DUAL COMBINED FINITE ELEMENT METHODS FOR NON-NEWTONIAN FLOW (II) PARAMETER-DEPENDENT PROBLEM

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Abstract. This is the second part of the paper for a Non-Newtonian flow. Dual combined Finite Element Methods are used to investigate the little parameter-dependent problem arising in a nonlinear three field version of the Stokes system for incompressible fluids, where the viscosity obeys a general law including the Carreau’s law and the Power law. Certain parameter-independent error bounds are obtained which solved the problem proposed by Baranger in [4] in a unifying way. We also give some stable finite element spaces by exemplifying the abstract $B-B$ inequality. The continuous approximation for the extra stress is achieved as a by-product of the new method.

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1. Introduction

In this section, we will give a brief description of a White-Metzner model type for the viscoelastic flow. We refer to [9] for more details.

Let $\Omega$ be a bounded convex polygon in $\mathbb{R}^n (n = 2, 3)$ with the Lipschitz boundary $\Gamma$. $\mathbb{R}^n$ is equipped with Cartesian coordinates $x_i, i = 1, \cdots, n$. For a function $u$, $\frac{\partial u}{\partial x_i}$ is written as $u_i$, the Einstein convention for a summation is used.

For a scalar function $p$, the gradient of $p$ is a vector $\nabla p$. $(\nabla p)_i = p_{,i}$. If $q$ is another scalar function, we denote $(p, q) = \int_{\Omega} pq$. For a vector function $u$, the gradient of $u$ is a tensor $\nabla u$. $(\nabla u)_{ij} = u_{i,j}$, div $u = u_{,i}$, $u \nabla = u_i \frac{\partial}{\partial x_i}$. For a tensor function $\sigma$, div $\sigma$ is a vector function, $(\text{div} \, \sigma)_i = (\sigma_{ij,j})$, $\tau = \sigma_{ij,j}$, $|\sigma|^2 = \sigma : \sigma$, and $(\sigma, \tau) = \int_{\Omega} \sigma : \tau$.

To describe the flow, we use the pressure $p$ (scalar), the velocity vector $u$ and the total stress tensor $\sigma_{\text{tot}}$. $\nabla u$ is the velocity gradient tensor, $d(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the rate of strain tensor, and $d_{\text{II}}(u) = \frac{1}{2}d_{ij}(u)d_{ij}(u)$ is the second invariant of $d(u)$.

A White-Metzner type model is described by the constitutive equations:

$$\sigma_{\text{tot}} = \sigma + \sigma_N - pI, \quad I = \delta_{ij}, \quad (1.1)$$
$$\sigma_N = 2\mu d(u), \quad (1.2)$$
$$\sigma = 2\eta(d_{\text{II}}(u))d(u), \quad (1.3)$$

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Lemma 1.1. There exists an operator $A : T \to T'$ such that $A(\sigma) = d(u)$.

The proof of this lemma only needs a simple manipulation, we omit it.

Some works have been done in the finite element approximation of the Problem 1.1. In [4], Baranger et al. gave the first approximation of the three-field Stokes flow of White-Metzner model. However, the abstract error bounds they obtained are $\mu$-dependent, i.e., the error bounds will be deteriorated as the Newtonian viscosity approaches zero. Furthermore, no concrete finite element space pair is available. In fact, the well-posedness of their variational formulation needs two $B-B$ inequalities, one for $(\sigma, u)$ and the other for $(u, p)$. The construction of finite element space pair satisfying these two $B-B$ inequalities simultaneously is not a trivial thing at all, even in the linear case (see [20]), since these two $B-B$ inequalities are interplaying. The relevant two-field problem has been studied in [21], but the abstract error bounds are also $\mu$-dependent, and still there is no concrete finite element space pair presented. Meanwhile, the continuous approximation for the extra stress is widely used in engineer literatures, however, a fairly large finite element space is needed to achieve this goal [13, 23] that would cause an extra cost and accuracy-losing. Recently, the so-called EVSS and its variants [12] are proposed to attack this problem, but it needs an extra variable that would increase the cost, and even seriously that would lead to unsymmetric algebraic equations.

In the following, dual combined finite element method [3, 18, 25] is proposed to solve the above problems.

To introduce the dual combined mixed formulation, we present some operators:

$$B : X \to X', \quad B(u) = 2\eta(d_1(u))d(u).$$

$$H_\alpha : T \times X \times M \to T' \times X' \times M'.$$

$$l : T \times X \times M \to T' \times X' \times M'.$$

For $x = (\sigma, u, p)$, $y = (\tau, v, q)$, we define

$$(H_\alpha(x), y) = \alpha(A(\sigma), \tau) - \alpha(\tau, d(u)) + \alpha(\sigma, d(u)) + (1 - \alpha)(B(u), d(v)) + 2\mu(d(u), d(v)) - (p, \text{div } v) + (q, \text{div } u), \text{ where } \alpha \in [0, 1],$$

and $(l, y) = (f, v)$. 

where $\mu$ is the Newtonian part of the viscosity, $\eta$ - the viscosity function for the viscoelastic part. The velocity $u$ must satisfy the incompressible condition

$$\text{div } u = 0.$$ 

In this paper we consider the stationary creeping flow [16]. The fluid is submitted to a density of forces $f$, then the momentum equation is written as

$$-\text{div} \sigma = f.$$ 

Two classical laws for $\eta$ are the Power law and Carreau’s law.

- Power law: $\eta_p(z) = \frac{1}{2}y^{(r-1)/r}, \ r > 1, \ g \geq 0$.
- Carreau’s law: $\eta_c(z) = (\eta_0 - \eta_\infty)(1 + \lambda z)^{\frac{r-1}{r}}, \ 0 \leq \eta_\infty < \eta_0, \ r > 1$.

Some Sobolev spaces [1] are needed. $T = [L^r(\Omega)]^{n(n+1)/2} = \{\tau = (\tau_{ij}) | \tau_{ij} = \tau_{ji}, \ \tau_{ij} \in L^r(\Omega), \ i, j = 1, \cdots, n\}$ with the norm $\|\tau\|_T = (\int_\Omega |\tau|^r)^{\frac{1}{r}}$. $(,)$ denotes the inner product of $X = [W_0^{1,r}(\Omega)]^n, M = L^r(\Omega) = \{q \in L^r(\Omega) | \int_\Omega q = 0\}$. $X, M$ are equipped with the norm $\|v\|_X = (\int_\Omega |d(v)|^r)^{\frac{1}{r}}, \ |q|_M = (\int_\Omega |q|^r)^{\frac{1}{r}}$ respectively. It is easy to see $\|\cdot\|_M$ is an equivalent norm over $X$. We also denote $T', X'$ the dual space of $T$ and $X$ respectively, and $(,)$ denotes the dual multiple between $X$ and $X'$. For $1 < r < 2$, we must modify the definition of $X$ and define $X_1 = X \cap [H^1_0(\Omega)]^n$, but for simplicity we still denote it as $X$. 


Problem 1.1. Find $x \in T \times X \times M$ such that

$$(H_\alpha(x), y) = \langle l, y \rangle \forall y \in T \times X \times M, \, \forall \alpha \in [0, 1].$$

This problem can be cast into the following saddle point problem.

Problem 1.2. Find $x \in T \times X \times M$ such that

$$\begin{align*}
\alpha(A(\sigma), \tau) - \alpha(\tau, d(u)) &= 0 & \forall \tau \in T, \\
\alpha(\sigma, d(v)) + (1 - \alpha)(B(u), d(v)) + 2\mu(d(u), d(v)) - (p, \text{div } v) &= (f, v) & \forall v \in X, \\
(\text{div } u, q) &= 0, & \forall q \in M,
\end{align*}$$

for some $\alpha \in [0, 1]$.

When $\alpha$ equal to 1 or 0, Problem 1.2 will degenerate to problems which have been discussed in [4] and [5], respectively. The case $\alpha \in (0, 1)$ is more interesting, we discuss it from now on.

An outline of this paper follows. Some basic inequalities and preliminaries are introduced in §2. In §3, dual combined FEM is proposed to deal with the non-Newtonian flow with Newtonian viscosity. The error bounds for all variables are given under some abstract assumptions. In the last section, we exemplify the abstract assumptions. Error estimates with respect to a strong norm are derived, and some stable finite element spaces are presented and justified.

Throughout this paper $C, C_1$ denote generic constants independent of $\mu$. In addition, $C(a_1, \ldots, a_m)$ denotes a constant depending on the non-negative parameter $a_i (1 \leq i \leq m)$ such that $C(a_1, \ldots, a_m) \leq C$.

2. SOME INEQUALITIES AND PRELIMINARIES

We use the following notations: $\mathbb{M}_n$ is the set of $n \times n$ real matrices. $\langle \cdot, \cdot \rangle$ denotes the inner product on $M_n$, i.e., for $X, Y \in \mathbb{M}_n$, $\langle X, Y \rangle = \sum_{i,j=1}^n X_{ij}Y_{ij}$, $|X| = \langle X, X \rangle^{1/2}$. $\theta = 0$ if $\eta = \eta_p$, $\theta = 1$ if $\eta = \eta_c$.

Lemma 2.1. [5] Let $K(|X|) = (\theta + |X|^2)^{r/2}$ and $r \in (1, \infty)$. Then for all $X, Y \in \mathbb{M}_n(n \geq 1)$ and $\delta \geq 0$, we have

$$\begin{align*}
|K(|X|)X - K(|Y|)Y| &\leq c|X - Y|^{1-\delta}(\theta + |X| + |Y|)^{r-2+\delta}, \\
|K(|X|)X - K(|Y|)Y, X - Y| &\geq C|X - Y|^{2\delta}(\theta + |X| + |Y|)^{r-2-\delta}.
\end{align*}$$

Lemma 2.2. [5] For all $r \in (1, \infty), a, \sigma_1, \sigma_2 \geq 0$, and $\varepsilon \in (0, 1)$, we have

$$\begin{align*}
(\theta + a + \sigma_1)^{r-2}\sigma_1\sigma_2 &\leq \varepsilon(\theta + a + \sigma_1)^{r-2}\sigma_1^2 + \varepsilon^{-\gamma}(\theta + a + \sigma_2)^{r-2}\sigma_2,
\end{align*}$$

where $\gamma = \max(1, r - 1)$ and $\theta$ is defined below.

We adopt the abstract quasi-norm introduced in [5], which is very useful for error estimates. Let $(\sigma, u) \in T \times X$ be the solution of Problem 1.1, then for $(\tau, v) \in T \times X$, we define the following quasi-norms:

$$\begin{align*}
|\tau|_{r'}^\rho &= \int_\Omega |\tau|^2(\theta + |\sigma| + |\tau|)^{r'-2}, \, \rho' = \max(2, r'), \\
|d(v)|_{r'}^\rho &= \int_\Omega |d(v)|^2(\theta + |d(u)| + |d(v)|)^{r'-2}, \, \rho = \max(2, r).
\end{align*}$$
Lemma 2.3. [5] Define

$$\gamma = \gamma(r,s) = \begin{cases} 2, & \text{if } r = 2, \\ \frac{s(2-r)}{2-s}, & \text{if } r \neq 2, s \neq 2, \\ \infty, & \text{if } r \neq 2, s = 2. \end{cases}$$

Then

1. For $1 < r \leq s \leq 2$, and $u, w \in [W^{1,\gamma}(\Omega)]^n$, we have

$$|w|_{1,s}^r \leq C(\theta + |u|_{1,\gamma} + |w|_{1,\gamma})^{2-r}|d(w)|_r^2. \quad (2.4)$$

2. For $r \in (1,2], \kappa \in [r, r + (2-r)\theta]$, and $w \in [W^{1,\kappa}(\Omega)]^n$, we have

$$|d(w)|_r^2 \leq C\|d(u)||\kappa^r < |w|_{1,r}^r. \quad (2.5)$$

3. For $r \in [2,\infty), \kappa \in [r + (2-r)\theta, r]$ and $w \in [W^{1,r}(\Omega)]^n$, we have

$$|w|_{1,s}^r \leq C|d(w)||\kappa^r \leq |d(w)|_r^r. \quad (2.6)$$

4. For $r \in [2,\infty)$, $2 \leq s \leq r < \infty$ and $u, w \in [W^{1,\gamma}(\Omega)]^n$, we have

$$|d(w)|_r^r \leq C(\theta + |u|_{1,\gamma} + |w|_{1,\gamma})^{r-2}|w|_{1,s}^2. \quad (2.7)$$

Lemma 2.4. [5] $|\cdot|_r$ and $|\cdot|_r$, are quasi-norms over $X, T$, respectively.

Though the explicit formula for $A(\sigma)$ is not available when $\eta = \eta_0$, we are still able to prove the monotony and continuity properties of $A(\sigma)$ in this case.

Theorem 2.1. There exists a constant $C$ independent of $\sigma, \tau$ such that for any $\delta > 0$

$$A(\sigma) - A(\tau), \sigma - \tau \geq C|\sigma - \tau|^{2+\delta}(1 + |\sigma| + |\tau|)^{r-2-\delta}, \quad (2.8)$$

$$|A(\sigma) - A(\tau)| \leq C|\sigma - \tau|^{1-\delta}(1 + |\sigma| + |\tau|)^{r-2+\delta}. \quad (2.9)$$

Proof. Note

$$(A(\sigma) - A(\tau), \sigma - \tau) = (d(u) - d(v), (1 + |d(u)|^2)^{\frac{r-2}{2}}d(u) - (1 + |d(v)|^2)^{\frac{r-2}{2}}d(v)).$$

Therefore, we only need to check the following two facts.

If $x, y \in \mathbb{R}^n, \phi(x) = (1 + |x|^2)^{\frac{r-2}{2}}x = \alpha, \phi(y) = \beta, x = \psi(\alpha), y = \psi(\beta)$, then

$$\langle \psi(\alpha) - \psi(\beta), \alpha - \beta \rangle \geq C|\alpha - \beta|^{2+\delta}(1 + |\alpha| + |\beta|)^{r-2+\delta}, \quad (2.10)$$

$$|\psi(\alpha) - \psi(\beta)| \leq C|\alpha - \beta|^{1-\delta}(1 + |\alpha| + |\beta|)^{r-2+\delta}. \quad (2.11)$$

We firstly prove that (2.10) holds for $\delta = 0$.

Define an operator $i : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}, i(x) = (x, 0), v \in \mathbb{R}^{n+1}, v_i = \delta_{i,n+1}$. Then

$$\alpha = \phi(x) = (1 + |x|^2)^{\frac{r-2}{2}}x = |v + i(x)|^{r-2}x, y = |v + i(y)|^{r-2}y.$$
We also have
\[
\begin{align*}
i(\alpha) &= |v + i(x)|^{r-2}i(x), \\
i(\beta) &= |v + i(y)|^{r-2}i(y), \\
i(\hat{\alpha}) &= |v + i(x)|^{r-2}(v + i(x)), \\
i(\hat{\beta}) &= |v + i(y)|^{r-2}(v + i(y)).
\end{align*}
\]

\[
\langle \psi(\alpha) - \psi(\beta), \alpha - \beta \rangle = \langle \phi(x) - \phi(y), x - y \rangle
\]
\[
= \langle |v + i(x)|^{r-2}(v + i(x)) - |v + i(y)|^{r-2}(v + i(y)), i(x) - i(y) \rangle
\]
\[
= \langle |v + i(x)|^{r-2}(v + i(x)) - |v + i(y)|^{r-2}(v + i(y)), (v + i(x)) - (v + i(y)) \rangle
\]
\[
= (i(\hat{\alpha}) - i(\hat{\beta}), |i(\hat{\alpha})|^{r-2} - |i(\hat{\beta})|^{r-2})i(\hat{\beta})
\]
\[
\geq C|i(\hat{\alpha}) - i(\hat{\beta})|^2|\hat{\beta}|^{r-2}.
\]

Note
\[
|i(\hat{\alpha}) - i(\hat{\beta})|^2 = |i(\alpha) - i(\beta)|^2 + |v + i(x)|^{r-2} - |v + i(y)|^{r-2}|^2 \geq |i(\alpha) - i(\beta)|^2.
\]

Furthermore, we have
\[
|i(\hat{\alpha})|^2 = |i(\alpha)|^2 + |v + i(x)|^{2r-4} = |i(\alpha)|^2 + |i(\hat{\alpha})|^{4-2r}.
\]

Therefore, we need to show that there exists a constant $C$ such that
\[
\left(|i(\hat{\alpha})| + |i(\hat{\beta})| \right)^{r-2} \geq (1 + |i(\alpha)| + |i(\beta)|)^{r-2}.
\] (2.12)

If $r' \in [2, \infty)$, i.e. $r \in (1, 2)$, we only need to prove
\[
|i(\hat{\alpha})| + |i(\hat{\beta})| \geq C(1 + |i(\alpha)| + |i(\beta)|).
\]

To do this, we show $|i(\hat{\alpha})| \geq C(1 + |i(\alpha)|)$ and $|i(\hat{\beta})| \geq C(1 + |i(\beta)|)$. These two inequalities can be cast into the following basic inequality:
\[
(1 + |x|^2)^{r-2} \geq (1 + (1 + |x|^2)^{r-2})^2 |x|.
\] (2.13)

Define a function $f(t) = \frac{(1+t^2)^{r-1}}{1+(1+t^2)^{r-1}}$ for $t > 0$, we show that $f(t)$ has a positive lower bound. In fact, $f(t)$ is a continuous function over interval $(0, \infty)$, and $f(0) = 1$, $\lim_{t \to \infty} f(t) = 1$. Furthermore, $f(t) > 0$ for $t \in (0, \infty)$, hence $f(t)$ must have a positive lower bound. Consequently, we have proved (2.13) for the case $r' \in [2, \infty)$. Similarly, we can prove it for the case $r' \in (1, 2)$. Hence (2.12) holds for all $r \in (1, \infty)$, therefore
\[
\langle A(\sigma) - A(\tau), \sigma - \tau \rangle \geq C|\sigma - \tau|^2(1 + |\sigma| + |\tau|)^{r-2}
\]
\[
\geq C|\sigma - \tau|^{2+\delta}(1 + |\sigma| + |\tau|)^{r-2-\delta},
\]

which is the first inequality (2.8) of the theorem.

Now it remains to show the continuity property of the operator $A$. We have
\[
(A(\sigma) - A(\tau), \sigma - \tau) = (d(u) - d(v), B(u) - B(v))
\]
\[
\geq C|d(u - v)|^2(1 + |d(u)| + |d(v)|)^{r-2}
\]
\[
= C|A(\sigma) - A(\tau)|^2(1 + |A(\sigma)| + |A(\tau)|)^{r-2}.
\]

Using Cauchy inequality yields
\[
|A(\sigma) - A(\tau)| \leq C|\sigma - \tau|(1 + |A(\sigma)| + |A(\tau)|)^{2-r}.
\]
On the other hand, we can prove that

\[(1 + |A(\sigma)| + |A(\tau)|)^{2-r} \leq C(1 + |\sigma| + |\tau|)^{r''-2}. \tag{2.14}\]

In fact, since \(A(\sigma) = d(u)\), (2.14) is equivalent to the following:

\[(1 + |d(u)| + |d(v)|)^{2-r} \leq C(1 + |B(u)| + |B(v)|)^{r''-2}. \tag{2.15}\]

As before, we only need to show

\[(1 + |x| + |y|)^{2-r} \leq C(1 + (1 + |x|^2)^{r''-2}|x| + (1 + |y|^2)^{r''-2}|y|)^{r''-2}, \tag{2.16}\]

which is just a direct consequence of the following inequality:

\[(1 + |x|)^{2-r} \leq (1 + (1 + |x|^2)^{r''-2}|x|)^{r''-2}, \quad \text{i.e.} \]

\[(1 + |x|)^{r''-1} \leq 1 + (1 + |x|^2)^{r''-2}|x|, \quad r \in (1, 2], \tag{2.17}\]

\[(1 + |x|)^{r''-1} \geq 1 + (1 + |x|^2)^{r''-2}|x|, \quad r \in [2, \infty). \tag{2.18}\]

Let \(t = |x|\), we define

\[G(t) = 1 + (1 + t^2)^{\frac{r''-2}{2}} t - (1 + t)^{r''-1}, \quad t \geq 0.\]

Then if \(r \in (1, 2]\),

\[G'(t) = (1 + t^2)^{\frac{r''-2}{2}}((1 + (r - 1)t^2) - (r - 1)(1 + t)^{r''-2}) \geq 0.\]

Hence \(G(t) \geq G(0)\) for \(r \in (1, 2]\), which means (2.17) holds. Similarly, (2.18) holds. Consequently, (2.14) and then (2.15), (2.16) hold.

Combining (2.15) and (2.16), we have

\[|A(\sigma) - A(\tau)| \leq C|\sigma - \tau|(1 + |\sigma| + |\tau|)^{r''-2} \leq C|\sigma - \tau|^{1-\delta}(1 + |\sigma| + |\tau|)^{r''-2+\delta}, \]

which completes the proof. \( \Box \)

**Corollary 2.1.** \((1 + |A(\sigma)|)^{2-r} \leq (1 + |\sigma|)^{r''-2} \).

**Corollary 2.2.** There exists a constants \(C\) such that

\[(A(\sigma), \sigma) \geq C \int_{\Omega} |\sigma|^2(1 + |\sigma|)^{r''-2}. \]
Using previous results we can prove the existence and uniqueness of solutions of Problem 1.1. Setting $V = \{ v \in X \mid \text{div} \ v = 0 \}$ and dropping the terms concerning the pressure, we introduce the following operator:

$$H_0^* : T \times V \rightarrow T \times V,$$

$$\quad \quad \quad (H_0^*(\sigma, u), (\tau, v)) = \alpha(A(\sigma), \tau) - \alpha(\tau, d(u)) + \alpha(\sigma, d(v)) + (1 - \alpha)(B(u), d(v)) + 2\mu(d(u), d(v)). \quad (2.19)$$

We consider the following two-field problem:

**Problem 2.1.** Find $(\sigma, u) \in T \times V$ such that

$$\quad \quad \quad (H_0^*(\sigma, u), (\tau, v)) = (f, v) \ \forall (\tau, v) \in T \times V. \quad (2.20)$$

**Lemma 2.5.** $H_0^*$ is an hemi-continuous, monotone and strong coercive operator.

**Lemma 2.6.** [2] There exists $\beta > 0$ such that

$$\inf_{q \in M} \sup_{v \in X} \frac{(\text{div} \ v, q)}{\|v\| X \|q\| M} \geq \beta.$$

**Theorem 2.2.** Problem 1.1 has a unique solution $(\sigma, u, p) \in T \times V \times M$ which admits the following estimate

$$\|\sigma\| T + \|u\| X + \|p\| M \leq C(\alpha, r, \|f\|_{-1,r}). \quad (2.21)$$

**Remark 2.1.** All the estimates are $\mu$-independent. Furthermore, we will obtain similar estimates for the finite element solution $(\sigma_h, u_h, p_h)$ in the next section.

3. **FINITE ELEMENT APPROXIMATION**

We assume that the triangulation $\mathcal{C}_h$ is a regular partition of $\Omega$ [8], the quasi-uniformity of $\mathcal{C}_h$ is not necessary unless we state it clearly. Let $T_h, X_h$ and $M_h$ be the finite element space: $T_h \subset T, X_h \subset X, M_h \subset M$. We denote $V_h = \{ v_h \in X_h \mid (\text{div} \ v_h, q_h) = 0 \ \forall q_h \in M_h\}.$

The corresponding finite element approximation for Problem 1.1 is:

**Problem 3.1.** Find $(\sigma_h, u_h, p_h) \in T_h \times X_h \times M_h$ such that

$$\alpha(A(\sigma_h), \tau) - \alpha(\tau, d(u_h)) = 0 \ \forall \tau \in T_h, \quad (3.1)$$

$$\alpha(\sigma_h, d(v)) + (1 - \alpha)(B(u_h), d(v)) + 2\mu(d(u_h), d(v)) - (p_h, \text{div} \ v) = (f, v) \ \forall v \in X_h, \quad (3.2)$$

$$\quad \quad \quad (\text{div} \ u_h, q) = 0 \ \forall q \in M_h. \quad (3.3)$$

Problem 3.1 can be reformulated as follows: Find $(\sigma_h, u_h) \in T_h \times V_h$ such that

$$\quad \quad \quad (H_0^*(\sigma_h, u_h), (\tau_h, v_h)) = (f, v) \ \forall (\tau_h, v_h) \in T_h \times V_h. \quad (3.4)$$

Assumption $\Theta$ (B-B inequality):

$$\exists \beta_h(r) > 0, \ \inf_{q_h \in M_h} \sup_{v_h \in X_h} \frac{(\text{div} \ v_h, q_h)}{\|v_h\| X \|q_h\| M} \geq \beta_h(r).$$

Assumption $\Delta$: for any $\nu \geq \mu \geq 1$ it holds:

$$\|u_h\|_{1,\nu} \leq C I h^n \left(\frac{1}{\nu} - \frac{1}{\mu}\right) \|u_h\|_{1,\mu} \ \forall u_h \in X_h,$$

$$\|\tau_h\|_{0,\nu} \leq C I h^n \left(\frac{1}{\nu} - \frac{1}{\mu}\right) \|\tau_h\|_{0,\mu} \ \forall \tau_h \in T_h.$$
Lemma 3.1. [21, Rem. 2.1]

\[ \inf_{v \in V_h} \|u - v\|_{X} \leq (1 + C) \inf_{w \in X_h} \|u - w\|_{X}. \]  

(3.5)

Remark 3.1. Note that the B-B inequality for \((\sigma, u)\) and \((\sigma_h, u_h)\) is no longer needed either for Problem 1.1 or for Problem 3.1. An enhanced K-ellipticity is introduced in a natural, reasonable and unifying way. It is known that there are many finite element space pairs satisfying the B-B inequality either for \((\sigma, u)\) or for \((u, p)\), but few of them satisfy the two B-B inequalities simultaneously [20].

Theorem 3.1. If Assumption \(\Theta\) holds, then Problem 3.1 admits a unique solution \((\sigma_h, u_h, p_h) \in T_h \times V_h \times M_h\), and the following a-priori estimate holds

\[ \|\sigma_h\|_X + \|u_h\|_X + \|p_h\|_M \leq C(\alpha, r, \beta_h(r), \|f\|_{1-r}). \]  

(3.6)

We define \(\kappa\) as that in Lemma 2.3, and \(\kappa'\) the conjugate exponent of \(\kappa\).

Theorem 3.2. If \((\sigma, u, p)\) and \((\sigma_h, u_h, p_h)\) are solutions of Problem 1.1 and Problem 3.1, respectively, then for all \(r \in (1, 2]\),

\[ \alpha |\sigma - \sigma_h|_{L^r} + (1 - \alpha) |d(u - u_h)|_{L^r} + \mu |u - u_h|_{1,2} \leq C(\alpha) \inf_{(r, v, q) \in T_h \times V_h \times M_h} [|\sigma - \tau|_{L^r}^2 + |d(u - v)|_{L^r}^2 + \beta \|u - v\|_{X}^2]
\]

\[ + \|\sigma - \tau\|_{L^{2-r}}^2 + (\theta + \|\sigma\|_{T} + \|\sigma_h\|_{T})^{2-r} |u - v|_{X}^2 + \mu |d(u - v)|_{1,2}^2 + \|p - q\|_{M} |u - v|_{X} + |p - q|_{L^{2-r}}, \]  

(3.7)

and for \(r \in [2, \infty)\),

\[ \alpha |\sigma - \sigma_h|_{L^r} + (1 - \alpha) |d(u - u_h)|_{L^r} + \mu |u - u_h|_{1,2} \leq C(\alpha) \inf_{(r, v, q) \in T_h \times V_h \times M_h} [|\sigma - \tau|_{L^r}^2 + |d(u - v)|_{L^r}^2 + \beta \|u - v\|_{X}^2]
\]

\[ + \|\sigma - \tau\|_{L^{2-r}}^2 + (\theta + \|\sigma\|_{T} + \|\sigma_h\|_{T})^{2-r} |u - v|_{X}^2 + \mu |d(u - v)|_{1,2}^2 + \|p - q\|_{M} |u - v|_{X} + |p - q|_{L^{2-r}}. \]

(3.8)

If the Assumption \(\Theta\) holds, then for \(r \in (1, 2]\),

\[ |p - p_h|_{L^{r'}} \leq C[(1 + \beta_h(r)^{-1}) \inf_{q \in M_h} |p - q|_{L^{r'}} + \beta_h(r)^{-1} (\alpha |\sigma - \sigma_h|_{L^{r'}}^2 + |d(u - u_h)|_{L^{r'}}^2 + \mu |u - u_h|_{1,2}^2)], \]

(3.9)

and for \(2 \leq s \leq r\),

\[ |p - p_h|_{L^{s'}} \leq C[(1 + \beta_h(r)^{-1}) \inf_{q \in M_h} |p - q|_{L^{s'}} + \beta_h(r)^{-1} (\theta + \|\sigma\|_{\gamma(r, r')} + \|\sigma_h\|_{\gamma(r, r')})^{\frac{2-s}{s-r'}} |\sigma - \sigma_h|_{L^{s'}}^2
\]

\[ + (\theta + |u|_{\gamma(r, s)} + |u_h|_{\gamma(r, s)})^{\frac{r'}{2-s}} |d(u - u_h)|_{L^{s'}}^2 + C \mu |u - u_h|_{1,2}^2], \]

(3.10)

Here \(\gamma(r, s)\) and \(\gamma(r', s')\) are defined as that in Lemma 2.3.

Proof. For \(x_h = (\sigma_h, u_h) \in T_h \times V_h\), we have

\[ (H'_\sigma(x_h), y_h) = (l, y_h) \forall y_h \in T_h \times V_h. \]
Let $x = ((\sigma, u), p)$ be the solution of Problem 1.1, then

$$(H^*_\alpha(x), y_h) = (l, y_h) + (p, \text{div} v_h) \forall y_h = (\tau, v) \in T_h \times V_h,$$

$$(H^*_\alpha(x) - H^*_\alpha(x_h), x - x_h) = (H^*_\alpha(x) - H^*_\alpha(x_h), x - y_h) + (H^*_\alpha(x) - H^*_\alpha(x_h), y_h - x_h)$$

$$= (H^*_\alpha(x) - H^*_\alpha(x_h), x - y_h) + (p - q_h, \text{div}(v_h - u_h)). \quad (3.11)$$

Set $\tilde{\sigma} = \sigma - \tau, \tilde{u} = u - v, \tilde{p} = p - q$, we have

$$(H^*_\alpha(x) - H^*_\alpha(x_h), x - y_h) = \alpha(A(\sigma) - A(\sigma_h), \tilde{\sigma}) + (1 - \alpha)(B(u) - B(u_h), d(\tilde{u}))$$

$$- \alpha(\tilde{\sigma}, d(u - u_h)) + \alpha(\sigma - \sigma_h, d(\tilde{u})) + 2\mu(d(\tilde{u}), d(u - u_h))$$

$$= I_1 + \cdots + I_5.$$ 

Let $I_6 = (\tilde{p}, \text{div}(v_h - u_h))$. Now we estimate the terms $I_1, \cdots, I_6$ one by one.

Using Lemma 2.2, we can estimate $I_1$ and $I_2$ as follows

$$|I_1| \leq C(\varepsilon|\sigma - \sigma_h|^p + \varepsilon^{-\gamma}||\tilde{\sigma}||^p).$$

$$|I_2| \leq C(1 - \alpha)(\varepsilon|d(u - u_h)|^p + \varepsilon^{-\gamma}|d(\tilde{u})|^p).$$

In light of Lemma 2.3, $I_3$ can be bounded as

$$|I_3| \leq \alpha\|u - u_h\|_X\|\tilde{\sigma}\|_T$$

$$\leq C\alpha(\theta + \|u\|_X + \|u_h\|_X)\frac{1}{\varepsilon}||d(u - u_h)||_T\|\tilde{\sigma}\|_T$$

$$\leq C\alpha(\theta + \|u\|_X + \|u_h\|_X)^{2-\gamma}\|\tilde{\sigma}\|^2_T \forall \gamma \in (1, 2].$$

$$|I_3| \leq \alpha\|d(u - u_h)||_{L^r}\|\tilde{\sigma}\|_{L^{r'}}.$$ 

$$\leq C\alpha(\varepsilon|d(u - u_h)|^r + C\varepsilon^{1-r'}||\tilde{\sigma}||^{r'}_{L^{r'}}) \forall r \in [2, \infty).$$

Following the same line, we estimate $I_4$ and $I_5$ as

$$|I_4| \leq C(\varepsilon|\sigma - \sigma_h|^p + \varepsilon^{-\gamma}||d(\tilde{u})||^p_{L^r}) \forall r \in (1, 2],$$

$$|I_4| \leq C(\varepsilon|\sigma - \sigma_h|^p + \frac{1}{4}\varepsilon(\theta + \|\sigma\|_T + \|\sigma_h\|_T)^{2-r'}||\tilde{u}||^2_X) \forall r \in [2, \infty),$$

$$|I_5| \leq \mu|u - u_h|_{1,2}|\tilde{u}|_{1,2} \leq \varepsilon\mu|u - u_h|_{1,2} + C(\varepsilon\mu|\tilde{u}|^2_{1,2}).$$

Note $I_6 = (\tilde{p}, \text{div} u) + (\tilde{p}, \text{div}(u - u_h))$, then

$$|I_6| \leq C(||\tilde{p}||_X||\tilde{u}||_X + \varepsilon|d(u - u_h)|^p + \varepsilon^{-1}||u||_X + \|u_h\|_X)^{2-r'||\tilde{p}||^2_{L^r}} \forall r \in (1, 2],$$

$$|I_6| \leq C(||\tilde{p}||_X||\tilde{u}||_X + \varepsilon|d(u - u_h)|^p + C\varepsilon^{1-r'}||\tilde{p}||^{r'}_{L^{r'}}) \forall r \in [2, \infty).$$

From Lemma 2.1, the left hand side of (3.11) has a lower bound:

$$(H^*_\alpha(x) - H^*_\alpha(x_h), x - x_h) \geq C(\alpha|\sigma - \sigma_h|^p + (1 - \alpha)|d(u - u_h)|^p + \mu|u - u_h|^2).$$

Noting the definition of $\kappa$, a careful choice of $\varepsilon$ gives (3.7) and (3.8).
By Assumption Θ, for $\nu \in (1, \infty)$, we have

$$
\beta_h(\nu)\|p_h - q_h\|_{L^{\nu}} \leq \sup_{v_h \in X_h} \frac{\langle \nabla v_h, p_h - q_h \rangle}{\|v_h\|_{1,\nu}}
$$

$$
= \sup_{v_h \in X_h} \frac{(H^*_h(x) - H^*_h(x_h), y_h) + \langle \nabla v_h, p - q_h \rangle}{\|v_h\|_{1,\nu}}
$$

$$
= \sup_{v_h \in X_h} \frac{1}{\|v_h\|_{1,\nu}} \left[ \alpha(\sigma - \sigma_h, d(v_h)) + (1 - \alpha)(B(u) - B(u_h), d(v_h))
+ 2\mu(d(u - u_h), d(v_h)) + \langle \nabla v_h, p - q_h \rangle \right].
$$

(3.12)

Denote terms on the right hand side of (3.12) by $J_1, \ldots, J_4$, and let $\nu$ be $\kappa$ and $s$ for the case $1 < r \leq 2$ and $2 < r < \infty$, respectively.

Then for $r \in (1, 2]$, we have

$$
|J_1| \leq C\alpha|\sigma - \sigma_h|_{\nu}^{\frac{\nu}{\nu'}}\|v_h\|_{1,\kappa}.
$$

If $r \in [2, \infty)$

$$
|J_1| \leq C\alpha(\theta + \|\sigma\|_{\gamma(r',\nu')} + \|\sigma_h\|_{\gamma(r',\nu')})^{2(\frac{\nu}{\nu'})} |\sigma - \sigma_h|_{\nu}^{\frac{\nu}{\nu'}}\|v_h\|_{1,\kappa}.
$$

As to $J_2$, for the case $r \in (1, 2]$

$$
|J_2| \leq C(1 - \alpha)\|v_h\|_{1,\kappa} \left( \int_\Omega |d(u - u_h)|^{\nu'}(\theta + |d(u)| + |d(u_h)|)^{(r-2)\kappa'} \right)^{\frac{\nu}{\nu'}}
$$

(3.13)

$$
\leq C(1 - \alpha)\|v_h\|_{1,\kappa} \left( \int_\Omega |d(u - u_h)|^{2}(\theta + |d(u)| + |d(u_h)|)^{r-2}|d(u - u_h)| \right)^{\kappa'-2}
$$

(3.14)

$$
\times \left( \theta + |d(u)| + |d(u_h)| \right)^{(r-2)(\kappa'-1)}^{\frac{\nu}{\nu'}}
$$

(3.15)

$$
\leq C(1 - \alpha)\|v_h\|_{1,\kappa} |d(u - u_h)|^{\frac{\nu}{\nu'},}
$$

(3.16)

where we have used the following inequality:

$$
|d(u - u_h)|^{\kappa'-2}(\theta + |d(u)| + |d(u_h)|)^{(r-2)(\kappa'-1)} \leq (\theta + |d(u)| + |d(u_h)|)^{\kappa'-2+(r-2)(\kappa'-1)} \leq 1.
$$

As $r \in [2, \infty)$

$$
|J_2| = C(1 - \alpha) \int_\Omega |d(v_h)||d(u - u_h)||\theta + |d(u)| + |d(u_h)| |^{\frac{\nu}{\nu'}}(\theta + |d(u)| + |d(u_h)|)^{\kappa'-2}
$$

$$
\leq C(1 - \alpha)\|v_h\|_{1,\kappa} |d(u - u_h)|^{\frac{\nu}{\nu'}} \left( \int_\Omega |d(u)| + |d(u_h)| \right)^{\frac{\nu}{\nu'}}
$$

$$
\leq C(1 - \alpha)\|v_h\|_{1,\kappa} (\theta + |u|_{1,\gamma} + |u_h|_{1,\gamma})^{\frac{\nu}{\nu'}} |d(u - u_h)|^{\frac{\nu}{\nu'}} \forall r \in [2, \infty).
$$

$J_3$ can be estimated by use of the inverse inequality

$$
|J_3| \leq C_1\mu h^{n(\frac{1}{2} - \frac{1}{\nu'})}|v_h|_{1,\gamma}|u - u_h|_{1,2}.
$$
Trivially, we get
\[ |J_4| \leq |v_{h1,\nu}||\bar{p}||_{L^{r\prime}}. \]
A combination of the above estimate for $J_1$ and $J_4$ yields (3.9) and (3.10).

**Corollary 3.1.** If $\theta = 0$, $\sigma, \sigma_h \in [L^{\infty}(\Omega)]^{\frac{m+1}{2}}$ and $u, u_h \in [W^{1,\infty}(\Omega)]^n$, then for the case $r \in (1,2]$ we have the following $\mu$-independent estimate for $u$:
\[ \|u - u_h\|_{1,2} \leq C|u - u_h|_{1,2} \leq C(|u|_{1,\infty} + |u_h|_{1,\infty})^\frac{2-\nu}{\nu} |d(u - u_h)|_r. \] (3.17)

If $r > 2$, then
\[ \|\sigma - \sigma_h\|_{0,2} \leq C(\|\sigma\|_{L^{\infty}} + \|\sigma_h\|_{L^{\infty}})^\frac{2-\nu}{\nu} |\sigma - \sigma_h|_{r\prime}. \]

**Proof.** If $r \in (1,2]$, we have
\[ |d(u - u_h)|_r^2 \geq C(|u|_{1,\infty} + |u_h|_{1,\infty})^{r-2} |d(u - u_h)|_{r}^2. \]
Then (3.17) follows immediately. The proof for $\|\sigma - \sigma_h\|_{L^2}$ is similar.

\[ \square \]

4. **Error bounds with respect to strong norm**

Our task in this section is to exemplify the abstract error bounds in Theorem 3.2. First of all, we give some approximation assumptions on $T_h, X_h$ and $M_h$.

- $A_1$: for $\sigma \in [W^{k,r\prime}(\Omega)]^{\frac{m+1}{2}}$, $\inf_{\tau \in T_h} \|\sigma - \tau\|_r \leq ch^k|\sigma|_{k,r\prime}$;
- $A_2$: for $u \in [W^{k+1,r}(\Omega)]^n \cap [W^{1,r}(\Omega)]^n$, $\inf_{v \in X_h} \|u - v\|_r \leq ch^k|u|_{k+1,r};$
- $A_3$: for $p \in W^{k,r}(\Omega) \cap L^{r\prime}(\Omega)$, $\inf_{q \in M_h} \|p - q\|_r \leq ch^k|p|_{k,r\prime}$.

**Theorem 4.1.** Let Assumptions $A_1, A_2$ and $\Theta$ hold, $\alpha \in (\alpha_0, \beta_0), \alpha_0, \beta_0 \in (0,1)$. Let $(\sigma_h, u_h) \in T_h \times V_h$ be the unique solution of Problem 3.1. Then

1. If $r \in (1,2]$, we have
\[ \|\sigma - \sigma_h\|_T \leq C_1 h^\frac{k}{r'}, \] (4.1)
and
\[ \|u - u_h\|_{1,r} \leq C_r \|u - u_h\|_X \leq C_r C_1 h^\frac{k}{r'}, \] (4.2)
where
\[ C_1 = C(|\sigma|_{k,r\prime}, \|f\|_{-1,r'}, |u|_{k+1,r}, |u|_{k+1,2}, |p|_{k,r\prime}, \beta_h(r)^{-1}, \beta_h(2)^{-1}), \]
and $C_r$ is the constant appeared in Korn inequality [17]. Moreover,
\[ \|u - u_h\|_{1,2} \leq C_2 h^\frac{k}{r'}, \] (4.3)
where
\[ C_2 = C(C_1, |u|_{1,\infty}, |u_h|_{1,\infty}). \]

In particular, when $\theta = 1$, we even have $\|\sigma - \sigma_h\|_{L^2} \leq C_1 h^k$. 
2. If \( r \in [2, \infty) \), we have

\[
\| \sigma - \sigma_h \|_{T} \leq C_3 h^{\frac{2r}{r-1}},
\]

and

\[
\| u - u_h \|_{1,r} \leq C_r \| u - u_h \|_{X} \leq C_r C_3 h^{\frac{2r}{r-1}},
\]

\[
C_3 = C(|\sigma|_{k,r}, |\sigma|_{k,2}, \| f \|_{-1,r}, |u|_{k+1,r}, |u|_{k+1,2}, |p|_{k,r}, |p|_{k,2}, \beta_h(r)^{-1}).
\]

When \( \theta = 1 \), we even have

\[
\| u - u_h \|_{1,2} \leq C_r C_3 h^k,
\]

\[
\| \sigma - \sigma_h \|_{L^2} \leq C_4 h^{\frac{k}{k+1}},
\]

\[
C_4 = (C_3, \| \sigma \|_{L^\infty}, \| \sigma_h \|_{L^\infty}).
\]

3. Furthermore, if Assumptions \( A_3 \) and \( \Delta \) hold, then

\[
\| p - p_h \|_{L^\infty} \leq C_5 \zeta, \quad r \in (1, 2],
\]

with

\[
\zeta = \min \left( \mu \frac{k}{k+1} + n \left( \frac{1}{2} \right), h^k (2-r) \right),
\]

and \( C_5 = C(C_1, |p|_{k,r}) \).

\[
\| p - p_h \|_{L^\infty} \leq C_6 h^{\frac{k}{k+1}}, \quad r \in [2, \infty),
\]

\[
C_6 = C(C_3, |p|_{k,r}).
\]

**Proof.** This is a direct consequence of Theorem 3.2 and Lemma 2.3. \( \square \)

**Remark 4.1.** When \( r \in (1, 2] \), the error bound (4.6) for the pressure will be deteriorated. After some careful analysis, we find that the deterioration happens only in the following limiting cases: (1) \( n = 2, k = 1 \), (2) \( n = 3, k = 1 \), (3) \( n = 3, k = 2 \). If we assume \( \mu = k^{(2-r)} \) in these three cases, the accuracy can be recovered. In fact, this kind of assumption on \( \mu \) is realistic when \( \mu \) is very small. Recalling that in this case the proper Sobolev space for the pressure is \( L_0^2(\Omega) \), thus we only need to derive error bounds in \( L_0^2(\Omega) \). Since the norm on \( L_0^2(\Omega) \) is weaker than that on \( L^\infty(\Omega) \) in the present situation, we can expect to get \( \mu \)-independent error bounds, it is just the case.

**Corollary 4.1.** With the same assumptions as in the third case of Theorem 4.1, we have

\[
\| p - p_h \|_{L^2} \leq C_5 h^{k(r-1)}, \quad \theta = 0,
\]

\[
\| p - p_h \|_{L^2} \leq C_5 h^k, \quad \theta = 1.
\]
Proof. (4.9) is a direct consequence of (4.7), but (4.8) needs a proof. We slightly modify the proof method in Section 3. The only difference is that here we use the Assumption Θ with the exponent 2. We still denote the right hand side of (3.12) by \( J_1, \cdots, J_4 \), then

\[
\begin{align*}
|J_1| & \leq C\alpha |\sigma - \sigma_h| r \|v_h\|_{1,r'} \leq C\alpha |\sigma - \sigma_h| r \|v_h\|_{1,2}, \\
|J_2| & \leq C(1 - \alpha)\|v_h\|_{1,r'} d(u - u_h) \|\nabla u\|_{r'} \leq C(1 - \alpha)\|d(u - u_h)\|_{r'} \|v_h\|_{1,2}, \\
|J_3| & \leq |u - u_h|_{1,2} \|v_h\|_{1,2},
\end{align*}
\]

and for any \( q \in M_h \),

\[
|J_4| \leq \|p - q\|_{L^2} \|v_h\|_{1,2}.
\]

Combining all these estimates and (4.1) and (4.2), we come to (4.8). By (4.8) and (4.9), we have the optimal error bounds for the pressure in the sense of [5].

If \( X_h \notin [W^1,\infty(\Omega)]^n \) and \( T_h \notin [L^\infty(\Omega)]^{\frac{n(n+1)}{2}} \), which would happen in some cases, we cannot obtain an \( \mu \)-independent estimate for \( |u - u_h|_{1,2} \) when \( r \in (1, 2] \), and for \( |\sigma - \sigma_h|_{L^2} \) when \( r \in [2, \infty) \). However, we compromise to obtain error bounds for all variables in a strong norm. In fact, by virtue of Theorem 4.2, it is seen that the only case for accuracy-losing is that the velocity is approximated by an element of a degree lower than 2.

**Theorem 4.2.** Let \((\sigma, u, p), (\sigma_h, u_h, p_h)\) be solutions of Problem 1.1 and Problem 3.1, respectively. If Assumptions A1, A2, and A3 hold, then

1. If \( r \in (1, 2] \) and Assumption Θ holds for the space \( X_h \) in the case of \( r \in (1, 2] \), then

\[
\|u - u_h\|_{1,s} \leq C_1 h^\frac{s}{s'} \quad \forall s \in [r, 2],
\]

where

\[
s = \begin{cases} 
\frac{r}{2}, & \text{if } k = 1, \\
\frac{2}{1 - \frac{r}{2}}, & \text{if } k \geq 2,
\end{cases}
\]

with

\[
C_1 = C(|\sigma|_{k,r'}, \|f\|_{-1,r'}, |u|_{k+1,2}, |p|_{k,r}, |u|_{k+1,2}, \beta_h(r)^{-1}, \beta_h(2)^{-1}).
\]

2. If \( r \in [2, \infty) \) and Assumption Θ holds for \( T_h \) in the case of \( s \in [2, r] \), \( s' \) and \( s \) are the conjugate exponents of each other, then

\[
\|\sigma - \sigma_h\|_{L^{r'}} \leq C_2 h^{\frac{s'}{s'}} \quad \forall s' \in [r', 2],
\]

where

\[
s' = \begin{cases} 
\frac{r'}{2}, & \text{if } k = 1, \\
\frac{2}{1 - \frac{r'}{2}}, & \text{if } k \geq 2,
\end{cases}
\]

with

\[
C_2 = C(|\sigma|_{k,s'}, \|f\|_{-1,r'}, |u|_{k+1,2}, |u|_{k+1,2}, |p|_{k,r'}, \beta_h(r)^{-1}).
\]
3. If Assumption $\Delta$ holds, and Assumption $\Theta$ holds for $T_h$ in the case of $r \in [2, s]$, then

$$\|p - p_h\|_{L^{r'}} \leq C_2 h^\frac{k}{r'},$$

$s'$ is defined as in the second case.

The Proof of (i) and (ii) is almost the same as that in [5], but the Proof of (iii) is beyond [5], we give all details.

**Proof of (iii).** A direct consequence of (i) and (ii) is that $\|\sigma_h\|_{0, \gamma'}$ can be bounded from above by

$$\gamma' = \frac{s'(2 - r')}{2 - s'} = \begin{cases} \frac{r'}{1 - \frac{s'}{r'}}, & \text{if } k = 1, \\ \infty, & \text{if } k \geq 2. \end{cases}$$

From Theorem 3.2, we have

$$\|p - p_h\|_{L^{r'}} \leq C(\beta_h(s)^{-1})(\|\hat{p}\|_{L^{r'}} + \|\sigma\|_{L^{\gamma'(r', r')}}) + \|\sigma_h\|_{L^{\gamma'(r', r')}})^{\frac{2 - r'}{2}}|\sigma - \sigma_h|_{r'}$$

$$+ (|u|_{1, \gamma(r, s)} + |u_h|_{1, \gamma(r, s)})^{\frac{r}{2}}|d(u - u_h)|_{\gamma}^{\frac{1}{2}} + C_{12}|u - u_h|_{1, 2}.$$

On the other hand, from Assumption $\Delta$ and Theorem 4.1, for $\gamma(0) \geq s(0) = r \geq 2$, we have

$$|u_h|_{1, \gamma(0)} \leq \inf_{v_h \in X_h} (|u_h - v_h|_{1, \gamma(0)} + |v_h|_{1, \gamma(0)})$$

$$\leq C_{13} h^n \left(\frac{1}{\gamma(0)} - \frac{1}{s}\right) \inf_{v_h \in X_h} (|u_h - v_h|_{1, r} + C|u|_{1, \gamma(0)})$$

$$\leq C_{14} h^n \left(\frac{1}{\gamma(0)} - \frac{1}{s}\right) \inf_{v_h \in X_h} (|u - u_h|_{1, r} + |u - v_h|_{1, r} + C|u|_{1, \gamma(0)})$$

$$\leq C_{15} h^n \left(\frac{1}{\gamma(0)} - \frac{1}{s}\right) \frac{1}{2} + C,$$

where $C$ depends on $|u|_{k+1, r}, |\sigma|_{k, r'}, |p|_{k, r'}, \beta_h(\nu), \nu \in [2, r]$. We can set

$$\gamma(0) = \begin{cases} \frac{r}{1 - \frac{s'}{r'}}, & \text{if } k = 1, \\ \infty, & \text{if } k \geq 2. \end{cases}$$

Observing that $s = s(r, \gamma) = \frac{2\gamma}{2 - s'}$, we have $s = s(r, \gamma(0)) = \frac{r}{1 + r - \frac{2\gamma}{2 - s'}}$. Thus $s' = \frac{r'}{1 + r - \frac{2\gamma}{2 - s'}}$. If $|\sigma_h|_{L^{\gamma'(r', r')}}$ can be bounded, we complete the proof. Using the definition of $\gamma(r', s') = \frac{s(0)(2 - r')}{2 - s(0)} = \frac{r'}{1 - \frac{s'}{r'}}$, the expected assertion is actually true.

Now we want to exemplify the abstract Assumptions $A_1, A_2, A_3, \Delta$ and $\Theta$. Indeed, Assumptions $A_1, A_2, A_3, \Delta$ are standard and easily be satisfied. For example, let $T_h$ be discontinuous(or continuous) piecewise polynomials of degree $k - 1 (k \geq 2)$, $X_h$- continuous piecewise polynomials of degree $k$, $M_h$- continuous(or discontinuous) piecewise polynomials of degree $k - 1$, then Assumptions $A_1, A_2, A_3$ will be satisfied. The Inverse Assumption $\Delta$ for $T_h, X_h$ also holds if we assume the mesh is quasi-uniform. However, choosing $X_h, M_h$ so that the Assumption $\Theta$ holds or not is a delicate task, though there are many examples for $r = 2$ (see [14]). Nevertheless, there exist by far less examples for the case $r \neq 2$ [22]. In the following, we will give some examples that can be justified strictly.

We will quote some finite element spaces without denotations, and refer to [7] and [14] for details. Let $P_m$ be the space of all polynomials of degree less than or equal to $m$, and $Q_m$ be the space of all polynomials of degree less than or equal to $m$ in each variable. Let $K$ be an arbitrary triangle of the partition $\mathcal{C}_h$ with vertices $a_1, a_2, a_3$, the middle point of each edge denote $a_{i,j}$ (oppose to $a_k$). In addition, we denote by $e_i$ the edge opposite to $a_i$ and by $n_i$ and $\tau_i$ the unit normal and tangent to $e_i$. 
Theorem 4.3. If \( X_h, \mathcal{M}_h \) are the following finite element space pair:

1. Crouzeix-Raviart finite element space. [10]
2. Mini finite element space [14] and \((\mathcal{P}^+_1, \mathcal{P}_0), (\mathcal{Q}^+_1, \mathcal{P}_0)\) finite element space. [11]
3. Hood-Taylor finite element space \((\mathcal{P}_2 - \mathcal{P}_1), (\mathcal{P}_3 - \mathcal{P}_2)\). [15]
4. rectangular Hood-Taylor finite element space \((\mathcal{Q}_h, \mathcal{Q}_{k-1})\). [6]

then the Assumption \( \Theta \) holds for \( r \in (1, \infty) \).

Proof. We only check Case 3. Case 1 and 2 can be proved following the procedure in [5]. For Case 3, we only deal with \( \mathcal{P}_2 - \mathcal{P}_1 \) in 2-\( \mathcal{D} \) and \( \mathcal{P}_3 - \mathcal{P}_2 \) without any difficulty.

By virtue of [18, Lem. 2.4], we have

\[
\inf_{p_h \in \mathcal{M}_h} \sup_{u_h \in X_h} \frac{(\text{div} \, u_h, \overline{v}_h)}{\|u_h\|_M \|p_h\|_M} \geq C,
\]

where \( \overline{v}_h \) is defined on \( K \), hence (4.11) is proved. For \( K \) of the standard Lagrangian function corresponding to those points and using the standard inverse estimate.

In fact, since \( 2 \) triangles of \( S \), let \( p_h \) be the remaining triangles. We construct \( z_h \in X_h \) with \( z_h = 0 \) at all vertices of the triangles, \( z_h \cdot n \) is zero at all nodes on edges of triangles, with the exception of nodes on the common edge of \( S_2 \cup S_3 \), and \( z_h \cdot \tau \) vanishes on the boundary of \( \Omega \). Define

\[
\begin{align*}
z_h \cdot \sigma(a_{ij}) &= -|e| r' |\nabla p_h \cdot \tau(a_{ij})| r' - 2 \nabla p_h(a_{ij}), \\
z_h \cdot n(a_{ij}) &= -\varepsilon |e| r' |\nabla p_h^2 \cdot n(a_{ij})| r' - 2 \nabla p_h \cdot n(a_{ij}),
\end{align*}
\]

where \( p_h^2 \) means \( p_h \) is defined on \( K \in S_2 \), \( \varepsilon \) is a constant to be chosen later. We first show

\[
\|z_h\|_X \leq C \|p_h - \overline{p}_h\|^{r'-1}_M. \tag{4.11}
\]

In fact, since \( a_{ij}, a_i \) is a unisolvent set of points for polynomials of degree less than 2 on \( K \), we write \( z_h \) in terms of the standard Lagrangian function corresponding to those points and using the standard inverse estimate. Then for \( K_1 \in S_1 \), we have

\[
\|z_h\|_{X, K_1} \leq C \sum |z_h(a_{ij})| |K|^{\frac{1}{2}} |e|^{-1} \leq C |K|^{\frac{1}{2}} |e|^{r'-1} \|\nabla (p_h - \overline{p}_h)\|_0, K_1^{r'-1}
\]

\[
\leq C |K|^{\frac{1}{2}} |e|^{r'-1} r \|\nabla (p_h - \overline{p}_h)\|_{0, r', K_1}^{r'-1} \leq C \|p_h - \overline{p}_h\|_{0, r', K_1}^{r'-1}.
\]

For \( K_1 \in S_2 \), the estimate is the same. For \( K_3 \in S_3 \), we have

\[
\|z_h\|_{X, K_3} \leq C \|p_h - \overline{p}_h\|_{0, r', K_2 \cup K_3}^{r'-1},
\]

hence (4.11) is proved.
Furthermore, $\forall K_1 \in \mathcal{S}_1$, we have
\[
\int_{K_1} \text{div} z_h p_h = -\int_{K_1} z_h \nabla p_h = -\frac{|K_1|}{3} \sum_{e \in \Gamma} \sum_{i \neq j} z_h \nabla p_h(a_{ij}) = \frac{|K_1|}{3} |e| |v| \sum_{e \in \Gamma} \sum_{i \neq j} |(\nabla p_h \cdot \tau)(a_{ij})| |v|'.
\] (4.12)

It is easy to check that the expression
\[
\left[ \sum_{e \in \Gamma} \sum_{i \neq j} |(\nabla p_h \cdot \tau)(a_{ij})| |v|' \right]^{\frac{1}{2}}
\] (4.13)

vanishes only if $p_h$ is a constant and hence is a norm over linear polynomials modulo constants on $K_1$. Using the scaling trick and (4.12), we can easily prove the following inequality
\[
\int_{K_1} \text{div} z_h p_h \geq C \|p_h - \overline{p}_h\|_{0, r', K_1}.
\] (4.14)

In fact,
\[
\|p_h - \overline{p}_h\|_{0, r', K_1} \leq C|K| \|\hat{p}_h - \overline{p}_h\|_{0, r', K} \leq C|K| \sum_{e \in \Gamma} \sum_{i \neq j} |\hat{\nabla} p_h(a_{ij})| |v|'
\]
\[
\leq C|K| \sum_{e \in \Gamma} |e| |v| \sum_{i \neq j} |\nabla p_h(a_{ij})| |v|' \leq C \int_{K_1} \text{div} z_h p_h.
\]

Now let $K_2 \in \mathcal{S}_2$ and $K_3 \in \mathcal{S}_3$ share a common edge $e_3$ and denote by $\tau^3$ and $\eta^3$ the counterclockwise unit tangent and outward unit normal vectors to $K_2$ along $e_3$. Therefore,
\[
\int_{K_2} \text{div} z_h p_h + \int_{K_3} \text{div} z_h p_h = -\int_{K_2} z_h \nabla p_h - \int_{K_3} z_h \nabla p_h
\]
\[
= \frac{|K_2|}{3} \sum_{e_3} \sum_{i \neq j} |\nabla p_h^3 \cdot \tau^3| |v|'(a_{ij}) |e| |v|' + \varepsilon |\nabla p_h^3 \cdot \eta^3| |v|'(a_{ij}) |e| |v|'
\]
\[
+ \frac{|K_3|}{3} \sum_{e_3} \sum_{i \neq j} |\nabla p_h^3 \cdot \tau^3| |v|'(a_{ij}) + \varepsilon \sum_{e_3} \sum_{i \neq j} |\nabla p_h^3 \cdot \eta^3| |v|' - 1 |\nabla p_h^3 \cdot \eta^3(a_{ij})|
\]
\[
\geq C \|p_h - \overline{p}_h\|_{0, r', K_2 \cup K_3},
\]

where we have used the Young inequality and the scaling trick. Consequently, we have
\[
(\text{div} z_h, p_h) \geq C \|p_h - \overline{p}_h\|_{M}.
\] (4.15)

Let $\tilde{v}_h = \overline{v}_h + \beta z_h$, then
\[
(\text{div} \tilde{v}_h, p_h) \geq C_1 \|\overline{v}_h\|_{M} - C_2 \|p_h - \overline{p}_h\|_{M} \|\overline{p}_h\|_{M}^{-1} + C_3 \beta \|p_h - \overline{p}_h\|_{M}
\]
\[
\geq C_4 \|\overline{v}_h\|_{M} + C_5 \|p_h - \overline{p}_h\|_{M}
\]
\[
\geq C_6 \|p_h\|_{M}.
\]

Noting $\|\tilde{v}_h\|_{X} \leq C \|p_h\|_{M}^{-1}$, therefore
\[
\frac{(\text{div} \tilde{v}_h, p_h)}{\|\tilde{v}_h\|_{X}} \geq C \|p_h\|_{M}.
\]

This means that the Assumption $\Theta$ holds for Hood-Taylor finite element space $(\mathcal{P}_2 - \mathcal{P}_1)$ in 2-D. \hfill \Box
Remark 4.2. Following almost the same line, we can prove the case 4. However, a more intricate scaling trick is involved and the tensor product of 1-D Gauss-Lobatto formula has to be used. Furthermore, the so-called Hierarchic shape function on rectangle \[24\] plays a fundamental role in the treatment of an expression like (4.10).

Remark 4.3. In [19], we proposed a Macro-element based trick to check the Assumption $\Theta$, which is very clear and clean.

REFERENCES