0}} \| b \|_{H^{s',0}}. \]

Then Bernstein’s inequality yields

\[ \| \Delta_k^v (a \Delta_k^v b) \|_{L^2(\mathbb{R}^3; H^{s+\sigma'-1})} \leq c_k^2 2^{-k(s+s_0)+\frac{s}{2}} \| a \|_{H^{s,0}} \| b \|_{H^{s',0}}. \]

This implies that

\[ \| \Delta_k^v R(a, b) \|_{L^2(\mathbb{R}^3; H^{s+\sigma'-1})} \leq c_k 2^{-k(s+s_0-\frac{s}{2})} \| a \|_{H^{s,0}} \| b \|_{H^{s',0}}, \]

and Lemma 1 is proved.

\[ \Box \]

2. Anisotropic energy estimates

We shall prove the following lemma, which is the core of the proof of Theorem 2.

**Lemma 3.** For any real numbers \( s_0 > 1/2 \) and \( s \geq s_0 \), a constant \( C \) exists such that for any vector fields \( a \) and \( b \), with a divergence free,

\[ |(\Delta_k^v (a \cdot \nabla b) \Delta_k^v b)_{L^2}| \leq C d_k 2^{-2ks} \| b \|_{H^{s',0}} \left( \| a \|_{H^{s,0}} \| \nabla_h b \|_{H^{s',0}} + \| a \|_{H^{s,0}} \| \nabla_h b \|_{H^{s',0}} \right) \]

\[ + \| \nabla_h a \|_{H^{s',0}} \| b \|_{H^{s',0}} + \| \nabla_h a \|_{H^{s',0}} \| b \|_{H^{s',0}} + \sum_{d_k=1} \]

**Proof of the lemma.** Let us define

\[ F_k^h \overset{\text{def}}{=} \Delta_k^v (a^h \cdot \nabla_h b) \quad \text{and} \quad F_k^v \overset{\text{def}}{=} \Delta_k^v (a^3 \partial_3 b). \]

The terms \( (F_k^h | b_k)_{L^2} \) and \( (F_k^v | b_k) \) are estimated in two different ways. The reason why is that the term \( F_k^h \) involves only horizontal derivatives which can be compensated by the horizontal viscosity as in the case of the usual Navier-Stokes system.
Let us start by proving the result for $F_k^v$. We have, using similar computations to those leading to Lemma 1,

$$A_k \overset{\text{def}}{=} \| \Delta_k^v (a \cdot \nabla_h b) \|_{L^2(\mathbb{R}^3)} \leq \sum_{i=1}^2 \| \Delta_k^v (T_i \partial_i a_i) + R^v (a_i, \partial_i b) \|_{L^2(\mathbb{R}^3; H^{-\frac{1}{2}}(\mathbb{R}^2))}$$

$$\leq C 2^{-k} \| S_k^v a_k \|_{L^\infty(\mathbb{R}^3; H^1(\mathbb{R}^2))} \| \nabla_h b \|_{L^2(\mathbb{R}^3; L^2(\mathbb{R}^2))}$$

$$+ C 2^{-k} \| S_k^v \nabla_h b \|_{L^\infty(\mathbb{R}^3; H^1(\mathbb{R}^2))} \| a \|_{L^2(\mathbb{R}^3; L^2(\mathbb{R}^2))}$$

$$+ \sum_{|k_i-k_j| \leq 1} \| \Delta_k^v a_k \|_{L^\infty(\mathbb{R}^3; H^1(\mathbb{R}^2))} \| \Delta_k^v \nabla_h b \|_{L^2(\mathbb{R}^3; L^2(\mathbb{R}^2))} \| \Delta_k^v \nabla_h b \|_{H^{1/2}(\mathbb{R}^2)}$$

$$\leq C 2^{-k} \| [a]_{H^{1/2}(\mathbb{R}^2)} \| \nabla_h b \|_{H^{1/2}(\mathbb{R}^2)} + \| a \|_{H^{1/2}(\mathbb{R}^2)} \| \nabla_h b \|_{H^{1/2}(\mathbb{R}^2)}.$$
That implies that
\[
\|F_k^{n,1}\|_{L^2(\mathbb{R}^3; H^s(\mathbb{R}^2))} \leq C 2^\frac{h}{2} \sum_{k' \geq k-N_0} 2^{k(1-s)}2^{-k'(1+s)}c_k \|\nabla_h a\|_{H^{\eta,0}} \|b\|_{H^\frac{1}{2},s},
\]
\[
\leq Cc_k 2^{-k}\|\nabla_h a\|_{H^{\eta,0}} \|b\|_{H^\frac{1}{2},s},
\]
since \(s_0 > 1/2\).

Up to that point, we did not actually use the energy estimate, but only laws of product or Sobolev embeddings.

The estimate of the term \((F_k^{n,2}|b_k)|_{L^2}\) will use in a crucial way the structure of the nonlinearity.

First of all, following a computation done in [5], we get that
\[
(F_k^{n,2}|b_k)|_{L^2} = (S_k^p(a^3)\partial_3 b_k|b_k)|_{L^2} + R_k(a,b)
\]
\[
R_k(a,b) \overset{\text{def}}{=} \sum_{|k' - k| \leq N_0} ([\Delta_k^n, S_{k'-1}^n a^3] \partial_3 \Delta_k^n b|b_k)|_{L^2} + \sum_{|k' - k| \leq N_0} (S_k^n - S_{k'-1}^n) a^3 \partial_3 \Delta_k^n b|b_k)|_{L^2}.
\]

Then, using an integration by parts, we infer that
\[
(S_k^n(a^3)\partial_3 b_k|b_k)|_{L^2} = -\frac{1}{2}(S_k^n(\partial_3 a^3)b_k)|_{L^2}.
\]

Thanks to the fact that \(a\) is divergence free, we deduce that
\[
(F_k^{n,2}|b_k)|_{L^2} = (b_k S_k^n \text{div}_h a^h|b_k)|_{L^2} + R_k(a,b).
\]

The term \((b_k S_k^n \text{div}_h a^h|b_k)|_{L^2}\) can be estimated as the term \((F_k^n|b_k)\) that appeared above. To estimate the term \(([\Delta_k^n, S_{k'-1}^n a^3] \partial_3 \Delta_k^n b|b_k)|_{L^2}\), we write that, for any function \(\omega\),
\[
([\Delta_k^n, S_{k'-1}^n a^3]\omega)(x_1, x_3) = 2^k \int h(2^k y_3)(S_{k'-1}^n a^3(x_1, x_3) - S_{k'-1}^n a^3(x_1, x_3 - y_3))\omega(x_1, x_3 - y_3) dy_3.
\]

Writing a Taylor formula, we get that
\[
([\Delta_k^n, S_{k'-1}^n a^3]\omega)(x_1, x_3) = \int_{\mathbb{R}^3} h_1(2^k y_3)(S_{k'-1}^n \partial_3 a^3)(x_1, x_3 + t(x_3 - y_3))\omega(x_1, x_3 - y_3) dy_3 dt
\]
with \(h_1(z) = z h(z)\). Thanks to the fact that \(a\) is divergence free, we get
\[
([\Delta_k^n, S_{k'-1}^n a^3]\omega)(x_1, x_3) = -\int_{\mathbb{R}^3} h_1(2^k y_3)(S_{k'-1}^n \text{div}_h v^h)(x_1, x_3 + t(x_3 - y_3))\omega(x_1, x_3 - y_3) dy_3 dt.
\]

Now, let us apply the rules of product of Sobolev spaces on \(\mathbb{R}^2\); this implies that
\[
\|([\Delta_k^n, S_{k'-1}^n a^3]\omega)(\cdot, x_3)\|_{H^{\frac{1}{2}}(\mathbb{R}^2)} \leq C \int |h_1(2^k y_3)| \|\text{div}_h v^h\|_{L^\infty(\mathbb{R}^3; L^2(\mathbb{R}^2))} \|\omega(\cdot, x_3 - y_3)\|_{H^{\frac{1}{2}}(\mathbb{R}^2)} dy_3.
\]

From this, we deduce that
\[
\|([\Delta_k^n, S_{k'-1}^n a^3]\omega)(\cdot, x_3)\|_{L^2(\mathbb{R}^3; H^{\frac{1}{2}}(\mathbb{R}^2))} \leq C 2^{-k} \|S_{k'-1}^n \text{div}_h a^h\|_{L^\infty(\mathbb{R}^3; L^2(\mathbb{R}^2))} \|\omega(\cdot, x_3 - y_3)\|_{L^2(\mathbb{R}^3; H^{\frac{1}{2}}(\mathbb{R}^2))}.
\]
So we get
\[
([\Delta_{k'}^v, S_{k'-1}^v a^3]\partial_3 b_k, |b_k|)_{L^2} \leq \|S_{k'-1}^v \text{div}_h a^4\|_{L^\infty(\mathbb{R}^2, L^2(\mathbb{R}^2))} \times 2^{k'-k} \|b_{k'}\|_{L^2(\mathbb{R}^2, H^{1/2}(\mathbb{R}^2))} \|b_k\|_{L^2(\mathbb{R}^2, H^{1/2}(\mathbb{R}^2))}.
\]

Finally, with Lemma 2 joint with the fact that \(|k' - k| \leq N_0\), we infer
\[
([\Delta_{k'}^v, S_{k'-1}^v a^3]\partial_3 b_k, |b_k|)_{L^2} \leq C_{dk}\|\nabla_h a\|_{H^{0, -k}} 2^{-k^2} \|b\|_{H^{1/2, -k}}^2.
\]

The term \((S_k^v - S_{k'-1}^v a^3)\partial_3 \Delta_{k'}^v b, b\) can be estimated exactly as the term \(F_{k, 1}^v\). That concludes the proof of Lemma 3.

\[\square\]

\textbf{Remark 5.} We are unable to estimate the term \((F_k^v|b_k|)_{L^2(\mathbb{R}^2, H^s(\mathbb{R}^2))}\), for any \(s \neq 0\): that is due to the fact that the estimate of \((S_k^v(a^3)\partial_3 b_k|b_k|)_{L^2(\mathbb{R}^2, H^s(\mathbb{R}^2))}\) which appears in (2.4) would lead to terms of the type \((\Delta_{k'}^v, S_{k'-1}^v a^3)|b_k|_{L^2(\mathbb{R}^2)}\). We do not see here any way of recovering the vertical derivative of \(b_k\) because of the lack of vertical viscosity.

### 3. Proof of Theorems 2 and 3

\textbf{Proof of Theorem 2.} It is obvious that for smooth initial data, smooth solutions exist (of course only for short times); we shall prove \textit{a priori} estimates for such smooth solutions.

As it occurs in the classical Navier-Stokes system, global results for small initial data are easier to prove than local results for large data. In any case, the proof relies strongly on Lemma 3 proved in the previous section; let us start by deducing from that lemma the global part of Theorem 2. Applying the operator \(\Delta_{k'}^v\) and using an \(L^2\) energy estimate gives, considering the fact that \(\nu_v \geq 0\),
\[
\frac{1}{2} \frac{d}{dt} \|v_k(t)\|_{L^2}^2 + \nu \|\nabla_h v_k(t)\|_{L^2}^2 \leq (\Delta_{k'}^v(v \cdot \nabla v)|v_k)_{L^2},
\]
where we denote by \(v_k\) the term \(\Delta_{k'}^v v\). So using the above Lemma 3 with \(s = s_0\) and \(a = b = v\), we get
\[
\frac{1}{2} \frac{d}{dt} \|v_k(t)\|_{L^2}^2 + \nu \|\nabla_h v_k(t)\|_{L^2}^2 \leq C_{dk} 2^{-2k \nu_0} \|v\|_{H^{1/2, -k_0}}^2 \|\nabla_h v\|_{H^{0, -k_0}}.
\]

Now multiplying this inequality by \(2^{2k \nu_0}\) and taking the sum over \(k\) gives
\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^{0, -k_0}}^2 + \nu \|\nabla_h v(t)\|_{H^{0, -k_0}}^2 \leq C \|v\|_{H^{1/2, -k_0}}^2 \|\nabla_h v\|_{H^{0, -k_0}}.
\]

An obvious interpolation inequality tells us that
\[
\|v\|_{H^{1/2, -k_0}}^2 \leq \|v\|_{H^{0, -k_0}} \|\nabla_h v\|_{H^{0, -k_0}}.
\]

So we infer that
\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^{0, -k_0}}^2 + \nu \|\nabla_h v(t)\|_{H^{0, -k_0}}^2 \leq C \|v(t)\|_{H^{0, -k_0}} \|\nabla_h v(t)\|_{H^{0, -k_0}}^2.
\]

So, if \(\|v_0\|_{H^{0, -k_0}}\) is small enough, the function \(\|v(t)\|_{H^{0, -k_0}}^2\) decreases; more precisely, if
\[
\|v(t)\|_{H^{0, -k_0}} \leq \frac{\nu}{2C},
\]

then we get
\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|^2_{H^{0,\alpha}} + \frac{\nu}{2} \|\nabla_h v(t)\|^2_{H^{0,\alpha}} \leq 0.
\]
So, we get that, for any positive \(t\),
\[
\|v(t)\|^2_{H^{0,\alpha}} + \frac{\nu}{2} \int_0^t \|\nabla_h v(t')\|^2_{H^{0,\alpha}} dt' \leq \|v_0\|^2_{H^{0,\alpha}}.
\]
Now standard compactness arguments give the global existence result.

The proof of the local part of Theorem 2 consists in several parts, considering differently the high vertical frequencies and the low ones.

The case of high vertical frequencies is treated thanks to the following lemma.

**Lemma 4.** For any \(s_0 > 1/2\), a constant \(C\) exists such that for any smooth solution \(v\) of \((NS_h)\) on \([0,T] \times \mathbb{R}^3\), for any integer \(N\), we have
\[
\|(\text{Id} - S_N^v) v\|_{L^\infty_t(H^{s_0,0})} + 2\nu \|\nabla_h (\text{Id} - S_N^v) v\|_{L^2_t(H^{0,0})} \leq C\|(\text{Id} - S_N^v) v_0\|_{H^{0,0}} + C \int_0^T \|v(t)\|_{H^1/2,0}^2 \|\nabla_h v(t)\|_{H^{0,0}} dt.
\]

**Proof of the lemma.** We just have to multiply inequality (3.5) by \(2^{2s_0}\) and take the sum for \(k\) bigger than \(N - 1\); we get
\[
\frac{1}{2} \frac{d}{dt} \left( \sum_{k \geq N-1} 2^{2s_0} \|v_k(t)\|_{L^2}^2 \right) + \nu \sum_{k \geq N-1} 2^{2s_0} \|\nabla_h v_k(t)\|_{L^2}^2 \leq C\|v(t)\|_{H^{1/2,0}}^2 \|\nabla_h v(t)\|_{H^{0,0}}.
\]
Then Lemma 4 follows by integration in time.

Let us now consider the low vertical frequencies and high horizontal frequencies.

**Lemma 5.** For any \(s_0 > 1/2\), a constant \(C\) exists such that we have the following properties. Let \(M\) and \(N\) be two integers such that \(M \geq N\), and let us define \(v_{M,N} \stackrel{\text{def}}{=} (\text{Id} - S_M^{h,v}) S_N^v\). Then
\[
\|v_{M,N}\|_{L^\infty_t(H^{s_0,0})} + 2\nu \|\nabla_h v_{M,N}\|_{L^2_t(H^{0,0})} \leq C\|v_{M,N}|_{t=0}\|_{H^{0,0}} + C \int_0^T \|v(t)\|_{H^1/2,0} \|\nabla_h v(t)\|_{H^{0,0}} dt.
\]

**Proof.** Again by standard energy estimates, we have
\[
\frac{1}{2} \frac{d}{dt} \|v_{M,N}(t)\|_{L^2}^2 + \nu \|\nabla_h v_{M,N}(t)\|_{L^2}^2 = (F_{M,N}^{h,v} |v_{M,N}|)_{L^2} + (F_{M,N}^w |v_{M,N}|)_{L^2}, \quad \text{with}
\]
\[
F_{M,N}^{h,v} \stackrel{\text{def}}{=} (\text{Id} - S_M^{h,v}) S_N^v (v_h \nabla_h v) \quad \text{and} \quad F_{M,N}^w \stackrel{\text{def}}{=} (\text{Id} - S_M^{h,v}) S_N^v (v^3 \partial_3 v).
\]
The term \(F_{M,N}^{h,v}\) is estimated easily, using the product rules given in Lemma 1. Using the law of product, we have
\[
\|F_{M,N}(\cdot, x_3)\|_{H^{-1/2}(\mathbb{R}^2)} \leq C\|v_h(\cdot, x_3)\|_{H^{1/2}(\mathbb{R}^2)} \|\nabla v(\cdot, x_3)\|_{L^2(\mathbb{R}^2)}.
\]
As \( s_0 > 1/2 \), we get
\[
\| F_{M,N}^h \|_{L^2(R^3;H^{-1/2}(R^3))} \leq C \| v^h \|_{H^{1/2-s_0}} \| \nabla_h v \|_{H^{s_0}},
\]
hence
\[
\|(F_{M,N}^h|v_{M,N})_{L^2(R^3)}\| \leq C \| v \|_{H^{1/2-s_0}} \| \nabla_h v \|_{H^{s_0}}.
\]
To estimate \( \| F_{M,N}^v \|_{L^2(R^3;H^{-1/2}(R^3))} \), we use the fact that \( v \) is divergence free: we can write that
\[
F_{M,N}^v = (\text{Id} - S_{M}^h)S_{N}^v (\partial_3 (v^3 v) + v \, \text{div}_h v^h).
\]
The term \( (\text{Id} - S_{M}^h)S_{N}^v (v \, \text{div}_h v^h) \) can be estimated exactly as \( F_{M,N}^h \). As to the other term, it is basic Fourier analysis to observe that
\[
(\text{Id} - S_{M}^h)S_{N}^v (\partial_3 (v^3 v)\|v_{M,N}\|_{L^2(R^3)} \leq \| v^3 v \|_{L^2(R^3)} 2^{N-M} \| \nabla_h v_{M,N} \|_{L^2(R^3)} \leq C \| v \|_{H^{1/2-s_0}} \| \nabla_h v_{M,N} \|_{H^{s_0}} \leq C \| v \|_{H^{1/2-s_0}} \| \nabla_h v \|_{H^{s_0}}.
\]
So finally
\[
(F_{M,N}|v_{M,N})_{L^2(R^3)} \leq C \| v \|_{H^{1/2-s_0}} \| \nabla_h v \|_{H^{s_0}},
\]
which proves the lemma in an obvious way. \( \square \)

Let us conclude the proof of the local existence part of Theorem 2. Let us observe that \( v_0 \) is in \( L^2(R^3) \), so we have the basic \( L^2 \) energy estimate
\[
\| v(t) \|_{L^2}^2 = 2 \nu \int_0^t \| \nabla_h v(t') \|_{L^2}^2 dt' \leq \| v_0 \|_{L^2}^2.
\]
So it is obvious that
\[
\| S_N S_{M}^h v(t) \|_{H^{s_0}}^2 + 2 \nu \int_0^t \| \nabla_h S_N S_{M}^h v(t') \|_{L^2}^2 dt' \leq C 2^{2M} 2^{2N \delta_0} T \| v_0 \|_{L^2}^2.
\]
So using an interpolation inequality and Lemmas 4 and 5, we infer that
\[
(2 \nu) \| v \|_{L^2((H^{s_0}) \cap L^2)}^2 + 2 \nu \| \nabla_h v \|_{L^2((H^{s_0}) \cap L^2)}^2 \leq C \left( \| (\text{Id} - S_{N}^v)v_0 \|_{H^{s_0}}^2 + \| S_{N}^v (\text{Id} - S_{M}^h)v_0 \|_{H^{s_0}}^2 + 2^{2M} 2^{2N \delta_0} T \| v_0 \|_{L^2}^2 + \int_0^T \| v(t') \|_{H^{s_0}}^2 dt' \right).
\]
So we deduce that
\[
\| v \|_{L^2((H^{s_0}) \cap L^2)}^2 + \nu \| \nabla_h v \|_{L^2((H^{s_0}) \cap L^2)}^2 \leq C \left( \| (\text{Id} - S_{N}^v)v_0 \|_{H^{s_0}}^2 + \| S_{N}^v (\text{Id} - S_{M}^h)v_0 \|_{H^{s_0}}^2 + 2^{2M} 2^{2N \delta_0} T \| v_0 \|_{L^2}^2 + \frac{C}{\nu} \| v \|_{L^2((H^{s_0}) \cap L^2)}^4 \right).
\]
Then choosing first an integer \( N \) such that
\[
\| (\text{Id} - S_{N}^v)v_0 \|_{H^{s_0}}^2 \leq \eta,
\]
then an integer $M$ greater than $N$ such that
\[ \| S_N (I - S_M^2) v_0 \|_{H^{n,0}}^2 \leq \eta, \]
and finally a strictly positive real, number $T_0$ such that
\[ 2^{2M} 2^{2N_0} T \| v_0 \|_{L^2}^2 \leq \eta, \]
we obtain that, for any $T \leq T_0$,
\begin{align*}
\nu^2 \| v \|_{L^2_t (H^{\frac{1}{2}, s_0})}^2 + \nu \| \nabla_h v \|_{L^2_t (H^{n, s_0})}^2 & \leq C \eta + \frac{C}{\nu} \| v \|_{L^2_t (H^{\frac{1}{2}, s_0})}^4, \\
\nu^2 \| v \|_{L^2_t (H^{\frac{1}{2}, s_0})}^2 + \nu \| \nabla_h v \|_{L^2_t (H^{n, s_0})}^2 & \leq \eta.
\end{align*}
So, for any positive real number $\eta$, a positive real number $T_1$ exists so that
\[ \nu^2 \| v \|_{L^2_t (H^{\frac{1}{2}, s_0})}^2 + \nu \| \nabla_h v \|_{L^2_t (H^{n, s_0})}^2 \leq \eta. \]
That yields the result, and proves Theorem 2.

**Proof of Theorem 3.** Let us start by proving the propagation result stated in Theorem 3, considering $v_0 \in H^{0,s}$ with $s > s_0$; we just have to follow the computations leading to the global existence result with data in $H^{0,s}$.
We have, as for (3.5) and using also Lemma 3,
\begin{align*}
\frac{d}{dt} \| v_h(t) \|_{L^2}^2 + \nu \| \nabla_h v_h(t) \|_{L^2}^2 & \leq Cd_2 2^{-2k_0} \| v \|_{H^{\frac{1}{2}, s_0}} (\| v \|_{H^{\frac{1}{2}, s_0}} \| \nabla_h v \|_{H^{n, s}} + \| v \|_{H^{\frac{1}{2}, s_0}} \| \nabla_h v \|_{H^{n, s}}), \\
\frac{1}{2} \frac{d}{dt} \| v(t) \|_{H^{0,s}}^2 + \nu \| \nabla_h v(t) \|_{H^{n,s}}^2 & \leq \frac{C}{\nu^2} \| v \|_{H^{\frac{1}{2}, s_0}}^4 \| v \|_{H^{\frac{1}{2}, s_0}}^2 + \frac{C}{\nu} \| \nabla_h v \|_{H^{n, s_0}}^2 \| v \|_{H^{n, s_0}}^2.
\end{align*}
By integration, we find that
\[ \| v \|_{L^2_t (H^{0,s})}^2 + 2 \nu \| \nabla_h v \|_{L^2_t (H^{n,s})}^2 \leq \exp \left( \frac{C}{\nu^2} \int_0^T \| v(t) \|_{H^{1/2, s_0}}^4 dt + \frac{C}{\nu} \int_0^T \| \nabla_h v(t) \|_{H^{n,s_0}}^2 dt \right). \]
So we find that the life span of the solution is controlled by the following norm:
\[ T < +\infty \quad \Rightarrow \quad \int_0^T \| v(t) \|_{H^{1/2, s_0}}^4 dt + \int_0^T \| \nabla_h v(t) \|_{H^{n,s_0}}^2 dt = +\infty. \]
Now we are left with the proof of the uniqueness of the solutions when $s > 3/2$. So let $v_1$ and $v_2$ be two solutions of $(NS_h)$, with initial data $v_0$, such that
\[ \forall i \in \{1, 2\}, \quad v_i \in L^\infty([0, T], H^{0,s}) \cap L^2([0, T], H^{1,s}), \quad s > \frac{3}{2}, \]
and let $w \overset{\text{def}}{=} v_1 - v_2$. We have
\[ \partial_t w - \nu \Delta_h w + v_1 \cdot \nabla w + w \cdot \nabla v_2 = -\nabla p, \]
hence using Lemma 3 and interpolation inequalities, we can write
\[
(v_1 \cdot \nabla w)_{H^{0,s}} \leq C \left( \| \nabla_h v_1 \|_{H^{0,s-1}} \| w \|_{H^{1/2,0}}^2 + \| v_1 \|_{H^{1/2,0}} \| \nabla_h w \|_{H^{0,s-1}} \right)
\leq \left( \| \nabla_h v_1 \|_{H^{0,s-1}} \| w \|_{H^{0,s-1}} \| \nabla_h w \|_{H^{0,s-1}} \right)
+ \| v_1 \|_{H^{1/2,0}} \| w \|_{H^{1/2,0}} \| \nabla_h w \|_{H^{0,s-1}}^2 \right)
\leq \frac{\nu}{4} \| \nabla_h w \|_{H^{0,s-1}}^2 + \frac{C}{\nu} \left( \| \nabla_h v_1 \|_{H^{1/2,0}}^2 + \| v_1 \|_{H^{1/2,0}}^2 \right) \| w \|_{H^{0,s-1}}.
\]

But with Lemma 1, we can write that
\[
\left| (\Delta (u^h \cdot \nabla v_2)) w_k \right| \leq C 2^{2(s-1)} k \| w \|_{H^{1/2,0}}^2 \| \nabla_h v_2 \|_{H^{0,s-1}}
\]
and
\[
\left| (\Delta (u^3 \partial_t v_2)) w_k \right| \leq C 2^{2(s-1)} k \| w \|_{H^{1/2,0}}^2 \| v_2 \|_{H^{0,s-1}}^2.
\]

It follows that
\[
\frac{d}{dt} \| w \|_{H^{0,s-1}}^2 + \nu \| \nabla_h w \|_{H^{0,s-1}}^2 \leq \frac{\nu}{4} \| \nabla_h w \|_{H^{0,s-1}}^2
+ \frac{C}{\nu} \left( \| \nabla_h v_1 \|_{H^{1/2,0}}^2 + \| v_1 \|_{H^{1/2,0}}^2 \right) \| w \|_{H^{0,s-1}}.
\]

The result follows by time integration, and Theorem 3 is proved.

4. THE CASE OF ROTATING FLUIDS: PROOF OF THEOREM 4

In this final section, we are going to prove Theorem 4. We shall in fact prove a more precise result that stated in Theorem 4, and in order to do so, we need to introduce some notation. We shall consider the following rotating fluid system:

\[
(RF^c) \quad \begin{cases}
\partial_t v - \nu \Delta_h v - \nu \cdot \partial^2_h v + v \cdot \nabla v + \frac{1}{\varepsilon} v \times e_3 = -\nabla p \\
v_{t=0} = v_0.
\end{cases}
\]

The elements of the kernel of the operator $Lv \overset{\text{def}}{=} P(v \times e_3)$, where $P$ is the projector onto divergence–free vector fields, are bidimensional vector fields; so it is natural to introduce the solution of the bidimensional Navier–Stokes equation:

\[
\begin{cases}
\partial_t \hat{u} - \nu \Delta_h \hat{u} + \hat{u} \cdot \nabla \hat{u} = -\nabla \hat{p} \\
\hat{u}_{t=0} = \hat{u}_0,
\end{cases}
\]

where $\hat{u}_0$ is a bidimensional, divergence–free vector field.

Finally in order to state the following theorem, we need to introduce the solution to the free wave equation associated with $(RF^c)$.

\[
\begin{cases}
\partial_t w_r - \nu \Delta_h w_r - \nu \cdot \partial^2_h w_r + \frac{1}{\varepsilon} w_r \times e_3 = -\nabla p_r \\
\text{div } w_r = 0 \\
w_{r|t=0} = w_0.
\end{cases}
\]

We shall prove the following result.
Theorem 6. Let $\nu > 0$ and $\nu_v \geq 0$ be two real numbers. Let $v_0 = \bar{u}_0 + w_0$, where $\bar{u}_0 \in L^2(\mathbb{R}^2)$ is a bidimensional vector field, and where $w_0 \in H^{0,s}$, with $s > 1/2$. We suppose both fields are divergence free. Then there exists $\varepsilon_0$, independent of $\nu$, such that for $\varepsilon \leq \varepsilon_0$, there is a global solution to the system $RF^e$, bounded in $L^\infty(\mathbb{R}^+, H^{0,s}) \cap L^2(\mathbb{R}^+, H^{1,s})$. Moreover, if $s > 3/2$, then the solution is unique. Finally when $\varepsilon$ goes to zero, $v$ satisfies

$$v - \bar{u} - w_p \to 0 \text{ in } L^\infty(\mathbb{R}^+, H^{0,s}) \cap L^2(\mathbb{R}^+, H^{1,s}).$$

The proof of that theorem mixes the results obtained in the previous sections with anisotropic Strichartz estimates on $w_p$, in the spirit of [6]: let us define

$$
\mathcal{F}(A_{np} f)(\xi) \equiv e^{i 2^{2j} \varepsilon^2 \kappa^2} \mathcal{F}(f)(\xi).
$$

We have the following estimate.

Theorem 7. For any vector field $f$ we have, for any $p \in [1, \infty]$,

$$
\|\Delta^j_k \Delta^e_k A_{np} f\|_{L^1_p(L^{\infty,2}(\mathbb{R}^2))} \leq C \nu^{2j(1 - \frac{1}{p})} \min \left(1, \left(\varepsilon 2^{2j}\right)^{\frac{1}{p}} 2^{\frac{1}{p}|j-k|}\right) \|\Delta^j_k \Delta^e_k f\|_{L^p(\mathbb{R}^3)}. \tag{4.6}
$$

Proof of Theorem 7. We follow exactly the same lines as in [6]. For the convenience of the reader, we present here a self-contained proof. Like all Strichartz-type inequalities, (4.6) is the consequence of dispersive estimates. Writing

$$
A_{np}(\tau, \theta, \tilde{\theta}) f \equiv \mathcal{F}^{-1} \left(e^{i 2^{2j} \varepsilon^2 \kappa^2} \tilde{f}(\xi)\right),
$$

we have the following dispersive lemma.

Lemma 6. There exists a constant $C$ such that the following holds. For any function $f$, we have

$$
\|\Delta^j_k \Delta^e_k A_{np}(\tau, \theta, \tilde{\theta}) f\|_{L^{\infty,2}(\mathbb{R}^2)} \leq C \min \left(1, \tau^{-\frac{1}{2}2^{j-k}}\right) 2^{2j} e^{-C \theta 2^{2j}} \|f\|_{L^1_{x,\xi}^p}. \tag{4.7}
$$

Proof of the lemma. Let us define

$$
I_{j,k}(\tau, \theta, x, \xi_3) \equiv \varphi(2^{-k} \xi_3) \int_{\mathbb{R}^2} e^{i \xi_3 \cdot \xi_3 + i \varepsilon \xi_3 - 2^{-j} \theta |\xi_3|} \varphi(2^{-j} |\xi_3|) \, d\xi_3.
$$

Then we have

$$
\|\Delta^j_k \Delta^e_k A_{np}(\tau, \theta, \tilde{\theta}) f\|_{L^{\infty,2}(\mathbb{R}^2)} \leq C e^{-C \theta 2^{2k}} \|I_{j,k}(\tau, \theta, \cdot, \cdot)\|_{L^{\infty,\infty}} \|f\|_{L^1_{x,\xi}^2},
$$

where we have used a Young inequality in the horizontal variables, and Hölder’s inequality in the vertical variable, associated with Plancherel’s formula.

Now all we need is an estimate of $I_{j,k}(\tau, \theta, \cdot, \cdot)$ in $L^{\infty,\infty}_{x,\xi_3}$. We can write

$$
I_{j,k}(\tau, x, \xi_3) = 2^{2k} \tilde{I}_{j,k}(\tau, 2^{2k} \theta, 2^{k} x, 2^{-k} \xi_3)
$$

with

$$
\tilde{I}_{j,k}(\tau, \eta, y, \xi_3) \equiv \varphi(\xi_3) \int_{\mathbb{R}^2} e^{i \eta \cdot \cdot + i \varepsilon \eta \cdot \cdot - 2^{-j} |\eta|} \varphi(2^{-j} |\eta|) \, d\eta.
$$

So we shall integrate by parts in the following integral:

$$
\Phi(\xi_3, \xi_2) \equiv \int_{\mathbb{R}} e^{i \varepsilon \xi_2} \varphi(2^{-j} |\xi_2|) \varphi(\xi_3) \, d\xi_2.
$$
Let us introduce the operator $L$ defined as

$$L a = \frac{1}{1 + \tau \alpha^2(\zeta)} (a - i \alpha(\zeta) \partial_{\zeta} a) \quad \text{with} \quad \alpha(\zeta) = \partial_{\zeta} \left( \frac{\zeta}{|\zeta|} \right).$$

An easy computation yields

$$L(e^{i t \hat{\alpha}_0}) = e^{i t \hat{\alpha}_0},$$

which implies immediately that

$$\Phi(\zeta_1, \zeta_3) = \int e^{i t \hat{\alpha}_0} \frac{1}{t L} \left( e^{-\eta |\zeta_1|^2} \varphi(|\zeta|) \right) \, d\zeta_2.$$

But we can compute, for any function $\Psi$,

$$t L (\Psi(\zeta)) = \left( \frac{1}{1 + \alpha^2 \tau} - i (\partial_{\zeta} \alpha) \frac{1 - \tau \alpha^2}{(1 + \tau \alpha^2)^2} \right) \Psi(\zeta) - \frac{i \alpha}{1 + \tau \alpha^2} \partial_{\zeta} \Psi(\zeta),$$

so using the fact that $\zeta$ is in a fixed ring of $\mathbb{R}^3$ and that $\varphi \in \mathcal{D}(\mathbb{R})$, we get after elementary computations

$$\left| t L \left( e^{-\eta |\zeta_1|^2} \varphi(2^{k-j} |\zeta_3|) \right) \right| \leq Ce^{-C \min(2^{k-j}, 1)} |\zeta_3|^{-\frac{3}{2}} e^{-\eta |\zeta_1|^2}.$$

It follows that

$$\| \mathcal{I}_{j,k}(\tau, \eta, \zeta_3) \|_{L_{\tau, \zeta_3}^{\infty, \infty}} \leq Ce^{-C \eta} \left\| \varphi(\zeta_3) \int_{|\zeta_1| \leq 2^{j-k}} \min(2^{k-j}, 1) \frac{d\zeta_2}{1 + \tau |\zeta_2|^2} \right\|_{L_{\tau}^{\infty}} \leq Ce^{-C \eta} t^{-\frac{3}{2}} \min(2^{j-k}, 1).$$

So finally we have obtained that

$$\| \Delta^h_k \Delta^v_k A_{\eta \mu}(\tau, \theta) f \|_{L_{\tau, \zeta_3}^{\infty, \infty}} \leq Ce^{-C \bar{\eta} 2^{2k}} e^{-C \theta^2 2^{2k} \min(2^{j-k}, 1)} \| f \|_{L_{\tau, \zeta_3}^{1, 2}}. \quad (4.8)$$

Now to conclude to (4.7), we note that by Sobolev embeddings,

$$\| \Delta^h_k \Delta^v_k A_{\eta \mu}(\tau, \theta) f \|_{L_{\tau, \zeta_3}^{\infty, \infty}} \leq C 2^j e^{-C \bar{\eta} 2^{2k}} e^{-C \theta^2 2^{2j}} \| \Delta^h_k \Delta_k^v f \|_{L^2} \leq C 2^j e^{-C \bar{\eta} 2^{2k}} e^{-C \theta^2 2^{2j}} \| \Delta^h_k \Delta_k^v f \|_{L_{\tau, \zeta_3}^{1, 2}},$$

which with (4.8) implies that

$$\| \Delta^h_k \Delta_k^v A_{\eta \mu}(\tau, \theta) f \|_{L_{\tau, \zeta_3}^{\infty, \infty}} \leq Ce^{-C \bar{\eta} 2^{2k}} e^{-C \theta^2 2^{2j}} \min \left( \frac{1}{\tau^2}, 2^{j-k}, 1 \right) \| f \|_{L_{\tau, \zeta_3}^{1, 2}}.$$

The lemma is proved.

\textbf{End of the proof of Theorem 7.} To prove the result, we are going to use duality arguments, based on the anisotropic estimate (4.7). In the following, we shall call

$$B \overset{\text{def}}{=} \left\{ \Psi \in \mathcal{D}(\mathbb{R}^3), \| \Psi \|_{L_{\tau}^{\infty}(L_{\zeta}^{1, 2})} \leq 1 \right\}.$$
The following set of equalities is standard:

\[
\| \Delta^h_j \Delta^v_k A_n r f \|_{L^1_n(L_{\infty,1})} = \sup_{\Psi \in \mathcal{B}} \left( 2\pi \right)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \varphi(2^{-j+1}|\xi|) e^{-\frac{1}{2} |\xi|^2 - |\xi|^2 \xi^2} \Psi(t, \xi) dt \xi,
\]

where we have defined \( f_{j,k} = \Delta^h_j \Delta^v_k f \) and \( \Psi_{j,k} = \Delta^h_j \Delta^v_k \Psi \). So it follows that

\[
\| \Delta^h_j A_n r f \|_{L^1_n(L_{\infty,1})} \leq \sup_{\Psi \in \mathcal{B}} \left( 2\pi \right)^{-\frac{3}{2}} \sum_{k \leq j} \| f_{j,k} \|_{L^2} \left\| \int_{\mathbb{R}^3} e^{-\frac{1}{2} |\xi|^2 - |\xi|^2 \xi^2} \Psi_{j,k}(t, \xi) dt \right\|_{L^2(\mathbb{R}^3)}.
\]

But we have

\[
\left\| \int_{\mathbb{R}^3} e^{-\frac{1}{2} |\xi|^2 - |\xi|^2 \xi^2} \Psi_{j,k}(t, \xi) dt \right\|_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \Psi_{j,k}(s, \xi) dt \xi.
\]

(4.10) yields finally

\[
\| \Delta^h_j \Delta^v_k A_n r f \|_{L^1_n(L_{\infty,1})} \leq C_\nu \min \left( 1, (2\nu)^{-\frac{1}{2}} \right) 2^{-j} \| \Delta^h_j A_n r f \|_{L^2(\mathbb{R}^3)}.
\]

But by Sobolev embeddings and energy estimates, we have

\[
\| \Delta^h_j \Delta^v_k A_n r f \|_{L^1_n(L_{\infty,1})} \leq C \| \Delta^h_j \Delta^v_k f \|_{L^2(\mathbb{R}^3)}
\]

and the theorem is proved by interpolation.

**Proof of Theorem 6.** The proof is going to use Theorem 7 proved above. Such an estimate is best used when the horizontal and vertical frequencies of \( w_r \) are fixed, and it is possible to do so for the following reason: it is clear that, as \( w_r \) satisfies a linear equation,

\[
\lim_{N, N_3 \to -\infty} (\text{Id} - S_N^h)(\text{Id} - S_{N_3}) w_r = w_r \quad \text{in} \quad H^{\sigma, \sigma'}, \forall \sigma, \sigma'.
\]
So for any $\eta > 0$, there exist integers $N, N', N_3, N_3'$ such that
\[
\|w_F - w_{F,m}\|_{L^\infty(\mathbb{R}^+; H^{0,s}) \cap L^2(\mathbb{R}^+, H^{1,s})} \leq \eta,
\]  
(4.12)
where we have noted
\[
w_{F,m} \overset{\text{def}}{=} (\text{Id} - S_N^h)S_{N'}^h (\text{Id} - S_{N_3}^w)S_{N_3'}^w w_F.
\]
Then (4.6) becomes simply
\[
\forall p \in [1, \infty], \quad \|w_{F,m}\|_{L^p_{\kappa}(L^{\infty,2}_{\kappa,\kappa})} \leq C_\eta \varepsilon^{1/4p} \|w_{n,m}\|_{L^2(\mathbb{R}^2)},
\]  
(4.13)
where here and in the following, we have written $C_\eta$ for any constant depending on $N$, $N'$, $N_3$ and $N_3'$, such that (4.12) is satisfied.

Let us recall the following classical estimates on the bidimensional Navier–Stokes equation, and on linear parabolic equations: we have
\[
\forall t \geq 0, \quad \|\bar{u}(t)\|_{L^2(\mathbb{R}^2)}^2 + 2\nu \int_0^t \|\nabla_h \bar{u}(t')\|_{L^2(\mathbb{R}^2)}^2 \, dt' = \|\bar{u}_0\|_{L^2(\mathbb{R}^2)}^2,
\]  
(4.14)
and
\[
\forall t \geq 0, \quad \|w_F(t)\|_{H^{p,s}}^p + 2\nu \int_0^t \|\nabla_h w_F(t')\|_{H^{p,s}} \, dt' = \|w_0\|_{H^{p,s}}^p.
\]  
(4.15)
Now if we define $w \overset{\text{def}}{=} v - \bar{u} - w_{F,m}$, it is enough to prove that $w$ is unique, and goes to zero in $L^\infty(\mathbb{R}^+; H^{0,s}) \cap L^2(\mathbb{R}^+, H^{1,s})$. The function $w$ satisfies the following equation:
\[
\partial_t w - \nu \Delta_h w - \nu \nu_\nu \partial_3^2 w + \frac{1}{8} w \times e_3 = -\nabla \tilde{p} + \sum_{j=1}^6 F_j
\]  
(4.16)
with
\[
F_1 \overset{\text{def}}{=} -w \cdot \nabla w,
F_2 \overset{\text{def}}{=} -\bar{u} \cdot \nabla w - w \cdot \nabla \bar{u},
F_3 \overset{\text{def}}{=} -w_{F,m} \cdot \nabla w,
F_4 \overset{\text{def}}{=} -w \cdot \nabla w_{F,m},
F_5 \overset{\text{def}}{=} -\text{div}_h(w_j^{F,m} \tilde{u}^h) - \text{div}_h(\tilde{u}^j w_{F,m}^h) - \text{div}_h(w_j^{F,m} w_{F,m}^h),
F_6 \overset{\text{def}}{=} -\tilde{u}^3 \partial_3 w_j^{F,m} - \tilde{u}^j \partial_3 w_{F,m}^3 - \partial_3 (w_j^{F,m} w_{F,m}^3),
\]
and
\[
w_{|t=0} = \left(\text{Id} - (\text{Id} - S_N^h)S_{N'}^h (\text{Id} - S_{N_3}^w)S_{N_3'}^w\right) w_0.
\]
Note that, according to (4.11), we have
\[
\forall \eta > 0, \quad \exists (N, N', N_3, N_3'), \quad \|w_{|t=0}\|_{H^{p,s}} \leq \eta.
\]  
(4.17)
The uniqueness result follows simply from Theorem 3, since the skew-symmetric term $\frac{1}{8} w \times e_3$ does not interfere in the energy estimates. So let us prove that for $\varepsilon$ small enough, there is a global solution $w$ to equation (4.16),
which goes to zero in $L^\infty(\mathbb{R}^+, H^{0,s}) \cap L^2(\mathbb{R}^+, H^{1,s})$. The proof will use estimates proved in the previous section, as well as the Strichartz estimate proved above.

First, using Lemma 3 proved in Section 2, we get

$$\left(\Delta_k^\nu (w \cdot \nabla w) |\Delta_k^\nu w\right)_{L^2(\mathbb{R}^3)} \leq C d_k 2^{-2ks} \|w\|_{H^{1/2,s}}^2 \|\nabla w\|_{H^{0,s}}.$$

Using the interpolation between $H^{0,s}$ and $H^{1,s}$, and taking the sum over $k$, we get

$$\left(F_1 |w\right)_{H^{0,s}} \leq C \|w\|_{H^{0,s}} \|\nabla w\|_{H^{0,s}}. \quad (4.18)$$

Now, let us estimate $(F_2 |w)_{H^{0,s}}$. Since $u$ does not depend on the third variable, we have

$$\left(\Delta_k^\nu (\tilde{u} \cdot \nabla w) |\Delta_k^\nu w\right)_{L^2(\mathbb{R}^3)} = \left(\tilde{u} \cdot \nabla |\Delta_k^\nu w\right)_{L^2(\mathbb{R}^3)} = -\frac{1}{2} \int_{\mathbb{R}^3} \text{div}_h \tilde{u} |\Delta_k^\nu w|^2 \, dx,$$

so as $H^{1/2}(\mathbb{R}^2)$ is embedded in $L^4(\mathbb{R}^2)$, we get

$$\left|\left(\Delta_k^\nu (\tilde{u} \cdot \nabla w) |\Delta_k^\nu w\right)_{L^2(\mathbb{R}^3)} \right| \leq C \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2)} \|\Delta_k^\nu w\|_{L^2(\mathbb{R}^2)} \|\Delta_k^\nu w\|_{L^2(\mathbb{R}^3)} + \frac{\nu}{16} \|\nabla \Delta_k^\nu w\|_{L^2(\mathbb{R}^3)}^2,$$

By interpolation between $L^2(\mathbb{R}^2)$ and $H^{1}(\mathbb{R}^2)$, that yields

$$\left|\left(\Delta_k^\nu (\tilde{u} \cdot \nabla w) |\Delta_k^\nu w\right)_{L^2(\mathbb{R}^3)} \right| \leq C \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2)} \|\Delta_k^\nu w\|_{L^2(\mathbb{R}^2)} \|\Delta_k^\nu w\|_{L^2(\mathbb{R}^3)} + \frac{\nu}{16} \|\nabla \Delta_k^\nu w\|_{L^2(\mathbb{R}^3)}^2,$$

and multiplying by $2^{2ks}$ and summing over $k$, we get finally

$$\left|\left(\tilde{u} \cdot \nabla w\right)_{H^{0,s}} \right| \leq C \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2)} \|w\|_{H^{0,s}}^2 + \frac{\nu}{16} \|\nabla w\|_{H^{0,s}}^2. \quad (4.19)$$

Moreover, we have

$$\left|\left(\Delta_k^\nu (w \cdot \nabla \tilde{u}) |\Delta_k^\nu w\right)_{L^2(\mathbb{R}^3)} \right| \leq C \|\Delta_k^\nu w\|_{L^2(\mathbb{R}^2)} \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2)} \|\nabla \Delta_k^\nu w\|_{L^2(\mathbb{R}^3)}^2,$$

so similarly

$$\left|\left(w \cdot \nabla \tilde{u}\right)_{H^{0,s}} \right| \leq C \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2)} \|w\|_{H^{0,s}}^2 + \frac{\nu}{16} \|\nabla w\|_{H^{0,s}}^2. \quad (4.20)$$

So, putting estimates (4.19) and (4.20), we get that

$$\left|\left(F_2 |w\right)_{H^{0,s}} \right| \leq C \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2)} \|w\|_{H^{0,s}}^2 + \frac{\nu}{16} \|\nabla w\|_{H^{0,s}}^2. \quad (4.21)$$

In order to estimate the term $(F_3 |w)_{H^{0,s}}$, we use Lemma 3 proved in Section 2, which yields

$$\left|\left(\Delta_k^\nu (w_{F,m} \cdot \nabla w) |\Delta_k^\nu w\right)_{L^2(\mathbb{R}^3)} \right| \leq C d_k 2^{-2ks} \|w\|_{H^{1/2,s}} \|\nabla w\|_{H^{0,s}} \|\nabla w\|_{H^{0,s}} + \|w\|_{H^{1/2,s}} \|\nabla w_{F,m}\|_{H^{0,s}},$$

which in turn implies that

$$\left|\left(F_3 |w\right)_{H^{0,s}} \right| \leq \frac{\nu}{16} \|\nabla w\|_{H^{0,s}}^2 + \frac{C}{\nu} \left(\|w_{F,m}\|_{H^{1/2,s}}^2 \|w\|_{H^{1/2,s}}^2 \|\nabla w_{F,m}\|_{H^{0,s}}^2 \right).$$
So finally, by interpolation, we infer

\[
|(F_4|w)_{H^{0.5}}| \leq \frac{\nu}{8} \|\nabla_h w\|_{H^{0.5}}^2 + \frac{C}{\nu} \|w\|_{H^{0.5}}^2 \left( \frac{C}{\nu^2} \|w_{F,m}\|_{H^{1/2},s}^4 + \|\nabla_h w_{F,m}\|_{H^{0.5}}^2 \right).
\]

(4.22)

Moreover, we can write

\[
|(F_4|w)_{H^{0.5}}| = \|(w \cdot \nabla w_{F,m}|w)_{H^{0.5}}\|
\leq \int_{\mathbb{R}^2} \left\| \left( w(x_h, \cdot) \cdot \nabla w_{F,m}(x_h, \cdot) \right)_{H^s(\mathbb{R}^3)} \right\| dx_h
\leq \int_{\mathbb{R}^2} \|w(x_h, \cdot)\|_{H^s(\mathbb{R}^3)}^2 \|\nabla w_{F,m}(x_h, \cdot)\|_{H^s(\mathbb{R}^3)} dx_h
\leq \|w\|_{H^{0.5}}^2 \|\nabla w_{F,m}\|_{L^\infty(\mathbb{R}^3;H^s(\mathbb{R}^3))}.
\]

But the spectral localization of \( w_{F,m} \) implies that, for any \( \eta \), a constant \( C_\eta \) exists such that

\[
\|\nabla w_{F,m}\|_{L^\infty(\mathbb{R}^3;H^s(\mathbb{R}^3))} \leq C_\eta \|w_{F,m}\|_{L^\infty_{x_h,t_3}},
\]

so finally

\[
|(F_4|w)_{H^{0.5}}| \leq C_\eta \|w_{F,m}\|_{L^\infty_{x_h,t_3}} \|w\|_{H^{0.5}}^2.
\]

(4.23)

To estimate \(|(F_5|w)_{H^{0.5}}|\), let us write that

\[
|(F_5|w)_{H^{0.5}}| \leq \|F_5\|_{H^{-1.5}} \|\nabla_h w\|_{H^{0.5}}
\leq \frac{C}{\nu} \|F_5\|_{H^{-1.5}} + \frac{\nu}{8} \|\nabla_h w\|_{H^{0.5}}^2.
\]

By definition of \( F_3 \), we have

\[
\|F_5\|_{H^{-1.5}} \leq C(\|w_{F,m} \phi\|_{H^{0.5}}^2 + \|w_{F,m} w_{F,m}\|_{H^{0.5}}^2).
\]

As the Fourier transform of \( w_{F,m} \) has its support in the vertical frequencies included in the ball of radius \( N_3^2 \), we have

\[
\|F_5\|_{H^{-1.5}}^2 \leq C N_3^{2s} (\|w_{F,m} \phi\|_{H^{2}(\mathbb{R}^3)}^2 + \|w_{F,m} w_{F,m}\|_{L^2(\mathbb{R}^3)})
\leq C N_3^{2s} (\|w\|_{L^2(\mathbb{R}^3)}^2 \|w_{F,m}\|_{L^\infty_{x_h,t_3}}^2 + \|w_{F,m}\|_{L^\infty_{x_h,t_3}}^2 \|w_{F,m}\|_{L^\infty_{x_h,t_3}}^2).
\]

Using again the spectral localization in the vertical variable and the energy estimate of the free rotating system, we get that

\[
\|w_{F,m}\|_{L^\infty_{x_h,t_3}}^2 \leq C N_3^{2s} \|w_{F,m}\|_{L^2(\mathbb{R}^3)}^2
\leq C N_3^{2s} \|w_{0,m}\|_{L^2(\mathbb{R}^3)}^2.
\]

So, using now the energy estimate in the \( 2 - D \) Navier-Stokes equation, we get that

\[
\|F_5\|_{H^{-1.5}} \leq C N_3^{2s} (\|u_0\|_{L^2(\mathbb{R}^3)}^2 + N_3^{2s} \|w_{0,m}\|_{L^2(\mathbb{R}^3)}^2 \|w_{F,m}\|_{L^\infty_{x_h,t_3}}^2).
\]
As in all that follows, let us denote by $C_\eta$ any constant depending only on $\eta$ and on the initial data $\bar{u}_0$ and $v_0$; we can rewrite the above estimate as

$$|(F_6|w)_{H^{0,\ast}}| \leq C_\eta \|w_{F,m}\|_{L^{\infty,2};x_3}^2 + \frac{\nu}{8} \|\nabla_h w\|_{H^{0,\ast}}$$  \hfill (4.24)

To estimate $|(F_6|w)_{H^{0,\ast}}|$, let us write that

$$|(F_6|w)_{H^{0,\ast}}| \leq \|F_6\|_{H^{0,\ast}} \|w\|_{H^{0,\ast}}.$$  

Using again the spectral localization in the vertical variable of $w_{F,m}$, the energy estimate in the $2 - D$ Navier-Stokes equation and the energy conservation for free rotating system, we get that

$$\|F_6\|_{H^{0,\ast}} \leq C N_3^{s+1} \left( \|\bar{u}_0\|_{L^2(\mathbb{R}^3)} + N_3 \|w_{0,m}\|_{L^2(\mathbb{R}^3)} \right) \|w_{F,m}\|_{L^{\infty,2};x_3}.$$  

As above, this can be written as

$$|(F_6|w)_{H^{0,\ast}}| \leq C_\eta \|w_{F,m}\|_{L^{\infty,2};x_3} \|w\|_{H^{0,\ast}}.$$  \hfill (4.25)

An energy estimate in $H^{0,\ast}$ on equation (4.16) yields, writing $F \overset{\text{def}}{=} \sum_{j=1}^6 F_j,$

$$\frac{1}{2} \frac{d}{dt} \|w(t)\|_{H^{0,\ast}}^2 + \nu \|\nabla_h w(t)\|_{H^{0,\ast}}^2 \leq (F(t)|w(t))_{H^{0,\ast}}.$$  

So using (4.18) and (4.21–4.25) yields

$$\frac{1}{2} \frac{d}{dt} \|w(t)\|_{H^{0,\ast}}^2 + \nu \|\nabla_h w(t)\|_{H^{0,\ast}}^2 \leq \Phi(t) \|w(t)\|_{H^{0,\ast}}^2 + G_\eta(t)$$

\[+ \left( \frac{\nu}{4} + C \|w(t)\|_{H^{0,\ast}} \left( 1 + \frac{1}{\nu} \|w(t)\|_{H^{0,\ast}} \right) \right) \|\nabla_h w(t)\|_{H^{0,\ast}}^2 \]

with

$$G_\eta(t) \overset{\text{def}}{=} C_\eta \|w_{F,m}\|_{L^{\infty,2};x_3}^2 + C_\eta \|w_{F,m}\|_{L^{\infty,2};x_3} \|w(t)\|_{H^{0,\ast}}$$

and

$$\Phi(t) \overset{\text{def}}{=} C \left( \|\nabla_h \bar{u}(t)\|_{L^2(\mathbb{R}^3)} + \|\nabla_h w_{F,m}(t)\|_{H^{0,\ast}} + \frac{C}{\nu^2} \|w_{F,m}(t)\|_{H^{0,\ast}}^4 + \nu C_\eta \|w_{F,m}(t)\|_{L^{\infty,2};x_3}^2 \right).$$

Note that according to (4.14), (4.15) and (4.13), the function $\Phi$ is in $L^1(\mathbb{R}^+)$ and

$$\|\Phi\|_{L^1(\mathbb{R}^+)} \leq C \left( \frac{2}{\nu} \|\bar{u}_0\|_{L^2}^2 + \|w_0\|_{H^{0,\ast}}^2 \left( 1 + \frac{C}{\nu} \|w_0\|_{H^{0,\ast}}^2 \right) + C_\eta \|w_{0,m}\|_{L^2(\mathbb{R}^3)} \right).$$  \hfill (4.26)

Let us fix $\eta$ such that

$$C \|w(0)\|_{H^{0,\ast}} \left( 1 + \frac{1}{\nu} \|w(0)\|_{H^{0,\ast}} \right) \leq \frac{\nu}{16}.$$  \hfill (4.27)

Let us consider the supremum $T^*$ of the set of the real numbers $T$ such that

$$\forall t \leq T, \ C \|w(t)\|_{H^{0,\ast}} \left( 1 + \frac{1}{\nu} \|w(t)\|_{H^{0,\ast}} \right) \leq \frac{\nu}{8}.$$  \hfill (4.28)
As long as the above condition (4.28) is satisfied, we have
\[
\frac{1}{2} \frac{d}{dt} \|w(t)\|_{H^{0,s}}^2 + \frac{\nu}{2} \|\nabla_h w(t)\|_{H^{0,s}}^2 \leq \Phi(t) \|w(t)\|_{H^{0,s}}^2 + C_\eta \|w_{F,m}\|_{L^{2,\infty,h,x_3}}^2 + C_\eta \|w_{F,m}\|_{L^{2,\infty,h,x_3}}.
\]

Let us define
\[
\tilde{w}(t) \overset{\text{def}}{=} w(t) \exp \left( - \int_0^t \Phi(t') dt' \right).
\]

We have
\[
\frac{1}{2} \frac{d}{dt} \|\tilde{w}(t)\|_{H^{0,s}}^2 + \frac{\nu}{2} \|\nabla_h \tilde{w}(t)\|_{H^{0,s}}^2 \leq \|w(t)\|_{H^{0,s}}^2 + C_\eta \|w_{F,m}\|_{L^{2,\infty,h,x_3}}^2 + C_\eta \|w_{F,m}\|_{L^{2,\infty,h,x_3}}.
\]

So by integration, we get, for any \(t \leq T^*\),
\[
\|\tilde{w}(t)\|_{H^{0,s}}^2 + \frac{\nu}{2} \int_0^t \|\nabla_h \tilde{w}(t')\|_{H^{0,s}}^2 dt' \leq \|w(0)\|_{H^{0,s}}^2 + C_\eta \int_0^t \left( \|w_{F,m}(t')\|_{L^{2,\infty,h,x_3}}^2 + C_\eta \|w_{F,m}(t')\|_{L^{2,\infty,h,x_3}} \right) dt'.
\]

From now on, we shall use the Strichartz’ type estimates. Using the estimate (4.13), we get
\[
C_\eta \int_0^t \left( \|w_{F,m}(t')\|_{L^{2,\infty,h,x_3}}^2 + C_\eta \|w_{F,m}(t')\|_{L^{2,\infty,h,x_3}} \right) dt' \leq C_\eta(\varepsilon)
\]
where \(C_\eta(\varepsilon)\) is a function of \(\varepsilon\) such that, for fixed \(\eta\), \(\lim_{\varepsilon \to 0} C_\eta(\varepsilon) = 0\).

So finally
\[
\|\tilde{w}(t)\|_{H^{0,s}}^2 + \frac{\nu}{2} \int_0^t \|\nabla_h \tilde{w}(t')\|_{H^{0,s}}^2 dt' \leq \|w(0)\|_{H^{0,s}}^2 + C_\eta(\varepsilon),
\]

hence
\[
\|w(t)\|_{H^{0,s}}^2 + \frac{\nu}{2} \int_0^t \|\nabla_h w(t')\|_{H^{0,s}}^2 dt' \leq \left( \|w(0)\|_{H^{0,s}}^2 + C_\eta(\varepsilon) \right) \exp \int_0^t \Phi(t') dt',
\]
and the theorem is proved using (4.26).

\[\square\]

\section*{References}


