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RIGOROUS DERIVATION OF KORTEWEG-DE VRIES-TYPE SYSTEMS FROM A GENERAL CLASS OF NONLINEAR HYPERBOLIC SYSTEMS

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Abstract. In this paper, we study the long wave approximation for quasilinear symmetric hyperbolic systems. Using the technics developed by Joly-Métivier-Rauch for nonlinear geometrical optics, we prove that under suitable assumptions the long wave limit is described by KdV-type systems. The error estimate if the system is coupled appears to be better. We apply formally our technics to Euler equations with free surface and Euler-Poisson systems. This leads to new systems of KdV-type.

Résumé. Dans cet article, nous étudions l'approximation de type ondes longues pour des systèmes hyperboliques quasi-linéaires symétriques. En utilisant des techniques développées par Joly-Métivier-Rauch pour l'optique géométrique non linéaire, nous montrons (sous des hypothèses convenables) que la limite onde longue est décrite par des systèmes de type KdV. L'estimation d'erreur est d'autant meilleure que l'on conserve les couplages dans ces systèmes. Nous appliquons formellement ensuite notre technique aux équations d'Euler avec surface libre et au système d'Euler-Poisson. Cela conduit à de nouveaux systèmes de type KdV.

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1. Introduction

1.1. Setting up the problem

This paper is mainly concerned with the exact derivation of Korteweg-de Vries type systems in one dimensional space, starting from generic quasilinear and symmetric hyperbolic systems. The Korteweg-de Vries systems are considered as asymptotical equations as the amplitude of the wave is considered small whereas the wavelength is large. The KdV equations occur in several physical situations such as plasma physics [23], meteorology and more importantly in the shallow water-waves context, which is the historical background in which Korteweg and de Vries obtained their result in 1895 [19].

As we said above, we present a systematic study of long wave approximation. More precisely, one considers:

$$\partial_t u^\epsilon + A(\partial_x)u^\epsilon + \frac{Eu^\epsilon}{\epsilon} = B(u^\epsilon)\partial_x u^\epsilon. \tag{1.1}$$

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The function $u^\epsilon(x,t)$ is a $\mathbb{R}^N$-valued function, where x lies in $\mathbb{R}$ and t is the time variable. The nonlinearity is taken to be as simple as possible in the quasi-linear context. $\forall u \in \mathbb{R}^N, B(u)$ is a symmetric matrix and $u \mapsto B(u)$ is linear so that the system has a quadratic nonlinearity. We assume that the $N \times N$ matrix A is symmetric and real and that E is a $N \times N$ skew-symmetric matrix. To finish, this description, let us suppose that E is non invertible in order to derive non trivial approximate solution.

Our aim is to derive from the hyperbolic equation (1.1) KdV systems. In order to do so, we keep in mind that we need our approximate solutions to approach small amplitude solutions with large wavelength and be governed by a system where nonlinear and dispersive effects exist at the same long time scale. There are two types of KdV systems: the coupled systems and the uncoupled ones which are nothing else but a pair of independent KdV equations each one of which describing a propagation in opposite directions. One of the motivation of this study is to establish a distinction between these two models as we prove that they do not approximate the exact solution of our problem (1.1) at the same level of accuracy with respect to the small parameter ϵ.

Note that the problem of the rigorous justification of the KdV equation from the Euler equations with free surface has been solved by Craig in [11]. Recently Schneider and Wayne [21] have extended this result to the case where two directions of propagation are present: they obtain a set of two uncoupled KdV equations. Here we study this problem, in a general framework, namely starting from system (1.1). And we derive systems of two uncoupled KdV equations as well as coupled systems of KdV type and we compare both approximations. Our results do not apply directly to the water-wave problem nor to the Euler-Poisson problem (both presented in the last section) since these systems can not be written under the simple form (1.1). We postpone this study for a latter work.

Notations

Within the course of this paper, the norm L^2 in space will be denoted as $\|.\|_2$, whereas the H^s norm of a function u will be denoted as $\|u\|_s = \|(1+\xi^2)^{s/2}\hat{u}\|_2$.

1.2. Formulating the ansatz

Our aim is to study the behavior of solutions for the system (1.1) for time scales where the nonlinearity and dispersion compete at the same order with respect to the small parameter ϵ in the leading order term of our approximate solution.

Following the work of [12,14,15,20], in the context of geometrical optics, one ought to set our ansatz *a priori* as follows,

$$\mathcal{U}^\epsilon(x,t) = \epsilon^p u(x,t,\epsilon^q t)$$

where ϵ^p is the size of the solutions and $t_1 = \epsilon^q t$ the long time variable at which our above intentions must meet their requirements. Note that we have here only two scales compared to the three scales of classical geometrical optics where one has to take into account the oscillatory nature of light by adding a scale for high frequencies and oscillatory modes, which does not fit our physical context here. From the degree of the nonlinearity in the set up of the problem and the presence of the parameter ϵ in the dispersive term of (1.1), (p,q) must satisfy $p+q = 2p \Rightarrow p = q$ to have the nonlinear contribution occurring at the long time scale t_1 and q must be equal to 2 for the third order dispersive term to be present at the same time scale, considering the nature of the nonlinear term.

Therefore, we start off with an ansatz, with t and $t_1 = \epsilon^2 t$, namely the short and long time variable, that reads,

$$\mathcal{U}^\epsilon(x,t) = \sum_{j=0}^{3} \epsilon^{j+2} u_j(x,t,\epsilon^2 t). \tag{1.2}$$

With this model, the nonlinear contribution occurs in large time scales of order $O(\frac{1}{\epsilon^2})$ along with the dispersive effects as it was our aim in the construction of (1.2).

Before going any further, our ansatz can be described as follows: u_0 is the leading order term whereas u_1, u_2 and u_3 are supposed to be correctors, which means that they remain smaller than u_0 for all times. Besides our intention is to study the behavior of the leading order term for large time scales of order $O(\frac{1}{\epsilon^2})$ which implies that the overall expansion (1.2) must be valid for such times. Hence, one must control, somehow, the growth in time of the corrector terms. Thus, to make sure that these terms are indeed correctors on time intervals of the form $[0; O(\frac{1}{\epsilon^2})]$, we assume that they satisfy *a priori* an analog of a sub-linear growth condition introduced in [14, 20], that reads for any function a sufficiently smooth in our case as:

Sub-squareroot growth condition

The function $a(x, t,_1)$ satisfies a sub-squareroot growth condition if only if

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} \|\partial^\alpha_{x,t,t_1} a(x,t,t_1)\|_2 = 0 \quad \text{for all} \quad \alpha \in \mathbb{N}^3. \tag{1.3}$$

Remark 1.1. In fact we will show in the course of this paper that the correctors are even better controlled since they are most of the time L^2-bounded in time.

We now plug in (1.1) the ansatz (1.2), assuming that the u_j are smooth enough and we get

$$\partial_t \mathcal{U}^\epsilon + A(\partial_x)\mathcal{U}^\epsilon + \frac{1}{\epsilon} E\mathcal{U}^\epsilon - B(\mathcal{U}^\epsilon)\partial_x \mathcal{U}^\epsilon = \sum_{j=1}^{10} \epsilon^j r_j \tag{1.4}$$

where the r_j are given by

$$\begin{cases} r_1 = Eu_0 \\ r_2 = \partial_t u_0 + A(\partial_x)u_0 + Eu_1 \\ r_3 = \partial_t u_1 + A(\partial_x)u_1 + Eu_2 \\ r_4 = \partial_{t_1} u_0 + \partial_t u_2 + A(\partial_x)u_2 + Eu_3 - B(u_0)\partial_x u_0 \\ r_5 = \partial_{t_1} u_1 + \partial_t u_3 + A(\partial_x)u_3 - B(u_1)\partial_x u_0 - B(u_0)\partial_x u_1 \\ r_6 = \partial_{t_1} u_2 - B(u_0)\partial_x u_2 - B(u_2)\partial_x u_0 - B(u_1)\partial_x u_1 \\ r_7 = \partial_{t_1} u_3 - B(u_3)\partial_x u_0 - B(u_0)\partial_x u_3 - B(u_1)\partial_x u_2 - B(u_2)\partial_x u_1 \\ r_8 = -B(u_3)\partial_x u_1 - B(u_1)\partial_x u_3 - B(u_2)\partial_x u_2 \\ r_9 = -B(u_3)\partial_x u_2 - B(u_2)\partial_x u_3 \\ r_{10} = -B(u_3)\partial_x u_3. \end{cases}$$

Our strategy to construct an approximate solution of (1.1) up to to the order 4 is to solve simultaneously the four equations $r_j = 0$ for $j = 1, 2, 3, 4$. These equations will be referred to as the profile equations and constitute a set of necessary conditions for $\mathcal{U}^\epsilon$ to be an approximate solution. They read

$$r_1 = 0 \quad \Longrightarrow \quad Eu_0 = 0 \tag{1.5}$$

$$r_2 = 0 \quad \Longrightarrow \quad \partial_t u_0 + A(\partial_x)u_0 + Eu_1 = 0 \tag{1.6}$$

$$r_3 = 0 \quad \Longrightarrow \quad \partial_t u_1 + A(\partial_x)u_1 + Eu_2 = 0 \tag{1.7}$$

$$r_4 = 0 \quad \Longrightarrow \quad \partial_{t_1} u_0 + \partial_t u_2 + A(\partial_x)u_2 + Eu_3 = B(u_0)\partial_x u_0. \tag{1.8}$$

The paper is organized as follows: in Section 2, we derive necessary conditions on the unknowns from equations (1.5)–(1.8) and establish the equations satisfied by the profiles u_0, u_1, u_2 and u_3. We show that $u_0 = u_{01} + u_{02}$, where each function u_{01}, u_{02} has to solve a KdV type equation.

In Section 3, we prove that the set of equations obtained in the second section can be solved and that the function $\epsilon^3 u_1 + \epsilon^4 u_2 + \epsilon^5 u_3$ is a corrector with respect to the first term of the expansion (1.2) and finally we prove in Theorem 3.1 that there exists a solution u^ϵ of (1.1) such that

$$\|u^\epsilon - \epsilon^2 u_0(x,t,\epsilon^2 t)\|_{L^\infty([0,\frac{T}{\epsilon^2}];H^s)} = o(\epsilon^2) \quad \text{as} \quad \epsilon \to 0.$$

In Section 4, we show in Theorem 4.1 that, if one modifies slightly the ansatz, on can find two functions (u_{01}, u_{02}) satisfying a system of KdV type such that

$$\|u^\epsilon - \epsilon^2 u_0\|_{L^\infty([0,\frac{T}{\epsilon^2}];H^s)} = O(\epsilon^3) \quad \text{as} \quad \epsilon \to 0.$$

The error estimate is therefore better if one keeps some coupling between the two components of u_0.

Finally in Section 5, we apply the second section to Euler-Poisson and Euler with free surface problems and derive new asymptotical models.

2. Equations for the profiles

2.1. Algebraic solvability conditions

Following the analysis used in [14,15,20], we introduce some formal operators in order to modify and simplify our set of profile equations and find a simplified set of equations satisfied by u_0, u_1, u_2 and u_3.

Definition 2.1. For $(\tau,\xi) \in \mathbb{R}\times\mathbb{R}$, let us denote by $L(\tau,\xi)$ and $L_1(\tau,\xi)$ the following maps

$$L(\tau,\xi) = \tau I + A\xi + \frac{E}{i} \quad \text{as well as} \quad L_1(\tau,\xi) = \tau I + A\xi$$

and we denote by $\Pi(\tau,\xi)$ the orthogonal projector on the Kernel of $L(\tau,\xi)$. We also define $Q(\tau,\xi)$ the partial inverse of $L(\tau,\xi)$ such that

$$Q(\tau,\xi)L(\tau,\xi) = L(\tau,\xi)Q(\tau,\xi) = I - \Pi(\tau,\xi)$$

and

$$Q(\tau,\xi)\Pi(\tau,\xi) = \Pi(\tau,\xi)Q(\tau,\xi) = 0.$$

Let us point out that along the course of this paper $L(0,0)$ will play an important role and will be denoted as L_0 along with $\Pi(0,0)$ as Π_0. Again Π_0 is nothing else but the projection on the Kernel of $\frac{1}{i}E = L_0$ which is symmetric.

Following [16], we first define the characteristic variety of the operator L, such as

$$\mathrm{Char}L = \left\{\beta = (\tau,\xi) \in \mathbb{R}\times\mathbb{R}/\det(\tau I + A\xi + \frac{E}{i}) = 0\right\}\cdot \tag{2.1}$$

Since the operator L is symmetric, we know that the polynomial equation in τ *i.e.* $\det L(\tau,\xi) = 0$ has only real roots for all ξ. Char$L$ can then be parametrized by a finite number of functions $\tau_i(\xi)$. Thereafter, following [20], $\beta_0 = (\tau_0,\xi_0) \in \mathrm{Char}L$ is called singular if it coincides with the intersection of different functions $\tau_i(\xi)$. β_0 is called regular otherwise.

The main assumption is the following one:

Assumption 2.1. $(0,0)$ *is an isolated singular point of* Char$L$ *of multiplicity* 2. *There exists a regular function* $\lambda(\xi)$ *defined on a neighborhood of* 0 *such that* $\lambda(0) = \lambda''(0) = 0$ *and* $\lambda'(0) \neq 0$ *with* $(\lambda(\xi),\xi) \in \mathrm{Char}L$ *and* $(-\lambda(\xi),\xi) \in \mathrm{Char}L$.

From Assumption 2.1, we denote by $\Pi_1(\xi)$ and $\Pi_2(\xi)$ the two projectors

$$\begin{cases} \Pi_1(\xi) = \Pi(\lambda(\xi),\xi) \\ \Pi_2(\xi) = \Pi(-\lambda(\xi),\xi) \end{cases} \quad \text{for } \xi \neq 0.$$

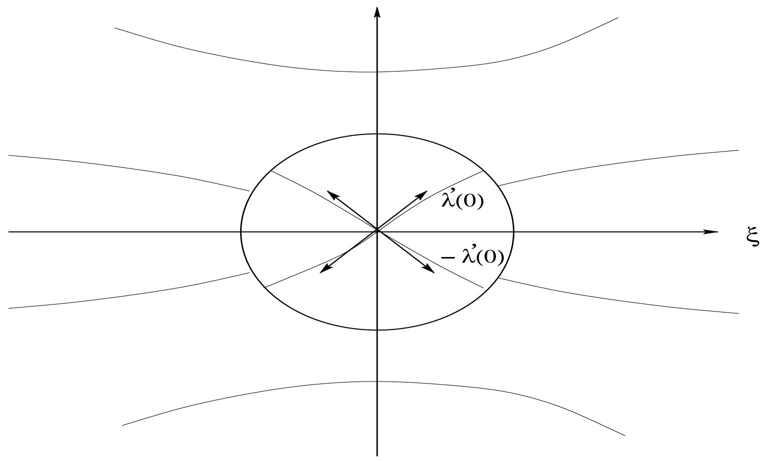

FIGURE 1. The characteristic variety around 0.

These two projectors are nonzero since λ and $-\lambda$ are eigenvalues of L. We denote also by $\Pi(\xi)$ the projector $\Pi(\xi) = \Pi_1(\xi) + \Pi_2(\xi)$ and one has that $\Pi_0 = \Pi(0)$ and $\Pi'_0 = \Pi'(0)$.

Remark 2.1. Since all the operators herein defined are analytical with respect to ξ around any point of the characteristic variety, these operators can be extended to 0 [17].

We intend now to use all these operators in order to solve the equations of the profiles (1.5)–(1.8), that are of the type $L(\tau, \xi)a = b$ for any a, b in $\mathbb{R}^N$. For that matter, we state the following straightforward lemma, that is easily deduced from the symmetry of the operators.

Lemma 2.1. *For any a, $b \in \mathbb{R}^N$*

$$L(\tau, \xi)a = b \Leftrightarrow \Pi(\tau, \xi)b = 0 \quad \textit{and} \quad a = \Pi(\tau, \xi)a + Q(\tau, \xi)b.$$

2.2. Consequences for the profile equations

One turns now to the resolution of the set of equations (1.5)–(1.8).

- The first equation (1.5): $Eu_0 = 0$, from Lemma 2.1 reads as

$$\Pi_0 u_0 = u_0. \tag{2.2}$$

This equation is non trivial since we assumed that L_0 is non invertible.

- The second equation (1.6): $\partial_t u_0 + A(\partial_x)u_0 + Eu_1 = 0$ reads as

$$L_0 u_1 = iL_1(\partial_t, \partial_x)u_0$$

which is equivalent from Lemma 2.1 to the following necessary solvability conditions

$$\begin{cases} 0 = i\Pi_0 L_1(\partial_t, \partial_x)u_0 = i\Pi_0 L_1(\partial_t, \partial_x)\Pi_0 u_0 \quad \text{thanks to (2.2)} \\ (I - \Pi_0)u_1 = iQ_0 L_1(\partial_t, \partial_x)u_0 = iQ_0 L_1(\partial_t, \partial_x)\Pi_0 u_0 \quad \text{thanks to (2.2).} \end{cases} \tag{2.3}$$

- The third equation (1.7): $\partial_t u_1 + A(\partial_x)u_1 + Eu_2 = 0$ reads as

$$L_0 u_2 = iL_1(\partial_t, \partial_x)u_1$$

which is equivalent, using Lemma 2.1 again, to

$$\begin{cases} i\Pi_0 L_1(\partial_t,\partial_x)u_1 = 0 \\ (I-\Pi_0)u_2 = iQ_0 L_1(\partial_t,\partial_x)u_1. \end{cases} \tag{2.4}$$

We decompose in the first equation $u_1 = \Pi_0 u_1 + (I-\Pi_0)u_1$ and use (2.3) to obtain the following equivalent solvability condition,

$$\begin{cases} \Pi_0 L_1(\partial_t,\partial_x)\Pi_0 u_1 = -i\Pi_0 L_1(\partial_t,\partial_x)Q_0 L_1(\partial_t,\partial_x)\Pi_0 u_0 \\ (I-\Pi_0)u_2 = iQ_0 L_1(\partial_t,\partial_x)u_1. \end{cases} \tag{2.5}$$

- Let us turn now to the fourth profile equation (1.8), where the nonlinearity and the long time evolution appear: $\partial_{t_1}u_0 + \partial_t u_2 + A(\partial_x)u_2 + Eu_3 = B(u_0)\partial_x u_0$, that reads

$$L_0 u_3 = i\partial_{t_1}u_0 + iL_1(\partial_t,\partial_x)u_2 - iB(u_0)\partial_x u_0$$

which is again equivalent to, thanks to Lemma 2.1

$$\begin{cases} \partial_{t_1}\Pi_0 u_0 + \Pi_0 L_1(\partial_t,\partial_x)u_2 = \Pi_0 B(u_0)\partial_x u_0 \\ (I-\Pi_0)u_3 = i\partial_{t_1}Q_0 u_0 + iQ_0 L_1(\partial_t,\partial_x)u_2 - iQ_0 B(u_0)\partial_x u_0. \end{cases} \tag{2.6}$$

Decomposing u_2 with the projector Π_0 and using (2.5), the first equation in the above system becomes,

$$\partial_{t_1}\Pi_0 u_0 + \Pi_0 L_1(\partial_t,\partial_x)\Pi_0 u_2 + i\partial_0 L_1(\partial_t,\partial_x)Q_0 L_1(\partial_t,\partial_x)u_1 = \Pi_0 B(u_0)\partial_x u_0$$

which again gives using (2.3) and writing $u_1 = \Pi_0 u_1 + (I-\Pi_0)u_1$, the following equivalent system to (1.8)

$$\begin{cases} \partial_{t_1}\Pi_0 u_0 + \Pi_0 L_1(\partial_t,\partial_x)\Pi_0 u_2 + i\Pi_0 L_1(\partial_t,\partial_x)Q_0 L_1(\partial_t,\partial_x)\Pi_0 u_1 \\ \quad -\Pi_0 L_1(\partial_t,\partial_x)Q_0 L_1(\partial_t,\partial_x)Q_0 L_1(\partial_t,\partial_x)\Pi_0 u_0 = \Pi_0 B(u_0)\partial_x u_0 \\ (I-\Pi_0)u_3 = i\partial_{t_1}Q_0 u_0 + iQ_0 L_1(\partial_t,\partial_x)u_2 - iQ_0 B(u_0)\partial_x u_0. \end{cases} \tag{2.7}$$

The equations obtained (2.2)–(2.7) constitute our set of solvability conditions on the profiles u_0, u_1, u_2 and u_3.

The last equation (2.7) is at this stage, the equation in u_0 (*e.g.* the principal term in the expansion) that contains nonlinear terms and dispersive third order terms in the long time evolution of u_0 and our ansatz was specifically constructed for this reason. In order to use the properties of our problem (*e.g.* the particular form of Char L), one needs to project this equation "on both branches of the characteristic variety", to be able to derive as claimed, KdV type systems either coupled or uncoupled with two components moving in two opposite directions, defined by each branch in Figure 2.1. We begin by describing the differential operators arising in (2.7).

The operator $\Pi_0 L_1(\partial_t,\partial_x)\Pi_0$

To begin with, it is essential to understand the operator $\Pi(\tau,\xi)L_1(\partial_t,\partial_x)\Pi(\tau,\xi)$ both in 0 (*e.g.* $(\tau,\xi)=(0,0)$ and $\Pi(0,0)=\Pi_0$) as well as on the branches of Char L on regular points. Indeed, when (τ,ξ) is not a singularity of the characteristic variety, $\Pi(\tau,\xi)L_1(\partial_t,\partial_x)\Pi(\tau,\xi)$ happen to be a simple scalar operator. This result is well known and proved in [14,15,20]. We give the proof here for the convenience of the reader and also because this proof leads to the result at $\xi=0$.

Lemma 2.2. *If Assumption 2.1 is satisfied, for all ξ in a neighborhood of 0 and $\xi \neq 0$, we have*

$$\begin{cases} \Pi_1(\xi)L_1(\partial_t,\partial_x)\Pi_1(\xi) = (\partial_t - \lambda'(\xi)\partial_x)\Pi_1(\xi) \\ \Pi_2(\xi)L_1(\partial_t,\partial_x)\Pi_2(\xi) = (\partial_t + \lambda'(\xi)\partial_x)\Pi_2(\xi) \end{cases} \tag{2.8}$$

and if we set, for any $\xi \in \mathbb{R}, \Pi(\xi) = \Pi_1(\xi) + \Pi_2(\xi)$, one has that

$$\Pi(\xi)L_1(\partial_t,\partial_x)\Pi(\xi) = \partial_t\Pi(\xi) - \lambda'(\xi)\partial_x(\Pi_1(\xi) - \Pi_2(\xi)) + 2\lambda(\xi)\left[\Pi_1(\xi)\Pi_2'(\xi) - \Pi_2(\xi)\Pi_1'(\xi)\right]. \tag{2.9}$$

Proof of Lemma 2.2. We recall that by definition one has

$$(\lambda(\xi) + A\xi + \frac{E}{i})\Pi_1(\xi) = 0 \quad \text{for } \xi \neq 0.$$

We differentiate this equation with respect to ξ and obtain,

$$(\lambda'(\xi) + A)\Pi_1(\xi) + (\lambda(\xi) + A\xi + \frac{E}{i})\Pi_1'(\xi) = 0. \tag{2.10}$$

Applying $\Pi_1(\xi)$ on the left side gives,

$$\Pi_1(\xi)(\lambda'(\xi) + A)\Pi_1(\xi) = 0$$

which yields the first relation in (2.8) and the second relation is obtained likewise with Π_2.

To derive the last relation (2.9), we develop the operator $\Pi(\xi)A\Pi(\xi)$:

$$\Pi(\xi)A\Pi(\xi) = \Pi_1(\xi)A\Pi_1(\xi) + \Pi_2(\xi)A\Pi_2(\xi) + \Pi_1(\xi)A\Pi_2(\xi) + \Pi_2(\xi)A\Pi_1(\xi).$$

In order to evaluate the crossed products, we apply the projector $\Pi_2(\xi)$ on (2.10),

$$\Pi_2(\xi)(\lambda'(\xi) + A)\Pi_1(\xi) + \Pi_2(\xi)(\lambda(\xi) + A\xi + \frac{E}{i})\Pi_1'(\xi) = 0.$$

Then,

$$\Pi_2(\xi)A\Pi_1(\xi) + 2\lambda(\xi)\Pi_2(\xi)\Pi_1'(\xi) = 0$$

and likewise, we have that

$$\Pi_1(\xi)A\Pi_2(\xi) - 2\lambda(\xi)\Pi_1(\xi)\Pi_2'(\xi) = 0.$$

Finally, we gather all the previous relations in the above development and obtain as claimed (2.9), which finishes the proof. □

Corollary 2.1. *At the singular point $(0,0)$ of Char L, one has that, under Assumption 2.1*

$$\Pi_0 L_1(\partial_t,\partial_x)\Pi_0 = \partial_t\Pi_0 - \lambda'(0)\partial_x\left(\Pi_1(0) - \Pi_2(0)\right). \tag{2.11}$$

Proof. The proof of this corollary is straightforward from the previous lemma since it is simply the value of the order 1 operator (2.9) extended to $\xi = 0$, using the fact that the projectors Π_1 and Π_2 are analytic on a neighborhood of 0. □

We therefore obtain from (2.3) and the previous Corollary 2.1, the following fundamental transport proposition for each component of u_0.

Proposition 2.1. *One has*

$$\begin{cases} (\partial_t - \lambda'(0)\partial_x)\Pi_1(0)u_0 = 0 \\ (\partial_t + \lambda'(0)\partial_x)\Pi_2(0)u_0 = 0. \end{cases} \tag{2.12}$$

The operator $\Pi_0 L_1(\partial_t, \partial_x)Q_0 L_1(\partial_t, \partial_x)\Pi_0$

At a regular point of the characteristic variety, one has that, as it is proved in [14],

$$\Pi(\lambda(\xi),\xi)AQ(\lambda(\xi),\xi)A\Pi(\lambda(\xi),\xi) = \lambda''(\xi)\Pi(\lambda(\xi),\xi).$$

Here, since $(0,0)$ is not regular and $\lambda''(0) = 0$, one has:

Proposition 2.2. *The matrix* $\Pi_0 AQ_0 A\Pi_0$ *is given by*

$$\Pi_0 AQ_0 A\Pi_0 = 2\lambda'(0)\left(\Pi_2(0)\Pi_1'(0) + \Pi_1'(0)\Pi_2(0)\right). \tag{2.13}$$

We deduce from this proposition the following corollary:

Corollary 2.2.

$$\begin{cases} \Pi_1(0)L_1(\partial_t, \partial_x)Q_0 L_1(\partial_t, \partial_x)\Pi_1(0) = 0 \\ \Pi_2(0)L_1(\partial_t, \partial_x)Q_0 L_1(\partial_t, \partial_x)\Pi_2(0) = 0. \end{cases} \tag{2.14}$$

Proof of Proposition 2.2. Let us introduce the ratio $\varphi(\xi)$:

$$\varphi(\xi) = \frac{\Pi(\xi)A\Pi(\xi) - \Pi_0 A\Pi_0}{\xi}.$$

The idea is to compute this ratio in two different manners as $\xi \to 0$ to derive the desired relation (2.13). One has, using Lemma 2.2, that

$$\varphi(\xi) = \frac{\Pi_1(\xi)A\Pi_2(\xi)}{\xi} + \frac{\Pi_2(\xi)A\Pi_1(\xi)}{\xi} - \frac{\lambda'(\xi)}{\xi}\Pi_1(\xi) + \frac{\lambda'(\xi)}{\xi}\Pi_2(\xi) + \frac{\lambda'(0)}{\xi}\Pi_1(0) - \frac{\lambda'(0)}{\xi}\Pi_2(0)$$

which can be written as

$$\begin{aligned}\varphi(\xi) = &-\left(\frac{\lambda'(\xi)-\lambda'(0)}{\xi}\right)\Pi_1(\xi) - \lambda'(0)\left(\frac{\Pi_1(\xi)-\Pi_1(0)}{\xi}\right) + \lambda'(0)\left(\frac{\Pi_2(\xi)-\Pi_2(0)}{\xi}\right) \\ &+ 2\left(\frac{\lambda(\xi)-\lambda(0)}{\xi}\right)\Pi_1(\xi)\Pi_2'(\xi) - 2\left(\frac{\lambda(\xi)-\lambda(0)}{\xi}\right)\Pi_2(\xi)\Pi_1'(\xi) + \left(\frac{\lambda'(\xi)-\lambda'(0)}{\xi}\right)\Pi_2(\xi).\end{aligned}$$

The operators $\Pi_1(\xi)$ and $\Pi_2(\xi)$ being analytical, they are, along with their derivative bounded around 0 and since we assumed (Assumption 2.1) that $\lambda(0) = \lambda''(0) = 0$, we let $\xi \to 0$ and obtain

$$\lim_{\xi\to 0}\varphi(\xi) = \lambda'(0)(\Pi_1'(0) + \Pi_2'(0)) + 2\lambda'(0)\left[\Pi_1(0)\Pi_2'(0) - \Pi_2(0)\Pi_1'(0)\right]. \tag{2.15}$$

We go back to $\varphi(\xi)$ and compute its limit in a different way. One can write

$$\varphi(\xi) = \left(\frac{\Pi(\xi)-\Pi_0}{\xi}\right)A\Pi(\xi) + \Pi_0 A\left(\frac{\Pi(\xi)-\Pi_0}{\xi}\right)$$

which gives $\lim_{\xi\to 0}\varphi(\xi) = \Pi_0' A\Pi_0 + \Pi_0 A\Pi_0'$. To evaluate the terms in the right-hand side, we differentiate the following quantity with respect to ξ, where $\Pi(\xi)$ is defined as in Lemma 2.2,

$$(\lambda(\xi) + A\xi + \frac{E}{i})(-\lambda(\xi) + A\xi + \frac{E}{i})\Pi(\xi) = 0$$

which gives

$$\begin{aligned}(\lambda'(\xi) + A)(-\lambda(\xi) + A\xi + \frac{E}{i})\Pi(\xi) + (\lambda(\xi) + A\xi + \frac{E}{i})(-\lambda'(\xi) + A)\Pi(\xi)\\ +(\lambda(\xi) + A\xi + \frac{E}{i})(-\lambda(\xi) + A\xi + \frac{E}{i})\Pi'(\xi) = 0.\end{aligned}$$

At $\xi = 0$, this reads as

$$\underbrace{(\lambda'(0) + A)\frac{E}{i}\Pi_0}_{=0} + \frac{E}{i}(-\lambda'(0) + A)\Pi_0 - E^2\Pi_0' = 0.$$

The first term is null since Π_0 is the projector on $\mathrm{Ker}\frac{E}{i}$. Thus,

$$EA\Pi_0 - iE^2\Pi_0' = 0$$

and applying Q_0 twice on the right side of the relation above gives

$$Q_0 A\Pi_0 + (I - \Pi_0)\Pi_0' = 0. \tag{2.16}$$

And likewise one obtains that

$$\Pi_0 A Q_0 + \Pi_0'(I - \Pi_0) = 0. \tag{2.17}$$

It follows that

$$\Pi_0' A\Pi_0 = -\Pi_0 A Q_0 A\Pi_0 - \lambda'(0)\Pi_0'(\Pi_1(0) - \Pi_2(0))$$

$$\Pi_0 A\Pi_0' = -\Pi_0 A Q_0 A\Pi_0 - \lambda'(0)(\Pi_1(0) - \Pi_2(0))\Pi_0'.$$

Equaling both expressions of $\lim_{\xi\to 0}\varphi(\xi)$, leads to

$$\begin{aligned}-2\Pi_0 A Q_0 A\Pi_0 - \lambda'(0)\Pi_0'(\Pi_1(0) - \Pi_2(0)) - \lambda'(0)(\Pi_1(0) - \Pi_2(0))\Pi_0'\\ = \lambda'(0)(\Pi_1'(0) + \Pi_2'(0)) + 2\lambda'(0)\left(\Pi_1(0)\Pi_2'(0) - \Pi_2(0)\Pi_1'(0)\right).\end{aligned}$$

This latter equation is simplified using straightforward algebraic relations on the projectors that we will constantly refer to, namely

$$\begin{cases}\Pi_1(\xi)\Pi_1'(\xi) + \Pi_1'(\xi)\Pi_1(\xi) = \Pi_1'(\xi)\\ \Pi_2(\xi)\Pi_2'(\xi) + \Pi_2'(\xi)\Pi_2(\xi) = \Pi_2'(\xi)\\ \Pi_1(\xi)\Pi_2'(\xi) + \Pi_1'(\xi)\Pi_2(\xi) = 0\\ \Pi_2'(\xi)\Pi_1(\xi) + \Pi_2(\xi)\Pi_1'(\xi) = 0\end{cases} \tag{2.18}$$

and the proof is complete. □

The corollary follows in a straightforward manner from Proposition 2.2 thanks to relations (2.18).

With all these tools in hand, we apply $\Pi_1(0)$ on the first equation of (2.7), which gives, thanks to Lemma 2.2 and Proposition 2.2 and their corresponding corollaries

$$\partial_{t_1}\Pi_1(0)u_0 + \Pi_1(0)L_1(\partial_t,\partial_x)\Pi_1(0)u_2 + i\Pi_1(0)L_1(\partial_t,\partial_x)Q_0L_1(\partial_t,\partial_x)\Pi_2(0)u_1 \\ -\Pi_1(0)L_1(\partial_t,\partial_x)Q_0L_1(\partial_t,\partial_x)Q_0L_1(\partial_t,\partial_x)\Pi_0u_0 = \Pi_0B(u_0)\partial_x u_0. \quad (2.19)$$

Going back to the solvability conditions established earlier, it is possible from (2.5) to solve exactly u_1 in terms of u_0. Indeed, applying successively $\Pi_1(0)$ and $\Pi_2(0)$ on (2.5), one recalls thanks to Lemma 2.2 that

$$\begin{aligned}(\partial_t - \lambda'(0)\partial_x)\Pi_1(0)u_1 &= -i\Pi_1(0)L_1(\partial_t,\partial_x)Q_0L_1(\partial_t,\partial_x)\Pi_0u_0 \\ &\underbrace{= -i\Pi_1(0)L_1(\partial_t,\partial_x)Q_0L_1(\partial_t,\partial_x)\Pi_2(0)u_0}_{\text{thanks to Corollary 2.2}}\end{aligned}$$

and likewise

$$(\partial_t + \lambda'(0)\partial_x)\Pi_2(0)u_1 \quad = -i\Pi_2(0)L_1(\partial_t,\partial_x)Q_0L_1(\partial_t,\partial_x)\Pi_1(0)u_0.$$

Thanks to Proposition 2.2, these two latter equations can be solved and one obtains:

$$\Pi_1(0)u_1 = \frac{i}{2\lambda'(0)}\Pi_1(0)AQ_0A\Pi_2(0)\partial_x u_0 + \Pi_1(0)v_1 \quad (2.20)$$

where v_1 is an unknown function such that

$$(\partial_t - \lambda'(0)\partial_x)\Pi_1(0)v_1 = 0.$$

And likewise for the second component

$$\Pi_2(0)u_1 = \frac{-i}{2\lambda'(0)}\Pi_2(0)AQ_0A\Pi_1(0)\partial_x u_0 + \Pi_2(0)v_1 \quad (2.21)$$

with $(\partial_t + \lambda'(0)\partial_x)\Pi_2(0)v_1 = 0$.

Plugging these values of $\Pi_1(0)u_1$ and $\Pi_2(0)u_1$ in (2.7) and applying the projector $\Pi_1(0)$ on the result yields

$$\partial_{t_1}\Pi_1(0)u_0 + \Pi_1(0)L_1(\partial_t,\partial_x)\Pi_1(0)u_2 + i\Pi_1(0)L_1Q_0L_1\Pi_2(0)v_1 + \frac{1}{2\lambda'(0)}\Pi_1(0)AQ_0A\Pi_2(0)AQ_0A\Pi_1(0)\partial_x^3 u_0 \\ -\Pi_1(0)AQ_0AQ_0A[\Pi_1(0)+\Pi_2(0)]\partial_x^3 u_0 - \lambda'(0)\Pi_1(0)AQ_0^2A\Pi_1(0)\partial_x^3 u_0 + \lambda'(0)\Pi_1(0)AQ_0^2A\Pi_2(0)\partial_x^3 u_0 \\ = \Pi_1(0)B([\Pi_1(0)+\Pi_2(0)]u_0)\partial_x\Pi_0u_0. \quad (2.22)$$

We replaced $L_1(\partial_t,\partial_x)$ by $\partial_t + A\partial_x$ in the previous calculation. As we developed $\partial_t + A\partial_x$ in some terms, the derivative with respect to t reads simply as either $\lambda'(0)\partial_x$ or $-\lambda'(0)\partial_x$ depending on the component of u_0 to which it is applied, thanks to the transport Proposition 2.12. We obtain likewise the second fundamental solvability equation for $\Pi_2(0)u_0$:

$$\partial_{t_1}\Pi_2(0)u_0 + \Pi_2(0)L_1(\partial_t,\partial_x)\Pi_2(0)u_2 + i\Pi_2(0)L_1Q_0L_1\Pi_1(0)v_1 - \frac{1}{2\lambda'(0)}\Pi_2(0)AQ_0A\Pi_1(0)AQ_0A\Pi_2(0)\partial_x^3 u_0 \\ -\Pi_2(0)AQ_0AQ_0A[\Pi_1(0)+\Pi_2(0)]\partial_x^3 u_0 - \lambda'(0)\Pi_2(0)AQ_0^2A\Pi_1(0)\partial_x^3 u_0 + \lambda'(0)\Pi_2(0)AQ_0^2A\Pi_2(0)\partial_x^3 u_0 \\ = \Pi_2(0)B([\Pi_1(0)+\Pi_2(0)]u_0)\partial_x\Pi_0u_0. \quad (2.23)$$

Transport operators

We introduce for convenience and clarity at this point some notations for the two transport operators that are scalar, corresponding respectively to the transport along the tangent space of both branches of the characteristic variety at 0:

$$\begin{cases} T_1(\partial_t, \partial_x) = \partial_t - \lambda'(0)\partial_x \\ \\ T_2(\partial_t, \partial_x) = \partial_t + \lambda'(0)\partial_x \end{cases} \tag{2.24}$$

and obviously one has, from Lemma 2.2, that

$$\begin{cases} \Pi_1(0)L_1(\partial_t, \partial_x)\Pi_1(0) = T_1(\partial_t, \partial_x)\Pi_1(0) \\ \\ \Pi_2(0)L_1(\partial_t, \partial_x)\Pi_2(0) = T_2(\partial_t, \partial_x)\Pi_2(0). \end{cases}$$

Comments on (2.22)–(2.23)

Let us make a few remarks on the previous equations (2.22) and (2.23). For large times of order $O(\frac{1}{\epsilon^2})$, both the nonlinearity and the dispersion occur in the evolution equations for u_0, which is separated in two waves $\Pi_1(0)u_0$ and $\Pi_2(0)u_0$ evolving in two opposite directions. As they are written in (2.22) and (2.23), these equations do not constitute exactly a system of KdV type, mainly because of the presence of the corrector u_2 that we need to get rid of somehow. We denote also the presence in both equations of dispersive terms of order 3 in both directions. Besides the nonlinearities in (2.22)–(2.23) are in both case coupled in the sense that we come across combination of derivatives of quadratic polynomials of terms moving in two different direction.

In order to simplify these equations and derive the KdV systems as claimed, we introduce average operators as in [20] to apply them on the two equations that govern the evolution of u_0. The aim of this technique is to derive supplementary necessary conditions that eliminate the corrector terms along with the dispersive terms moving in the wrong direction. After this operation, the system (2.22)–(2.23) turns into as claimed, a pair of two independent KdV equations for each component $\Pi_1(0)$ and $\Pi_2(0)$ moving in two different directions.

2.3. Average operators

We must keep in mind that these operators are constructed in order to eliminate u_2 from the equations (2.22)–(2.23) governing the profile u_0. We recall that u_2 was supposed to respect some growth condition.

As in [20], an average operator is defined relatively to a transport operator. Hence for T_1 and T_2, we define two average operators G_{T_1} and G_{T_2}:

Definition 2.2. For $h > 0$ and w sufficiently smooth,

$$G^h_{T_1} w(x, t, \tau) = \frac{1}{h}\int_0^h w(x - \lambda'(0)s, t + s, \tau)\,\mathrm{d}s$$

$$G^h_{T_2} w(x, t, \tau) = \frac{1}{h}\int_0^h w(x + \lambda'(0)s, t + s, \tau)\,\mathrm{d}s$$

and

$$\begin{cases} G_{T_1} w = \lim\limits_{h\to\infty} G^h_{T_1} \\ \\ G_{T_2} w = \lim\limits_{h\to\infty} G^h_{T_2} \end{cases} \tag{2.25}$$

when this limit exists.

These operators were described and introduced in detail in [20]. We recall their properties and refer to [20] for the corresponding proofs.

Proposition 2.3 (Properties of the average operator)**.** *Let T be a transport operator such that $T(\partial_t, \partial_x) = \partial_t - c\partial_x$, then*

(i) *If w satisfies $T(\partial_t, \partial_x)w = 0$, then $G_T w$ exists and $G_T w = w$.*
(ii) *If w satisfies $T'(\partial_t, \partial_x)w = 0$ where $T'(\partial_t, \partial_x) = \partial_t - c'\partial_x$ and if $c \neq c'$ then $G_T w$ exists and $G_T w = 0$.*
(iii) *If w respects a sub-squareroot growth condition (1.3), then $G_T T(\partial_t, \partial_x)w$ is well defined and $G_T T(\partial_t, \partial_x)w = 0$.*
(iv) *Let $W := ww'$ where w and w' are such that $T(\partial_t, \partial_x)w = 0$ and $T'(\partial_t, \partial_x)w' = 0$. If $T(\partial_t, \partial_x) = T'(\partial_t, \partial_x)$, then $G_T W = W$. In any other case $G_T W = 0$.*

The first two properties mean that when we apply G_T to the linear terms of the equations, it leaves only those transported by $T(\partial_t, \partial_x)$ and eliminates the rest. The third property allows us to get rid of the correctors in the equations as it was the motivation in the construction of these operators. And the important last property allows us to eliminate all the product terms where the factors are transported by different operators. And as we said earlier, it is thanks to this last property that we will reduce dramatically the nonlinear terms and thus uncouple the system (2.22)–(2.23) in order to derive a pair of independent KdV equations for the evolution of each component of u_0.

2.4. **Consequence for the profile equations**

Obtaining the uncoupled system.

As we are looking for solvability condition on the system (2.22)–(2.23), let us apply the operator G_{T_1} on (2.22) and G_{T_2} on (2.23), which gives thanks to the properties of these operators

$$
\begin{aligned}
&\partial_{t_1}\Pi_1(0)u_0 + \underbrace{G_{T_1}(T_1(\partial_t, \partial_x)u_2)}_{=0\,\text{property (iii)}} + \frac{1}{2\lambda'(0)}\Pi_1(0)AQ_0A\Pi_2(0)AQ_0A\Pi_1(0)\partial_x^3 u_0 \\
&-\Pi_1(0)AQ_0AQ_0A\Pi_1(0)\partial_x^3 u_0 - \lambda'(0)\Pi_1(0)AQ_0^2A\Pi_1(0)\partial_x^3 u_0 \\
&-\Pi_1(0)AQ_0AQ_0A\underbrace{G_{T_1}(\Pi_2(0)\partial_x^3 u_0)}_{=0\,\text{property (ii)}} + \lambda'(0)\Pi_1(0)AQ_0^2A\underbrace{G_{T_1}(\Pi_2(0)\partial_x^3 u_0)}_{=0\,\text{property (ii)}} \\
&-\Pi_1(0)AQ_0A\underbrace{G_{T_1}(\Pi_2(0)\partial_x^2 v_1)}_{=0\,\text{property (ii)}} = G_{T_1}(\Pi_1(0)B(\Pi_0 u_0)\partial_x \Pi_0 u_0).
\end{aligned}
\tag{2.26}
$$

In the nonlinear terms, only the terms polarized in the direction of $\Pi_1(0)$ remain thanks to Property (iv), and therefore one has that, $G_{T_1}(\Pi_1(0)B(\Pi_0 u_0)\partial_x \Pi_0 u_0) = \Pi_1(0)B(\Pi_1(0)u_0)\partial_x \Pi_1(0)u_0)$. Each component of u_0 being either transported by T_1 or T_2, some remain unchanged and other disappear thanks to Properties (i) and (ii).

We obtain similarly an analog equation governing $\Pi_2(0)u_0$. Our system (2.22)–(2.23), reduces to the following system for u_0

$$\begin{cases} \partial_{t_1}\Pi_1(0)u_0 + \Big(\frac{1}{2\lambda'(0)}\Pi_1(0)AQ_0A\Pi_2(0)AQ_0A\Pi_1(0) - \Pi_1(0)AQ_0AQ_0A\Pi_1(0) \\ -\lambda'(0)\Pi_1(0)AQ_0^2A\Pi_1(0)\Big)\,\partial_x^3\Pi_1(0)u_0 = \Pi_1(0)B(\Pi_1(0)u_0)\partial_x\Pi_1(0)u_0 \\ \partial_{t_1}\Pi_2(0)u_0 + \Big(-\frac{1}{2\lambda'(0)}\Pi_2(0)AQ_0A\Pi_1(0)AQ_0A\Pi_2(0) \\ -\Pi_2(0)AQ_0AQ_0A\Pi_2(0) + \lambda'(0)\Pi_2(0)AQ_0^2A\Pi_2(0)\Big)\,\partial_x^3\Pi_2(0)u_0 \\ = \Pi_2(0)B(\Pi_2(0)u_0)\partial_x\Pi_2(0)u_0. \end{cases} \tag{2.27}$$

This system (2.27) is indeed uncoupled and corresponds to a pair of independent KdV equations governing each component of u_0 moving in opposite directions and u_2 whose supposed to be a corrector verifies

$$\begin{cases} T_1(\partial_t,\partial_x)\Pi_1(0)u_2 = \Pi_1(0)B(\Pi_0(u_0))\partial_x\Pi_0u_0 - \Pi_1(0)B(\Pi_1(0)u_0)\partial_x\Pi_1u_0 \\ \qquad - \big(\Pi_1(0)AQ_0AQ_0A\Pi_2(0) + \lambda'(0)\Pi_1(0)AQ_0^2A\Pi_2(0)\big)\,\partial_x^3u_0 \\ T_2(\partial_t,\partial_x)\Pi_2(0)u_2 = \Pi_2(0)B(\Pi_0(u_0))\partial_x\Pi_0u_0 - \Pi_2(0)B(\Pi_2(0)u_0)\partial_x\Pi_2u_0 \\ \qquad - \big(\Pi_2(0)AQ_0AQ_0A\Pi_1(0) + \lambda'(0)\Pi_2(0)AQ_0^2A\Pi_1(0)\big)\,\partial_x^3u_0. \end{cases} \tag{2.28}$$

Remark 2.2. One can set $v_1 = 0$ (the initial condition as we solved (2.5)) with no loss of generality since it appears in the equation (2.22) polarized such as it ends up in the equation describing the corrector term u_2 (4.3).

2.5. Main algebraic lemma

The system (2.27) will read as KdV type system in a more obvious way, thanks to the following algebraic lemma, regarding the operators of order 3, namely the dispersive terms, that gives:

Lemma 2.3 (Main lemma)**.** *One has the following relations*

$$\frac{1}{2\lambda'(0)}\Pi_1(0)AQ_0A\Pi_2(0)AQ_0A\Pi_1(0) - \Pi_1(0)AQ_0AQ_0A\Pi_1(0) - \lambda'(0)\Pi_1(0)AQ_0^2A\Pi_1(0) = \frac{\lambda'''(0)}{6}\Pi_1(0) \tag{2.29}$$

and likewise,

$$-\frac{1}{2\lambda'(0)}\Pi_2(0)AQ_0A\Pi_1(0)AQ_0A\Pi_2(0) - \Pi_2(0)AQ_0AQ_0A\Pi_2(0) + \lambda'(0)\Pi_2(0)AQ_0^2A\Pi_2(0) = -\frac{\lambda'''(0)}{6}\Pi_2(0). \tag{2.30}$$

Proof of the main lemma. Let us start by proving the first relation. Use will be made in this proof of the previous lemmas and in particular we start by a proposition concerning the behavior of the operators $Q_1(\xi)$ and $Q_2(\xi)$ as ξ tends to 0. Note that these two operators are defined as expected as $Q_1(\xi) = Q(\lambda(\xi), \xi)$ and $Q_2(\xi) = Q(-\lambda(\xi), \xi)$ and are meromorph with respect to the variable ξ as a straightforward consequence of the analycity of the projector operators. □

Proposition 2.4. *Q_1 and Q_2 admit the following expansion around* 0 *with respect to ξ,*

$$Q_1(\xi) = Q_0 + \frac{1}{2\lambda(\xi)}\Pi_2(\xi) + O(\lambda(\xi)) \tag{2.31}$$

$$Q_2(\xi) = Q_0 - \frac{1}{2\lambda(\xi)}\Pi_1(\xi) + O(\lambda(\xi)). \tag{2.32}$$

Proof of Proposition 2.4. From our original hypothesis laid out in the set up of the problem, one has that $A\xi + \frac{E}{i}$ is symmetric and real for any ξ in $\mathbb{R}$. Therefore, there exists $P(\xi)$ an orthogonal $N \times N$ matrix such that

$$A\xi + \frac{E}{i} = P^{-1}(\xi)\begin{pmatrix} -\lambda(\xi) & & & \\ & \lambda(\xi) & & \\ & & \ddots & \\ & & & \lambda_N(\xi) \end{pmatrix} P(\xi) \tag{2.33}$$

where the first two eigenvalues are those of interest in this paper and the matrix $P(\xi)$ is analytical with respect to ξ. Thereafter, $L(\xi)$ namely $L(\pm\lambda(\xi), \xi)$ reads as, on the same basis,

$$L(\xi) = \lambda(\xi) + A\xi + \frac{E}{i} = P^{-1}(\xi)\begin{pmatrix} 0 & & & & \\ & 2\lambda(\xi) & & & \\ & & \lambda_3(\xi)+\lambda(\xi) & & \\ & & & \ddots & \\ & & & & \lambda_N(\xi)+\lambda(\xi) \end{pmatrix} P(\xi)$$

and consequently

$$Q_1(\xi) = P^{-1}(\xi)\ \mathrm{Diag}\ \left[0, \frac{1}{2\lambda(\xi)}, \frac{1}{\lambda_3(\xi)+\lambda(\xi)}, \dots, \frac{1}{\lambda_N(\xi)+\lambda(\xi)}\right] P(\xi)$$

and similarly one has that

$$Q_2(\xi) = P^{-1}(\xi)\ \mathrm{Diag}\ \left[-\frac{1}{2\lambda(\xi)}, 0, \frac{1}{\lambda_3(\xi)-\lambda(\xi)}, \dots, \frac{1}{\lambda_N(\xi)-\lambda(\xi)}\right] P(\xi)$$

whereas Q_0 which is the partial inverse of $\frac{1}{i}E$ reads in a very straightforward manner from (2.33) and Assumption 2.1, as

$$Q_0 = P^{-1}(0)\begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & \frac{1}{\lambda_3(0)} & & \\ & & & \ddots & \\ & & & & \frac{1}{\lambda_N(0)} \end{pmatrix} P(0).$$

As we have expressed explicitly all the operators involved in Proposition 2.4, it is a straightforward task to finish the proof. □

We denote by **I**, **II** and **III** the three terms in the left-hand side of (2.29).

- From (2.13) in Lemma 2.2, the first term I gives immediately, using the algebraical relations (2.18), that

$$\mathbf{I} = \frac{1}{2\lambda'(0)}\Pi_1(0)AQ_0A\Pi_2(0)AQ_0A\Pi_1(0) = 2\lambda'(0)\Pi_1(0)\Pi_2'^2(0)\Pi_1(0). \tag{2.34}$$

- For the second term **II**, one needs to compute the value of the operators of order 2 and order 3 at a regular point of the characteristic variety. For that matter, we state the following proposition:

Proposition 2.5. *If $(\lambda(\xi), \xi)$ is a regular point of the characteristic variety Char L, which in our case means that $\xi \neq 0$ in a ball near 0, then,*

$$\Pi_1(\xi)AQ_1(\xi)A\Pi_1(\xi) = \frac{\lambda''(\xi)}{2}\Pi_1(\xi) \tag{2.35}$$

$$\Pi_1(\xi)AQ_1(\xi)AQ_1(\xi)A\Pi_1(\xi) = -\frac{\lambda'''(\xi)}{6}\Pi_1(\xi) - \lambda'(\xi)\Pi_1(\xi)\Pi'^2_1(\xi)\Pi_1(\xi) \tag{2.36}$$

and likewise for the second branch of the characteristic relatively to the projector $\Pi_2(\xi)$.

Proof of Proposition 2.5. The first relation is not difficult to establish and its complete proof can be found in [14]. Briefly, one differentiates the relation

$$I - \Pi_1(\xi) = Q_1(\xi)(\lambda(\xi) + A\xi + \frac{E}{i}) \tag{2.37}$$

and apply $\Pi_1(\xi)$ on the right-hand side to obtain $\Pi_1'(\xi)\Pi(\xi) = -Q_1(\xi)A\Pi_1(\xi)$ and likewise $\Pi_1(\xi)\Pi_1'(\xi) = -\Pi_1(\xi)AQ_1(\xi)$. Then differentiating the first order relation, one gets

$$\Pi_1(\xi)A\Pi_1(\xi) = -\lambda'(\xi)\Pi_1(\xi),$$

which gives the first relation displayed in Proposition 2.5.

Let us turn now to the third order operator and prove the second relation in Proposition 2.5. We start by differentiating the first relation which gives, at all regular point ξ,

$$\frac{\lambda'''}{2}\Pi_1 + \frac{\lambda''}{2}\Pi_1' = \Pi_1'AQ_1A\Pi_1 + \Pi_1AQ_1'A\Pi_1 + \Pi_1AQ_1A\Pi_1'.$$

We apply $\Pi_1(\xi)$ both on the left and right side of the relation, which yields

$$-2\Pi_1AQ_1AQ_1A\Pi_1 + \Pi_1AQ_1'A\Pi_1 = \frac{\lambda'''}{2}\Pi_1 + \frac{\lambda''}{2}\underbrace{\Pi_1\Pi_1'\Pi_1}_{=0}.$$

In order to evaluate $Q_1'(\xi)$, we differentiate the relation (2.37) and apply $Q_1(\xi)$ in order to obtain:

$$Q_1'(\xi) = -\Pi_1'(\xi)Q_1(\xi) + Q_1'(\xi)\Pi_1(\xi) - \lambda'(\xi)Q_1^2(\xi) - Q_1(\xi)AQ_1(\xi).$$

Thereafter, using the fact that $Q_1'\Pi_1 + Q_1\Pi_1' = 0$, one has that

$$\Pi_1AQ_1'A\Pi_1 = -\Pi_1A\Pi_1'Q_1A\Pi_1 + \Pi_1AQ_1'\Pi_1A\Pi_1 - \lambda'\Pi_1AQ_1^2A\Pi_1 - \Pi_1AQ_1AQ_1A\Pi_1,$$

and using $\Pi_1'(\xi)\Pi(\xi) = -Q_1(\xi)A\Pi_1(\xi)$ and $\Pi_1(\xi)\Pi_1'(\xi) = -\Pi_1(\xi)AQ_1(\xi)$, we obtain

$$\Pi_1AQ_1'A\Pi_1 = \Pi_1A\Pi'^2_1\Pi_1 + \Pi_1\Pi'^2_1A\Pi_1 - \lambda'\Pi_1\Pi'^2_1\Pi_1 - \Pi_1AQ_1AQ_1A\Pi_1.$$

Now thanks to the algebraic relations (2.18) and Lemma 2.2, we get

$$\Pi_1A\Pi'^2_1\Pi_1 = -\lambda'\Pi_1\Pi'^2_1\Pi_1$$

as well as

$$\Pi_1{\Pi'}_1^2 A\Pi_1 = -\lambda'\Pi_1{\Pi'}_1^2\Pi_1.$$

Gathering all the terms together gives the second relation of Proposition 2.5 and finishes the proof. □

We go back to the computation of **II**. Our strategy is to evaluate the third order operator $\Pi_1(\xi)AQ_1(\xi)AQ_1(\xi)A\Pi_1(\xi)$ at 0 by using Proposition 2.4 and letting ξ tend to 0. Hence one has that

$$\Pi_1(\xi)AQ_1(\xi)AQ_1(\xi)A\Pi_1(\xi) =\Pi_1(\xi)A\left[Q_0+\frac{1}{2\lambda(\xi)}\Pi_2(\xi)+O(\lambda)\right]A\left[Q_0+\frac{1}{2\lambda(\xi)}\Pi_2(\xi)+O(\lambda)\right]A\Pi_1(\xi).$$

As we develop the quantity in the right-hand side, nine terms appear, most of which tend to 0 as ξ tends to 0. Indeed, the five terms that contain $O(\lambda)$, in the development can be crossed out since everything else is bounded and the singularity $\frac{1}{2\lambda(\xi)}$ as ξ tends to 0 is controlled by either $\Pi_1(\xi)A\Pi_2(\xi) = 2\lambda(\xi)\Pi_1(\xi)\Pi_2'(\xi)$ or $\Pi_2(\xi)A\Pi_1(\xi) = -2\lambda(\xi)\Pi_2(\xi)\Pi_1'(\xi)$, in each of these terms.

Thus, after developing, we are left with the following four terms:

$$\lim_{\xi\to 0}\Pi_1(\xi)AQ_1(\xi)AQ_1(\xi)A\Pi_1(\xi) = \lim_{\xi\to 0}\left\{\underbrace{\Pi_1(\xi)AQ_0AQ_0A\Pi_1(\xi)}_{[1]}+\underbrace{\frac{1}{2\lambda(\xi)}\Pi_1(\xi)AQ_0A\Pi_2(\xi)A\Pi_1(\xi)}_{[2]}\right.$$

$$\left.+\underbrace{\frac{1}{2\lambda(\xi)}\Pi_1(\xi)A\Pi_2(\xi)AQ_0A\Pi_1(\xi)}_{[3]}+\underbrace{\frac{1}{4\lambda^2(\xi)}\Pi_1(\xi)A\Pi_2(\xi)A\Pi_2(\xi)A\Pi_1(\xi)}_{[4]}\right\}.$$

As ξ tends to 0, thanks to Lemma 2.3 along with the projectors properties (2.18) and the two previous relations for the crossed products $\Pi_1(\xi)A\Pi_2(\xi)$ and $\Pi_2(\xi)A\Pi_1(\xi)$, each of the four above limit reads as

$$\lim_{\xi\to 0}[1] = -\mathbf{II}\,; \qquad \lim_{\xi\to 0}[2] = -2\lambda'(0)\Pi_1(0){\Pi'}_2^2(0)\Pi_1(0);$$

$$\lim_{\xi\to 0}[3] = -2\lambda'(0)\Pi_1(0){\Pi'}_2^2(0)\Pi_1(0)\,; \qquad \lim_{\xi\to 0}[4] = \lambda'(0)\Pi_1(0){\Pi'}_2^2(0)\Pi_1(0).$$

Now that we have the limit of the third order operator, we identify it with the second relation of Proposition 2.5 at $\xi=0$ and obtain

$$-\frac{\lambda'''(0)}{6}\Pi_1-\lambda'(0)\Pi_1(0){\Pi'}_1^2(0)\Pi_1(0) = -\mathbf{II}-3\lambda'(0)\Pi_1(0){\Pi'}_2^2(0)\Pi_1(0)$$

which yields

$$\mathbf{II} = \frac{\lambda'''(0)}{6}\Pi_1+\lambda'(0)\Pi_1(0){\Pi'}_1^2(0)\Pi_1(0)-3\lambda'(0)\Pi_1(0){\Pi'}_2^2(0)\Pi_1(0). \tag{2.38}$$

- We are left now with **III**. As for **II**, we use Proposition 2.4 to compute the limit as ξ tends to 0 of the operator $\Pi_1(\xi)AQ_1^2(\xi)A\Pi_1(\xi)$.

We develop the latter operator as suggested by Proposition 2.4, which gives

$$\lim_{\xi\to 0}\Pi_1(\xi)A\left[Q_0+\frac{1}{2\lambda(\xi)}\Pi_2(\xi)+O(\lambda)\right]A\Pi_1 =\lim_{\xi\to 0}\left[\Pi_1(\xi)AQ_0^2A\Pi_1(\xi)+\frac{1}{4\lambda^2(\xi)}\Pi_1(\xi)A\Pi_2(\xi)A\Pi_1(\xi)\right].$$

The other terms in the development cancel out as ξ tends to 0 either because of the presence of $O(\lambda)$ or because of the projectors $\Pi_1(0)$ and $\Pi_2(0)$ applied to Q_0.

On the other hand, one has that $\Pi_1(\xi)AQ_1^2(\xi)A\Pi_1(\xi) = \Pi_1(\xi){\Pi'}_1^2(\xi)\Pi_1(\xi)$ and therefore identifying the two limit as ξ tends to 0 gives

$$\Pi_1(0){\Pi'}_1^2(0)\Pi_1(0) = \Pi_1(0)AQ_0^2A\Pi_1(0) + \Pi_1(0){\Pi'}_2^2(0)\Pi_1(0)$$

which gives

$$\mathbf{III} = -\lambda'(0)\Pi_1(0){\Pi'}_1^2(0)\Pi_1(0) + \lambda'(0)\Pi_1(0){\Pi'}_2^2(0)\Pi_1(0) \tag{2.39}$$

and finally as we sum $\mathbf{I} + \mathbf{II} + \mathbf{III}$, (2.29) holds. The proof of (2.30) is exactly the same. □

Thanks to the previous lemmas, the uncoupled system derived earlier (2.27) read in a much simpler way, as an obvious KdV type system:

$$\begin{cases} \partial_{t_1}\Pi_1(0)u_0 + \dfrac{\lambda'''(0)}{6}\partial_x^3\Pi_1(0)u_0 = \Pi_1(0)B(\Pi_1(0)u_0)\partial_x\Pi_1(0)u_0 \\ \partial_{t_1}\Pi_2(0)u_0 - \dfrac{\lambda'''(0)}{6}\partial_x^3\Pi_2(0)u_0 = \Pi_2(0)B(\Pi_2(0)u_0)\partial_x\Pi_2(0)u_0. \end{cases} \tag{2.40}$$

3. Convergence in the uncoupled case

In the preceding section, we have obtained a set of necessary conditions on u_0, u_1, u_2 and u_3 in order that $\mathcal{U}^\epsilon$ given by (1.2) is an approximate solution of (1.1). The aim of this section is to show that one can solve simultaneously equations (2.12), (2.40) and (2.28) and that there exists a solution to (1.1) which is indeed asymptotic to the approximate solution thus constructed. One of the key argument will be that the correctors u_2 and u_3 given (2.28) and (2.7) satisfy the sub-squareroot condition (1.3).

In order to be able to state our theorem, one needs to prove the following proposition regarding the existence for large times of order $O(\frac{1}{\epsilon^2})$ of the exact solution of (1.1).

Proposition 3.1. *For any $s > \frac{3}{2}$ and for any $u_{t=0} = \epsilon^2 u_{in}$ such that $u_{in} \in H^s(\mathbb{R})$, there exists $T > 0$ such that there is a unique solution u of (1.1) lying in the space $C([0, \frac{T}{\epsilon^2}], H^s) \cap C^1([0, \frac{T}{\epsilon^2}], H^{s-1})$.*

Proof of Proposition 3.1. The proof relies mainly on the fact that $B(u)$ is symmetric and follows the existence proof for quasilinear symmetric systems [1]. The only non trivial thing here that needs to be proved is that the H^s-norm of $u(t)$ remains bounded for large time scales of order $O(\frac{1}{\epsilon^2})$. Let us briefly sketch the proof: as we multiply the equation (1.1) by $\partial^{2s}u$ and integrate with respect to the space variable, one obtains that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} |\partial^s u|^2 \,\mathrm{d}x = \int_{\mathbb{R}} \partial^{2s}uB(u)\partial u \,\mathrm{d}x.$$

As usual, we manage to estimate the right-hand side as follows

$$\left|\int_{\mathbb{R}} \partial^{2s}uB(u)\partial u \,\mathrm{d}x\right| \le c\|u\|_s^3$$

and we conclude by applying Gronwall's lemma, that gives

$$\|u\|_s \le \frac{\|u_{t=0}\|_s}{2 - ct\epsilon^2\|u_{in}\|_s} \le \|u_{t=0}\|_s \quad \text{for} \quad t \le c\frac{T}{\epsilon^2}.$$

Therefore our H^s-bound does not blow up for times of order $O(\frac{1}{\epsilon^2})$ and the natural local existence theorem for (1.1) as a hyperbolic system extends itself to the interval $[0, \frac{T}{\epsilon^2}]$, which finishes the proof. □

Thanks to Proposition 3.1, let us introduce u^ϵ, for any f lying in H^s with s strictly greater than $\frac{3}{2}$, solution of

$$\begin{cases} \left(\partial_t + A(\partial_x) + \dfrac{E}{\epsilon}\right) u^\epsilon = B(u^\epsilon)\partial_x u^\epsilon \\ \\ u^\epsilon(x,0) = \epsilon^2 f(x) \end{cases} \tag{3.1}$$

defined on $[0, \frac{T_1}{\epsilon^2}]$ for $T_1 > 0$.

Our result reads as follows.

Theorem 3.1. *Let $s > \frac{3}{2}$ and $f \in H^\sigma$ (σ sufficiently large) such that $\Pi_0 f = f$. Under Assumption 2.1, there exists $T_1 > 0$ and a unique $u^\epsilon(x,t)$ in $L^\infty([0, \frac{T_1}{\epsilon^2}], H^s)$ solution of (3.1) as well as $T_2 > 0$ and $u_{01}(X_1, t_1)$ and $u_{02}(X_2, t_1)$ solutions of*

$$\begin{cases} \partial_{t_1} u_{01} + \dfrac{\lambda'''(0)}{6}\partial^3_{X_1} u_{01} = \Pi_1(0)B(u_{01})\partial_{X_1} u_{01} \\ \\ u_{01}(X_1, 0) = \Pi_1(0) f(X_1) \end{cases} \tag{3.2}$$

and

$$\begin{cases} \partial_{t_1} u_{02} - \dfrac{\lambda'''(0)}{6}\partial^3_{X_2} u_{02} = \Pi_2(0)B(u_{02})\partial_{X_2} u_{02} \\ \\ u_{02}(X_2, 0) = \Pi_2(0) f(X_2) \end{cases} \tag{3.3}$$

both lying in $L^\infty([0, T_2], H^s)$.

Moreover there exists $T_0 > 0$ (s.t. $T_0 \le \min(T_1, T_2)$) such that

$$\left\| \frac{u^\epsilon(x,t)}{\epsilon^2} - \left[u_{01}(x + \lambda'(0)t, \epsilon^2 t) + u_{02}(x - \lambda'(0)t, \epsilon^2 t)\right] \right\|_{L^\infty([0, \frac{T_0}{\epsilon^2}]; H^s)} = o(1)$$

as ϵ tends to 0.

The strategy to prove this theorem relies on three points. We start by introducing

$$\mathcal{U}^\epsilon(t,x) = \epsilon^2 \left[u_{01}(x + \lambda'(0)t, \epsilon^2 t) + u_{02}(x - \lambda'(0)t, \epsilon^2 t)\right] + \epsilon^3 u_1^\epsilon(x,t) + \epsilon^4 u_2^\epsilon(x,t) + \epsilon^5 u_3^\epsilon(x,t). \tag{3.4}$$

Then we prove the following three points:

1. The equations for u_{01}, u_{02} as well as those determining u_1^ϵ, u_2^ϵ and u_3^ϵ are well posed and all these terms exist for time scales of order $O(\frac{1}{\epsilon^2})$ and lie in $L^\infty([0, \frac{T_2}{\epsilon^2}]; H^s)$.
2. In the expression of $\mathcal{U}^\epsilon(t,x)$, $\epsilon^3 u_1^\epsilon + \epsilon^4 u_2^\epsilon + \epsilon^5 u_3^\epsilon$ is indeed a corrector of the principal term that is

$$\|\epsilon^3 u_1^\epsilon + \epsilon^4 u_2^\epsilon + \epsilon^5 u_3^\epsilon\|_{L^\infty([0, \frac{T}{\epsilon^2}]; H^s)} = O(\epsilon^3).$$

3. We obtain an estimate of the residues r_j for $j \ge 5$ and we finish the proof by performing a standard energy estimate on $\frac{u^\epsilon}{\epsilon^2} - \frac{\mathcal{U}^\epsilon}{\epsilon^2}$.

3.1. Properties of the approximate solution

One first has to solve the following set of equations:

$$\begin{cases} (\partial_t - \lambda'(0)\partial_x)u_{01} = 0 \\ (\partial_t + \lambda'(0)\partial_x)u_{02} = 0 \end{cases}$$

and an uncoupled system of KdV equations for the long time evolution

$$\begin{cases} \partial_{t_1} u_{01} + \dfrac{\lambda'''(0)}{6}\partial_x^3 u_{01} = \Pi_1(0)B(u_{01})\partial_x u_{01} \\ \partial_{t_1} u_{02} - \dfrac{\lambda'''(0)}{6}\partial_x^3 u_{02} = \Pi_1(0)B(u_{02})\partial_x u_{02} \end{cases}$$

with $u_{01} = \Pi_1(0)f$ and $u_{02} = \Pi_2(0)f$.
For each component we have a global existence theorem in $L^\infty(\mathbb{R}, H^\sigma(\mathbb{R}))$ (with $\sigma \geq 0$) for f in H^s, since the long time evolution is governed by a classical KdV whose Cauchy problem in H^s is well known (see [18] for example) and the short time evolution is compatible with the long time KdV. Note that if the initial condition is polarized by the projectors $\Pi_1(0)$ and $\Pi_2(0)$, the solution remains likewise.

As u_0 is uniquely determined, it is an easy task to find the remaining terms of the expansion (1.2) from the solvability conditions that are all satisfied in the uncoupled case. Indeed, one recalls from (2.20)–(2.21) that for $\Pi_0 u_1^\epsilon$, we have that

$$\begin{cases} \Pi_1(0)u_1^\epsilon = \dfrac{i}{2\lambda'(0)}\Pi_1 A Q_0 A \Pi_2 \partial_x u_0 \\ \Pi_2(0)u_1^\epsilon = -\dfrac{i}{2\lambda'(0)}\Pi_2 A Q_0 A \Pi_1 \partial_x u_0 \end{cases}$$

if we choose to set v_1 equal to 0. As for the remaining component of u_0 (*e.g.* $(I - \Pi_0)u_0$), it is given by (2.3).

We turn now to u_2^ϵ, whose components on Π_0 are given by (2.28). These equations for $\Pi_1(0)u_2^\epsilon$ and $\Pi_2(0)u_2^\epsilon$ are very important in order to determine the growth of u_2^ϵ with respect to time. The more terms we put at the right-hand side of these equations, the more it affects the final result of convergence. These hyperbolic equations for each component of $\Pi_0 u_2^\epsilon$ can be solved and thus determine $\Pi_0 u_2^\epsilon$. As for the remaining component $(I - \Pi_0)u_2^\epsilon$, it is given by (2.5) as we already found u_1^ϵ.

We are left with u_3^ϵ that we set as equal to $(I - \Pi_0)u_3^\epsilon$ which is given by (2.7) as we already know u_0 and u_2^ϵ.

Moreover since all the operators involved in the description of u_1^ϵ, u_2^ϵ and u_3^ϵ from u_0 are bounded, one concludes that theses terms are not only determined from u_0 but lie also in $L^\infty(\mathbb{R}, H^\sigma)$ as u_0 and T_2 can be chosen as large as we want (recall that σ is large enough). The existence of T_1 is clear from Proposition 3.1.

3.2. Correctors

To construct our approximate solution, we have assumed as we have set up our ansatz that the term $\epsilon^3 u_1^\epsilon + \epsilon^4 u_2^\epsilon + \epsilon^5 u_3^\epsilon$ was a corrector of the leading order term, which in other words means that we control the growth in time of u_1^ϵ, u_2^ϵ and u_3^ϵ.

Let us check each term separately. For u_1^ϵ, since u_0 is bounded in H^σ for σ sufficiently large, both $\Pi_0 u_1^\epsilon$ and $(I - \Pi_0)u_1^\epsilon$ are bounded in $H^{\sigma-1}$ from the previous relations that we used to determine u_1^ϵ.

Again u_2^ϵ decomposes itself in two parts. The component $(I - \Pi_0)u_2^\epsilon$ is bounded in $H^{\sigma-2}$ since u_1^ϵ is bounded in $H^{\sigma-1}$ thanks to (2.5). For $\Pi_0 u_2^\epsilon$, the general results of [20] give that u_2^ϵ has a sublinear growth in time. This is not enough in our case since this gives $\epsilon^4 u_2^\epsilon = o(\epsilon^2)$ on $[0, \frac{T}{\epsilon^2}]$. We give below more precise results.

Proposition 3.2. *Let $f(x,t)$ be a sufficiently smooth function such that*

$$T_1(\partial_t, \partial_x)f = \partial_x g$$

where $T_2(\partial_t, \partial_x)g = 0$ and $g \in L^\infty(\mathbb{R}; L^2)$ then $f \in L^\infty(\mathbb{R}; L^2)$.

Proposition 3.3. *Let $u(x,t)$ be a sufficiently smooth function such that*

$$T_1(\partial_t, \partial_x)u = gh$$

where g and h are such that $T_1(\partial_t, \partial_x)h = 0$ and $T_2(\partial_t, \partial_x)g = 0$ with $g, h \in L^\infty(\mathbb{R}; L^2)$ then u respects sub-square root growth condition as defined in (1.3) that is

$$\lim_{t\to 0} \frac{1}{\sqrt{t}}\|f\|_2 = 0.$$

Proof of Proposition 3.2. Since g is transported by T_2, one can write the relation in Proposition 3.2, as

$$T_1(\partial_t, \partial_x)f(x,t) = \partial_x g(x - \lambda'(0)t)$$

which leads to

$$T_1(\partial_t, \partial_x)f(x,t) = \int_{\mathbb{R}} \mathrm{e}^{ix\xi} i\xi \mathrm{e}^{-i\lambda'(0)t\xi}\widehat{g}(\xi)\,\mathrm{d}\xi.$$

Then,

$$\widehat{f}(\xi,t) = \mathrm{e}^{i\lambda'(0)\xi t}\widehat{g}(\xi,0) + \int_0^t \mathrm{e}^{i\lambda'(0)(t-s)\xi}\mathrm{e}^{-\lambda'(0)s\xi} i\xi\widehat{g}(\xi)\,\mathrm{ds}.$$

Therefore,

$$\begin{aligned}\widehat{f}(\xi,t) &= \mathrm{e}^{i\lambda'(0)\xi t}\widehat{g}(\xi,0) + i\xi\widehat{g}(\xi)\mathrm{e}^{i\lambda'(0)t\xi}\textstyle\int_0^t \mathrm{e}^{-2i\lambda'(0)s\xi}\,\mathrm{ds}\\ &= \mathrm{e}^{i\lambda'(0)\xi t}\widehat{g}(\xi,0) + i\widehat{g}(\xi)\mathrm{e}^{i\lambda'(0)t\xi}\left[\frac{1-\mathrm{e}^{-2i\lambda'(0)\xi t}}{2i\lambda'(0)}\right].\end{aligned}$$

It follows that

$$\|\widehat{f}\|_2(t) \le 2\|\widehat{g}\|_2.$$

□

Remark 3.1. The crucial point in the previous proof is the presence of the ∂_x in the right-hand side.

Proof of Proposition 3.3. Since g and h are transported, the relation in Proposition 3.3 can be written as follows

$$(\partial_t - \lambda'(0)\partial_x)u = g(x - \lambda'(0)t)h(x + \lambda'(0)t).$$

We perform the change of function $u(x,t) = v(x + \lambda'(0)t, t)$ and set $X = x + \lambda'(0)t$, the equation becomes

$$\partial_t v(X,t) = h(X)g(X - 2\lambda'(0)t) \tag{3.5}$$

and therefore

$$v(X,t) = v_0(X) + h(X)\int_0^t g(X - 2\lambda'(0)s)\,\mathrm{ds}.$$

Cauchy-Schwartz inequality gives

$$|v(X,t)| \le |v_0| + t^{1/2}|h(X)|\|g\|_2$$

which leads to

$$\frac{\|u(t)\|_2}{t^{1/2}} \le \frac{\|u_0\|_2}{t^{1/2}} + \|h\|_2\|g\|_2. \tag{3.6}$$

Introduce, as in [15], the dense subset $\mathcal{A}$ of L^2 given by

$$\mathcal{A} = \left\{ f \in L^2 \ / \ \widehat{f} \in C_0^\infty(\mathbb{R} - \{0\}) \right\}.$$

Then let u_n be a sequence in $\mathcal{A}$ be such that u_n tends to u in L^2 and such that for each n

$$(\partial_t - \lambda'(0)\partial_x)u_n(x,t) = g_n(x - \lambda'(0)t)h_n(x + \lambda'(0)t)$$

and where h_n and g_n belonging to $\mathcal{A}$ tend respectively to h and g in L^2. u_n is given by

$$\widehat{u_n}(\xi,t) = \mathrm{e}^{i\lambda'(0)\xi t}\widehat{u}_{n0}(\xi) + \mathrm{e}^{i\lambda'(0)\xi t}\int_{\mathbb{R}} \widehat{h_n}(\eta)\widehat{g_n}(\xi-\eta)\frac{1-\mathrm{e}^{2i\lambda'(0)\eta t}}{2i\lambda'(0)\eta}\,\mathrm{d}\eta.$$

Since the denominator is bounded away from 0 on the support of f_n and g_n, it follows that

$$\lim_{t\to 0} \frac{1}{\sqrt{t}}\|u_n\|_2 = 0. \tag{3.7}$$

Then, one has that

$$\frac{1}{\sqrt{t}}\|u\|_2 \le \frac{1}{\sqrt{t}}\|u_n - u\|_2(t) + \frac{1}{\sqrt{t}}\|u_n\|_2(t).$$

Applying the inequality (3.6) to $u_n - u$ that verifies

$$\partial_t(u_n - u) = (g_n - g)f_n + (f_n - f)g$$

gives for n sufficiently large such that $\|f_n - f\|_2 \le \epsilon$ and $\|g_n - g\| \le \epsilon$, that

$$\frac{1}{\sqrt{t}}\|u\|_2 \le \epsilon + \frac{1}{\sqrt{t}}\|u_n\|_2(t) + \frac{\|u_0\|_2}{\sqrt{t}}$$

and now taking the limit in t as it tends to ∞ gives the desired result thanks to (3.7). □

Proposition 3.4. *The solutions* $\Pi_1(0)u_2^\epsilon$ *and* $\Pi_2(0)u_2^\epsilon$ *to (2.28) satisfy a sub-square root growth condition (1.3) that is*

$$\lim_{t\to 0} \frac{1}{\sqrt{t}}\|\Pi_1(0)u_2^\epsilon\|_{H^s} = 0$$

and likewise for $\Pi_2(0)u_2^\epsilon$.

Proof. We first write equations (2.28) in a simplified way. Indeed, since we have that $T_1(\partial_t,\partial_x)\Pi_1 u_0 = 0$ and $T_2(\partial_t,\partial_x)\Pi_2 u_0 = 0$, the two components of u_0 read as $\Pi_1(0)u_0(x+\lambda'(0)t)$ and $\Pi_2(0)u_0(x-\lambda'(0)t)$ with the variable t_1 taken as a parameter. Thereafter, we simply write the first equation above with the right-hand side being the sum of two generic terms, such as

$$(\partial_t - \lambda'(0)\partial_x)u = \partial_x f(x - \lambda'(0)t) + g(x - \lambda'(0)t)h(x + \lambda'(0)t)$$

where f, g and h are L^2-bounded functions and u any function of x and t sufficiently smooth. We have from Proposition 3.2 that the first term in the right-hand side gives in u a bounded contribution in time, and from Proposition 3.3, that the second term implies that u respects a sub-squareroot growth in time. This holds exactly the same for the second component and one has, as claimed, that $\Pi_0 u_2^\epsilon$ respects the growth condition (1.3). □

Finally for u_3^ϵ we deduce the same growth control in time as for u_2^ϵ from the solvability condition (2.7). These two conditions give then that $\|\epsilon^4 u_2^\epsilon\|_2 = o(\epsilon^3)$ and $\|\epsilon^5 u_3^\epsilon\|_2 = o(\epsilon^4)$ and we can state the following proposition.

Proposition 3.5. *The corrector term* $\epsilon^3 u_1^\epsilon + \epsilon^4 u_2^\epsilon + \epsilon^5 u_3^\epsilon$ *is indeed a corrector in (3.4) and one has that*

$$\|\epsilon^3 u_1^\epsilon + \epsilon^4 u_2^\epsilon + \epsilon^5 u_3^\epsilon\|_{L^\infty([0,\frac{T}{\epsilon^2}];H^s)} = O(\epsilon^3).$$

3.3. Estimate for the residue and end of the proof

Before proving the convergence result, we first estimate the residue. Since one has $r_j = 0$ for $j = 1$ to $j = 4$, the residue reads as the remaining terms:

$$\mathrm{Res}(x,t,t_1,\epsilon) = \epsilon^5 r_5 + \epsilon^6 r_6 + \epsilon^7 r_7 + \epsilon^8 r_8 + \epsilon^9 r_9 + \epsilon^{10} r_{10}.$$

Only the first two summands of this residue play a role. To estimate the H^s-norm L^2 of this residue, we use the fact that u_1^ϵ is H^s-bounded and that u_2^ϵ and u_3^ϵ are controlled in time as proved from Proposition 3.2 and Proposition 3.3.

The first term for instance is estimated as follows, using the Sobolev embeddings where $H^s \hookrightarrow L^\infty$ and $H^{s-1} \hookrightarrow L^\infty$ for $s > \frac{3}{2}$. And we have

$$\|\epsilon^5 r_5\|_2 \le \sqrt{t}\epsilon^5 \left(\frac{1}{\sqrt{t}} \|\partial_{t_1} u_3^\epsilon\|_2 + \quad \text{bounded terms} \right) \cdot$$

We have then:

Proposition 3.6. *The residue can be estimated as follows in the norm* $L^\infty([0,\frac{T}{\epsilon^2}];L^2)$*:*

$$\|\mathrm{Res}\|_{L^\infty([0,\frac{T}{\epsilon^2}];L^2)} = o(\epsilon^4).$$

Remark 3.2. If u_2^ϵ verifies only a sub-linear growth condition, we would have concluded using the same arguments that $\|\mathrm{Res}\|_2 = o(\epsilon^3)$ which would not had been enough to establish our theorem.

As we have estimated the residue, we have that our approximate solution $\mathcal{U}^\epsilon$ satisfies

$$\partial_t \mathcal{U}^\epsilon + A(\partial_x)\mathcal{U}^\epsilon + \frac{E\mathcal{U}^\epsilon}{\epsilon} - B(\mathcal{U}^\epsilon)\partial_x \mathcal{U}^\epsilon = o(\epsilon^4) \tag{3.8}$$

where $o(\epsilon^4)$ is in $L^\infty([0,\frac{T}{\epsilon^2}];H^s)$ norm.

Let us turn to the final proof of our convergence result that can be compared to the stability results displayed in [14,20]. We denote by u^ϵ the exact solution of (1.1) and $\mathcal{U}^\epsilon$ both lying in $C([0,\frac{T}{\epsilon^2}];H^s)$, for some $T > 0$.

We denote by $\tilde{u}$ the difference

$$\tilde{u} = \mathcal{U}^\epsilon - u^\epsilon \quad \text{with} \quad \tilde{u}(x,0) = 0.$$

Thus the equation satisfied by $\tilde{u}$ reads as

$$\partial_t \tilde{u} + A\partial_x \tilde{u} + \frac{E\tilde{u}}{\epsilon} - B(u)\partial_x u + B(\mathcal{U}^\epsilon)\partial_x \mathcal{U}^\epsilon = o(\epsilon^4)$$

which can be written

$$\partial_t \tilde{u} + A\partial_x \tilde{u} + \frac{E\tilde{u}}{\epsilon} + B(\tilde{u})\partial_x u + B(\mathcal{U}^\epsilon)\tilde{u}_x = o(\epsilon^4).$$

Multiplying by $\partial^{2s}\tilde{u}$ and integrating with respect to the space variable gives

$$(-1)^s \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}} \int_{\mathbb{R}} |\partial^s \tilde{u}|_2^2 \,\mathrm{d}x + \underbrace{\int_{\mathbb{R}} \partial^{2s}\tilde{u} B(\tilde{u}) u_x \,\mathrm{d}x}_{\mathbf{A}} + \underbrace{\int_{\mathbb{R}} \partial^{2s}\tilde{u} B(\mathcal{U}^\epsilon)\tilde{u}_x \,\mathrm{d}x}_{\mathbf{B}} = o(\epsilon^4).$$

As in the proof of Proposition 3.1, if s is strictly greater than $\frac{3}{2}$, one can bound the two terms **A** and **B**, such as

$$\mathbf{A} \leq \|\mathcal{U}^\epsilon\|_s \|\tilde{u}\|_s^2$$
$$\mathbf{B} \leq \ \|u\|_s \|\tilde{u}\|_s^2$$

which finally gives that

$$\frac{\mathrm{d}}{\mathrm{dt}}\|\tilde{u}\|_s^2 \leq C\left(\|\mathcal{U}^\epsilon\|_s + \|u\|_s\right)\|\tilde{u}\|_s^2 + o(\epsilon^4)$$

and Gronwall's lemma gives with $\|\tilde{u}\|_s(0) = 0$

$$\|\tilde{u}\|_s^2 \leq (\mathrm{e}^{c\epsilon^2 t} - 1)o(\epsilon^2) \quad \text{for} \quad t \leq \frac{T}{\epsilon^2}$$

and it is thus straightforward to conclude the proof of Theorem 3.1.

3.4. Higher order terms

A natural question that arises at this point is to push further the formal expansion and check if a new term in the expansion provides a better precision. In the previous expansion, we have set the ansatz to be $\mathcal{U}^\epsilon$ as described in (3.4). In the expression (3.4), u_3 was set as equal to its component $(I - \Pi_0)u_3$ and we verified that the corrector terms were indeed correcting the leading order term. Let us start off now with the following ansatz that has one more term

$$\begin{aligned}\mathcal{U}^\epsilon(t,x) = \epsilon^2 &\left[u_{01}(x + \lambda'(0)t, \epsilon^2 t) + u_{02}(x - \lambda'(0)t, \epsilon^2 t)\right] \\ &+ \epsilon^3 u_1(x,t,\epsilon^2 t) + \epsilon^4 u_2(x,t,\epsilon^2 t) + \epsilon^5 u_3(x,t,\epsilon^2 t) + \epsilon^6 u_4(x,t,\epsilon^2 t).\end{aligned} \tag{3.9}$$

Again, we plug this ansatz (3.9) in (1.1) and obtain the same profile equations (1.5), (1.6), (1.7) and (1.8) as in the first section up to the order 4 ($r_j = 0$). At the order 5, annihilating r_5 gives

$$\partial_{t_1} u_1 + \partial_t u_3 + A(\partial_x)u_3 + Eu_4 = B(u_0)\partial_x u_1 + B(u_1)\partial_x u_0. \tag{3.10}$$

From the above equation and thanks to Lemma (2.1), we deduce the following solvability conditions

$$\begin{cases}\partial_{t_1}\Pi_0 u_1 + \Pi_0 L_1 u_3 = \Pi_0 B(u_0)\partial_x u_1 + \Pi_0 B(u_1)\partial_x u_0 \\ (I - \Pi_0)u_4 = i\partial_{t_1} Q_0 u_1 + iQ_0 L_1 u_3 - iQ_0 B(u_0)\partial_x u_1 - iQ_0 B(u_1)\partial_x u_0.\end{cases} \tag{3.11}$$

Now thanks to (2.5) and (2.6), we deduce the following long time evolution equation for the corrector u_1, where $\Pi_0 u_3$ is not null anymore

$$\begin{aligned}\partial_{t_1}\Pi_0 u_1 + \Pi_0 L_1 \Pi_0 u_3 + i\Pi_0 L_1 Q_0 L_1 \Pi_0 u_2 - \Pi_0 L_1 Q_0 L_1 Q_0 L_1 \Pi_0 u_1 - i\Pi_0 L_1 Q_0 L_1 Q_0 L_1 Q_0 L_1 \Pi_0 u_0 \\ = \Pi_0 L_1 Q_0 B(u_0)\partial_x u_0 + \Pi_0 B(u_0)\partial_x u_1 + \Pi_0 B(u_1)\partial_x u_0.\end{aligned} \tag{3.12}$$

From this equation (3.12), as earlier, we apply successively the projectors $\Pi_1(0)$ and $\Pi_2(0)$. Then by using the average operators, G_{T_1} and G_{T_2}, we deduce the equation governing the corrector $\Pi_0 u_3$ that appeared in this new formal expansion. It reads

$$\begin{aligned}T_1(\partial_t, \partial_x)\Pi_1(0)u_3 &= -i\Pi_1(0)L_1 Q_0 L_1 \Pi_2(0)u_2 + F(u_1, u_0) \\ T_2(\partial_t, \partial_x)\Pi_2(0)u_3 &= -i\Pi_2(0)L_1 Q_0 L_1 \Pi_1(0)u_2 + G(u_1, u_0)\end{aligned} \tag{3.13}$$

where F and G are some bounded functions depending only on u_0 and u_1. We have proved previously that the corrector u_1 was bounded in $L^\infty(\mathbb{R}, L^2)$ and that u_2 respected a sub-squareroot growth condition (1.3) along with $(I - \Pi_0)u_3$. As one solves (3.13) by integrating the right-hand side term, one realises that the two components of $\Pi_0 u_3$ cannot respect any more a sub-squareroot growth (1.3) but in fact verify at the most a sublinear growth condition (5.4), which implies that the term $\epsilon^5 \Pi_0 u_3$ is not a corrector of the term $\epsilon^4 u_2$ in the expansion (3.9). Indeed one has that $\|\epsilon^4 u_2\|_2 = o(\epsilon^3)$ and at the most $\|\epsilon^5 u_3\|_2 = o(\epsilon^3)$ for large times of order $O(\frac{1}{\epsilon^2})$.

It is thereafter clear that we cannot push the expansion any further as it does not provide us with terms that improve the accuracy. Nevertheless, with some manipulations in the previous expansion, we derive in the next section, coupled KdV type systems for which we obtain a better error estimate.

4. The coupled system: derivation and convergence

4.1. Derivation of the system and statement of the result

The way we derive the coupled system of KdV type relies on the following remark. The convergence result in the previous section shows that the error between the approximate solution and the exact solution of (1.1) is $o(1)$ rather than $O(\epsilon)$ as one could expect. This is mainly due to the fact that when we constructed u_2^ϵ, the contribution of the coupled nonlinear terms in (2.28) yields a sub-square root growth in time. In order to avoid this fact, one can impose to conserve all the nonlinear terms in the equations satisfied by $\Pi_1(0)u_0$ and $\Pi_2(0)u_0$ in the previous analysis, which gives

$$\begin{cases} \partial_{t_1}\Pi_1(0)u_0 + \Big(\dfrac{1}{2\lambda'(0)}\Pi_1(0)AQ_0A\Pi_2(0)AQ_0A\Pi_1(0) \\ -\Pi_1(0)AQ_0AQ_0A\Pi_1(0) - \lambda'(0)\Pi_1(0)AQ_0^2A\Pi_1(0)\Big)\,\partial_x^3\Pi_1(0)u_0 = \Pi_1(0)B(\Pi_0(0)u_0)\partial_x\Pi_0(0)u_0 \\ \partial_{t_1}\Pi_2(0)u_0 + \Big(-\dfrac{1}{2\lambda'(0)}\Pi_2(0)AQ_0A\Pi_1(0)AQ_0A\Pi_2(0) - \Pi_2(0)AQ_0AQ_0A\Pi_2(0) \\ +\lambda'(0)\Pi_2(0)AQ_0^2A\Pi_2(0)\Big)\,\partial_x^3\Pi_2(0)u_0 = \Pi_2(0)B(\Pi_0(0)u_0)\partial_x\Pi_0(0)u_0. \end{cases} \tag{4.1}$$

Thanks to the main Lemma 2.3, the above system reduce to

$$\begin{cases} \partial_{t_1}\Pi_1(0)u_0 + \dfrac{\lambda'''(0)}{6}\partial_x^3\Pi_1(0)u_0 = \Pi_1(0)B(\Pi_0u_0)\partial_x\Pi_0u_0 \\[2ex] \partial_{t_1}\Pi_2(0)u_0 - \dfrac{\lambda'''(0)}{6}\partial_x^3\Pi_2(0)u_0 = \Pi_2(0)B(\Pi_0u_0)\partial_x\Pi_0u_0. \end{cases} \tag{4.2}$$

Then u_2^ϵ is given by

$$\begin{aligned} T_1(\partial_t,\partial_x)\Pi_1(0)u_2^\epsilon &= -\Pi_1(0)AQ_0AQ_0A\Pi_2(0)\partial_x^3u_0 - \lambda'(0)\Pi_1(0)AQ_0^2A\Pi_2(0)\partial_x^3u_0 \\ T_2(\partial_t,\partial_x)\Pi_2(0)u_2^\epsilon &= -\Pi_2(0)AQ_0AQ_0A\Pi_1(0)\partial_x^3u_0 - \lambda'(0)\Pi_2(0)AQ_0^2A\Pi_1(0)\partial_x^3u_0. \end{aligned} \tag{4.3}$$

We have to keep in mind that $\Pi_1(0)u_0$ and $\Pi_2(0)u_0$ have also to satisfy the equations of transport (2.12). Obviously this last set of equations (2.12) **is not** compatible with (4.2).

In order to overcome this difficulty, the crucial point is to modify the ansatz (1.2): we do not consider any more functions depending on two scales in time but only functions under the form

$$\mathcal{U}^\epsilon(t,x) = \epsilon^2[u_{01}^\epsilon(x,t) + u_{02}^\epsilon(x,t)] + \epsilon^3u_1^\epsilon(t,x) + \epsilon^4u_2^\epsilon(t,x) + \epsilon^5u_3^\epsilon(t,x). \tag{4.4}$$

We impose that $(u^\epsilon_{01}, u^\epsilon_{02})$ satisfies

$$\begin{cases} \partial_t u^\epsilon_{01} - \lambda'(0)\partial_x u^\epsilon_{01} + \epsilon^2 \left[\dfrac{\lambda'''(0)}{6}\partial_x^3 u^\epsilon_{01} - \Pi_1(0)B(u^\epsilon_{01}+u^\epsilon_{02})\partial_x(u^\epsilon_{01}+u^\epsilon_{02})\right] = 0 \\ \partial_t u^\epsilon_{02} + \lambda'(0)\partial_x u^\epsilon_{02} - \epsilon^2 \left[\dfrac{\lambda'''(0)}{6}\partial_x^3 u^\epsilon_{02} + \Pi_2(0)B(u^\epsilon_{01}+u^\epsilon_{02})\partial_x(u^\epsilon_{01}+u^\epsilon_{02})\right] = 0. \end{cases} \tag{4.5}$$

Again this system is not compatible with the set of transport equations (2.12) which is satisfied only at the order $O(\epsilon^2)$

$$\begin{cases} (\partial_t - \lambda'(0)\partial_x)u^\epsilon_{01} = O(\epsilon^2) \\ (\partial_t + \lambda'(0)\partial_x)u^\epsilon_{02} = O(\epsilon^2). \end{cases} \tag{4.6}$$

Remark 4.1. If we look at the system (4.5) as non homogeneous linear system, we have that u_{01} and u_{02} remain polarized with respect to $\Pi_1(0)$ and $\Pi_2(0)$ as long as they do respect this polarisation condition at $t = 0$. This is easily deduced from the presence of $\Pi_1(0)$ and $\Pi_2(0)$ in front of the non linear terms.

We still define u^ϵ_1 by

$$\begin{cases} \Pi_1(0)u^\epsilon_1(x,t) = \dfrac{i}{2\lambda'(0)}\Pi_1(0)AQ_0A\Pi_2(0)\partial_x u^\epsilon_{02} \\ \Pi_2(0)u^\epsilon_1(x,t) = -\dfrac{i}{2\lambda'(0)}\Pi_2(0)AQ_0A\Pi_1(0)\partial_x u^\epsilon_{01} \end{cases} \tag{4.7}$$

and for the remaining part $(I - \Pi_0)u^\epsilon_1$ we maintain the second equation in (2.3). For u^ϵ_2 we set

$$\begin{cases} T_1(\partial_t, \partial_x)\Pi_1(0)u^\epsilon_2 = -\Pi_1(0)AQ_0(\partial_t + A\partial_x)Q_0A\Pi_2(0)\partial_x^2 u^\epsilon_{02} \\ T_2(\partial_t, \partial_x)\Pi_2(0)u^\epsilon_2 = -\Pi_2(0)AQ_0(\partial_t + A\partial_x)Q_0A\Pi_1(0)\partial_x^2 u^\epsilon_{01} \end{cases} \tag{4.8}$$

and again for the remaining part $(I - \Pi_0)u^\epsilon_2$, we maintain the solvability condition in (2.5). To finish our set of conditions for our ansatz, we set $\Pi_0 u^\epsilon_3 = 0$ and the remaining part differs from (2.7) as we have eliminated in our ansatz the variable t_1, as is

$$(I - \Pi_0)u^\epsilon_3 = iQ_0L_1(\partial_t, \partial_x)u^\epsilon_2 - iQ_0B(u^\epsilon_0)\partial_x u^\epsilon_0. \tag{4.9}$$

Our result reads as follows:

Theorem 4.1 (Coupled system)**.** *Let $s > \frac{3}{2}$ and f in H^σ (σ large enough) be such that $\Pi_0 f = f$. Under Assumption 2.1, there exists $T_1 > 0$ and a unique solution $u^\epsilon(x,t)$ of (3.1) bounded in $L^\infty([0, \frac{T_1}{\epsilon^2}]; H^s)$ as well as $T_2 > 0$ such a unique couple $(u^\epsilon_{01}(x,t), u^\epsilon_{02}(x,t))$ bounded (with respect to ϵ) in $L^\infty([0, \frac{T_2}{\epsilon^2}]; H^s)$ solution of*

$$\begin{cases} \partial_t u^\epsilon_{01} - \lambda'(0)\partial_x u^\epsilon_{01} + \epsilon^2 \dfrac{\lambda'''(0)}{6}\partial_x^3 u^\epsilon_{01} = \epsilon^2 \Pi_1(0)B(u^\epsilon_{01}+u^\epsilon_{02})\partial_x(u^\epsilon_{01}+u^\epsilon_{02}) \\ \partial_t u^\epsilon_{02} + \lambda'(0)\partial_x u^\epsilon_{02} - \epsilon^2 \dfrac{\lambda'''(0)}{6}\partial_x^3 u^\epsilon_{02} = \epsilon^2 \Pi_2(0)B(u^\epsilon_{01}+u^\epsilon_{02})\partial_x(u^\epsilon_{01}+u^\epsilon_{02}) \end{cases} \tag{4.10}$$

with $u^\epsilon_{01}(x,0) = \Pi_1(0)f$ and $u^\epsilon_{02}(x,0) = \Pi_2(0)f$. Moreover, there exists $T_0 > 0$ ($T_0 \le \min(T_1, T_2)$) such that

$$\left\| \frac{u^\epsilon(x,t)}{\epsilon^2} - [u^\epsilon_{01}(x,t) + u^\epsilon_{02}(x,t)] \right\|_{L^\infty([0,\frac{T_0}{\epsilon^2}];H^s)} = O(\epsilon).$$

The strategy for proving this theorem is the same that for the previous one. However, the proofs are slightly different.

4.2. **Properties of the approximate solution**

We have a local existence theorem for this coupled system (see [3]) that can be viewed as a dispersive perturbation of a symmetric hyperbolic system. The solution is defined on $[0, \frac{T}{\epsilon^2}]$ thanks to the presence of ϵ^2 in front of the nonlinear terms (as in Prop. 3.1). Therefore all the terms of the ansatz are well defined and u_{01}^ϵ and u_{02}^ϵ are bounded in $L^\infty([0, \frac{T}{\epsilon^2}]; H^s)$. The crucial point is now to prove that $u_1^\epsilon, u_2^\epsilon$ and u_3^ϵ are bounded.

Furthermore, as we have remarked in Remark 4.1, u_{01}^ϵ and u_{02}^ϵ remain polarized respectively to $\Pi_1(0)$ and $\Pi_2(0)$ for all times as it is the case at $t = 0$.

4.3. **Properties of the corrector**

u_0^ϵ and u_1^ϵ are indeed bounded as in the previous proof. For u_2^ϵ we improve the previous results and those displayed in [14,15] in similar cases. We prove that u_2^ϵ is bounded in time on $[0, \frac{T}{\epsilon^2}]$. Let us recall the equations defining u_2^ϵ

$$\begin{cases} T_1(\partial_t, \partial_x)\Pi_1(0)u_2^\epsilon = -\Pi_1(0)AQ_0(\partial_t + A\partial_x)Q_0A\Pi_2(0)\partial_x^2 u_{02}^\epsilon \\ T_2(\partial_t, \partial_x)\Pi_2(0)u_2^\epsilon = -\Pi_2(0)AQ_0(\partial_t + A\partial_x)Q_0A\Pi_1(0)\partial_x^2 u_{01}^\epsilon. \end{cases} \tag{4.11}$$

Recall that u_{01}^ϵ and u_{02}^ϵ satisfy

$$\begin{cases} (\partial_t - \lambda'(0)\partial_x)u_{01}^\epsilon = O(\epsilon^2) \\ (\partial_t + \lambda'(0)\partial_x)u_{02}^\epsilon = O(\epsilon^2). \end{cases} \tag{4.12}$$

We prove the following proposition.

Proposition 4.1. *u_2^ϵ is bounded independently of ϵ in $L^\infty([0, \frac{T}{\epsilon^2}]; H^s)$.*

Proof of Proposition 4.1. We prove the result for the first component $\Pi_1(0)u_2^\epsilon$ of u_2^ϵ. The proof is similar for $\Pi_2(0)u_2^\epsilon$. Let us rewrite the first equation in (4.11) in a simplified way:

$$T_1(\partial_t, \partial_x)\Pi_1(0)u_2^\epsilon = M\partial_x^2 u_{02}^\epsilon$$

where M is a $N \times N$ matrix and u_{02}^ϵ lies in $L^\infty([0, \frac{T}{\epsilon^2}]; H^s)$. Then, one has

$$\widehat{\Pi_1(0)u_2^\epsilon}(\xi, t) = \mathrm{e}^{i\lambda'(0)\xi t} M \underbrace{\widehat{u_2^\epsilon}(\xi, 0)}_{=0} + \xi^2 \mathrm{e}^{i\lambda'(0)t\xi} \underbrace{\int_0^t \mathrm{e}^{-i\lambda'(0)s\xi} M \widehat{u_{02}^\epsilon}(\xi, s)\,\mathrm{d}s}_{\mathbf{A}}.$$

We integrate by parts $\mathbf{A}$, which gives

$$\mathbf{A} = \left[-\frac{1}{i\lambda'(0)\xi}\mathrm{e}^{-i\lambda'(0)s\xi} M\widehat{u_{02}^\epsilon}\right]_0^t + \int_0^t \frac{1}{i\lambda'(0)\xi}\mathrm{e}^{-i\lambda'(0)s\xi} M\partial_s\widehat{u_{02}^\epsilon}\,\mathrm{d}s$$

and now from (4.12), we have that $\partial_s \widehat{u_{02}^\epsilon} = -i\lambda'(0)\xi\widehat{u_{02}^\epsilon} + O(\epsilon^2)$, which gives

$$2\widehat{\Pi_1(0)u_2^\epsilon}(\xi, t) = M\xi\left(-\frac{\widehat{u_{02}^\epsilon}}{i\lambda'(0)} + \frac{\mathrm{e}^{i\lambda'(0)t\xi}}{i\lambda'(0)}\widehat{u_{02}^\epsilon}\right) + O(1)$$

for times of order $O(\frac{1}{\epsilon^2})$, which gives that $\Pi_1(0)u_2^\epsilon$ is bounded in H^s on $[0, \frac{T}{\epsilon^2}]$ and since $(I-\Pi_0)u_2^\epsilon$ is bounded from (2.5), we conclude the proof. □

The fact that u_3^ϵ is also bounded is easily deduced from (4.9) and the fact that u_2^ϵ is bounded. We therefore have proved:

Proposition 4.2. *The corrector term $\epsilon^3 u_1^\epsilon + \epsilon^4 u_2^\epsilon + \epsilon^5 u_3^\epsilon$ is indeed a corrector in (4.4) and one has that*

$$\|\epsilon^3 u_1^\epsilon + \epsilon^4 u_2^\epsilon + \epsilon^5 u_3^\epsilon\|_{L^\infty([0,\frac{T}{\epsilon^2}];H^s)} = O(\epsilon^3).$$

4.4. **Estimate for the residue and end of the proof**

As earlier, we start by estimating the residue. It is more complicated than in the previous proof since the conditions we have chosen on the terms of the ansatz (3.4) do not imply $r_i^\epsilon = 0$ for $i = 1, 2, 3, 4$. For the moment, we can only write the residue as

$$\mathrm{Res}(x, t, t_1, \epsilon) = \sum_{i=1}^{10} \epsilon^i r_i^\epsilon$$

and perform the asymptotic expansion with respect to ϵ. Note here that the ansatz is expressed only in the variable t and x and therefore the values of r_i^ϵ displayed at the beginning of this section do not hold anymore, in particular, the variable t_1 is not used anymore. Therefore, one has that

$$r_1^\epsilon = Eu_0^\epsilon \quad \text{with} \quad u_0^\epsilon = u_{01}^\epsilon + u_{02}^\epsilon; \qquad r_2^\epsilon = \partial_t u_0^\epsilon + A(\partial_x)u_0^\epsilon + Eu_1^\epsilon$$

$$r_3^\epsilon = \partial_t u_1^\epsilon + A(\partial_x)u_1^\epsilon + Eu_3^\epsilon; \qquad r_4^\epsilon = \partial_t u_2^\epsilon + A(\partial_x)u_2^\epsilon + Eu_3^\epsilon - B(u_0^\epsilon)\partial_x u_0^\epsilon.$$

From the conditions imposed on each term of the ansatz at the end of Section 1.2 for the coupled system, we deduce, as part of the ansatz is constructed for that matter, that

- $r_1^\epsilon = 0$ since $u_0^\epsilon = \Pi_0 u_0^\epsilon$;
- $(I - \Pi_0)r_2^\epsilon = 0$ from the second equation in (2.3);
- $\Pi_0 r_3^\epsilon = 0$ from the expressions of $\Pi_0 u_1^\epsilon$ in (4.7) and $(I-\Pi_0)r_3^\epsilon = 0$ from the expression of $(I-\Pi_0)u_2^\epsilon$ in (2.5);
- $(I - \Pi_0)r_4^\epsilon = 0$ from the expression of $(I - \Pi_0)u_3^\epsilon$ in (4.9).

Up to the order 5, we are *a priori*, only left with $\epsilon^2\Pi_0 r_2^\epsilon + \epsilon^4 \Pi_0 r_4^\epsilon$ which reduces to, for its first component,

$$\Pi_1(0)r_2^\epsilon + \epsilon^2\Pi_1(0)r_4^\epsilon = \partial_t u_{01}^\epsilon - \lambda'(0)\partial_x u_{01}^\epsilon + \epsilon^2\Big(T_1(\partial_t, \partial_x)\Pi_1(0)u_2^\epsilon + \frac{1}{2\lambda'(0)}\Pi_1(0)AQ_0A\Pi_2(0)AQ_0A\Pi_1(0)\partial_x^3 u_{01}^\epsilon$$
$$- \Pi_1(0)AQ_0(\partial_t + A\partial_x)Q_0A(\Pi_1(0) + \Pi_2(0))\partial_x^2 u_0^\epsilon - \Pi_1(0)B(u_{01}^\epsilon + u_{02}^\epsilon)\partial_x(u_{01}^\epsilon + u_{02}^\epsilon)\Big).$$

Now from the system (4.5) verified by u_{01}^ϵ and u_{02}^ϵ, and the main algebraic lemma 2.3, one has that the previous equation reduces to, using also the condition verified by u_2^ϵ, namely (4.8),

$$\Pi_1(0)r_2^\epsilon + \epsilon^2\Pi_1(0)r_4^\epsilon = \epsilon^2(\partial_t - \lambda'(0)\partial_x)\Pi_1(0)AQ_0^2A\Pi_1(0)\partial_x^2 u_{01}^\epsilon$$

and from (4.12), the term $\epsilon^2\Pi_0 r_2^\epsilon + \epsilon^4\Pi_0 r_4^\epsilon$ is nothing else but a residue at the order 6. Then it is obvious to deduce the following proposition since all the terms u_1, u_2^ϵ and u_3 are bounded.

Proposition 4.3. *We have that* $\|\mathrm{Res}\|_{L^\infty([0,\frac{T}{\epsilon^2}];L^2)} = O(\epsilon^5)$.

Thus our approximate solution solves (1.1) such as

$$\partial_t \mathcal{U}^\epsilon + A(\partial_x)\mathcal{U}^\epsilon + \frac{E\mathcal{U}^\epsilon}{\epsilon} - B(\mathcal{U}^\epsilon)\partial_x \mathcal{U}^\epsilon = O(\epsilon^5). \tag{4.13}$$

Following then the argument laid out earlier, we obtain in the same manner, the following estimation on the norm H^s of the difference $\tilde{u}$ between the exact solution and our approximate solution, that reads

$$\|\tilde{u}\|_s^2 \leq (\mathrm{e}^{c\epsilon^2 t} - 1)O(\epsilon^3) \quad \text{for} \quad t \leq \frac{T}{\epsilon^2}$$

which finishes the proof of Theorem 4.1.

5. Comparison between both models

The two convergence theorems presented in this paper rise a few questions. As the error estimate between the approximate solution and the exact solution is improved in the second theorem, one ought to think that the second approximation is more accurate. It is in fact not clear as we do not exhibit a lower bound estimate of the error between the exact solution and the approximate solution in both cases.

Nevertheless, we want in this section to establish a link between the two models that partially enlight their comparison. Indeed, our purpose here is to show that, in large time scales, the solution of the coupled system converges to the solution of an uncoupled pair of KdV type equations.

We rewrite both systems (4.10) and (3.2)–(3.3) in the variable (t, x). The relevant small parameter reads as ϵ (we replace ϵ^2 by ϵ in this section). This gives:

$$\begin{cases} \partial_t u + \partial_x u + \epsilon\partial_x^3 u + \epsilon\partial_x\partial_u P(u,v) = 0 \\ \partial_t v - \partial_x v - \epsilon\partial_x^3 v + \epsilon\partial_x\partial_v P(u,v) = 0 \end{cases} \tag{5.1}$$

for the coupled system and

$$\begin{cases} \partial_t u + \partial_x u + \epsilon\partial_x^3 u + \epsilon\partial_x\partial_u P(u,0) = 0 \\ \partial_t v - \partial_x v - \epsilon\partial_x^3 v + \epsilon\partial_x\partial_v P(0,v) = 0 \end{cases} \tag{5.2}$$

for the uncoupled system, where $P(u, v)$ is an homogeneous polynomial of degree 3.

We intend to prove in this section the following theorem:

Theorem 5.1. *Let $s > \frac{3}{2}$. There exists $T_{\max} > 0$ (independent of ϵ) such that there exists $(u^\epsilon, v^\epsilon) \in C([0, \frac{T_{\max}}{\epsilon}], H^s)$ solution of (5.1) and $(\mathcal{U}^\epsilon, \mathcal{V}^\epsilon) \in C([0, \frac{T_{\max}}{\epsilon}], H^s)$ solution of (5.2) with $\mathcal{U}^\epsilon(x, 0) = u(x, 0)$ and $\mathcal{V}^\epsilon(x, 0) = v(x, 0)$. Moreover*

$$\left.\begin{array}{l} \|\mathcal{U}^\epsilon - u^\epsilon\|_{L^\infty([0,\frac{T_{\max}}{\epsilon}];H^s)} \longrightarrow 0 \\[2ex] \|\mathcal{V}^\epsilon - v^\epsilon\|_{L^\infty([0,\frac{T_{\max}}{\epsilon}];H^s)} \longrightarrow 0 \end{array}\right| \; \text{as } \epsilon \to 0.$$

Proof. As it is proved in Proposition 3.1, we have a local existence theorem for (u^ϵ, v^ϵ) solution of (5.1), valid for times of order $O(\frac{T}{\epsilon})$. We remind the reader that the proof of this proposition relies on the fact that (5.1) is a symmetric hyperbolic system with regards to the nonlinear terms.

The local existence for $(\mathcal{U}^\epsilon, \mathcal{V}^\epsilon)$ is obvious from the global existence theorem available for the Korteweg-de Vries equation.

The idea to prove the convergence result is to seek approximate solution of the system (5.1) as an asymptotic expansion with respect to ϵ. This approximation reads as the following ansatz:

$$\begin{aligned} \mathcal{U}^\epsilon(x,t) &= u_0(x,t,\epsilon t) + \epsilon u_1(x,t,\epsilon t) \\ \mathcal{V}^\epsilon(xt) &= v_0(x,t,\epsilon t) + \epsilon v_1(x,t,\epsilon t) \end{aligned} \tag{5.3}$$

and we denote by $\tau = \epsilon t$. These expansions are *a priori* valid for times of order $O(\frac{1}{\epsilon})$ which is consistence with the existence in time of the exact solution (u^ϵ, v^ϵ) of (5.1). u_0 and v_0 correspond to the leading order terms in the expansion where u_1 and v_1 are meant to be correctors. The same formal expansion as in the previous section leads to a proof of Theorem 5.1.

We introduce as in [15, 20] a sublinear growth condition that ought to be satisfied by (u_1,v_1) in order to be correctors. This sublinear growth condition is weaker than the sub-squareroot condition introduced earlier (1.3) but is enough for this proof.

Sublinear growth condition

For w sufficiently smooth

$$\lim_{t\to\infty} \frac{1}{t} \|\partial^\alpha_{t,x,\tau} w(x,t,\tau)\|_2 = 0 \quad \text{for all } \alpha \in \mathbb{N}^3. \tag{5.4}$$

Plugging our ansatz (5.3) in (5.2) gives

$$\begin{cases} (\partial_x + \partial_t)\mathcal{U}^\epsilon + \epsilon\left(\partial_x^3 \mathcal{U}^\epsilon + \partial_x \partial_u P(\mathcal{U}^\epsilon, \mathcal{V}^\epsilon)\right) = \displaystyle\sum_{i=0}^{3} \epsilon^j r_j \\ (\partial_x - \partial_t)\mathcal{V}^\epsilon - \epsilon\left(\partial_x^3 \mathcal{V}^\epsilon - \partial_x \partial_v P(\mathcal{U}^\epsilon, \mathcal{V}^\epsilon)\right) = \displaystyle\sum_{i=0}^{3} \epsilon^j s_j. \end{cases} \tag{5.5}$$

We solve simultaneously ($r_i = 0$,$s_i = 0$) for $i = 0, 1$, which gives the following set of necessary equations

$$\begin{cases} (\partial_t + \partial_x)u_0 = 0 \\ (\partial_t - \partial_x)v_0 = 0 \end{cases} \quad \text{and} \quad \begin{cases} \partial_\tau u_0 + \partial_t u_1 + \partial_x u_1 + \partial_x^3 u_0 + \partial_x \partial_u P(u_0, v_0) = 0 \\ \partial_\tau v_0 + \partial_t v_1 - \partial_x v_1 - \partial_x^3 v_0 + \partial_x \partial_v P(u_0, v_0) = 0. \end{cases}$$

For $(\mathcal{U}^\epsilon, \mathcal{V}^\epsilon)$ to be an approximate solution of (5.2), the two above systems constitute a set of necessary solvability conditions.

In an analog manner as in the second section, we denote by T_+ and T_- the two transport operators $T_+(\partial_t, \partial_x) = \partial_t - \partial_x$ and $T_-(\partial_t, \partial_x) = \partial_t + \partial_x$. We introduce the corresponding average operators G_{T_+} and G_{T_-} as defined in Section 2.3. We apply these operators to the long time profile equations. Note that Property (iii) (Prop. 2.3) holds with a sublinear growth condition (5.4). Then applying G_{T_-} for example gives

$$\begin{aligned} G_{T_-}(\partial_t u_1 + \partial_x u_1) &= G_{T_-}(T_- u_1) = 0 \\ G_{T_-}(\partial_\tau u_0 + \partial_x^3 u_0) &= \partial_\tau u_0 + \partial_x^3 u_0 \\ G_{T_-}(\partial_x \partial_u P(u_0, v_0)) &= \partial_x \partial_u P(u_0, 0). \end{aligned}$$

Note that $P(u_0, 0)$ gathers the only terms in $P(u_0, v_0)$ that are polarized with respect to $T_-(\partial_t, \partial_x)$ and that are left unchanged by the action of G_{T_-}. Analog actions of G_{T+} on the other equation hold likewise.

We obtain the new solvability conditions for u_0 and v_0.

$$\begin{cases} (\partial_t + \partial_x)u_0 = 0 \\ \\ (\partial_t - \partial_x)v_0 = 0 \end{cases} \quad \text{and} \quad \begin{cases} \partial_\tau u_0 + \partial_x^3 u_0 + \partial_x \partial_u P(u_0, 0) = 0 \\ \\ \partial_\tau v_0 - \partial_x^3 v_0 + \partial_x \partial_v P(u_0, 0) = 0 \end{cases}$$

and for the correctors, one has

$$\begin{cases} (\partial_t + \partial_x)u_1 = \partial_x \partial_u P(u_0, 0) - \partial_x \partial_u P(u_0, v_0) \\ \\ (\partial_t - \partial_x)v_1 = \partial_x \partial_v P(0, v_0) - \partial_x \partial_v P(u_0, v_0). \end{cases} \tag{5.6}$$

From Proposition 3.3, we easily deduce that u_1 and v_1 verify a sublinear growth condition (5.4).

At this point u_0 and v_0 are completely determined by the above solvability conditions and therefore lie in $C(\mathbb{R}, H^s)$ together with the correctors whose growth is correctly controlled. Thereafter $\mathcal{U}^\epsilon$ and $\mathcal{V}^\epsilon$ lie in $C([0, \frac{T}{\epsilon}]; H^s)$ as u^ϵ and v^ϵ.

Our proof of Theorem 5.1 ends with a stability result for which we ought to estimate the residue in (5.5). The latter reads as $\left| \begin{matrix} \epsilon^2 r_2 + \epsilon^3 r_3 \\ \epsilon^2 s_2 + \epsilon^3 s_3 \end{matrix} \right.$, and one has from the sublinear growth condition verified by u_1 and v_1 along with the boundedness of u_0 and v_0 in $C([0, \frac{T}{\epsilon}]; H^s)$ that

$$\begin{cases} (\partial_x + \partial_t)\mathcal{U}^\epsilon + \epsilon \left(\partial_x^3 \mathcal{U}^\epsilon + \partial_x \partial_u P(\mathcal{U}^\epsilon, \mathcal{V}^\epsilon) \right) = o(\epsilon) \\ \\ (\partial_x - \partial_t)\mathcal{V}^\epsilon - \epsilon \left(\partial_x^3 \mathcal{V}^\epsilon - \partial_x \partial_v P(\mathcal{U}^\epsilon, \mathcal{V}^\epsilon) \right) = o(\epsilon). \end{cases} \tag{5.7}$$

It is afterwords easy as in Section 3.3 to finish the proof by estimating $\|\mathcal{U}^\epsilon - u^\epsilon\|_s$ and $\|\mathcal{V}^\epsilon - v^\epsilon\|_s$ and thus conclude. □

Remark 5.1. One must be careful if we try to interpret this result. It is indeed a decoupling result that enables us to compare the two models but we have to keep in mind that the H^s norms $\|\mathcal{U}^\epsilon - u^\epsilon\|_s$ and $\|\mathcal{V}^\epsilon - v^\epsilon\|_s$ are *only* of order $o(1)$ and not $O(\epsilon)$. This means that for relatively large ϵ (physically $\epsilon = 10^{-1}$ is relevant in the water waves context), the discrepancy between the two models can be large. For instance, in the case of interactions of solitary waves, the interaction is definitely nonlinear and the coupled system is a better model as it is clear in the simulations conducted in [4, 6].

6. Examples

In this section, we present the derivation of KdV coupled systems in two physical cases. We recall that our convergence results do not apply in these cases.

6.1. **The Euler-Poisson equations**

In this section, we investigate the Euler-Poisson equations that occur in the context of ion acoustic waves. Consider a plasma of electrons and ions, where the inertia of the electrons can be neglected unlike the electrostatic effects of the electron charges. The electrons are modelized as a gas. Expressing the Boltzmann equation of state along with the conservation of mass, with ϕ being the electrostatic potential, η the density of electrons and v their velocity, one obtains the simplified dimensionless equations, namely the Euler Poisson system, that reads as

$$\begin{cases} \eta_t + (\eta v)_x = 0 \\ v_t + v v_x = -\phi_x \\ \phi_{xx} = \mathrm{e}^\phi - \eta. \end{cases} \tag{6.1}$$

We refer to Dodd [13] for a detailed derivation of (6.1).

We will apply the second section, in this particular physical context, that is starting from (6.1), we give a derivation of KdV type systems as an asymptotical equation describing (6.1) for long waves and small amplitudes.

If one linearizes this system, we obtain describing the potential, the following equation in ϕ

$$\partial_x^2 \partial_t^2 \phi + \partial_x^2 \phi - \partial_t^2 \phi = 0$$

which gives the relation of dispersion $w^2 = k^2(1+k^2)^{-1}$ whose shape near 0 (long wave approximation) meets the requirements of the preceding general study as in figure 2.1. As we set up our ansatz, we derive necessary conditions on the approximate solution and obtain KdV systems. The system (6.1) if obviously not of the form (1.1). However, we will show that the second section applies in this case. We first make the following remark.

Remark 6.1. If $u^\epsilon(t,x)$ is a solution of (1.1) then $v^\epsilon(t,x) = u^\epsilon(\epsilon t, \epsilon x)$ is a solution to

$$\partial_t v^\epsilon + A\partial_x v^\epsilon + E v^\epsilon = B(v^\epsilon)\partial_x v^\epsilon. \tag{6.2}$$

Then the ansatz that has to be used for (6.2) is

$$\mathcal{U}^\epsilon = \sum_{j=0}^{3} \epsilon^{j+2} u_j(\epsilon x, \epsilon t, \epsilon^3 t) \tag{6.3}$$

in order to pursue the same analysis as in Section 2.

System (6.1) can be seen as belonging to a class of pseudo-differential systems that generalize (6.2). We therefore use the same ansatz.

The ansatz

We seek approximate solutions for (6.1) of the form

$$\begin{cases} \eta^\epsilon = 1 + \epsilon^2 \eta_0(\epsilon x, \epsilon t, \epsilon^3 t) + \epsilon^3 \eta_1 + \epsilon^4 \eta_2 \\ \phi^\epsilon = \epsilon^2 \phi_0(\epsilon x, \epsilon t, \epsilon^3 t) + \epsilon^3 \phi_1 + \epsilon^4 \phi_2 \\ v^\epsilon = \epsilon^2 v_0(\epsilon x, \epsilon t, \epsilon^3 t) + \epsilon^3 v_1 + \epsilon^4 v_2. \end{cases} \tag{6.4}$$

Plugging the ansatz (6.4) into (6.1), one obtains the following expansion with respect to ϵ,

$$\begin{cases} \eta_t^\epsilon + (\eta^\epsilon v^\epsilon)_x = \displaystyle\sum_{i=0}^{2} r_i \epsilon^{i+3} + O(\epsilon^6) \\ v_t^\epsilon + \phi_x^\epsilon + v^\epsilon v_x^\epsilon = \displaystyle\sum_{i=0}^{2} s_i \epsilon^{i+3} + O(\epsilon^6) \\ \phi_{xx}^\epsilon - \mathrm{e}^{\phi^\epsilon} + \eta^\epsilon = \displaystyle\sum_{i=0}^{2} q_i \epsilon^{i+2} + O(\epsilon^5). \end{cases}$$

Since we intend to solve this system up to the order ϵ^5, we obtain the following set of equations

- for the first equation,

$$\begin{cases} \partial_{t_1}\eta_0 + \partial_x v_0 = 0 & (r_0 = 0) \\ \partial_{t_1}\eta_1 + \partial_x v_1 = 0 & (r_1 = 0) \\ \partial_{t_2}\eta_0 + \partial_{t_1}\eta_2 + \partial_x v_2 + \partial_x(v_0\eta_0) = 0 & (r_2 = 0) \end{cases} \tag{6.5}$$

- for the second equation,

$$\begin{cases} \partial_{t_1}v_0 + \partial_x\phi_0 = 0 & (s_0 = 0) \\ \partial_{t_1}v_1 + \partial_x\varphi_1 = 0 & (s_1 = 0) \\ \partial_{t_2}v_0 + \partial_{t_1}v_2 + \partial_x\phi_2 + v_0\partial_x v_0 = 0 & (s_2 = 0) \end{cases} \tag{6.6}$$

- and for the last equation, we have

$$\begin{cases} \eta_0 - \phi_0 = 0 & (q_0 = 0) \\ \eta_1 - \varphi_1 = 0 & (q_1 = 0) \\ \partial_x^2\phi_0 - \phi_2 + \eta_2 - \dfrac{(\phi_0)^2}{2} = 0 & (q_2 = 0). \end{cases} \tag{6.7}$$

Since from (6.7), $\eta_0 = \phi_0$, both these variables solve a classical wave equation $(\partial_t^2 - \partial_x^2)u = 0$ and read as the sum of two functions moving at the speed ± 1. Let us then define the two transport operators as previously such as $T_1(\partial_t, \partial_x) = \partial_{t_1} - \partial_x$ and $T_2(\partial_t, \partial_x) = \partial_{t_1} + \partial_x$ and the associated average operators G_{T_1} and G_{T_2} that are in this context nothing else but the projectors on the kernels of respectively $\Pi_1(0)$ and $\Pi_2(0)$ and that we will denote from now on, P_1 and P_2. With these notations, we deduce that η_0 and v_0 read as follows

$$\eta_0 = P_1\eta_0 + P_2\eta_0$$

and

$$v_0 = P_1 v_0 + P_2 v_0$$

and from the wave equation in η_0 and v_0, one has that $P_1\eta_0 = -P_1 v_0$ and $P_2\eta_0 = P_2 v_0$. Hence applying both projectors P_1 and P_2 on every equations will lead to the desired results. Let us point out beforehand that these two average projectors can be applied on each terms of the equations. For η_0 and v_0, it is clear since they are transported by the scalar operators T_1 and T_2 and for the other terms in the expansions indexed by 2 (the ones indexed by 1 do not play any role - see below -), we assume that their growth in time is controlled and is at least sub-linear and Property (iii) of the average operators allows us to conclude. Naturally, this hypothesis needs to be verified once we have derived necessary conditions on the corrector terms.

Now the equations satisfied by η_1, v_1 and φ_1 are the same than those satisfied by η_0, v_0 and φ_0 and are solved in the same way. Moreover, since the unknowns η_1, v_1 and φ_1 do not appear in the equations $r_2 = 0, s_2 = 0$ and $q_2 = 0$, we can set them to zero.

In order to obtain the profile equations for η_0, v_0 and φ_0, we start by applying P_2 on the equations (6.5) and (6.6) and look at the evolution of the profile moving in the right direction

$$\partial_{t_2}P_2\eta_0 + (P_2\partial_{t_1}\eta_2 + P_2\partial_x v_2) + 2P_2\eta_0\partial_x P_2\eta_0 = 0 \tag{6.8}$$

and

$$\partial_{t_2} P_2\eta_0 + (P_2\partial_x\phi_2 + P_2\partial_{t_1} v_2) + P_2\eta_0\partial_x P_2\eta_0 - P_2\eta_0\partial_x P_1\eta_0 = 0. \tag{6.9}$$

Now summing (6.8) and (6.9) and differentiate the second equation in (6.7) in order to replace $\partial_x P_2\phi_2$ in the equation, gives

$$\partial_{t_2} P_2\eta_0 + \underbrace{P_2\eta_0\partial_x P_2\eta_0}_{\text{nonlinear term}} - \underbrace{P_2\eta_0\partial_x P_1\eta_0}_{\text{nonlinear coupled term}} + \underbrace{1/2\partial_x^3 P_2\eta_0}_{\text{dispersive term}} + \underbrace{P_2(\partial_{t_1} + \partial_x)v_2}_{\text{corrector term}} + \underbrace{P_2(\partial_{t_1} + \partial_x)\eta_2}_{\text{corrector term}} = 0. \tag{6.10}$$

We obtain in an analog manner the second equation governing the long time evolution of $P_1\eta_0$

$$\partial_{t_2} P_1\eta_0 - \frac{1}{2}\partial_x^3 P_1\eta_0 - \frac{3}{2}P_1\eta_0\partial_x P_1\eta_0 - \frac{1}{2}P_2\eta_2\partial_x P_2\eta_2 - \frac{1}{2}\partial_x(P_1\eta_0 P_2\eta_2) - P_1(\partial_{t_1} - \partial_x)v_2 + P_1(\partial_{t_1} - \partial_x)\eta_2 = 0 \tag{6.11}$$

as we set now the corrector in the equation to be such that

$$\left| \begin{array}{l} P_2(\partial_{t_1} + \partial_x)(v_2 + \eta_2) = 0 \\ \\ P_1(\partial_{t_1} - \partial_x)(\eta_2 - v_2) = 0 \end{array} \right. \tag{6.12}$$

we obtain the following coupled KdV system as an asymptotic limit to our problem that reads for the long time evolution, with $u = P_2\eta_0$ and $v = P_1\eta_0$,

$$\begin{cases} \partial_{t_2} u + \dfrac{1}{2}\partial_x^3 u + u\partial_x u - u\partial_x v = 0 \\ \\ \partial_{t_2} v - \dfrac{1}{2}\partial_x^3 v - \dfrac{3}{2}v\partial_x v - \dfrac{1}{2}u\partial_x u - \partial_x(uv) = 0. \end{cases} \tag{6.13}$$

Since we kept coupled terms, we recall from the previous general discussion that (6.13) is not compatible with the 1-dimensional wave equation solved by η_0 and v_1. Therefore as for the derivation of the coupled system laid out earlier, we consider u and v, back in the variable (x,t) solutions of

$$\begin{cases} \partial_t u + \partial_x u + \epsilon\left[\dfrac{1}{2}\partial_x^3 u + u\partial_x u - u\partial_x v\right] = 0 \\ \\ \partial_t v - \partial_x v + \epsilon\left[-\dfrac{1}{2}\partial_x^3 v - \dfrac{3}{2}v\partial_x v - \dfrac{1}{2}u\partial_x u - \partial_x(uv)\right] = 0. \end{cases} \tag{6.14}$$

Note that we actually obtain a whole class of limit systems, since we can eliminate or add nonlinear term in the equation as long as they can be compensated by the same terms in the correctors with a contribution that must remain bounded in order not to affect the convergence result. For instance we add or subtract terms of the form $v\partial_x v$ in the first equation and terms of the form $u\partial_x u$ in the second equation. Any of these terms thanks to Proposition 3.2, implies a bounded contribution in the correctors and therefore does not affect the convergence. We can for instance set up a combination of such terms in order to obtain a limit system with a symmetric nonlinearity. This operation could very well be baptized as a "symmetrisation" process. The motivation for applying such a process is double: first of all it assures that the limit system has at least an L^2 invariant, which is physically important and secondly that the limit is well posed and has a solution that exists for large time scales of order $O(\frac{1}{\epsilon})$, which is crucial in the scope of a convergence theorem.

In this case, the "symmetrisation" process gives that we need to add $v\partial_x v$ in the first equation and $-\frac{1}{2}u\partial_x u$ in the second and modify consequently the expression for the correctors. We then obtain, for the correctors,

$$\left| \begin{array}{l} P_2(\partial_{t_1} + \partial_x)(v_2 + \eta_2) = -v\partial_x v \\ \\ P_1(\partial_{t_1} - \partial_x)(\eta_2 - v_2) = \dfrac{1}{2}u\partial_x u \end{array} \right. \tag{6.15}$$

which from proposition 3.2 remain bounded. The proposition can be applied here, thanks to Remark 6.1, as we used the proper ansatz for which the previous general theory stands and as u^2 and v^2 are indeed L^2-bounded since u and v lie in the proper H^s (for $s > \frac{1}{2}$). The final limit system for the Euler-Poisson problem read in its vectorial form as

$$\partial_t U + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x U + \epsilon \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x^3 U + \epsilon M(U)\partial_x U = 0$$

where U is now the vector $\left| \begin{array}{l} u \\ v \end{array} \right.$ and $M(U)$ the symmetric matrix

$$M(U) = \begin{pmatrix} -u & v-u \\ v-u & u - \frac{3}{2}v \end{pmatrix}.$$

We do not go further into the analysis of this model as we do not intend to prove in the scope of this paper, a convergence result for this example. This convergence result may be obtained using technics of Cordier-Grenier [10]. We postpone this study for a further work.

6.2. **Water waves**

The Korteweg-de Vries equation was first derived in the context of surface water waves after Russel's observation of a soliton. From Bona-Chen's derivation displayed in [5, 6], one can derive a large class of KdV type systems modeling counter-propagating water waves.

Indeed, starting from the Euler equation for an irrotational and incompressible flow, associated to the appropriate boundary conditions at the bottom and no surface tension at the surface lead to the Laplace equation in the flow domain. Then designating by $\phi(x, y, t)$ the velocity potential where x is the horizontal variable and y the vertical variable, $\eta(x, t)$ being the water elevation lead to the Euler equation with free boundary conditions, that read in its classical dimensionless form as

$$\left\{ \begin{array}{ll} \beta\phi_{xx} + \phi_{yy} = 0 & 0 < y < 1 + \alpha\eta \\ \phi_y = 0 & y = 0 \\ \left. \begin{array}{l} \eta_t + \alpha\phi_x\eta_x - \dfrac{1}{\beta}\phi_y = 0 \\ \eta + \phi_t + \dfrac{1}{2}\alpha\phi_x^2 + \dfrac{1}{2}\dfrac{\alpha}{\beta}\phi_y^2 = 0 \end{array} \right\} & y = 1 + \alpha\eta \end{array} \right. \tag{6.16}$$

where $\alpha = \frac{\text{amplitude}}{\text{depth}}$ and $\beta = \left(\frac{\text{depth}}{\text{wavelength}}\right)^2$ that we suppose to be the small parameters of the system, *e.g.* we place ourselves in the framework of large wavelength with small amplitude. Furthermore, one assumes that $\alpha \sim \beta$. We will not recall in detail their derivation. Let us just say that it relies on an expansion of the potential of velocity with respect to the vertical variable in order to derive the shallow water system. Taking w as the horizontal velocity at a certain water level θ, one obtains a class of systems as it formulated in [6, 7]

$$\left\{ \begin{array}{l} \eta_t + w_x + \alpha(w\eta)_x + \beta\left(\dfrac{\theta^2}{2} - \dfrac{1}{6}\right)(\lambda w_{xxx} - (1-\lambda)\eta_{xxt}) = O(\alpha^2, \beta^2) \\ w_t + \eta_x + \alpha w w_x - \beta\left(\dfrac{1}{2} - \dfrac{\theta^2}{2}\right)(-\mu\eta_{xxx} + (1-\mu)w_{xxt}) = O(\alpha^2, \beta^2). \end{array} \right. \tag{6.17}$$

Depending on the choice of $(\lambda, \mu) \in \mathbb{R}^2$, the system above describes a class of systems that are all equivalent to each other and the crucial point in their derivation holds in the system written at the first order such as

$$\begin{cases} \eta_t = -w_x + O(\alpha, \beta) \\ w_t = -\eta_x + O(\alpha, \beta) \end{cases} \tag{6.18}$$

which means for these authors that a derivative with respect to t, ∂_t can be replaced by a derivative with respect to x, ∂_x as long as w is replaced by η with no loss of precision.

Out of that large class of systems thus defined, two of them stand out as the KdV type system and the BBM type system

$$\text{KdV type} \begin{cases} \eta_t + w_x + \alpha(wn)_x + \dfrac{\beta}{6} w_{xxx} = 0 \\ w_t + \eta_x + \alpha w w_x + \dfrac{\beta}{6} \eta_{xxx} = 0 \end{cases}$$

$$\text{BBM type} \begin{cases} \eta_t + w_x + \alpha(wn)_x - \dfrac{\beta}{6} \eta_{xxt} = 0 \\ w_t + \eta_x + \alpha w w_x - \dfrac{\beta}{6} w_{xxt} = 0. \end{cases}$$

There exists numerous discussions regarding the comparison between these two models especially for the single KdV equation compared to the BBM equation. The most fruitful and detailed one can be found in [2], where the authors explain how the regularized model fits better with regards to the various drawbacks of the KdV equation. However, these two systems nor any of the systems that can be derived from the class described above, are satisfactory from our point of view, as in all cases the nonlinearity is not symmetric unlike in the original system.

In this section, we propose a more satisfactory and "rigorous" derivation of KdV systems in the context of water waves which gives a new class of such equivalent system including symmetric systems of KdV type that do *a priori* hold the same approximation properties. Besides, in our derivation, the small parameter appear to be unique and the arguments used are no different from those used in the general theory displayed in this chapter. Let us rewrite the Euler system with free boundary conditions, with a unique small parameter ϵ,

$$\begin{cases} \epsilon\phi_{xx} + \phi_{yy} = 0 & 0 < y < 1 + \epsilon\eta \\ \phi_y = 0 & y = 0 \\ \left.\begin{array}{l} \eta_t + \epsilon\phi_x\eta_x - \dfrac{1}{\epsilon}\phi_y = 0 \\ \eta + \phi_t + \dfrac{1}{2}\epsilon\phi_x^2 + \dfrac{1}{2}\phi_y^2 = 0 \end{array}\right\} & y = 1 + \epsilon\eta. \end{cases} \tag{6.19}$$

Before setting up our ansatz and plugging it in (6.19), let us expand $\phi(x, y, t)$ with respect to the second variable around $y = 1$ such as

$$\phi(x, y, t)|_{y=1+\epsilon\eta} = \phi(x, y, t)|_{y=1} + \epsilon\eta\partial_y\phi(x, y, t)|_{y=1} + \frac{1}{2}\epsilon^2\eta^2\partial_y^2\phi(x, y, t)|_{y=1} + O(\epsilon^3). \tag{6.20}$$

We set $\varphi(x, t) = \phi(x, 1, t)$ the profile at the undisturbed water level, and from the following system,

$$\begin{cases} \epsilon\phi_{xx} + \phi_{yy} = 0 \quad 0 < y < 1 + \epsilon\eta \\ \phi_y = 0 \quad \text{at} \quad y = 0 \\ \phi(x, 1, t) = \varphi(x, t) \end{cases} \tag{6.21}$$

we solve ϕ in the y variable with respect to the other variables, using Fourier transforms,

$$\widehat{\phi}(\xi, y, t) = \frac{\widehat{\varphi}(\xi, t)}{ch(\sqrt{\epsilon}\xi)} ch(\sqrt{\epsilon}\xi y)$$

and thereafter one has that

$$\left| \begin{aligned} &\partial_y \widehat{\phi}(\xi, y, t)|_{y=1} = \sqrt{\epsilon}\xi\widehat{\varphi}(\xi, t) th(\sqrt{\epsilon}\xi) \\ &\partial_y^2 \widehat{\phi}(\xi, y, t)|_{y=1} = \epsilon\xi^2\widehat{\varphi}(\xi, t) \end{aligned} \right.$$

which gives the first terms of the expansion of these quantities with respect to ϵ,

$$\left| \begin{aligned} &\partial_y \phi(x, y, t)|_{y=1} = -\epsilon\partial_x^2\varphi(x, t) - \frac{\epsilon^2}{3}\partial_x^4\varphi(x, t) + O(\epsilon^3) \\ &\partial_y^2 \phi(x, y, t)|_{y=1} = -\epsilon\partial_x^2\varphi(x, t). \end{aligned} \right. \tag{6.22}$$

Now plugging the expansion of $\phi(x, y, t)$ in the last two equations of (6.19) at $y = 1 + \epsilon\eta$ gives, using (6.22):

$$\begin{cases} \eta_t + \partial_x^2\varphi + \epsilon\eta_x\partial_x\varphi + \epsilon\eta\partial_x^2\varphi + \dfrac{\epsilon}{3}\partial_x^4\varphi - \epsilon^2\eta\partial_x^3\varphi = 0(\epsilon^3) \\ \partial_t\varphi + \eta + \dfrac{\epsilon}{2}(\partial_x\varphi)^2 + \dfrac{\epsilon^2}{2}\partial_x^2\varphi - \epsilon^2\eta\partial_t\partial_x^2\varphi + \dfrac{\epsilon^2}{2}\eta^2\varphi = 0(\epsilon^3). \end{cases} \tag{6.23}$$

In order to use the same ansatz as in the general theory, we change ϵ by ϵ^2 in the above system and make the following change of unknowns $\tilde{\eta} = \frac{\eta}{\epsilon^2}$ and $\tilde{\varphi} = \frac{\varphi}{\epsilon^2}$. System (6.23) gives omitting the $\tilde{}$,

$$\begin{cases} \eta_t + \partial_x^2\varphi + \eta_x\partial_x\varphi + \eta\partial_x^2\varphi + \dfrac{\epsilon^2}{3}\partial_x^4\varphi - \epsilon^2\eta\partial_x^3\varphi = 0(\epsilon^5) \\ \partial_t\varphi + \eta + \dfrac{1}{2}(\partial_x\varphi)^2 + \dfrac{\epsilon^4}{2}\partial_x^2\varphi - \epsilon^2\eta\partial_t\partial_x^2\varphi + \dfrac{\epsilon^2}{2}\eta^2\varphi = 0(\epsilon^5). \end{cases} \tag{6.24}$$

We seek now an ansatz as follows

$$\begin{cases} \eta(x, t) = \epsilon^2\eta_0(x, t, \epsilon^2 t) + \epsilon^3\eta_1(x, t, \epsilon^2 t) + \epsilon^4\eta_2(x, t, \epsilon^2 t) \\ \varphi(x, t) = \epsilon^2\varphi_0(x, t, \epsilon^2 t) + \epsilon^3\varphi_1(x, t, \epsilon^2 t) + \epsilon^4\varphi_2(x, t, \epsilon^2 t). \end{cases} \tag{6.25}$$

We now plug (6.25) in (6.24). It gives as we identify the terms at each order of ϵ:

- at the order $O(\epsilon^2)$

$$\begin{cases} \partial_t\eta_0 + \partial_x^2\varphi_0 = 0 \\ \partial_t\varphi_0 + \eta_0 = 0; \end{cases} \tag{6.26}$$

- at the order $O(\epsilon^3)$

$$\begin{cases} \partial_t\eta_1 + \partial_x^2\varphi_1 = 0 \\ \partial_t\varphi_1 + \eta_1 = 0; \end{cases} \tag{6.27}$$

- at the order $O(\epsilon^4)$

$$\begin{cases} \partial_{t_1}\eta_0 + \partial_t\eta_2 + \partial_x\eta_0\partial_x\varphi_0 + \partial_x^2\varphi_2 + \dfrac{1}{3}\partial_x^4\varphi_0 + \eta_0\partial_x^2\varphi_0 = 0 \\ \partial_{t_1}\varphi_0 + \partial_t\varphi_2 + \eta_2 + \dfrac{1}{2}(\partial_x\varphi_0)^2 = 0. \end{cases} \tag{6.28}$$

First of all, note that the equations satisfied by φ_1 and η_1 are the same as those satisfied by η_0 and φ_0 in (6.26). Besides η_1 and φ_1 do not appear in the long time evolution equations (6.28). One can therefore set them to zero with no loss of generality.

These sets of equations suggest us to consider $g = \partial_x \varphi$ as an auxiliary unknown. Then, as in the first example, solving (6.26) gives that

$$\begin{cases} \eta_0(x,t) = \eta_{01}(x-t) + \eta_{02}(x+t) \\ g_0(x,t) = g_{01}(x-t) + g_{02}(x+t) \end{cases}$$

and naturally from the wave equation (6.26), one has that, $\eta_{01} = g_{01}$ and $\eta_{02} = -g_{02}$. Thereafter, if we set $u = \eta_{01}$ and $v = \eta_{02}$, and rewrite the system (6.28) as follows, one obtains,

$$\begin{cases} \partial_{t_1}\eta_0 + \partial_t \eta_2 + \partial_x(g_0\eta_0) + \partial_x g_2 + \dfrac{1}{3}\partial_x^3 g_0 = 0, \\ \\ \partial_{t_1} g_0 + \partial_t g_2 + \partial_x \eta_2 + g_0 \partial_x g_0 = 0. \end{cases} \tag{6.29}$$

Now summing and subtracting the above equations gives the following system,

$$\begin{cases} 2\partial_{t_1} u + \dfrac{1}{3}\partial_x^3(u-v) + 3u\partial_x u - v\partial_x v - \partial_x(uv) + (\partial_t + \partial_x)\eta_2 + (\partial_t + \partial_x) g_2 = 0, \\ \\ 2\partial_{t_1} v + \dfrac{1}{3}\partial_x^3(u-v) + u\partial_x u - 3v\partial_x v + \partial_x(uv) + (\partial_t - \partial_x)\eta_1 - (\partial_t - \partial_x) g_1 = 0. \end{cases} \tag{6.30}$$

We know from the previous general theory described in the previous sections, that we need to get rid of the corrector terms along with terms whose contributions in the corrector terms will keep them bounded. In that case and only in that case, we do not affect the convergence result. For that matter, we set the corrector terms to be such that the nonlinearity in the final system is symmetric. One needs afterwards to verify, that the corrector terms remain bounded. This gives, as a necessary condition, that

$$\left| \begin{array}{l} T_1(\partial_t, \partial_x)(\eta_2 + g_2) = -\dfrac{1}{3}\partial_x^3 v - \underbrace{2v\partial_x v}_{\text{added term}} \\ \\ T_2(\partial_t, \partial_x)(\eta_2 - g_2) = \dfrac{1}{3}\partial_x^3 u + \underbrace{2u\partial_x u}_{\text{added term}} \end{array} \right. \tag{6.31}$$

and the final system read then as a KdV type system whose nonlinearity derives from a gradient. Indeed one has

$$\begin{cases} \partial_{t_1} u + \dfrac{1}{6}\partial_x^3 u + \dfrac{1}{2}\partial_x \underbrace{\left[\dfrac{3u^2}{2} + \dfrac{v^2}{2} - uv\right]}_{\frac{\partial V(u,v)}{\partial u}} = 0 \\ \\ \partial_{t_1} v - \dfrac{1}{6}\partial_x^3 v + \dfrac{1}{2}\partial_x \underbrace{\left[-\dfrac{3v^2}{2} - \dfrac{u^2}{2} + uv\right]}_{\frac{\partial V(u,v)}{\partial v}} = 0 \end{cases} \tag{6.32}$$

with $V(u,v) = \frac{u^3}{2} - \frac{v^3}{2} + \frac{v^2 u}{2} - \frac{u^2 v}{2}$. The crucial point now is to verify that our correctors are indeed bounded from (6.31), which is a straightforward task as it is already established in Proposition 3.2, that holds since u and v lie in H^s (for $s > \frac{1}{2}$).

As for the Euler-Poisson example earlier and as for the coupled KdV system derived in the general case, the above system (6.32) is not compatible with the wave equation verified by η_0 and g_0. Then, as usual, we come back to the (x,t) variable, and consider our approximate (u^ϵ, v^ϵ) solution to solve

$$\begin{cases} \partial_t u^\epsilon + \partial_x u^\epsilon + \epsilon \frac{1}{6} \partial_x^3 u^\epsilon + \epsilon \frac{1}{2} \partial_x \frac{\partial V(u^\epsilon, v^\epsilon)}{\partial u^\epsilon} = 0 \\ \partial_t v^\epsilon - \partial_x v^\epsilon - \epsilon \frac{1}{6} \partial_x^3 v^\epsilon + \epsilon \frac{1}{2} \partial_x \frac{\partial V(u^\epsilon, v^\epsilon)}{\partial v^\epsilon} = 0 \end{cases} \tag{6.33}$$

along with the condition (6.31) on the correctors.

Finally we have at hand asymptotic systems of KdV type with a non linearity deriving from a gradient that compete as models for the propagation of counter-propagating hydrodynamic surface waves, exactly at the same level of approximation as those displayed in the literature. For comparison purposes, let us rewrite the system with the physical unknowns η_0 and g_0 being respectively the water elevation and the horizontal velocity. This gives

$$\begin{cases} \partial_t \eta_0 + \partial_x g_0 + \frac{\epsilon}{6} \partial_x^3 g_0 + \frac{\epsilon}{2} \partial_x (\eta_0 g_0) = 0 \\ \partial_t g_0 + \partial_x \eta_0 + \frac{\epsilon}{6} \partial_x^3 \eta_0 + \frac{\epsilon}{4} \partial_x (\eta_0^2 + 3 g_0^2) = 0. \end{cases} \tag{6.34}$$

Again our intention, is not in the scope of this paper to prove a convergence theorem as it is anyhow a difficult task. The purpose of this example was only meant to convince the reader of the relevance of such models in the context of water waves and show as well how the methods deriving from geometrical optics provides a rather rigorous framework for our problem.

References

[1] S. Alinhac and P. Gerard, *Opérateurs pseudo-différentiels et thérorème de Nash-Moser.* Interéditions/Éditions du CNRS (1991).

[2] T.B. Benjamin, J.L. Bona and J.J. Mahony, Model equations for long waves in nonlinear, dispersive systems. *Philos. Trans. Roy. Soc. Lond. A* **272** (1972) 47–78.

[3] W. Ben Youssef, *The global Cauchy problem for Korteweg-de Vries type systems describing counter-propagating waves.* MAB, Université Bordeaux I, preprint (1999).

[4] W. Ben Youssef, *Conservative, high order schemes and numerical study of a coupled system of Korteweg-de Vries type.* Université de Bordeaux I, preprint (1999).

[5] J.L. Bona and H. Chen, *Lecture notes in Austin.* Texas Institute for Computational and Applied Mathematics (1997).

[6] J.L. Bona and H. Chen, A Boussinesq system for two-way propagation of nonlinear dispersive waves. *Physica D* **116** (1998) 191–224.

[7] J.L. Bona, H. Chen and J.C. Saut, personal communications.

[8] J.L. Bona and R. Smith, The initial-value problem for the Korteweg-de Vries equation. *Philos. Trans. Roy. Soc. Lond. A* **278** (1975) 555–604.

[9] J. Bourgain, Fourier transform restriction phenomena for certain lattice substes and applications to nonlinear evolution equations. II. The Korteweg-de Vries equation. *Geom. Funct. Anal.* **3** (1993) 209–262.

[10] S. Cordier and E. Grenier, *Quasineutral limit of Euler-Poisson system arising from plasma physics.* Université de Paris VI, preprint (1997).

[11] W. Craig, An existence theory for water waves and the Boussinesq and the Korteweg-de Vries scaling limits. *Comm. Partial Differential Equations* **10** (1985) 787–1003.

[12] T. Colin, *Rigorous derivation of the nonlinear Schrodinger equation and Davey-Stewartson systems from quadratic hyperbolic systems.* Université de Bordeaux I, preprint No. 99001 (1999).

[13] R.K. Dodd, J.C. Eilbeck, J.D. Gibbon and H.C. Morris, *Solitons and nonlinear wave equations.* Academic Press (1982).

[14] P. Donnat, J.L. Joly, G. Metivier and J. Rauch, *Diffractive nonlinear geometric optics. I. Séminaire équations aux dérivées partielles.* École Polytechnique, Palaiseau, exposé No. XVII-XVIII (1995–1996).

[15] J.L. Joly, G. Metivier and J. Rauch, Diffractive nonlinear geometric optics with rectification. *Indiana Univ. Math. J.* **47** (1998) 1167–1241.

[16] J.L. Joly, G. Metivier and J. Rauch, Generic rigorous asymptotic expansions for weakly nonlinear multidimensional oscillatory waves. *Duke Math. J.* **70** (1993) 373–404.
[17] T. Kato, Perturbation theory for linear operators. *Grundlehren Math. Wiss.* **132** (1966).
[18] C.E. Kenig, G. Ponce and L. Vega, Well posedness and scaterring results for the generalized Korteweg-de Vries equation *via* the contraction principle. *Comm. Pure Appl. Math.* **XLVI** (1993) 527–620.
[19] D.J. Korteweg and G. de Vries, On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary wave. *Phil. Mag.* **39** (1895) 422–443.
[20] D. Lannes, Dispersive effects for nonlinear geometrical optics with rectification. *Asymptot. Anal.* **18** (1998) 111–146.
[21] G. Schneider and C.E. Wayne, *The long wave limit for the water wave problem. I. The case of zero surface tension.* University of Bayreuth, preprint (1999).
[22] G.B. Whitham, *Linear and nonlinear waves.* J. Wiley, New York (1974).
[23] N.J. Zabusky and M.D. Kruskal, Interaction of solitons in a collisionless plasma and the recurrence of initial states. *Phys. Rev. Lett.* **15** (1965) 240.