Mathematical Modelling and Numerical Analysis ESAIM: M2AN
Modélisation Mathématique et Analyse Numérique Vol. 34, N° 5, 2000, p. 1051–1067

DUAL COMBINED FINITE ELEMENT METHODS FOR NON-NEWTONIAN FLOW (II) PARAMETER-DEPENDENT PROBLEM

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Abstract. This is the second part of the paper for a Non-Newtonian flow. Dual combined Finite Element Methods are used to investigate the little parameter-dependent problem arising in a nonliner three field version of the Stokes system for incompressible fluids, where the viscosity obeys a general law including the Carreau's law and the Power law. Certain parameter-independent error bounds are obtained which solved the problem proposed by Baranger in [4] in a unifying way. We also give some stable finite element spaces by exemplifying the abstract *B-B* inequality. The continuous approximation for the extra stress is achieved as a by-product of the new method.

Mathematics Subject Classification. 65N30.

Received: January 15, 1999.

1. Introduction

In this section, we will give a brief description of a White-Metzner model type for the viscoelastic flow. We refer to [9] for more details.

Let Ω be a bounded convex polygon in $\mathbb{R}^n (n = 2, 3)$ with the Lipschitz boundary Γ. $\mathbb{R}^n$ is equipped with Cartesian coordinates $x_i, i = 1, \cdots, n$. For a function u, $\frac{\partial u}{\partial x_i}$ is written as $u_{,i}$, the Einstein convention for a summation is used.

For a scalar function p, the gradient of p is a vector $\nabla p, (\nabla p)_i = p_{,i}$. If q is another scalar function, we denote $(p, q) = \int_\Omega pq$. For a vector function u, the gradient of $\boldsymbol{u}$ is a tensor $\nabla \boldsymbol{u}, (\nabla \boldsymbol{u})_{ij} = \boldsymbol{u}_{i,j}, \operatorname{div} \boldsymbol{u} = \boldsymbol{u}_{i,i}, \boldsymbol{u}\cdot\nabla = \boldsymbol{u}_i \frac{\partial}{\partial x_i}$. For a tensor function $\boldsymbol{\sigma}, \operatorname{div}\boldsymbol{\sigma}$ is a vector function, $(\operatorname{div}\boldsymbol{\sigma})_i = \boldsymbol{\sigma}_{ij,j}, \boldsymbol{\sigma} : \boldsymbol{\tau} = \boldsymbol{\sigma}_{i,j}\boldsymbol{\tau}_{i,j}, |\boldsymbol{\sigma}|^2 = \boldsymbol{\sigma} : \boldsymbol{\sigma}$, and $(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}$.

To describe the flow, we use the pressure p (scalar), the velocity vector $\boldsymbol{u}$ and the total stress tensor $\boldsymbol{\sigma}_{tot}$. $\nabla\boldsymbol{u}$ is the velocity gradient tensor, $d(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$ is the rate of strain tensor, and $d_{\rm II}(\boldsymbol{u}) = \frac{1}{2}d_{ij}(\boldsymbol{u})d_{ij}(\boldsymbol{u})$ is the second invariant of $d(\boldsymbol{u})$.

A White-Metzner type model is described by the constitutive equations:

$$\boldsymbol{\sigma}_{tot} = \boldsymbol{\sigma} + \boldsymbol{\sigma}_N - p\boldsymbol{I},\ \boldsymbol{I} = \delta_{ij}, \tag{1.1}$$
$$\boldsymbol{\sigma}_N = 2\mu d(\boldsymbol{u}), \tag{1.2}$$
$$\boldsymbol{\sigma} = 2\eta(d_{\rm II}(\boldsymbol{u}))d(\boldsymbol{u}), \tag{1.3}$$

Keywords and phrases. Dual combined FEM, non-Newtonian flow, parameter-independent error bounds.

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where μ is the Newtonian part of the viscosity, η - the viscosity function for the viscoelastic part. The velocity $\boldsymbol{u}$ must satisfy the incompressible condition

$$\operatorname{div} \boldsymbol{u} = 0. \tag{1.4}$$

In this paper we consider the stationary creeping flow [16]. The fluid is submitted to a density of forces $\boldsymbol{f}$, then the momentum equation is written as

$$-\operatorname{div} \boldsymbol{\sigma}_{tot} = \boldsymbol{f}. \tag{1.5}$$

Two classical laws for η are the Power law and Carreau's law.

- Power law: $\eta_p(z) = \frac{1}{2} g z^{\frac{(r-2)}{2}},\ r > 1,\ g \geq 0.$
- Carreau's law: $\eta_c(z) = (\eta_0 - \eta_\infty)(1 + \lambda z)^{\frac{r-2}{2}},\ 0 \leq \eta_\infty < \eta_0, r > 1.$

Some Sobolev spaces [1] are needed. $T = [L^{r'}(\Omega)]^{\frac{n(n+1)}{2}} = \{\boldsymbol{\tau} = (\boldsymbol{\tau}_{ij}) \mid \boldsymbol{\tau}_{ij} = \boldsymbol{\tau}_{ji},\ \boldsymbol{\tau}_{ij} \in L^{r'}(\Omega),\ i, j = 1, \cdots, n\}$ with the norm $\|\boldsymbol{\tau}\|_{\boldsymbol{T}} = (\int_\Omega |\boldsymbol{\tau}|^{r'})^{\frac{1}{r'}}$. $(,)$ denotes the inner product of $X = [W_0^{1,r}(\Omega)]^n, M = L_0^{r'}(\Omega) = \{q \in L^{r'}(\Omega) \mid \int_\Omega q = 0\}$. X, M are equipped with the norm $\|\boldsymbol{v}\|_{\boldsymbol{X}} = (\int_\Omega |d(\boldsymbol{v})|^r)^{\frac{1}{r}},\ \|q\|_M = (\int_\Omega |q|^{r'})^{\frac{1}{r'}}$ respectively. It is easy to see $\|\cdot\|_M$ is an equivalent norm over $\boldsymbol{X}$. We also denote $\boldsymbol{T}', \boldsymbol{X}'$ the dual space of $\boldsymbol{T}$ and $\boldsymbol{X}$ respectively, and $\langle , \rangle$ denotes the dual multiple between $\boldsymbol{X}$ and $\boldsymbol{X}'$. For $1 < r < 2$, we must modify the definition of $\boldsymbol{X}$ and define $\boldsymbol{X}_1 = \boldsymbol{X} \cap [H_0^1(\Omega)]^n$, but for simplicity we still denote it as $\boldsymbol{X}$.

Lemma 1.1. *There exists an operator $A : \boldsymbol{T} \to \boldsymbol{T}'$ such that $A(\boldsymbol{\sigma}) = d(\boldsymbol{u})$.*

The proof of this lemma only needs a simple manipulation, we omit it.

Some works have been done in the finite element approximation of the Problem 1.1. In [4], Baranger *et al.* gave the first approximation of the three-field Stokes flow of White-Metzner model. However, the abstract error bounds they obtained are μ-dependent, *i.e.*, the error bounds will be deteriorated as the Newtonian viscosity approaches zero. Furthermore, no concrete finite element space pair is available. In fact, the well-posedness of their variational formulation needs two *B-B* inequalities, one for $(\boldsymbol{\sigma}, \boldsymbol{u})$ and the other for $(\boldsymbol{u}, p)$. The construction of finite element space pair satisfying these two *B-B* inequalities simultaneously is not a trivial thing at all, even in the linear case (see [20]), since these two *B-B* inequalities are interplaying. The relevant two-field problem has been studied in [21], but the abstract error bounds are also μ-dependent, and still there is no concrete finite element space pair presented. Meanwhile, the continuous approximation for the extra stress is widely used in engineer literatures, however, a fairly large finite element space is needed to achieve this goal [13, 23] that would cause an extra cost and accuracy-losing. Recently, the so-called EVSS and its variants [12] are proposed to attack this problem, but it needs an extra variable that would increase the cost, and even seriously that would lead to unsymmetric algebraic equations.

In the following, dual combined finite element method [3, 18, 25] is proposed to solve the above problems.

To introduce the dual combined mixed formulation, we present some operators:

$$\begin{aligned} B : \boldsymbol{X} \to \boldsymbol{X}', &\qquad B(\boldsymbol{u}) = 2\eta(d_{\mathrm{II}}(\boldsymbol{u}))d(\boldsymbol{u}). \\ H_\alpha : \boldsymbol{T} \times \boldsymbol{X} \times M &\quad \to \quad \boldsymbol{T}' \times \boldsymbol{X}' \times M', \\ l : \boldsymbol{T} \times \boldsymbol{X} \times M &\quad \to \quad \boldsymbol{T}' \times \boldsymbol{X}' \times M'. \end{aligned}$$

For $\boldsymbol{x} = (\boldsymbol{\sigma}, \boldsymbol{u}, p),\ \boldsymbol{y} = (\boldsymbol{\tau}, \boldsymbol{v}, q)$, we define

$$\begin{aligned} (H_\alpha(\boldsymbol{x}), \boldsymbol{y}) \quad = \quad & \alpha(A(\boldsymbol{\sigma}), \boldsymbol{\tau}) - \alpha(\boldsymbol{\tau}, d(\boldsymbol{u})) + \alpha(\boldsymbol{\sigma}, d(\boldsymbol{u})) + (1 - \alpha)(B(\boldsymbol{u}), d(\boldsymbol{v})) \\ & + 2\mu(d(\boldsymbol{u}), d(\boldsymbol{v})) - (p, \operatorname{div} \boldsymbol{v}) + (q, \operatorname{div} \boldsymbol{u}), \text{ where } \alpha \in [0, 1], \end{aligned}$$

and $\langle \boldsymbol{l}, \boldsymbol{y} \rangle = \langle \boldsymbol{f}, \boldsymbol{v} \rangle$.

Problem 1.1. Find $\boldsymbol{x} \in \boldsymbol{T} \times \boldsymbol{X} \times M$ such that

$$(H_\alpha(\boldsymbol{x}), \boldsymbol{y}) = \langle l, y \rangle \ \forall \boldsymbol{y} \in \boldsymbol{T} \times \boldsymbol{X} \times M, \ \forall \alpha \in [0,1].$$

This problem can be cast into the following saddle point problem.

Problem 1.2. Find $\boldsymbol{x} \in \boldsymbol{T} \times \boldsymbol{X} \times M$ such that

$$\alpha(A(\boldsymbol{\sigma}), \boldsymbol{\tau}) - \alpha(\boldsymbol{\tau}, d(\boldsymbol{u})) = 0 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{T}, \tag{1.6}$$

$$\alpha(\boldsymbol{\sigma}, d(\boldsymbol{v})) + (1-\alpha)(B(\boldsymbol{u}), d(\boldsymbol{v})) + 2\mu(d(\boldsymbol{u}), d(\boldsymbol{v})) - (p, \operatorname{div} \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall \boldsymbol{v} \in \boldsymbol{X}, \tag{1.7}$$

$$(\operatorname{div} \boldsymbol{u}, q) = 0, \qquad \forall q \in M, \tag{1.8}$$

for some $\alpha \in [0,1]$.

When α equal to 1 or 0, Problem 1.2 will degenerate to problems which have been discussed in [4] and [5], respectively. The case $\alpha \in (0,1)$ is more interesting, we discuss it from now on.

An outline of this paper follows. Some basic inequalities and preliminaries are introduced in §2. In §3, dual combined FEM is proposed to deal with the non-Newtonian flow with Newtonian viscosity. The error bounds for all variables are given under some abstract assumptions. In the last section, we exemplify the abstract assumptions. Error estimates with respect to a strong norm are derived, and some stable finite element spaces are presented and justified.

Throughout this paper C, C_i denote generic constants independent of μ. In addition, $C(a_1, \dots, a_m)$ denotes a constant depending on the non-negative parameter $a_i (1 \le i \le m)$ such that $C(a_1, \cdots, a_m) \le C$.

2. Some inequalities and preliminaries

We use the following notations: $\mathbb{M}_n$ is the set of $n \times n$ real matrices. $\langle \cdot, \cdot \rangle$denotes the inner product on M_n, *i.e.*, for $X, Y \in \mathbb{M}_n, \langle X, Y \rangle = \sum_{i,j=1}^{n} X_{ij} Y_{ij}$, $|X| = \langle X, X \rangle^{\frac{1}{2}}$. $\theta = 0$ if $\eta = \eta_p$, $\theta = 1$ if $\eta = \eta_c$.

Lemma 2.1. *[5] Let $K(|X|) = (\theta + |X|^2)^{\frac{r-2}{2}}$ and $r \in (1, \infty)$. Then for all $X, Y \in \mathbb{M}_n (n \ge 1)$ and $\delta \ge 0$, we have*

$$|K(|X|)X - K(|Y|)Y| \le c|X - Y|^{1-\delta}(\theta + |X| + |Y|)^{r-2+\delta}, \tag{2.1}$$

$$\langle K(|X|)X - K(|Y|)Y, X - Y \rangle \ \ge C|X - Y|^{2+\delta}(\theta + |X| + |Y|)^{r-2-\delta}. \tag{2.2}$$

Lemma 2.2. *[5] For all $r \in (1, \infty), a, \sigma_1, \sigma_2 \ge 0$, and $\varepsilon \in (0,1)$, we have*

$$(\theta + a + \sigma_1)^{r-2}\sigma_1\sigma_2 \le \varepsilon(\theta + a + \sigma_1)^{r-2}\sigma_1^2 + \varepsilon^{-\gamma}(\theta + a + \sigma_2)^{r-2}\sigma_2, \tag{2.3}$$

where $\boldsymbol{\gamma} = \max(1, r-1)$ and θ is defined below.

We adopt the abstract quasi-norm introduced in [5], which is very useful for error estimates. Let $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{T} \times \boldsymbol{X}$ be the solution of Problem 1.1, then for $(\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{T} \times \boldsymbol{X}$, we define the following quasi-norms:

$$\begin{aligned} |\boldsymbol{\tau}|_{r'}^{\rho'} &= \int_\Omega |\boldsymbol{\tau}|^2(\theta + |\boldsymbol{\sigma}| + |\boldsymbol{\tau}|)^{r'-2}, \ \rho' = \max(2, r'), \\ |\mathrm{d}(\boldsymbol{v})|_r^\rho &= \int_\Omega |\mathrm{d}(\boldsymbol{v})|^2(\theta + |\mathrm{d}(\boldsymbol{u})| + |\mathrm{d}(\boldsymbol{v})|)^{r'-2}, \ \rho = \max(2, r). \end{aligned}$$

Lemma 2.3. *[5] Define*

$$\gamma = \gamma(r,s) = \begin{cases} 2, & \text{if } r = 2, \\ \frac{s(2-r)}{2-s}, & \text{if } r \neq 2,\ s \neq 2, \\ \infty, & \text{if } r \neq 2,\ s = 2. \end{cases}$$

Then

1. *For* $1 < r \le s \le 2$, *and* $\boldsymbol{u}, \boldsymbol{w} \in [W^{1,\gamma}(\Omega)]^n$, *we have*

$$|\boldsymbol{w}|_{1,s}^2 \le C(\theta + |\boldsymbol{u}|_{1,\gamma} + |\boldsymbol{w}|_{1,\gamma})^{2-r} |d(\boldsymbol{w})|_r^2. \tag{2.4}$$

2. *For* $r \in (1,2]$, $\kappa \in [r, r + (2-r)\theta]$, *and* $\boldsymbol{w} \in [W^{1,\kappa}(\Omega)]^n$, *we have*

$$|d(\boldsymbol{w})|_r^2 \le C\|d(\boldsymbol{w})\|_{L^\kappa}^\kappa < |\boldsymbol{w}|_{1,r}^r. \tag{2.5}$$

3. *For* $r \in [2,\infty), \kappa \in [r + (2-r)\theta, r]$ *and* $\boldsymbol{w} \in [W_0^{1,r}(\Omega)]^n$, *we have*

$$|\boldsymbol{w}|_{1,\kappa}^\kappa \le C\|d(\boldsymbol{w})\|_{L^\kappa}^\kappa \le |d(\boldsymbol{w})|_r^r. \tag{2.6}$$

4. *For* $r \in [2,\infty)$, $2 \le s \le r < \infty$ *and* $\boldsymbol{u}, \boldsymbol{w} \in [W^{1,\gamma}(\Omega)]^n$, *we have*

$$|d(\boldsymbol{w})|_r^r \le C(\theta + |\boldsymbol{u}|_{1,\gamma} + |\boldsymbol{w}|_{1,\gamma})^{r-2} |\boldsymbol{w}|_{1,s}^2. \tag{2.7}$$

Lemma 2.4. *[5]* $|\cdot|_r$ *and* $|\cdot|_{r'}$ *are quasi-norms over* $\boldsymbol{X}, \boldsymbol{T}$, *respectively.*

Though the explicit formula for $A(\boldsymbol{\sigma})$ is not available when $\eta = \eta_c$, we are still able to prove the monotony and continuity properties of $A(\boldsymbol{\sigma})$ in this case.

Theorem 2.1. *There exists a constant* C *independent of* $\boldsymbol{\sigma}, \boldsymbol{\tau}$ *such that for any* $\delta > 0$

$$(A(\boldsymbol{\sigma}) - A(\boldsymbol{\tau}), \boldsymbol{\sigma} - \boldsymbol{\tau}) \ \ge\ C|\boldsymbol{\sigma} - \boldsymbol{\tau}|^{2+\delta}(1 + |\boldsymbol{\sigma}| + |\boldsymbol{\tau}|)^{r'-2-\delta}, \tag{2.8}$$

$$|A(\boldsymbol{\sigma}) - A(\boldsymbol{\tau})| \ \le\ C|\boldsymbol{\sigma} - \boldsymbol{\tau}|^{1-\delta}(1 + |\boldsymbol{\sigma}| + |\boldsymbol{\tau}|)^{r'-2+\delta}. \tag{2.9}$$

Proof. Note

$$(A(\boldsymbol{\sigma}) - A(\boldsymbol{\tau}), \boldsymbol{\sigma} - \boldsymbol{\tau}) = (d(\boldsymbol{u}) - d(\boldsymbol{v}), (1 + |d(\boldsymbol{u})|^2)^{\frac{r-2}{2}} d(\boldsymbol{u}) - (1 + |d(\boldsymbol{v})|^2)^{\frac{r-2}{2}} d(\boldsymbol{v})).$$

Therefore, we only need to check the following two facts.

If $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n, \phi(\boldsymbol{x}) = (1 + |\boldsymbol{x}|^2)^{\frac{r-2}{2}}\boldsymbol{x} = \boldsymbol{\alpha}, \phi(\boldsymbol{y}) = \boldsymbol{\beta}, \boldsymbol{x} = \psi(\boldsymbol{\alpha}), \boldsymbol{y} = \psi(\boldsymbol{\beta})$, then

$$\langle \psi(\boldsymbol{\alpha}) - \psi(\boldsymbol{\beta}), \boldsymbol{\alpha} - \boldsymbol{\beta} \rangle \ge C|\boldsymbol{\alpha} - \boldsymbol{\beta}|^{2+\delta}(1 + |\boldsymbol{\alpha}| + |\boldsymbol{\beta}|)^{r'-2+\delta}, \tag{2.10}$$

$$|\psi(\boldsymbol{\alpha}) - \psi(\boldsymbol{\beta})| \le C|\alpha - \beta|^{1-\delta}(1 + |\boldsymbol{\alpha}| + |\boldsymbol{\beta}|)^{r'-2+\delta}. \tag{2.11}$$

We firstly prove that (2.10) holds for $\delta = 0$.

Define an operator $i : \mathbb{R}^n \to \mathbb{R}^{n+1}, i(\boldsymbol{x}) = (\boldsymbol{x}, 0), \boldsymbol{v} \in \mathbb{R}^{n+1}, \boldsymbol{v}_i = \delta_{i,n+1}$. Then

$$\boldsymbol{\alpha} = \phi(\boldsymbol{x}) = (1 + |\boldsymbol{x}|^2)^{\frac{r-2}{2}}\boldsymbol{x} = |\boldsymbol{v} + i(\boldsymbol{x})|^{r-2}\boldsymbol{x}, \boldsymbol{y} = |\boldsymbol{v} + i(\boldsymbol{y})|^{r-2}\boldsymbol{y}.$$

We also have

$$\begin{array}{ll} i(\alpha) = |\boldsymbol{v} + i(\boldsymbol{x})|^{r-2} i(\boldsymbol{x}), & i(\boldsymbol{\beta}) = |\boldsymbol{v} + i(\boldsymbol{y})|^{r-2} i(\boldsymbol{y}), \\ i(\hat{\boldsymbol{\alpha}}) = |\boldsymbol{v} + i(\boldsymbol{x})|^{r-2}(\boldsymbol{v} + i(\boldsymbol{x})), & i(\hat{\boldsymbol{\beta}}) = |\boldsymbol{v} + i(\boldsymbol{y})|^{r-2}(\boldsymbol{v} + i(\boldsymbol{y})). \end{array}$$

$$\begin{aligned} \langle \psi(\boldsymbol{\alpha}) - \psi(\boldsymbol{\beta}), \boldsymbol{\alpha} - \boldsymbol{\beta} \rangle &= \langle \phi(\boldsymbol{x}) - \phi(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle \\ &= \langle |\boldsymbol{v} + i(\boldsymbol{x})|^{r-2}(\boldsymbol{v} + i(\boldsymbol{x})) - |\boldsymbol{v} + i(\boldsymbol{y})|^{r-2}(\boldsymbol{v} + i(\boldsymbol{y})), i(\boldsymbol{x}) - i(\boldsymbol{y}) \rangle \\ &= \langle |\boldsymbol{v} + i(\boldsymbol{x})|^{r-2}(\boldsymbol{v} + i(\boldsymbol{x})) - |\boldsymbol{v} + i(\boldsymbol{y})|^{r-2}(\boldsymbol{v} + i(\boldsymbol{y})), (\boldsymbol{v} + i(\boldsymbol{x})) - (\boldsymbol{v} + i(\boldsymbol{y})) \rangle \\ &= \langle i(\hat{\boldsymbol{\alpha}}) - i(\hat{\boldsymbol{\beta}}), |i(\hat{\boldsymbol{\alpha}})|^{r'-2} i(\hat{\boldsymbol{\alpha}}) - |i(\hat{\boldsymbol{\beta}})|^{r'-2} i(\hat{\boldsymbol{\beta}}) \rangle \\ &\geq C |i(\hat{\boldsymbol{\alpha}}) - i(\hat{\boldsymbol{\beta}})|^2 (|i(\hat{\boldsymbol{\alpha}})| + |i(\boldsymbol{\beta})|)^{r'-2}. \end{aligned}$$

Note

$$|i(\hat{\boldsymbol{\alpha}}) - i(\hat{\boldsymbol{\beta}})|^2 = |i(\alpha) - i(\beta)|^2 + |\ |\boldsymbol{v} + i(\boldsymbol{x})|^{r-2} - |\boldsymbol{v} + i(\boldsymbol{y})|^{r-2}|^2 \geq |i(\alpha) - i(\beta)|^2.$$

Furthermore, we have $|i(\hat{\boldsymbol{\alpha}})|^2 = |i(\alpha)|^2 + |\boldsymbol{v} + i(\boldsymbol{x})|^{2r-4} = |i(\alpha)|^2 + |i(\hat{\boldsymbol{\alpha}})|^{4-2r'}$.
Therefore, we need to show that there exists a constant C such that

$$\left(|i(\hat{\boldsymbol{\alpha}})| + |i(\hat{\boldsymbol{\beta}})| \right)^{r'-2} \geq (1 + |i(\alpha)| + |i(\beta)|)^{r'-2}. \tag{2.12}$$

If $r' \in [2, \infty)$, *i.e.* $r \in (1, 2)$, we only need to prove

$$|i(\hat{\boldsymbol{\alpha}})| + |i(\hat{\boldsymbol{\beta}})| \geq C(1 + |i(\boldsymbol{\alpha})| + |i(\boldsymbol{\beta})|).$$

To do this, we show $|i(\hat{\boldsymbol{\alpha}})| \geq C(1 + |i(\boldsymbol{\alpha})|)$ and $|i(\hat{\boldsymbol{\beta}})| \geq C(1 + |i(\boldsymbol{\beta})|)$. These two inequalities can be cast into the following basic inequality:

$$(1 + |\boldsymbol{x}|^2)^{\frac{r-1}{2}} \geq C(1 + (1 + |\boldsymbol{x}|^2)^{\frac{r-2}{2}} |\boldsymbol{x}|). \tag{2.13}$$

Define a function $f(t) = \frac{(1+t^2)^{\frac{r-1}{2}}}{1+(1+t^2)^{\frac{r-2}{2}} t} (t > 0)$, we show that $f(t)$ has a positive lower bound. In fact, $f(t)$ is a continuous function over interval $(0, \infty)$, and $f(0) = 1$, $\lim_{t \to \infty} f(t) = 1$. Furthermore, $f(t) > 0$ for $t \in (0, \infty)$, hence $f(t)$ must have a positive lower bound. Consequently, we have proved (2.13) for the case $r' \in [2, \infty)$. Similarly, we can prove it for the case $r' \in (1, 2)$. Hence (2.12) holds for all $r \in (1, \infty)$, therefore

$$\begin{aligned} \langle A(\boldsymbol{\sigma}) - A(\boldsymbol{\tau}), \boldsymbol{\sigma} - \boldsymbol{\tau} \rangle &\geq C|\boldsymbol{\sigma} - \boldsymbol{\tau}|^2 (1 + |\boldsymbol{\sigma}| + |\boldsymbol{\tau}|)^{r'-2} \\ &\geq C|\boldsymbol{\sigma} - \boldsymbol{\tau}|^{2+\delta} (1 + |\boldsymbol{\sigma}| + |\boldsymbol{\tau}|)^{r'-2-\delta}, \end{aligned}$$

which is the first inequality (2.8) of the theorem.

Now it remains to show the continuity property of the operator A. We have

$$\begin{aligned} (A(\boldsymbol{\sigma}) - A(\boldsymbol{\tau}), \boldsymbol{\sigma} - \boldsymbol{\tau}) &= (d(\boldsymbol{u}) - d(v), B(u) - B(v)) \\ &\geq C|d(u - v)|^2 (1 + |d(\boldsymbol{u})| + |d(v)|)^{r'-2} \\ &= C|A(\boldsymbol{\sigma}) - A(\boldsymbol{\tau})|^2 (1 + |A(\boldsymbol{\sigma})| + |A(\boldsymbol{\tau})|)^{r'-2}. \end{aligned}$$

Using Cauchy inequality yields

$$|A(\boldsymbol{\sigma}) - A(\boldsymbol{\tau})| \leq C|\boldsymbol{\sigma} - \boldsymbol{\tau}|(1 + |A(\boldsymbol{\sigma})| + |A(\boldsymbol{\tau})|)^{2-r}.$$

On the other hand, we can prove that

$$(1+|A(\boldsymbol{\sigma})|+|A(\boldsymbol{\tau})|)^{2-r} \leq C(1+|\boldsymbol{\sigma}|+|\boldsymbol{\tau}|)^{r'-2}. \tag{2.14}$$

In fact, since $A(\boldsymbol{\sigma}) = d(\boldsymbol{u})$, (2.14) is equivalent to the following:

$$(1+|d(\boldsymbol{u})|+|d(\boldsymbol{v})|)^{2-r} \leq C(1+|B(\boldsymbol{u})|+|B(\boldsymbol{v})|)^{r'-2}. \tag{2.15}$$

As before, we only need to show

$$(1+|\boldsymbol{x}|+|\boldsymbol{y}|)^{2-r} \leq C(1+(1+|\boldsymbol{x}|^2)^{\frac{r-2}{2}}|\boldsymbol{x}|+(1+|\boldsymbol{y}|^2)^{\frac{r-2}{2}}|\boldsymbol{y}|)^{r'-2}, \tag{2.16}$$

which is just a direct consequence of the following inequality:

$$(1+|\boldsymbol{x}|)^{2-r} \leq (1+(1+|\boldsymbol{x}|^2)^{\frac{r-2}{2}}|\boldsymbol{x}|)^{r'-2},$$

i.e.

$$\begin{aligned} (1+|\boldsymbol{x}|)^{r-1} &\leq 1+(1+|\boldsymbol{x}|^2)^{\frac{r-2}{2}}|\boldsymbol{x}|, \; r \in (1,2], & (2.17)\\ (1+|\boldsymbol{x}|)^{r-1} &\geq 1+(1+|\boldsymbol{x}|^2)^{\frac{r-2}{2}}|\boldsymbol{x}|, \; r \in [2,\infty). & (2.18) \end{aligned}$$

Let $t = |\boldsymbol{x}|$, we define

$$G(t) = 1+(1+t^2)^{\frac{r-2}{2}}t-(1+t)^{r-1}, \; t \geq 0.$$

Then if $r \in (1.2]$,

$$\begin{aligned} G'(t) &= (1+t^2)^{\frac{r-4}{2}}(1+(r-1)t^2)-(r-1)(1+t)^{r-2}\\ &\geq (1+t^2)^{\frac{r-4}{2}}(r-1+(r-1)t^2)-(r-1)(1+t)^{r-2}\\ &\geq (r-1)((1+t^2)^{\frac{r-2}{2}}-(1+t)^{r-2})\\ &\geq 0. \end{aligned}$$

Hence $G(t) \geq G(0)$ for $r \in (1,2]$, which means (2.17) holds. Similarly, (2.18) holds. Consequently, (2.14) and then (2.15), (2.16) hold.

Combining (2.15) and (2.16), we have

$$\begin{aligned} |A(\boldsymbol{\sigma})-A(\boldsymbol{\tau})| &\leq C|\boldsymbol{\sigma}-\boldsymbol{\tau}|(1+|\boldsymbol{\sigma}|+|\boldsymbol{\tau}|)^{r'-2}\\ &\leq C|\boldsymbol{\sigma}-\boldsymbol{\tau}|^{1-\delta}(1+|\boldsymbol{\sigma}|+|\boldsymbol{\tau}|)^{r'-2+\delta}, \end{aligned}$$

which completes the proof. □

Corollary 2.1. $(1+|A(\boldsymbol{\sigma})|)^{2-r} \leq (1+|\boldsymbol{\sigma}|)^{r'-2}$.

Corollary 2.2. *There exists a constants C such that*

$$(A(\boldsymbol{\sigma}),\boldsymbol{\sigma}) \geq C\int_\Omega |\boldsymbol{\sigma}|^2(1+|\boldsymbol{\sigma}|)^{r'-2}.$$

Using previous results we can prove the existence and uniqueness of solutions of Problem 1.1. Setting $\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{X} \mid \operatorname{div} \boldsymbol{v} = 0\}$ and dropping the terms concerning the pressure, we introduce the following operator: $H^{\star}_{\alpha} : \boldsymbol{T} \times \boldsymbol{V} \to \boldsymbol{T} \times \boldsymbol{V}$,

$$(H^{\star}_{\alpha}(\boldsymbol{\sigma}, \boldsymbol{u}), (\boldsymbol{\tau}, \boldsymbol{v})) = \alpha(A(\boldsymbol{\sigma}), \boldsymbol{\tau}) - \alpha(\boldsymbol{\tau}, d(\boldsymbol{u})) + \alpha(\boldsymbol{\sigma}, d(\boldsymbol{v})) + (1-\alpha)(B(\boldsymbol{u}), d(\boldsymbol{v})) + 2\mu(d(\boldsymbol{u}), d(\boldsymbol{v})). \quad (2.19)$$

We consider the following two-field problem:

Problem 2.1. Find $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{T} \times \boldsymbol{V}$ such that

$$(H^{\star}_{\alpha}(\boldsymbol{\sigma}, \boldsymbol{u}), (\boldsymbol{\tau}, \boldsymbol{v})) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \ \forall (\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{T} \times \boldsymbol{V}. \quad (2.20)$$

Lemma 2.5. *$H^{\star}_{\alpha}$ is an hemi-continuous, monotone and strong coercive operator.*

Lemma 2.5 can easily be derived from Theorem 2.1 and Lemma 2.1.

From Lemma 2.5 it follows the existence and uniqueness of the Problem 2.1.

Lemma 2.6. *[2] There exists $\beta > 0$ such that*

$$\inf_{q \in M} \sup_{\boldsymbol{v} \in \boldsymbol{X}} \frac{(\operatorname{div} \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\boldsymbol{X}} \|q\|_M} \geq \beta.$$

Theorem 2.2. *Problem 1.1 has a unique solution $(\boldsymbol{\sigma}, \boldsymbol{u}, p) \in \boldsymbol{T} \times \boldsymbol{V} \times M$ which admits the following estimate*

$$\|\boldsymbol{\sigma}\|_{\boldsymbol{T}} + \|\boldsymbol{u}\|_{\boldsymbol{X}} + \|p\|_M \leq C(\alpha, r, \|\boldsymbol{f}\|_{-1,r'}). \quad (2.21)$$

Remark 2.1. All the estimates are μ-independent. Furthermore, we will obtain similar estimates for the finite element solution $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, p_h)$ in the next section.

3. Finite element approximation

We assume that the triangulation $\mathcal{C}_h$ is a regular partition of Ω [8], the quasi-uniformity of $\mathcal{C}_h$ is not necessary unless we state it clearly. Let $\boldsymbol{T}_h, \boldsymbol{X}_h$ and M_h be the finite element space: $\boldsymbol{T}_h \subset \boldsymbol{T}, \boldsymbol{X}_h \subset \boldsymbol{X}, M_h \subset M$. We denote $\boldsymbol{V}_h = \{\boldsymbol{v}_h \in \boldsymbol{X}_h \mid (\operatorname{div} \boldsymbol{v}_h, q_h) = 0 \ \forall q_h \in M_h\}$.

The corresponding finite element approximation for Problem 1.1 is:

Problem 3.1. Find $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, p_h) \in \boldsymbol{T}_h \times \boldsymbol{X}_h \times M_h$ such that

$$\alpha(A(\boldsymbol{\sigma}_h), \boldsymbol{\tau})) - \alpha(\boldsymbol{\tau}, d(\boldsymbol{u}_h)) = 0 \ \forall \boldsymbol{\tau} \in \boldsymbol{T}_h, \quad (3.1)$$

$$\alpha(\boldsymbol{\sigma}_h, d(\boldsymbol{v})) + (1-\alpha)(B(\boldsymbol{u}_h), d(\boldsymbol{v})) + 2\mu(d(\boldsymbol{u}_h), d(\boldsymbol{v})) - (p_h, \operatorname{div} \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \ \forall \boldsymbol{v} \in \boldsymbol{X}_h, \quad (3.2)$$

$$(\operatorname{div} \boldsymbol{u}_h, q) = 0 \ \forall q \in M_h. \quad (3.3)$$

Problem 3.1 can be reformulated as follows: Find $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \boldsymbol{T}_h \times \boldsymbol{V}_h$ such that

$$(H^{\star}_{\alpha}(\boldsymbol{\sigma}_h, \boldsymbol{u}_h), (\boldsymbol{\tau}_h, \boldsymbol{v}_h)) = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle \ \forall (\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in \boldsymbol{T}_h \times \boldsymbol{V}_h. \quad (3.4)$$

Assumption Θ (*B*-*B* inequality):

$$\exists\, \beta_h(r) > 0, \inf_{q_h \in M_h} \sup_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \frac{(\operatorname{div} \boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_X \|q_h\|_M} \geq \beta_h(r).$$

Assumption Δ: for any $\nu \geq \mu \geq 1$ it holds:

$$\begin{aligned} \|\boldsymbol{w}_h\|_{1,\nu} &\leq C_I h^{n(\frac{1}{\nu} - \frac{1}{\mu})} \|\boldsymbol{w}_h\|_{1,\mu} \ \forall w_h \in X_h, \\ \|\boldsymbol{\tau}_h\|_{0,\nu} &\leq C_I h^{n(\frac{1}{\nu} - \frac{1}{\mu})} \|\boldsymbol{\tau}_h\|_{0,\mu} \ \forall \boldsymbol{\tau}_h \in \boldsymbol{T}_h. \end{aligned}$$

Lemma 3.1. *[21, Rem. 2.1]*

$$\inf_{\boldsymbol{v}\in\boldsymbol{V}_h}\|\boldsymbol{u}-\boldsymbol{v}\|_{\boldsymbol{X}}\le(1+C)\inf_{\boldsymbol{w}\in\boldsymbol{X}_h}\|\boldsymbol{u}-\boldsymbol{w}\|_{\boldsymbol{X}}. \tag{3.5}$$

Remark 3.1. Note that the B-B inequality for $(\boldsymbol{\sigma},\boldsymbol{u})$ and $(\boldsymbol{\sigma}_h,\boldsymbol{u}_h)$ is no longer needed either for Problem 1.1 or for Problem 3.1. An enhanced K-ellipticity is introduced in a natural, reasonable and unifying way. It is known that there are many finite element space pairs satisfying the B-B inequality either for $(\boldsymbol{\sigma},\boldsymbol{u})$ or for $(\boldsymbol{u},p)$, but few of them satisfy the two B-B inequalities simultaneously [20].

Theorem 3.1. *If Assumption Θ holds, then Problem 3.1 admits a unique solution $(\boldsymbol{\sigma}_h,\boldsymbol{u}_h,p_h)\in\boldsymbol{T}_h\times\boldsymbol{V}_h\times M_h$, and the following a-priori estimate holds*

$$\|\boldsymbol{\sigma}_h\|_{\boldsymbol{T}}+\|\boldsymbol{u}_h\|_{\boldsymbol{X}}+\|p_h\|_M\le C(\alpha,r,\beta_h(r),\|\boldsymbol{f}\|_{-1,r'}). \tag{3.6}$$

We define κ as that in Lemma 2.3, and κ' the conjugate exponent of κ.

Theorem 3.2. *If $(\boldsymbol{\sigma},u,p)$ and $(\boldsymbol{\sigma}_h,\boldsymbol{u}_h,p_h)$ are solutions of Problem 1.1 and Problem 3.1, respectively, then for all $r\in(1,2]$,*

$$\begin{aligned}\alpha|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h|_{r'}^{r'}+(1-\alpha)|d(\boldsymbol{u}-\boldsymbol{u}_h)|_r^2+\mu|\boldsymbol{u}-\boldsymbol{u}_h|_{1,2}^2\le C(\alpha)&\inf_{(\boldsymbol{\tau},\boldsymbol{v},q)\in\boldsymbol{T}_h\times\boldsymbol{V}_h\times M_h}(|\boldsymbol{\sigma}-\boldsymbol{\tau}|_{r'}^{r'}+|d(\boldsymbol{u}-\boldsymbol{v})|_r^2\\&+(\theta+\|\boldsymbol{u}\|_{\boldsymbol{X}}+\|\boldsymbol{u}_h\|_{\boldsymbol{X}})^{2-r}\|\boldsymbol{\sigma}-\boldsymbol{\tau}\|_{\boldsymbol{T}}^2\\&+\|d(\boldsymbol{u}-\boldsymbol{v})\|_{L^\kappa}^{\kappa}+\mu|\boldsymbol{u}-\boldsymbol{v}|_{1,2}^2+\|p-q\|_M\|\boldsymbol{u}-\boldsymbol{v}\|_{\boldsymbol{X}}\\&+(\theta+\|\boldsymbol{u}\|_{\boldsymbol{X}}+\|\boldsymbol{u}_h\|_{\boldsymbol{X}})^{2-r}\|p-q\|_M^2),\end{aligned} \tag{3.7}$$

and for $r\in[2,\infty)$,

$$\begin{aligned}\alpha|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h|_{r'}^2+(1-\alpha)|d(\boldsymbol{u}-\boldsymbol{u}_h)|_r^r+\mu|\boldsymbol{u}-\boldsymbol{u}_h|_{1,2}^2\le C(\alpha)&\inf_{(\boldsymbol{\tau},\boldsymbol{v},q)\in\boldsymbol{T}_h\times\boldsymbol{V}_h\times M_h}(|\boldsymbol{\sigma}-\boldsymbol{\tau}|_{r'}^2+|d(\boldsymbol{u}-\boldsymbol{v})|_r^r\\&+\|\boldsymbol{\sigma}-\boldsymbol{\tau}\|_{L^{\kappa'}}^{\kappa'}+(\theta+\|\boldsymbol{\sigma}\|_{\boldsymbol{T}}+\|\boldsymbol{\sigma}_h\|_{\boldsymbol{T}})^{2-r'}\|\boldsymbol{u}-\boldsymbol{v}\|_{\boldsymbol{X}}^2\\&+\mu|d(\boldsymbol{u}-\boldsymbol{v})|_{1,2}^2+\|p-q\|_M\|\boldsymbol{u}-\boldsymbol{v}\|_{\boldsymbol{X}}+\|p-q\|_{L^{\kappa'}}^{\kappa'}).\end{aligned} \tag{3.8}$$

If the Assumption Θ holds, then for $r\in(1,2]$,

$$\begin{aligned}\|p-p_h\|_{L^{\kappa'}}\le C[(1+\beta_h(r)^{-1})\inf_{q\in M_h}\|p-q\|_{L^{\kappa'}}+\beta_h(r)^{-1}(\alpha|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h|_{r'}^{\frac{r'}{\kappa'}}\\+|d(\boldsymbol{u}-\boldsymbol{u}_h)|_r^{\frac{2}{\kappa'}}+\mu C_I h^{n(\frac12-\frac1\kappa)}|\boldsymbol{u}-\boldsymbol{u}_h|_{1,2})],\end{aligned} \tag{3.9}$$

and for $2\le s\le r$,

$$\begin{aligned}\|p-p_h\|_{L^{s'}}\le C[(1+\beta_h(r)^{-1})\inf_{q\in M_h}\|p-q\|_{L^{s'}}+\beta_h(r)^{-1}((\theta+\|\boldsymbol{\sigma}\|_{\boldsymbol{\gamma}(r',s')}+\|\boldsymbol{\sigma}_h\|_{\boldsymbol{\gamma}(r',s')})^{\frac{2-r'}{2}}|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h|_{r'}\\+(\theta+|\boldsymbol{u}|_{1,\boldsymbol{\gamma}(r,s)}+|\boldsymbol{u}_h|_{1,\boldsymbol{\gamma}(r,s)})^{\frac{r-2}{2}}|d(\boldsymbol{u}-\boldsymbol{u}_h)|_r^{\frac{r}{2}}+C_I\mu|\boldsymbol{u}-\boldsymbol{u}_h|_{1,2})].\end{aligned} \tag{3.10}$$

Here $\boldsymbol{\gamma}(r,s)$ and $\boldsymbol{\gamma}(r',s')$ are defined as that in Lemma 2.3.

Proof. For $\boldsymbol{x}_h=(\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\in\boldsymbol{T}_h\times\boldsymbol{V}_h$, we have

$$(H_\alpha^\star(\boldsymbol{x}_h),\boldsymbol{y}_h)=\langle\boldsymbol{l},\boldsymbol{y}_h\rangle\ \forall\boldsymbol{y}_h\in\boldsymbol{T}_h\times\boldsymbol{V}_h.$$

Let $\boldsymbol{x} = ((\boldsymbol{\sigma}, \boldsymbol{u}), p)$ be the solution of Problem 1.1, then

$$(H_\alpha^\star(\boldsymbol{x}), \boldsymbol{y}_h) = \langle \boldsymbol{l}, \boldsymbol{y}_h\rangle + (p, \operatorname{div} \boldsymbol{v}_h)\ \forall \boldsymbol{y}_h = (\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{T}_h \times \boldsymbol{V}_h,$$

$$\begin{aligned}(H_\alpha^\star(\boldsymbol{x}) - H_\alpha^\star(\boldsymbol{x}_h), \boldsymbol{x} - \boldsymbol{x}_h) &= (H_\alpha^\star(\boldsymbol{x}) - H_\alpha^\star(\boldsymbol{x}_h), \boldsymbol{x} - \boldsymbol{y}_h) + (H_\alpha^\star(\boldsymbol{x}) - H_\alpha^\star(\boldsymbol{x}_h), \boldsymbol{y}_h - \boldsymbol{x}_h)\\ &= (H_\alpha^\star(\boldsymbol{x}) - H_\alpha^\star(\boldsymbol{x}_h), \boldsymbol{x} - \boldsymbol{y}_h) + (p - q_h, \operatorname{div}(\boldsymbol{v}_h - \boldsymbol{u}_h)). \qquad (3.11)\end{aligned}$$

Set $\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \boldsymbol{\tau}, \hat{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{v}, \hat{p} = p - q$, we have

$$\begin{aligned}(H_\alpha^\star(\boldsymbol{x}) - H_\alpha^\star(\boldsymbol{x}_h), \boldsymbol{x} - \boldsymbol{y}_h) &= \alpha(A(\boldsymbol{\sigma}) - A(\boldsymbol{\sigma}_h), \hat{\boldsymbol{\sigma}}) + (1-\alpha)(B(\boldsymbol{u}) - B(\boldsymbol{u}_h), d(\hat{\boldsymbol{u}})\\ &\quad - \alpha(\hat{\boldsymbol{\sigma}}, d(\boldsymbol{u} - \boldsymbol{u}_h)) + \alpha(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, d(\hat{\boldsymbol{u}})) + 2\mu(d(\hat{\boldsymbol{u}}), d(\boldsymbol{u} - \boldsymbol{u}_h))\\ &= I_1 + \cdots + I_5.\end{aligned}$$

Let $I_6 = (\hat{p}, \operatorname{div}(\boldsymbol{v}_h - \boldsymbol{u}_h))$. Now we estimate the terms $I_1, \cdots, I_6$ one by one.

Using Lemma 2.2, we can estimate I_1 and I_2 as follows

$$\begin{aligned}|I_1| &\leq C\alpha(\varepsilon|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h|_{r'}^{\rho'} + \varepsilon^{-\gamma}|\hat{\boldsymbol{\sigma}}|_{r'}^{\rho'}),\\ |I_2| &\leq C(1-\alpha)(\varepsilon|d(\boldsymbol{u} - \boldsymbol{u}_h)|_r^{\rho} + \varepsilon^{-\gamma}|d(\hat{\boldsymbol{u}})|_r^{\rho}).\end{aligned}$$

In light of Lemma 2.3, I_3 can be bounded as

$$\begin{aligned}|I_3| &\leq \alpha\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{X}}\|\hat{\boldsymbol{\sigma}}\|_{\boldsymbol{T}}\\ &\leq C\alpha(\theta + \|\boldsymbol{u}\|_{\boldsymbol{X}} + \|\boldsymbol{u}_h\|_{\boldsymbol{X}})^{\frac{2-r}{2}}|d(\boldsymbol{u} - \boldsymbol{u}_h)|_r\|\hat{\boldsymbol{\sigma}}\|_{\boldsymbol{T}}\\ &\leq C\alpha\varepsilon|d(\boldsymbol{u} - \boldsymbol{u}_h)|_r^2 + C\frac{\alpha}{4\varepsilon}(\theta + \|\boldsymbol{u}\|_{\boldsymbol{X}} + \|\boldsymbol{u}_h\|_{\boldsymbol{X}})^{2-r}\|\hat{\boldsymbol{\sigma}}\|_{\boldsymbol{T}}^2\ \forall r \in (1,2],\\ |I_3| &\leq \alpha\|d(\boldsymbol{u} - \boldsymbol{u}_h)\|_{L^\kappa}\|\hat{\boldsymbol{\sigma}}\|_{L^{\kappa'}}\\ &\leq C\alpha|d(\boldsymbol{u} - \boldsymbol{u}_h)|_r^{\frac{r}{\kappa}}\|\hat{\boldsymbol{\sigma}}\|_{L^{\kappa'}}\\ &\leq C\alpha(\varepsilon|d(\boldsymbol{u} - \boldsymbol{u}_h)|_r^r + C\varepsilon^{1-\kappa'}\|\hat{\boldsymbol{\sigma}}\|_{L^{\kappa'}}^{\kappa'})\ \forall r \in [2,\infty).\end{aligned}$$

Following the same line, we estimate I_4 and I_5 as

$$\begin{aligned}|I_4| &\leq C\alpha(\varepsilon|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h|_{r'}^{r'} + \varepsilon^{1-\kappa'}\|d(\hat{\boldsymbol{u}})\|_{L^\kappa}^{\kappa})\ \forall r \in (1,2],\\ |I_4| &\leq C\alpha(\varepsilon|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h|_{r'}^2 + \frac{1}{4}\varepsilon(\theta + \|\boldsymbol{\sigma}\|_{\boldsymbol{T}} + \|\boldsymbol{\sigma}_h\|_{\boldsymbol{T}})^{2-r'}\|\hat{\boldsymbol{u}}\|_{\boldsymbol{X}}^2)\ \forall r \in [2,\infty),\\ |I_5| &\leq \mu|\boldsymbol{u} - \boldsymbol{u}_h|_{1,2}|\hat{\boldsymbol{u}}|_{1,2} \leq \varepsilon\mu|\boldsymbol{u} - \boldsymbol{u}_h|_{1,2}^2 + C(\varepsilon)\mu|\hat{\boldsymbol{u}}|_{1,2}^2.\end{aligned}$$

Note $I_6 = -(\hat{p}, \operatorname{div}\hat{\boldsymbol{u}}) + (\hat{p}, \operatorname{div}(\boldsymbol{u} - \boldsymbol{u}_h))$, then

$$\begin{aligned}|I_6| &\leq C(\|\hat{p}\|_M\|\hat{\boldsymbol{u}}\|_{\boldsymbol{X}} + \varepsilon|d(\boldsymbol{u} - \boldsymbol{u}_h)|_r^2 + \varepsilon^{-1}(\|\boldsymbol{u}\|_{\boldsymbol{X}} + \|\boldsymbol{u}_h\|_{\boldsymbol{X}})^{2-r}\|\hat{p}\|_M^2)\ \forall r \in (1,2],\\ |I_6| &\leq C(\|\hat{p}\|_M\|\hat{\boldsymbol{u}}\|_{\boldsymbol{X}} + \varepsilon|d(\boldsymbol{u} - \boldsymbol{u}_h)|_r^r + C\varepsilon^{1-\kappa'}\|\hat{p}\|_{L^{\kappa'}}^{\kappa'})\ \forall r \in [2,\infty).\end{aligned}$$

From Lemma 2.1, the left hand side of (3.11) has a lower bound:

$$(H_\alpha^\star(\boldsymbol{x}) - H_\alpha^\star(\boldsymbol{x}_h), \boldsymbol{x} - \boldsymbol{x}_h) \geq C(\alpha|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h|_{r'}^{\rho'} + (1-\alpha)|d(\boldsymbol{u} - \boldsymbol{u}_h)|_r^{\rho} + \mu|\boldsymbol{u} - \boldsymbol{u}_h|_{1,2}^2).$$

Noting the definition of κ, a careful choice of ε gives (3.7) and (3.8).

By Assumption Θ, for $\nu \in (1,\infty)$, we have

$$\begin{aligned}
\beta_h(\nu)\|p_h-q_h\|_{L^{\nu'}} &\le \sup_{\boldsymbol{v}_h\in\boldsymbol{X}_h}\frac{(\operatorname{div}\boldsymbol{v}_h,p_h-q_h)}{\|\boldsymbol{v}_h\|_{1,\nu}}\\
&= \sup_{\boldsymbol{v}_h\in\boldsymbol{X}_h}\frac{(H^\star_\alpha(\boldsymbol{x})-H^\star_\alpha(\boldsymbol{x}_h),\boldsymbol{y}_h)+(\operatorname{div}\boldsymbol{v}_h,p-q_h)}{\|\boldsymbol{v}_h\|_{1,\nu}}\\
&= \sup_{\boldsymbol{v}_h\in\boldsymbol{X}_h}\frac{1}{\|\boldsymbol{v}_h\|_{1,\nu}}[\alpha(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,d(\boldsymbol{v}_h))+(1-\alpha)(B(\boldsymbol{u})-B(\boldsymbol{u}_h),d(\boldsymbol{v}_h))\\
&\quad+2\mu(d(\boldsymbol{u}-\boldsymbol{u}_h),d(\boldsymbol{v}_h))+(\operatorname{div}\boldsymbol{v}_h,p-q_h)]. \qquad (3.12)
\end{aligned}$$

Denote terms on the right hand side of (3.12) by $J_1,\cdots,J_4$, and let ν be κ and s for the case $1<r\le 2$ and $2<r<\infty$, respectively.

Then for $r\in(1,2]$, we have

$$|J_1|\le C\alpha|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h|_{r'}^{\frac{r'}{\kappa'}}\|\boldsymbol{v}_h\|_{1,\kappa}.$$

If $r\in[2,\infty)$

$$|J_1|\le C\alpha(\theta+\|\boldsymbol{\sigma}\|_{\boldsymbol{\gamma}(r',s')}+\|\boldsymbol{\sigma}_h\|_{\boldsymbol{\gamma}(r',s')})^{\frac{2-r'}{2}}|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h|_{r'}\|\boldsymbol{v}_h\|_{1,s}.$$

As to J_2, for the case $r\in(1,2]$

$$\begin{aligned}
|J_2| &\le C(1-\alpha)\|\boldsymbol{v}_h\|_{1,\kappa}\left(\int_\Omega |d(\boldsymbol{u}-\boldsymbol{u}_h|^{\kappa'}(\theta+|d(\boldsymbol{u})|+|d(\boldsymbol{u}_h)|)^{(r-2)\kappa'}\right)^{\frac{1}{\kappa'}} &(3.13)\\
&\le C(1-\alpha)\|\boldsymbol{v}_h\|_{1,\kappa}\left(\int_\Omega |d(\boldsymbol{u}-\boldsymbol{u}_h)|^2(\theta+|d(\boldsymbol{u})|+|d(\boldsymbol{u}_h)|)^{r-2}|d(\boldsymbol{u}-\boldsymbol{u}_h)|\right)^{\kappa'-2} &(3.14)\\
&\quad\times\left(\theta+|d(\boldsymbol{u})|+|d(\boldsymbol{u}_h)|)^{(r-2)(\kappa'-1)}\right)^{\frac{1}{\kappa'}} &(3.15)\\
&\le C(1-\alpha)\|v_h\|_{1,\kappa}|d(\boldsymbol{u}-\boldsymbol{u}_h)|_r^{\frac{2}{\kappa'}}, &(3.16)
\end{aligned}$$

where we have used the following inequality:

$$|d(\boldsymbol{u}-\boldsymbol{u}_h)|^{\kappa'-2}(\theta+|d(\boldsymbol{u})|+|d(\boldsymbol{u}_h)|)^{(r-2)(\kappa'-1)}\le(\theta+|d(\boldsymbol{u})|+|d(\boldsymbol{u}_h)|)^{\kappa'-2+(r-2)(\kappa'-1)}\le 1.$$

As to $r\in[2,\infty)$

$$\begin{aligned}
|J_2| &= C(1-\alpha)\int_\Omega |d(\boldsymbol{v}_h)||d(\boldsymbol{u}-\boldsymbol{u}_h)|(\theta+|d(\boldsymbol{u})|+d(\boldsymbol{u}_h)|)^{\frac{r-2}{2}}(\theta+|d(\boldsymbol{u})|+|d(\boldsymbol{u}_h)|)^{\frac{r-2}{2}}\\
&\le C(1-\alpha)\|\boldsymbol{v}_h\|_{1,s}|d(\boldsymbol{u}-\boldsymbol{u}_h)|_r^{\frac{r}{2}}\left(\int_\Omega(\theta+|d(\boldsymbol{u})|+|d(\boldsymbol{u}_h)|)^\gamma\right)^{\frac{r-2}{2\gamma}}\\
&\le C(1-\alpha)\|\boldsymbol{v}_h\|_{1,s}(\theta+|\boldsymbol{u}|_{1,\boldsymbol{\gamma}}+|\boldsymbol{u}_h|_{1,\boldsymbol{\gamma}})^{\frac{r-2}{2}}|d(\boldsymbol{u}-\boldsymbol{u}_h)|_r^{\frac{r}{2}}\ \forall r\in[2,\infty).
\end{aligned}$$

J_3 can be estimated by use of the inverse inequality

$$|J_3|\le C_I\mu h^{n(\frac{1}{2}-\frac{1}{\kappa})}|\boldsymbol{v}_h|_{1,\kappa}|\boldsymbol{u}-\boldsymbol{u}_h|_{1,2}.$$

Trivially, we get

$$|J_4| \leq |\boldsymbol{v}_h|_{1,\nu} \|\hat{p}\|_{L^{\nu'}}.$$

A combination of the above estimate for J_1 and J_4 yields (3.9) and (3.10). □

Corollary 3.1. *If $\theta = 0, \boldsymbol{\sigma}, \boldsymbol{\sigma}_h \in [L^\infty(\Omega)]^{\frac{n(n+1)}{2}}$ and $\boldsymbol{u}, \boldsymbol{u}_h \in [W^{1,\infty}(\Omega)]^n$, then for the case $r \in (1,2]$ we have the following μ-independent estimate for $\boldsymbol{u}$:*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,2} \leq C|\boldsymbol{u} - \boldsymbol{u}_h|_{1,2} \leq C(|\boldsymbol{u}|_{1,\infty} + |\boldsymbol{u}_h|_{1,\infty})^{\frac{2-r}{2}} |d(\boldsymbol{u} - \boldsymbol{u}_h)|_r. \tag{3.17}$$

If $r > 2$, then

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,2} \leq C(\|\boldsymbol{\sigma}\|_{L^\infty} + \|\boldsymbol{\sigma}_h\|_{L^\infty})^{\frac{2-r'}{2}} |\boldsymbol{\sigma} - \boldsymbol{\sigma}_h|_{r'}.$$

Proof. If $r \in (1,2]$, we have

$$|d(\boldsymbol{u} - \boldsymbol{u}_h)|_r^2 \geq C(|\boldsymbol{u}|_{1,\infty} + |\boldsymbol{u}_h|_{1,\infty})^{r-2} \|d(\boldsymbol{u} - \boldsymbol{u}_h)\|_{L^2}^2.$$

Then (3.17) follows immediately. The proof for $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2}$ is similar. □

4. Error bounds with respect to strong norm

Our task in this section is to exemplify the abstract error bounds in Theorem 3.2. First of all, we give some approximation assumptions on $\boldsymbol{T}_h, \boldsymbol{X}_h$ and M_h.

A_1: for $\boldsymbol{\sigma} \in [W^{k,r'}(\Omega)]^{\frac{n(n+1)}{2}}$, $\inf_{\boldsymbol{\tau} \in \boldsymbol{T}_h} \|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_{\boldsymbol{T}} \leq ch^k |\boldsymbol{\sigma}|_{k,r'}$,

A_2: For $u \in [W^{k+1,r}(\Omega)]^n \cap [W_0^{1,r}(\Omega)]^n$, $\inf_{\boldsymbol{v} \in \boldsymbol{X}_h} \|\boldsymbol{u} - \boldsymbol{v}\|_{\boldsymbol{X}} \leq ch^k |\boldsymbol{u}|_{k+1,r}$,

A_3: For $p \in W^{k,r'}(\Omega) \cap L_0^{r'}(\Omega)$, $\inf_{q \in M_h} \|p - q\|_M \leq ch^k |p|_{k,r'}$.

Theorem 4.1. *Let Assumptions A_1, A_2 and Θ hold, $\alpha \in (\alpha_0, \beta_0), \alpha_0, \beta_0 \in (0,1)$. Let $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \boldsymbol{T}_h \times \boldsymbol{V}_h$ be the unique solution of Problem 3.1. Then*

1. *If $r \in (1,2]$, we have*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\boldsymbol{T}} \leq C_1 h^{k\frac{\kappa}{r'}}, \tag{4.1}$$

and

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,r} \leq C_r \|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{X}} \leq C_r C_1 h^{\frac{k\kappa}{2}}, \tag{4.2}$$

where

$$C_1 = C(|\boldsymbol{\sigma}|_{k,r'}, \|\boldsymbol{f}\|_{-1,r'}, |\boldsymbol{u}|_{k+1,r}, |\boldsymbol{u}|_{k+1,2}, |p|_{k,r'}, \beta_h(r)^{-1}, \beta_h(2)^{-1}),$$

and C_r is the constant appeared in korn inequality [17]. Moreover,

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,2} \leq C_2 h^{\frac{k\kappa}{2}}, \tag{4.3}$$

where

$$C_2 = C(C_1, |\boldsymbol{u}|_{1,\infty}, |\boldsymbol{u}_h|_{1,\infty}).$$

In particular, when $\theta = 1$, we even have $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2} \leq C_1 h^k$.

2. *If $r \in [2,\infty)$, we have*

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\boldsymbol{T}} \le C_3 h^{\frac{k\kappa'}{2}}, \tag{4.4}$$

and

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,r} \le C_r\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{X}} \le C_r C_3 h^{\frac{k\kappa'}{r}}, \tag{4.5}$$

$$C_3 = C(|\boldsymbol{\sigma}|_{k,r'}, |\boldsymbol{\sigma}|_{k,2}, \|\boldsymbol{f}\|_{-1,r'}, |\boldsymbol{u}|_{k+1,r}, |\boldsymbol{u}|_{k+1,2}, |p|_{k,r'}, |p|_{k,2}, \beta_h(r)^{-1}).$$

When $\theta = 1$, we even have

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,2} &\le C_r C_3 h^k,\\ \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L^2} &\le C_4 h^{\frac{k\kappa'}{2}},\end{aligned}$$

$$C_4 = (C_3, \|\boldsymbol{\sigma}\|_{L^\infty}, \|\boldsymbol{\sigma}_h\|_{L^\infty}).$$

3. *Furthermore, if Assumptions A_3 and Δ hold, then*

$$\|p-p_h\|_{L^{\kappa'}} \le C_5\zeta,\ r\in(1,2], \tag{4.6}$$

with

$$\zeta = \min\,(\mu^{\frac12} h^{\frac{k\kappa}{2}+n(\frac12-\frac1\kappa)}, h^{k(\kappa-1)}),$$

and $C_5 = C(C_1, |p|_{k,\kappa'})$.

$$\|p-p_h\|_{L^{s'}} \le C_6 h^{\frac{k\kappa'}{2}},\ r\in[2,\infty), \tag{4.7}$$

$C_6 = C(C_3, |p|_{k,\kappa'})$.

Proof. This is a direct consequence of Theorem 3.2 and Lemma 2.3. □

Remark 4.1. When $r \in (1,2]$, the error bound (4.6) for the pressure will be deteriorated. After some careful analysis, we find that the deterioration happens only in the following limiting cases: (1) $n=2, k=1$, (2) $n=3, k=1$, (3)$n=3, k=2(r\in(1,\frac32))$. If we assume $\mu = h^{(\frac{n}{r}-k)(2-r)}$ in these three cases, the accuracy can be recovered. In fact, this kind of assumption on μ is realistic when μ is very small. Recalling that in this case the proper Sobolev space for the pressure is $L_0^2(\Omega)$, thus we only need to derive error bounds in $L_0^2(\Omega)$. Since the norm on $L_0^2(\Omega)$ is weaker than that on $L_0^{r'}(\Omega)$ in the present situation, we can expect to get μ-independent error bounds, it is just the case.

Corollary 4.1. *With the same assumptions as in the third case of Theorem 4.1, we have*

$$\|p-p_h\|_{L^2} \le C_5 h^{k(r-1)},\ \theta=0, \tag{4.8}$$

$$\|p-p_h\|_{L^2} \le C_5 h^{k},\ \theta=1. \tag{4.9}$$

Proof. (4.9) is a direct consequence of (4.7), but (4.8) needs a proof. We slightly modify the proof method in Section 3. The only difference is that here we use the Assumption Θ with the exponent 2. We still denote the right hand side of (3.12) by $J_1, \cdots, J_4$, then

$$\begin{aligned} |J_1| &\le C\alpha|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h|_{r'}\|v_h\|_{1,r'} \le C\alpha|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h|_{r'}\|v_h\|_{1,2}, \\ |J_2| &\le C(1-\alpha)\|\boldsymbol{v}_h\|_{1,r'}|d(\boldsymbol{u}-\boldsymbol{u}_h)|_r^{\frac{2}{r'}} \le C(1-\alpha)|d(\boldsymbol{u}-\boldsymbol{u}_h)|_r^{\frac{2}{r'}}\|\boldsymbol{v}_h\|_{1,2}, \\ |J_3| &\le \mu|\boldsymbol{u}-\boldsymbol{u}_h|_{1,2}|v_h|_{1,2}, \end{aligned}$$

and for any $q \in M_h$,

$$|J_4| \le \|p-q\|_{L^2}|\boldsymbol{v}_h|_{1,2}.$$

Combining all these estimates and (4.1) and (4.2), we come to (4.8). By (4.8) and (4.9), we have the optimal error bounds for the pressure in the sense of [5].

If $\boldsymbol{X}_h \notin [W^{1,\infty}(\Omega)]^n$ and $\boldsymbol{T}_h \notin [L^\infty(\Omega)]^{\frac{n(n+1)}{2}}$, which would happen in some cases, we cannot obtain an $\mu-$independent estimate for $|\boldsymbol{u}-\boldsymbol{u}_h|_{1,2}$ when $r \in (1,2]$, and for $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L^2}$ when $r \in [2,\infty)$. However, we compromise to obtain error bounds for all variables in a strong norm. In fact, by virtue of Theorem 4.2, it is seen that some $\mu-$independent estimates can be obtained in these two cases for $k \ge 2$. Together with Remark 4.1, it is seen that the only case for accuracy-losing is that the velocity is approximated by an element of a degree lower than 2.

Theorem 4.2. *Let $(\boldsymbol{\sigma},\boldsymbol{u},p),(\boldsymbol{\sigma}_h,\boldsymbol{u}_h,p_h)$ be solutions of Problem 1.1 and Problem 3.1, respectively. If Assumptions A_1, A_2, and A_3 hold, then*

1. *If $r \in (1,2]$ and Assumption Θ holds for the space $\boldsymbol{X}_h$ in the case of $r \in (1,2]$, then*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,s} \le C_1 h^{\frac{k\kappa}{2}} \ \forall s \in [r,2],$$

where

$$s = \begin{cases} \frac{r}{1-\frac{(2-r)\kappa}{2n}}, & \text{if } k=1, \\ 2, & \text{if } k \ge 2, \end{cases}$$

with

$$C_1 = C(|\boldsymbol{\sigma}|_{k,r'}, \|\boldsymbol{f}\|_{-1,r'}, |\boldsymbol{u}|_{k+1,s}, |p|_{k,r}, |\boldsymbol{u}|_{k+1,2}, \beta_h(r)^{-1}, \beta_h(2)^{-1}).$$

2. *If $r \in [2,\infty)$ and Assumption Θ holds for $\boldsymbol{T}_h$ in the case of $s \in [2,r]$, s' and s are the conjugate exponents of each other, then*

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L^{s'}} \le C_2 h^{\frac{kr'}{2}}, \quad \forall s' \in [r',2],$$

where

$$s' = \begin{cases} \frac{r'}{1-\frac{(2-r')\kappa'}{2n}}, & \text{if } k=1, \\ 2, & \text{if } k \ge 2, \end{cases}$$

with

$$C_2 = C(|\boldsymbol{\sigma}|_{k,s'}, \|\boldsymbol{f}\|_{-1,r'}, |\boldsymbol{u}|_{k+1,r}, |\boldsymbol{u}|_{k+1,2}, |p|_{k,r'}, \beta_h(r)^{-1}).$$

3. *If Assumption Δ holds, and Assumption Θ holds for $\boldsymbol{T}_h$ in the case of $r \in [2, s]$, then*

$$\|p - p_h\|_{L^{s'}} \le C_2 h^{\frac{k\kappa'}{2}},$$

s' is defined as in the second case.

The Proof of (i) and (ii) is almost the same as that in [5], but the Proof of (iii) is beyond [5], we give all details.

Proof of (iii). A direct consequence of (i) and (ii) is that $\|\boldsymbol{\sigma}_h\|_{0,\boldsymbol{\gamma}'}$ can be bounded from above by

$$\boldsymbol{\gamma}' = \frac{s^{'}(2-r')}{2-s'} = \begin{cases} \frac{r'}{1-\frac{\kappa'}{n}}, & \text{if } k = 1, \\ \infty, & \text{if } k \ge 2. \end{cases}$$

From Theorem 3.2, we have

$$\begin{aligned}\|p - p_h\|_{L^{s'}} \le C(\beta_h(s)^{-1})(\|\hat{p}\|_{L^{s'}} &+ (\|\boldsymbol{\sigma}\|_{L^{\boldsymbol{\gamma}(r',s')}} + \|\boldsymbol{\sigma}_h\|_{L^{\boldsymbol{\gamma}(r',s')}})^{\frac{2-r'}{2}} |\boldsymbol{\sigma} - \boldsymbol{\sigma}_h|_{r'} \\ &+ (|\boldsymbol{u}|_{1,\boldsymbol{\gamma}(r,s)} + |\boldsymbol{u}_h|_{1,\boldsymbol{\gamma}(r,s)})^{\frac{r-2}{2}} |d(\boldsymbol{u} - \boldsymbol{u}_h)|_r^{\frac{r}{2}} + C_I\mu|\boldsymbol{u} - \boldsymbol{u}_h|_{1,2}).\end{aligned}$$

On the other hand, from Assumption Δ and Theorem 4.1, for $\boldsymbol{\gamma}(0) \ge s(0) = r \ge 2$, we have

$$\begin{aligned} |\boldsymbol{u}_h|_{1,\boldsymbol{\gamma}(0)} &\le \inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h} (|\boldsymbol{u}_h - \boldsymbol{v}_h|_{1,\boldsymbol{\gamma}(0)} + |\boldsymbol{v}_h|_{1,\boldsymbol{\gamma}(0)}) \\ &\le C_I h^{n(\frac{1}{\boldsymbol{\gamma}(0)} - \frac{1}{s(0)})} \inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h} |\boldsymbol{u}_h - \boldsymbol{v}_h|_{1,r} + C|\boldsymbol{u}|_{1,\boldsymbol{\gamma}(0)} \\ &\le C_I h^{n(\frac{1}{\boldsymbol{\gamma}(0)} - \frac{1}{s(0)})} \inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h} (|\boldsymbol{u} - \boldsymbol{u}_h|_{1,r} + |\boldsymbol{u} - \boldsymbol{v}_h|_{1,r}) + C|\boldsymbol{u}|_{1,\boldsymbol{\gamma}(0)} \\ &\le C h^{n(\frac{1}{\boldsymbol{\gamma}(0)} - \frac{1}{s(0)}) + k\frac{\kappa'}{r}} + C, \end{aligned}$$

where C depends on $|\boldsymbol{u}|_{k+1,r}, |\boldsymbol{\sigma}|_{k,r'}, |p|_{k,r'}, \beta_h(\nu), \nu \in [2, r]$. We can set

$$\boldsymbol{\gamma}(0) = \begin{cases} \frac{r}{1-\frac{\kappa'}{n}}, & \text{if } k = 1, \\ \infty, & \text{if } k \ge 2. \end{cases}$$

Observing that $s = s(r, \boldsymbol{\gamma}) = \frac{2\boldsymbol{\gamma}}{2-r+\boldsymbol{\gamma}}$, we have $s = s(r, \boldsymbol{\gamma}(0)) = \frac{r}{1+\frac{(r-2)\kappa'}{2n}}$. Thus $s' = \frac{r'}{1-\frac{(2-r')\kappa'}{2n}}$. If $|\boldsymbol{\sigma}_h|_{L^{\boldsymbol{\gamma}(r',s')}}$ can be bounded, we complete the proof. Using the definition of $\boldsymbol{\gamma}(r', s') = \frac{s(0)^{'}(2-r')}{2-s(0)'} = \frac{r'}{1-\frac{\kappa'}{n}}$, the expected assertion is actually true. □

Now we want to exemplify the abstract Assumptions A_1, A_2, A_3, Δ and Θ. Indeed, Assumptions A_1, A_2, A_3, Δ are standard and easily be satisfied. For example, let $\boldsymbol{T}_h$ be discontinuous(or continuous) piecewise polynomials of degree $k-1 (k \ge 2)$, $\boldsymbol{X}_h$- continuous piecewise polynomials of degree k, M_h- continuous(or discontinuous) piecewise polynomials of degree $k-1$, then Assumptions A_1, A_2, A_3 will be satisfied. The Inverse Assumption Δ for $\boldsymbol{T}_h, \boldsymbol{X}_h$ also holds if we assume the mesh is quasi-uniform. However, choosing $\boldsymbol{X}_h, M_h$ so that the Assumption Θ holds or not is a delicate task, though there are many examples for $r = 2$ (see [14]). Nevertheless, there exist by far less examples for the case $r \ne 2$ [22]. In the following, we will give some examples that can be justified strictly.

We will quote some finite element spaces without denotations, and refer to [7] and [14] for details. Let $\mathcal{P}_m$ be the space of all polynomials of degree less than or equal to m, and $\mathcal{Q}_m$ be the space of all polynomials of degree less than or equal to m in each variable. Let K be an arbitrary triangle of the partition $\mathcal{C}_h$ with vertices a_1, a_2 and a_3, the middle point of each edge denote $a_{i,j}$(oppose to a_k). In addition, we denote by e_i the edge opposite to a_i and by $\boldsymbol{n}_i$ and $\boldsymbol{\tau}_i$ the unit normal and tangent to e_i.

Theorem 4.3. *If $\boldsymbol{X}_h, M_h$ are the following finite element space pair:*

1. *Crouzeix-Raviart finite element space. [10]*
2. *Mini finite element space [14] and $(\mathcal{P}_1^+, \mathcal{P}_0), (\mathcal{Q}_1^+, \mathcal{P}_0)$ finite element space. [11]*
3. *Hood-Taylor finite element space $(\mathcal{P}_2 - \mathcal{P}_1)$, $(\mathcal{P}_3 - \mathcal{P}_2)$. [15]*
4. *rectangular Hood-Taylor finite element space $(\mathcal{Q}_k, \mathcal{Q}_{k-1})$, [6]*

then the Assumption Θ holds for $r \in (1, \infty)$.

Proof. We only check Case 3. Case 1 and 2 can be proved following the procedure in [5]. For Case 3, we only deal with $\mathcal{P}_2 - \mathcal{P}_1$ in 2-$\mathcal{D}$, but it can be extended to 3-$\mathcal{D}$ and $\mathcal{P}_3 - \mathcal{P}_2$ without any difficulty.

By virtue of [18, Lem. 2.4], we have

$$\inf_{p_h \in M_h} \sup_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \frac{(\operatorname{div} \boldsymbol{v}_h, \overline{p}_h)}{\|\boldsymbol{v}_h\|_{\boldsymbol{X}} \|p_h\|_M} \geq C,$$

where $\overline{p}_{h|_K} = \frac{1}{|K|} \int_K p_h$,

$$\begin{aligned} (\operatorname{div} \boldsymbol{v}_h, p_h) &= (\operatorname{div} \boldsymbol{v}_h, \overline{p}_h) + (\operatorname{div} \boldsymbol{v}_h, p_h - \overline{p}_h) \\ &\geq C\|\boldsymbol{v}_h\|_{\boldsymbol{X}}(\|\overline{p}_h\|_M - \|p_h - \overline{p}_h\|_M). \end{aligned}$$

Let $\overline{\boldsymbol{v}}_{h|_K} = \frac{\boldsymbol{v}_h}{\|\boldsymbol{v}_h\|_{\boldsymbol{X}}} \|\overline{p}_h\|_M^{r'-1}$, then $\|\overline{\boldsymbol{v}}_h\|_{\boldsymbol{X}} = \|\overline{p}_h\|_M^{r'-1}$,

$$(\operatorname{div} \overline{\boldsymbol{v}}_h, p_h) \geq C_1 \|\overline{p}_h\|_M^{r'} - C_2 \|p_h - \overline{p}_h\|_M \|\overline{p}_h\|_M^{r'-1}. \tag{4.10}$$

Now let $\mathcal{S}_2$ be the set of triangles with two sides on Γ, $\mathcal{S}_3$ be the set of triangles sharing a common edge with triangles of $\mathcal{S}_2$, and $\mathcal{S}_1$ be the remaining triangles. We construct $\boldsymbol{z}_h \in \boldsymbol{X}_h$ with $\boldsymbol{z}_h = 0$ at all vertices of the triangles, $\boldsymbol{z}_h \cdot \boldsymbol{n}$ is zero at all nodes on edges of triangles, with the exception of nodes on the common edge of $\mathcal{S}_2 \cup \mathcal{S}_3$, and $\boldsymbol{z}_h \cdot \boldsymbol{\tau}$ vanishes on the boundary of Ω. Define

$$\begin{aligned} \boldsymbol{z}_h \cdot \boldsymbol{\tau}(a_{ij}) &= -|e|^{r'} |\nabla p_h \cdot \boldsymbol{\tau}(a_{ij})|^{r'-2} \nabla p_h(a_{ij}), \\ \boldsymbol{z}_h \cdot \boldsymbol{n}(a_{ij}) &= -\varepsilon |e|^{r'} |\nabla p_h^2 \cdot \boldsymbol{n}(a_{ij})|^{r'-2} \nabla p_h \cdot \boldsymbol{n}(a_{ij}), \end{aligned}$$

where p_h^2 means that p_h is defined on $K \in \mathcal{S}_2$, ε is a constant to be chosen later. We first show

$$\|\boldsymbol{z}_h\|_{\boldsymbol{X}} \leq C \|p_h - \overline{p}_h\|_M^{r'-1}. \tag{4.11}$$

In fact, since a_{ij}, a_i is a unisolvent set of points for polynomials of degree less than 2 on K, we write $\boldsymbol{z}_h$ in terms of the standard Lagrangian function corresponding to those points and using the standard inverse estimate. Then for $K_1 \in \mathcal{S}_1$, we have

$$\begin{aligned} \|\boldsymbol{z}_h\|_{\boldsymbol{X}, K_1} \leq C \sum |\boldsymbol{z}_h(a_{ij})| |K|^{\frac{1}{r}} |e|^{-1} &\leq C |K|^{\frac{1}{r}} |e|^{r'-1} \|\nabla (p_h - \overline{p_h})\|_{0,\infty,K_1}^{r'-1} \\ &\leq C |K|^{\frac{1}{r}} |e|^{r'-1} |e|^{\frac{2(1-r')}{r'}} \|\nabla (p_h - \overline{p}_h)\|_{0,r',K_1}^{r'-1} \\ &\leq C \|p_h - \overline{p}_h\|_{0,r',K_1}^{r'-1}. \end{aligned}$$

For $K_1 \in \mathcal{S}_2$, the estimate is the same. For $K_3 \in \mathcal{S}_3$, we have

$$\|\boldsymbol{z}_h\|_{\boldsymbol{X}, K_3} \leq C \|p_h - \overline{p}_h\|_{0,r',K_2 \cup K_3}^{r'-1},$$

hence (4.11) is proved.

Furthermore, $\forall K_1 \in \mathcal{S}_1$, we have

$$\int_{K_1} \operatorname{div} z_h p_h = -\int_{K_1} \boldsymbol{z}_h \nabla p_h = -\frac{|K_1|}{3} \sum_{e \notin \Gamma} \sum_{i \neq j} \boldsymbol{z}_h \nabla p_h(a_{ij}) = \frac{|K_1|}{3} |e|^{r'} \sum_{e \notin \Gamma} \sum_{i \neq j} |(\nabla p_h \cdot \boldsymbol{\tau})(a_{ij})|^{r'}. \tag{4.12}$$

It is easy to check that the expression

$$\left[\sum_{e \notin \Gamma} \sum_{i \neq j} |(\nabla p_h \cdot \boldsymbol{\tau})(a_{ij})|^{r'} \right]^{\frac{1}{r}} \tag{4.13}$$

vanishes only if p_h is a constant and hence is a norm over linear polynomials modulo constants on K_1. Using the scaling trick and (4.12), we can easily prove the following inequality

$$\int_{K_1} \operatorname{div} \boldsymbol{z}_h p_h \geq C \|p_h - \overline{p}_h\|_{0,r',K_1}^{r'}. \tag{4.14}$$

In fact,

$$\begin{aligned} \|p_h - \overline{p}_h\|_{0,r',K_1}^{r'} &\leq C|K| \|\hat{p}_h - \overline{\hat{p}}\|_{0,r',\hat{K}}^{r'} \leq C|K| \sum_{e \notin \Gamma} \sum_{i \neq j} |\hat{\nabla} \hat{p}_h(\hat{a}_{ij})|^{r'} \\ &\leq C|K| \sum_{e \notin \Gamma} |e|^{r'} \sum_{i \neq j} |\nabla p_h(a_{ij})|^{r'} \leq C \int_{K_1} \operatorname{div} z_h p_h. \end{aligned}$$

Now let $K_2 \in \mathcal{S}_2$ and $K_3 \in \mathcal{S}_3$ share a common edge e_3 and denote by $\boldsymbol{\tau}^3$ and η^3 the counterclockwise unit tangent and outward unit normal vectors to K_2 along e_3. Therefore,

$$\begin{aligned} \int_{K_2} \operatorname{div} \boldsymbol{z}_h p_h + \int_{K_3} \operatorname{div} \boldsymbol{z}_h p_h &= -\int_{K_2} z_h \nabla p_h - \int_{K_3} \boldsymbol{z}_h \nabla p_h \\ &= \frac{|K_2|}{3} \sum_{e_3} \sum_{i \neq j} [|\nabla p_h^2 \cdot \boldsymbol{\tau}^3|^{r'}(a_{ij})|e|^{r'} + \varepsilon |\nabla p_h^2 \cdot \eta^3|^{r'}(a_{ij})|e|^{r'}] \\ &\quad + \frac{K_3}{3} \sum_{e \notin \Gamma} \sum_{i \neq j} [\ |\nabla p_h^3 \cdot \boldsymbol{\tau}|^{r'}(a_{ij}) + \varepsilon \sum_{e_3} \sum_{i \neq j} |\nabla p_h^3 \cdot \eta^3|^{r'-1} \nabla p_h^2 \cdot \eta^3(a_{ij})] \\ &\geq C \|p_h - \overline{p}_h\|_{0,r',K_2 \cup K_3}^{r'}, \end{aligned}$$

where we have used the Young inequality and the scaling trick. Consequently, we have

$$(\operatorname{div} \boldsymbol{z}_h, p_h) \geq C \|p_h - \overline{p}_h\|_M^{r'}. \tag{4.15}$$

Let $\tilde{v}_h = \overline{\boldsymbol{v}}_h + \beta \boldsymbol{z}_h$, then

$$\begin{aligned} (\operatorname{div} \tilde{\boldsymbol{v}}_h, p_h) &\geq C_1 \|\overline{p}_h\|_M^{r'} - C_2 \|p_h - \overline{p}_h\|_M \|\overline{p}_h\|_M^{r'-1} + C_3 \beta \|p_h - \overline{p}_h\|_M^{r'} \\ &\geq C_4 \|\overline{p}_h\|_M^{r'} + C_5 \|p_h - \overline{p}_h\|_M^{r'} \\ &\geq C_6 \|p_h\|_M^{r'}. \end{aligned}$$

Noting $\|\tilde{\boldsymbol{v}}_h\|_{\boldsymbol{X}} \leq C \|p_h\|_M^{r'-1}$, therefore

$$\frac{(\operatorname{div} \tilde{\boldsymbol{v}}_h, p_h)}{\|\tilde{\boldsymbol{v}}_h\|_{\boldsymbol{X}}} \geq C \|p_h\|_M.$$

This means that the Assumption Θ holds for Hood-Taylor finite element space $(\mathcal{P}_2 - \mathcal{P}_1)$ in 2-$\mathcal{D}$. □

Remark 4.2. Following almost the same line, we can prove the case 4. However, a more intricate scaling trick is involved and the tensor product of 1-$\mathcal{D}$ Gauss-Lobatto formula has to be used. Furthermore, the so-called *Hierarchic shape function on rectangle* [24] plays a fundamental role in the treatment of an expression like (4.10).

Remark 4.3. In [19], we proposed a Macro-element based trick to check the Assumption Θ, which is very clear and clean.

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