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BIPOLAR BAROTROPIC NON-NEWTONIAN COMPRESSIBLE FLUIDS

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Abstract. We are interested in a barotropic motion of the non-Newtonian bipolar fluids . We consider a special case where the stress tensor is expressed in the form of potentials depending on e_{ii} and $(\frac{\partial e_{ij}}{\partial x_k})$. We prove the asymptotic stability of the rest state under the assumption of the regularity of the potential forces.

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Introduction

There are many substances which are capable of flowing but which exhibit flow characteristics that cannot be adequately described by the classical linearly viscous fluid model. In order to describe some of the departures from Newtonian behaviour (rheological properties, elastic features such as yield stress, stress relaxation and non-zero normal stress differences) many idealized material models have been suggested. During the last decades, mathematicians have also started to pay attention to these models and several results concerning the existence, uniqueness and stability of solutions have appeared, see [1–3, 5, 8–19, 22–24, 26–28].

We will deal with compressible non-Newtonian fluids. The models which describe their rheological properties were studied from the mathematical point of view by Matušů-Nečasová , Lukáčová-Medviďová see [14,16]. They proved the existence and uniqueness of a weak solution. All of these results were studied on bounded domains. In the case of an isothermal process the stability of the rest state has been proved [16]. The global existence of equations of non-Newtonian compressible fluids when the coefficients of viscosity depend on the invariants of velocity field where the growth of these coefficients is not polynomial but exponential was proved in [12, 13]. In the case of viscoelastic compressible fluids the existence of a classical solution of steady motion in the bounded domain was proved by Sy [27]. This result was extended to an exterior domain by Matušů-Nečasová *et al.* [15]. Both results are valid for small data only. One of the very interesting problem is the stability of the rest state. The crucial point of every proof of stability is the uniqueness of the steady state solutions. Beirão da

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Veiga [4] obtained a necessary and sufficient condition for the existence of a strictly positive solution of the following problem

$$
\begin{aligned}
&\partial_i p(\rho) = \rho \partial_i \xi_i, \qquad i = 1, ..., d \\
&\rho > 0, \int_\Omega \rho(x) \mathrm{d}x = m \\
&v = 0,
\end{aligned} \tag{1.1}
$$

where ρ is the density, $m > 0$ the total mass conserved by the flow, $p(\rho)$ is the pressure, ξ is the potential of external forces which is locally Lipschitz on Ω, v is the velocity. It is easy to show that such a solution is necessarily unique. On the other hand, this restriction excludes a class of solutions with vacuum state ($\rho = 0$). The optimal condition for the solution of (1.1) to be unique is shown by Feireisl and Petzeltová [6, 7].

Here, our objective is to study the stability of the rest state of the barotropic motion. This model will be mathematically formulated in Chapter 1. Chapter 2 is devoted to mathematical preliminaries and known results. In Chapter 3 we will prove the asymptotic stability of the rest state.

1. Formulation of the problem

The barotropic motion of the bipolar non-Newtonial fluids is governed by the following system of the equations

$$
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0, \tag{1.2}
$$

$$
\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) - \frac{\partial}{\partial x_j}\tau_{ij}^V(v) = -\frac{\partial p}{\partial x_i} + \rho b_i, \ i = 1, 2, ..., d, \tag{1.3}
$$

where ρ denotes the density, $v = (v_1, ..., v_d)$ is the velocity vector, $b = (b_1, ..., b_d)$ denotes the density of external forces and $p = p(\rho)$ is the pressure. We assume that $p \in C^2(0, +\infty)$. The equations (1.2), (1.3) are solved in the time–space cylinder $Q_T := I \times \Omega$, $I = (0, T)$, where $\Omega \subset \mathbb{R}^d$ is a bounded domain with a smooth infinitely differentiable boundary $\partial\Omega$. Let us define

$$
\begin{aligned}
P(\rho) &= \rho \int_1^\rho \frac{p(\sigma)}{\sigma^2}, \quad \rho > 0, \\
P(0) &= \lim_{\rho \to 0_+} P(\rho).
\end{aligned} \tag{1.3'}
$$

It may be verified (see [21]) that

$$
\begin{aligned}
\rho P'(\rho) - P(\rho) &= p(\rho), \quad \rho > 0, \\
P''(\rho) &= \tfrac{p'(\rho)}{\rho}, \quad \rho > 0, \\
P(\rho) &\geq -k_1, \quad \rho \geq 0.
\end{aligned}
$$

We suppose that the body forces are given and satisfy

$$
b \in L^\infty(Q_T). \tag{1.4}
$$

Now, we can specify the assumptions on the stress tensor τ^V_{ij}. Namely:

$$\tau^V_{ij} = \frac{\partial V(\mathrm{Tr}e)}{\partial e_{ii}}\delta_{ij} - \frac{\partial}{\partial x_k}\Big(\frac{\partial W(De)}{\partial(\frac{\partial e_{ij}}{\partial x_k})}\Big), \tag{1.5}$$

$$De = \left(\frac{\partial e_{ij}}{\partial x_k}\right)^d_{i,j,k=1}, \quad e = (e_{ij})^d_{i,j=1}, \quad e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad \mathrm{Tr}e = (e_{ii})^d_{i=1}. \tag{1.6}$$

Also, we consider the third stress tensor τ^V_{ijk}, which has the following form

$$\tau^V_{ijk} = \frac{\partial W(De)}{\partial(\frac{\partial e_{ij}}{\partial x_k})}\,. \tag{1.7}$$

Moreover, we will assume that the potentials V, W satisfy the following conditions:

$$\begin{aligned} C_1(1+|De|)^{q-2}|\xi|^2 &\le \frac{\partial^2 W(De)}{\partial(\frac{\partial e_{ij}}{\partial x_k})\partial(\frac{\partial e_{i_1j_1}}{\partial x_{k_1}})}\xi^k_{ij}\xi^{k_1}_{i_1j_1} \\ &\le C_2(1+|De|)^{q-2}|\xi|^2, \end{aligned} \tag{1.8}$$

$$\begin{aligned} C_3(1+|\mathrm{Tr}e|)^{q-2}|\xi|^2 &\le \frac{\partial^2 V(\mathrm{Tr}e)}{\partial e_{ii}\partial e_{i_1i_1}}\xi_{ii}\xi_{i_1i_1} \\ &\le C_4(1+|\mathrm{Tr}e|)^{q-2}|\xi|^2, \end{aligned} \tag{1.9}$$

where C_1, C_2, C_3, C_4 are positive constants, $q > d$, and $|\cdot|$ is the usual Euclidean norm of a vector. Let

$$W(0) = 0,\ V(0) = 0, \tag{1.10}$$

$$\frac{\partial W}{\partial(\frac{\partial e_{ij}}{\partial x_k})}(0) = 0,\ \frac{\partial V}{\partial e_{ii}}(0) = 0. \tag{1.11}$$

The system (1.2), (1.3) is completed by the initial conditions

$$v(0) = v_0,\ \rho(0) = \rho_0,\ \rho_0 > 0 \quad \text{in}\ \ \Omega; \tag{1.12}$$

and the boundary conditions

$$\tau^V_{ijk}\nu_j\nu_k = 0 \quad \text{on}\ (0,T)\times\partial\Omega \tag{1.13}$$

(ν is an outer normal to $\partial\Omega$) ,

$$v = 0 \quad \text{on}\ \ (0,T)\times\partial\Omega. \tag{1.14}$$

2. Mathematical preliminaries

By $L^p(\Omega)$ and $W^{l,p}(\Omega)$, $0 \leq p, l < \infty$, we denote the Lebesgue and Sobolev spaces, respectively, equipped with the standard norm.

First, we give the well-known results on the equilibrium solution, where we do not consider the cavitation of density. In the work of Beirão da Veiga [4] the necessary and sufficient conditions for the existence of the equilibrium solutions for an arbitrary $b \in L^\infty(\Omega)$ was proved. Let p be a continuously differentiable real function defined on $\mathbb{R}^+ = \{s \in R : s > 0\}$, such that $p'(s) > 0, \forall s \in \mathbb{R}^+$. Let

$$0 < \operatorname{ess}\inf_{x\in\Omega} \rho(x),\ \operatorname{ess}\sup_{x\in\Omega} \rho(x) < +\infty \tag{2.1}$$

and

$$\frac{1}{|\Omega|}\int_\Omega \rho(x)\mathrm{d}x = m \tag{2.2}$$

for a fixed $m > 0$, we define

$$\pi(s) = \int_0^s \frac{p'(t)}{t}\mathrm{d}t, \quad \forall s \in \mathbb{R}^+. \tag{2.3}$$

We denote (a, f) the range of π, $(a, f) = \pi(\mathbb{R}^+)$.

Let us define $\phi = \pi^{-1}$. Clearly, $\phi((a, f)) = \mathbb{R}^+$. Put $\phi(a) = 0$, $\phi(f) = +\infty$.

Definition 2.1. Let $b \in L^\infty(\Omega)$. A function ρ is called an equilibrium solution of (1.1) if $\rho \in L^\infty(\Omega)$ and if

$$\pi(\rho(x)) = \xi(x) + c \quad \text{a.e. in } \Omega$$

and (2.1), (2.2) hold.

We set $n_o = \operatorname{ess}\inf b$ in Ω, $N_0 = \operatorname{ess}\sup b$ in Ω.

Theorem 2.1. *Let $b \in L^\infty(\Omega)$ be given. There exists an equilibrium solution $\rho(x)$ if and only if there exists a constant*

$$c \in (a - n_0, f - N_0), \tag{2.4}$$

such that

$$\frac{1}{|\Omega|}\int_\Omega \phi(c + \xi(x))\mathrm{d}x = m. \tag{2.5}$$

If such a constant exists then the (unique) equilibrium solution is given by

$$\rho(x) = \phi(c + \xi(x)), \forall x \in \Omega. \tag{2.6}$$

Proof. see [4].

Theorem 2.2. *Under the assumptions of Theorem 2.1 there exists an equilibrium solution $\rho(x)$ if and only if*

$$a - n_0 < f - N_0, \tag{2.7}$$

and

$$\frac{1}{|\Omega|}\int_{\infty}\phi(a-n_0+\xi(x))\mathrm{d}x < m < \frac{1}{|\Omega|}\phi(f-N_0+\xi(x))\mathrm{d}x. \tag{2.8}$$

In this case the equilibrium solution $\rho(x)$ is given by (2.6), where c is the (unique) solution of (2.4)–(2.5).

Proof. see [4].

We mention a weak formulation of the problem (1.2), (1.3), (1.12)–(1.14).

Definition 2.2. A pair (ρ, v) is said to be a weak solution of the problem (1.2), (1.3), (1.12)–(1.14), if the following conditions are satisfied

(i) $\rho \in L^\infty(I; W^{1,q}(\Omega))$,

(ii) $\frac{\partial\rho}{\partial t} \in L^\infty(I; L^q(\Omega))$,

(iii) $v \in L^\infty(I; W^{2,q}(\Omega) \cap W_0^{1,2}(\Omega))$,

(iv) $\frac{\partial v}{\partial t} \in L^2(Q_T)$,

(v) the continuity equation (1.2) is satisfied in the sense of distributions on Q_T,

(vi) $$\int_\Omega \frac{\partial}{\partial t}(\rho v_i)\varphi_i - \int_\Omega \rho v_i v_j \frac{\partial\varphi_i}{\partial x_j} - \int_\Omega p(\rho)\frac{\partial\varphi_i}{\partial x_i} + \int_\Omega \frac{\partial V}{\partial e_{ii}}(\mathrm{Tr}e(v))e_{ii}(\varphi)\delta_{ij} + \frac{\partial W(De(v))}{\partial(\frac{\partial e_{ij}}{\partial x_k})}\frac{\partial e_{ij}}{\partial x_k}(\varphi) = \int_\Omega \rho b_i\varphi_i$$
holds for a.e. $t \in I$ and for every $\varphi = (\varphi_1, ..., \varphi_d) \in W^{2,q}(\Omega) \cap W_0^{1,2}(\Omega)$,

(vii) the initial conditions (1.12) with $\rho_0 \in C^1(\overline{\Omega})$ and $v_0 \in W^{2,q}(\Omega) \cap W_0^{1,2}(\Omega)$ are fulfilled.

In [14] authors proved the existence and uniqueness of weak solution of the similar problem except the nonlinear potential V, which depends on the whole tensor $e = (e_{ij})_{i,j=1}^d$ instead of $\mathrm{Tr}e = (e_{ii})_{i=1}^d$ as in (1.5). Nevertheless the existence and uniqueness can be proved in the same way. In what follows we only point out the parts which differ from results stated in [14].

Theorem 2.3. *(Existence of a weak solution).*
Let $\rho_0 \in C^1(\overline{\Omega}), \rho_0 > 0$ in $\overline{\Omega}$, and $v_0 \in W^{2,q}(\Omega) \cap W_0^{1,2}(\Omega)$. Let the assumptions (1.8)–(1.11) *hold. Then there is at least one weak solution (ρ, v) to the problem* (1.2), (1.3), (1.12)–(1.14) *such that*

$$\begin{aligned} &\rho \in L^\infty(I; W^{1,q}(\Omega)),\ \frac{\partial\rho}{\partial t} \in L^\infty(I; L^q(\Omega)), \\ &v \in L^\infty(I; W^{2,q}(\Omega) \cap W_0^{1,2}(\Omega)),\ \frac{\partial v}{\partial t} \in L^2(Q_T). \end{aligned} \tag{2.9}$$

Proof. First step is based on the modified Galerkin method and the method of characteristics, which give us the following identities see [14]:

$$\int_{\Omega_t} \rho_m \mathrm{d}x = \int_\Omega \rho_0 \mathrm{d}x = m_0, \tag{2.10}$$

$$\frac{1}{2}\int_{Q_T}\frac{\partial}{\partial t}(\rho_m|v^m|^2) + \int_{Q_T}\frac{\partial}{\partial t}(P(\rho_m)) + \int_{Q_T}\frac{\partial V}{\partial e_{ii}}(\mathrm{Tr}e(v^m))e_{ii}(v^m) + \frac{\partial W}{\partial(\frac{e_{ij}}{\partial x_k})}\frac{\partial e_{ij}}{\partial x_k}(v^m) = \int_{Q_T}\rho_m b_i v_i^m, \tag{2.11}$$

$$\begin{aligned} &\int_{Q_T}\rho_m\left(\frac{\partial v_i^m}{\partial t}\right)^2 + \int_{Q_T}\rho_m\frac{\partial v_i^m}{\partial x_l}v_j^m\frac{\partial v_i^m}{\partial t} + \int_{\Omega_t} V(\mathrm{Tr}e(v^m)) + W(De(v^m)) + \int_{Q_T}\frac{\partial p(\rho_m)}{\partial x_i}\frac{\partial v_i^m}{\partial t} = \\ &\qquad \int_{\Omega_0} V(\mathrm{Tr}e(v^m(0))) + W(De(v^m(0))) + \int_{Q_T}\rho_m b_i\frac{\partial v_i^m}{\partial t}\cdot \end{aligned} \tag{2.12}$$

In the second step the limiting process is performed. We mention only the convergence in the nonlinear term which is different now and also was not clearly explained in [14]. It reads

$$\int_\Omega \frac{\partial V}{\partial \mathrm{Tr} e}(\mathrm{Tr} e(v^m))e_{ii}(v^m) + \frac{\partial W}{\partial(\frac{\partial e_{ij}}{\partial x_k})}(De(v^m))\frac{\partial e_{ij}(v^m)}{\partial x_k}\cdot$$

Let us define

$$G_1(v^m) = \frac{\partial V}{\partial \mathrm{Tr} e}(\mathrm{Tr} e(v^m))\mathrm{Tr} e(v^m)$$

and

$$G_2(v^m) = \frac{\partial W}{\partial(\frac{\partial e_{ij}}{\partial x_k})}(De(v^m))\frac{\partial e_{ij}(v^m)}{\partial x_k}\cdot$$

We set

$$X_m^s = \int_0^s (G_1(v^m) - G_1(\varphi), e(v^m) - e(\varphi))\mathrm{d}t + \int_0^s (G_2(v^m) - G_2(\varphi), De(v^m) - De(\varphi))\mathrm{d}t \ + \frac{1}{2}|\rho_m v^m(s)|^2.$$

Choosing a subsequence v_{n_k} which we denote as v_n and using monotonicity of X_m^s we obtain

$$\liminf_{m\to+\infty} X_m^s \geq \frac{1}{2}|\rho v(s)|^2.$$

Thus, it follows from (2.11) that

$$\begin{aligned}
X_m^s &= \int_0^s (\rho_m b, v^m)\mathrm{d}t + \frac{1}{2}|\rho_{0m} v_0^m|^2 - \int_0^s (G_2(v^m), De(\varphi))\mathrm{d}t - \int_0^s (G_2(\varphi), De(v^m - \varphi))\mathrm{d}t \\
&\quad - \int_0^s (G_1(v^m), e(\varphi))\mathrm{d}t - \int_0^s (G_1(\varphi), e(v^m - \varphi))\mathrm{d}t \\
&\to \int_0^s (\rho b, v) + \frac{1}{2}|\rho_0 v_0|^2 - \int_0^s (\xi_1, De(\varphi)) - \int_0^s (G_2(\varphi), De(v - \varphi)) - \int_0^s (\xi_2, e(\varphi)) \\
&\quad - \int_0^s (G_1(\varphi), e(v - \varphi)) \geq \frac{1}{2}|\rho v(s)|^2.
\end{aligned}$$

Finally, we get

$$\int_0^t (\xi_1 - G_2(\varphi), De(v) - De(\varphi)) + (\xi_2 - G_1(\varphi), e(v - \varphi)) \geq 0$$

which implies that

$$\xi_1 = G_2(v), \quad \xi_2 = G_1(v).$$

This concludes the proof. □

Theorem 2.4. *(Uniqueness of the weak solution).*
Let the assumptions of Theorem 2.3 *be fulfilled. Then the weak solution obtained in Theorem* 2.3 *is unique.*

Proof. Is analogous to the proof of uniqueness in [14].

3. Unconditional asymptotic stability of the rest state

The goal of this section is to prove the stability of the rest state $(\rho, 0)$, characterized by (1.1). Let $b = \frac{\partial \xi}{\partial x_i}$ with $\xi \in W^{2,\infty}(\Omega)$, $\frac{\mathrm{d}p}{\mathrm{d}\rho} > 0$ and let (ρ_0, v_0) satisfy conditions $\rho_0 \in C^1(\overline{\Omega})$, $\rho_0 > 0$ in $\overline{\Omega}$, $v_0 \in W^{2,q}(\Omega) \cap W_0^{1,2}(\Omega)$.We specify the class of perturbated flows where the rest state will be stable:

$$\begin{aligned}\mathcal{J} = \{(\tilde{\rho}, \tilde{v}) : \tilde{\rho} \in L^\infty(I, W^{1,q}(\Omega)), \tilde{v} \in L^\infty(I, W^{2,q}(\Omega)) \cap W_0^{1,2}(\Omega)),\\ \frac{\partial \tilde{\rho}}{\partial t} \in L^\infty(I, L^q(\Omega)), \frac{\partial \tilde{v}}{\partial t} \in L^2(Q_T), \text{for any T, and there exist}\\ \delta_1, \delta_2 \text{ such that } , 0 < \delta_1 \leq \tilde{\rho} < \delta_2 \text{ uniformly in } (0, \infty) \times \bar{\Omega},\\ (\tilde{\rho}, \tilde{v}) \text{ is a weak solution of } (1.2), (1.3)\}.\end{aligned}$$

Multiplying (1.3) by v we obtain

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega [\rho |v|^2 + \mathcal{P}(\rho) + \xi \rho] \mathrm{d}x + \|u\|_{W^{2,q}} = 0. \tag{3.1}$$

Let us denote $\sigma(t) = \left(\int_{t-1}^t \|v\|^q_{W^{2,q}(\Omega)} \right)^{1/q}$. From (3.1) we have that

$$\lim_{t \to \infty} \sigma(t) = 0. \tag{3.2}$$

Moreover, we would like to prove that

$$v(t) \to 0 \quad \text{strongly in } L^2(\Omega) \text{ as } t \to \infty.$$

It holds

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho |v|^2 + \|v\|_{W^{2,q}(\Omega)} + \|\nabla \cdot v\|_{L^q(\Omega)} - \int_\Omega p(\rho) \nabla \cdot v = \int_\Omega \rho b v \mathrm{d}x.$$

By integrating the above relation with respect to t from s to t, $0 \leq s \leq t < \infty$ and with respect to s from $t-1$ to t we obtain

$$\begin{aligned}\frac{1}{2} \|\sqrt{\rho(t)} v(t)\|^2_{L^2(\Omega)} + \int_{t-1}^t \int_s^t \|v\|_{W^{2,q}(\Omega)} + \int_{t-1}^t \int_s^t \|\nabla \cdot v\|_{L^q(\Omega)} =\\ \frac{1}{2} \int_{t-1}^t \|\sqrt{\rho(t)} v(t)\|^2_{L^2(\Omega)} + \int_{t-1}^t \int_s^t \int_\Omega |\rho b v + p(\rho) \nabla \cdot v| \mathrm{d}x \mathrm{d}\tau \mathrm{d}s.\end{aligned} \tag{3.3}$$

Further,

$$\int_{t-1}^t \int_s^t \int_\Omega |\rho b v| \leq \|\rho\|_{L^\infty(0,T; L^q(\Omega))} \|b\|_{L^\infty(Q_T)} \left(\int_{t-1}^t \|v\|^q_{W^{2,q}(\Omega)} \mathrm{d}\tau \right) \leq c \|b\|_{L^\infty(Q_T)} \sigma(t),$$

$$\int_{t-1}^t \int_s^t \int_\Omega p(\rho) \nabla \cdot v \leq c \sup_{\delta_1 \leq r \leq \delta_2} p(r) \sigma(t).$$

Then

$$\|\sqrt{\rho(t)} v(t)\|^2_{L^2(\Omega)} \leq c(1 + \|b\|_{L^\infty(Q_T)}) \sigma(t).$$

From (3.2), using the fact that $\rho > 0$, we get that

$$v(t) \to 0 \quad \text{strongly in } L^2(\Omega) \text{ as } t \to \infty.$$

Now, we define a function $w(t)$ in Q_T as the solution of the following problem

$$\begin{aligned} &\Delta w = \nabla \cdot (\widetilde{\rho} b) \quad \text{ in } Q_T, \\ &\frac{\partial w}{\partial n} = \widetilde{\rho} b \cdot n \quad \text{ on } \partial\Omega \text{ x } [0,\infty), \\ &\int_\Omega w \, \mathrm{d}x = 0. \end{aligned} \tag{3.4}$$

The goal is to prove that

$$\widetilde{\rho}(t) \to \rho \text{ in } L^2(\Omega) \quad \text{as } t \to \infty \tag{3.5}$$

where

$$\begin{aligned} &\nabla p(\rho) = \rho b \text{ in } \Omega, \\ &\int_\Omega \rho \mathrm{d}x = \int_\Omega \rho_0 \mathrm{d}x. \end{aligned} \tag{3.6}$$

Firstly, we shall prove that

$$\left| \int_\Omega [p(\tilde{\rho}(t)) - w(t)](h - Mh)\mathrm{d}x \right| \le c\sigma(t)\|h\|_{L^2(\Omega)}, \quad h \in L^2(\Omega), \tag{3.7}$$

with a constant c independent of w. We set

$$Mh = |\Omega|^{-1} \int_\Omega h \mathrm{d}x.$$

For this purpose we shall firstly estimate the integral

$$I(t) = \int_{t-1}^t \varphi(s-t) \int_\Omega [p(\tilde{\rho}(s)) - w(s)] h \mathrm{d}x \mathrm{d}s, \quad \varphi \in C_0^\infty(-1,0), \tag{3.8}$$

where φ is a fixed function such that $\int_{-1}^0 \varphi(\tau)\mathrm{d}\tau = 1$.

Let ψ be a solution to the following problem for arbitrary $h \in L^2(\Omega)$

$$\begin{aligned} &\Delta \psi = h - Mh \quad \text{ in } \Omega \\ &\frac{\partial \psi}{\partial n} = 0 \quad \text{ on } \partial\Omega \\ &\int_\Omega \psi \mathrm{d}x = 0. \end{aligned} \tag{3.9}$$

Applying the classical results on boundary-value problems for elliptic equations it yields

$$\|\psi\|_{W^{2,2}(\Omega)} \le c\|h - Mh\|_{L^2(\Omega)}, \tag{3.10}$$

$$\|h - Mh\|_{L^2(\Omega)} \le \|h\|_{L^2(\Omega)}, \tag{3.11}$$

with a constant $c > 0$ independent of h. Substituting (3.9) into (3.8) we find

$$\begin{aligned}
I(t) &= -\int_{t-1}^{t} \varphi(s-t) \int_\Omega [w(s) - p(\tilde{\rho}(s))] \Delta\psi \mathrm{d}x\mathrm{d}s \\
&= \int_{t-1}^{t} \varphi(s-t) \int_\Omega [\tilde{\rho}(s) b - \nabla p(\tilde{\rho}(s))] \nabla\psi \mathrm{d}x\mathrm{d}s \\
&= \int_{t-1}^{t} \varphi(s-t) \int_\Omega \left[\tilde{\rho}\frac{\partial v(s)}{\partial s} + \tilde{\rho}(s)v(s)\nabla v(s) - div\tau_{ij}\right] \nabla\psi \mathrm{d}x\mathrm{d}s \\
&= \int_{t-1}^{t} \varphi(s-t) \int_\Omega \frac{\partial\tilde{\rho}v}{\partial s}\nabla\psi + \int_{t-1}^{t} \varphi(s-t) \int_\Omega \nabla\cdot(\tilde{\rho}(s)v(s)v(s))\nabla\psi \\
&\quad + \int_{t-1}^{t} \varphi(s-t) \int_\Omega \tau_{ij}\nabla\nabla\psi \\
&= -\int_{t-1}^{t} \varphi'(s-t) \int_\Omega \tilde{\rho}(s)v(s)\nabla\psi - \int_{t-1}^{t} \varphi(s-t) \int_\Omega \tilde{\rho}(s)v(s)v(s)\nabla\nabla\psi \\
&\quad + \int_{t-1}^{t} \varphi(s-t) \int_\Omega \frac{\partial^2 W}{\partial(\frac{\partial e_{ij}}{\partial x_k})\partial(\frac{\partial e_{ij}}{\partial x_l})}(\frac{\partial e_{ij}}{\partial x_l})\nabla\nabla\psi + \int_{t-1}^{t} \varphi(s-t) \int_\Omega \frac{\partial V(\mathrm{Tr}e)}{\partial e_{ii}}\Delta\psi \\
&\leq \sup_{t\in(0,T)} \|\tilde{\rho}(t)\|_{L^q(\Omega)} \left(\int_{-1}^{0} \varphi'(\tau)^{q'}\mathrm{d}\tau\right)^{1/q'} \|\nabla\psi\|_{L^2(\Omega)} \left(\int_{t-1}^{t} \|v(s)\|^q_{W^{2,q}(\Omega)}\right)^{1/q} \\
&\quad + \sup_{t\in(0,T)} \|\tilde{\rho}(t)\|_{L^q(\Omega)} \left(\int_{-1}^{0} \varphi(\tau)^{q'}\mathrm{d}\tau\right)^{1/q'} \|\psi\|_{W^{2,2}(\Omega)} \left(\int_{t-1}^{t} \|v(s)\|^q_{W^{2,q}(\Omega)}\right)^{2/q} \\
&\quad + \left(\int_{-1}^{0} \varphi(\tau)^{q'}\mathrm{d}\tau\right)^{1/q'} \|\psi\|_{W^{2,2}(\Omega)} \left(\int_{t-1}^{t} \|v\|^q_{W^{2,q}(\Omega)}\right)^{1/q} \\
&\quad + \left(\int_{-1}^{0} \|\varphi(\tau)\|^{q'}\mathrm{d}\tau\right)^{1/q'} \left(\int_{0}^{t} \|v\|^q_{W^{2,q}(\Omega)}\right)^{1/q} \|h\|_{L^2(\Omega)} \leq \|\varphi\|_{1,q'}\|h\|_{L^2(\Omega)}\sigma(t).
\end{aligned}$$

However, φ is an arbitrary fixed function, which implies that

$$I(t) \leq c\sigma(t)\|h\|_{L^2(\Omega)}, \quad t \geq 0. \tag{3.12}$$

Now, we go back and prove (3.7).

$$\begin{aligned}
\left|\int_\Omega [p(\tilde{\rho}(t)) - w(t)](h - Mh)\mathrm{d}x\right| &= \left|\int_{t-1}^{t} \varphi(s-t) \int_\Omega (\nabla p(\tilde{\rho}(t)) - \tilde{\rho}(t)b)\nabla\psi \mathrm{d}x\mathrm{d}s\right| \\
&\leq |I(t)| + \left|\int_{t-1}^{t} \varphi(s-t) \int_\Omega [p(\tilde{\rho}(t)) - p(\tilde{\rho}(s))][h - Mh]\mathrm{d}x\mathrm{d}s\right| \\
&\quad + \left|\int_{t-1}^{t} \varphi(s-t) \int_\Omega [\tilde{\rho}(t) - \tilde{\rho}(s)]b\nabla\psi \mathrm{d}x\mathrm{d}s\right|.
\end{aligned} \tag{3.13}$$

We have

$$\tilde{\rho}(t) - \tilde{\rho}(s) = \int_s^t \frac{\partial\tilde{\rho}}{\partial\tau}\mathrm{d}\tau = -\int_s^t \nabla\cdot(\tilde{\rho}(\tau)u(\tau))\mathrm{d}\tau. \tag{3.14}$$

Hence

$$\begin{aligned}\left|\int_{t-1}^{t}\varphi(s-t)\int_\Omega[\tilde\rho(t)-\tilde\rho(s)]b\cdot\nabla\psi\right| &= \left|\int_{t-1}^{t}\int_s^t\varphi(s-t)\int_\Omega\tilde\rho(\tau)u(\tau)\nabla(b\cdot\nabla\psi)\mathrm{d}x\mathrm{d}\tau\mathrm{d}s\right| \\ &\le \|\tilde\rho\|_{L^\infty(0,T;L^q(\Omega))}\|\varphi\|_{L^{q'}(0,T)}\|\nabla b\|_{L^\infty(0,T;L^\infty(\Omega)}\|\nabla\psi\|_{L^2(\Omega)} \\ &\quad+\|b\|_{L^\infty(0,T;L^\infty(\Omega)}\|\psi\|_{W^{2,2}(\Omega)}\left(\int_{t-1}^{t}\|v\|^q_{W^{2,q}(\Omega)}\mathrm{d}s\right)^{1/q} \\ &\le c\|b\|_{W^{1,\infty}(\Omega)}\|\psi\|_{W^{2,2}(\Omega)}\sigma(t).\end{aligned}\tag{3.15}$$

The second term of the RHS of (3.13) can be rewriten in the following form

$$\int_{t-1}^{t}\varphi(s-t)\int_\Omega[p(\tilde\rho(t))-p(\tilde\rho(s))](h-Mh)\mathrm{d}x\mathrm{d}s=\int_{t-1}^{t}\varphi(s-t)\int_\Omega\int_s^t d_\tau p(\tilde\rho(\tau))(h-Mh)\mathrm{d}\tau\mathrm{d}s\mathrm{d}x.\tag{3.16}$$

From the continuity equation it follows that

$$\frac{\partial p(\tilde\rho)}{\partial t}=-(vp'(\tilde\rho)\nabla\tilde\rho+\tilde\rho p'(\tilde\rho)\nabla\cdot v).$$

Thus, we can estimate (3.16) from above

$$\begin{aligned}\left|\int_{t-1}^{t}\varphi(s-t)\int_s^t\int_\Omega d_\tau p(\tilde\rho(\tau))(h-Mh)\mathrm{d}x\mathrm{d}\tau\mathrm{d}s\right| &\le \left|\int_{t-1}^{t}\varphi(s-t)\int_s^t\int_\Omega v(\tau)p'(\tilde\rho(\tau))\nabla\tilde\rho(\tau)(h-Mh)\mathrm{d}x\mathrm{d}\tau\mathrm{d}s\right| \\ &\quad+\left|\int_{t-1}^{t}\varphi(s-t)\int_s^t\int_\Omega\tilde\rho(\tau)p'(\tilde\rho)\nabla\cdot v(h-Mh)\mathrm{d}x\mathrm{d}\tau\mathrm{d}s\right| \\ &\le\|\varphi\|_{L^\infty(0,T)}\sup_{\delta_1\le r\le\delta_2}p'(r)\sigma(t)\left(\int_{t-1}^{t}\|\nabla\tilde\rho(\tau)\|^q_{L^q(\Omega)}\right)^{1/q}\|h\|_{L^2(\Omega)} \\ &\quad+c\|\varphi\|_{L^{q'}(0,T)}\sup_{\delta_1\le r\le\delta_2}rp'(r)\|h\|_{L^2(\Omega)}\sigma(t)\le c\sigma(t)\|h\|_{L^2(\Omega)},\end{aligned}\tag{3.17}$$

which together with (3.12), (3.13) and (3.15) leads to (3.7). But it follows from (3.7) that

$$p(\tilde\rho(t))-w(t)-Mp(\tilde\rho(t))\to 0\quad\text{ in } L^2(\Omega)\quad t\to\infty.\tag{3.18}$$

In fact, there exists a subsequence $t_n\to\infty$ such that $\tilde\rho(t_n)\to\rho$ weakly in L^2, where $w(t)$ is a solution of (3.4) for any t. The sequence $w(t_n)$ is compact in $L^2(\Omega)$. Choosing a subsequence converging to $w_\infty\in L^2(\Omega)$ we find that $w_\infty=A(\rho b)$, where A stands for the solution operator of (3.4). Thus, $p(\widetilde\rho(t_n))-Mp(\widetilde\rho(t_n))\to w_\infty$ in $L^2(\Omega)$ and a.e. in Ω. Since $\widetilde\rho$ belongs to $\mathcal{J}$, we get that $Mp(\tilde\rho(t_n))\to p_\infty$. This yields

$$\widetilde\rho(t_n)\to p^{-1}(w_\infty+p_\infty)\text{ a.e. in }\Omega\text{ and in }L^2(\Omega)\quad\text{strongly.}\tag{3.19}$$

Hence $\rho=p^{-1}(w_\infty+p_\infty)$ or $p(\rho)=A(\rho b)+p_\infty$. This is equivalent to

$$\int_\Omega[\nabla p(\rho)-\rho b]\nabla\theta\mathrm{d}x=0\quad\forall\quad\theta\in C_0^\infty(\Omega),\quad M\theta=0\tag{3.20}$$

which yields

$$\nabla p(\rho)=(Id-R)(\rho b),\tag{3.21}$$

where R is the projection of $L^2(\Omega)$ onto the closure in $L^2(\Omega)$ of the space of divergence free vector functions. Now, we want to prove that $R(\rho b) = 0$. To this purpose let us take an arbitrary $\theta \in C_0^\infty(\Omega), \eta = R\theta$. Then we have

$$\int_\Omega R(\widetilde{\rho}(s)b)\theta = \int_\Omega \left[\widetilde{\rho}(s)\frac{\partial v}{\partial s} + \widetilde{\rho}(s)v(s)\nabla v(s) - \mathrm{div}\tau_{ij}\right]\eta \mathrm{d}x. \tag{3.22}$$

In the same way as before we can prove

$$\left|\int_{t-1}^t \varphi(s-t)\int_\Omega R(\widetilde{\rho}(s)b)\theta \mathrm{d}x\mathrm{d}s\right| \le c\|\varphi'\|_{L^q(0,T)}\|\eta\|_{W^{1,2}(\Omega)}\sigma(t)$$

which implies that

$$\left|\int_\Omega R(\widetilde{\rho}(t)b)\theta \mathrm{d}x\right| \le \|\varphi'\|_{L^q(0,T)}\|\eta\|_{W^{1,2}(\Omega)}\sigma(t), \quad t \ge 0.$$

Since $\|\eta\|_{W^{1,2}(\Omega)} \le \|\theta\|_{W^{1,2}(\Omega)}$, we will obtain that

$$R(\widetilde{\rho}(t)b) \to 0 \text{ in } W^{-1,2}(\Omega) \text{ strongly.} \tag{3.23}$$

Then $R(\widetilde{\rho}(t_n)b) \to 0$ in $W^{-1,2}(\Omega)$ and $R(\widetilde{\rho}(t_n)b) \to R(\rho b)$ in $L^2(\Omega)$. Thus, $R(\rho b) = 0$ and

$$\nabla p(\rho) = \rho b \text{ in } \Omega. \tag{3.24}$$

Now, if π is such a function that $\pi'(r) = r^{-1}p'(r)$ then, $\nabla\pi(\rho) = \nabla\xi$. Hence, we have $\pi(\rho) = g + h$ with some constant h or $\rho = \pi^{-1}(g+h)$. It holds that

$$\int_\Omega \widetilde{\rho}(x,t_n)\mathrm{d}x = \int_\Omega \widetilde{\rho}_0 \mathrm{d}x, \tag{3.25}$$
$$\int_\Omega \widetilde{\rho}(x)\mathrm{d}x = \int_\Omega \widetilde{\rho}_0(x)\mathrm{d}x.$$

Since the function $a(h) = \int_\Omega \pi^{-1}(g+h)\mathrm{d}x$ is increasing, the function ρ is uniquely determined by a unique constant h^* satisfying $\int_\Omega \pi^{-1}(g+h^*)\mathrm{d}x = \int_\Omega \widetilde{\rho}_0\mathrm{d}x$. This implies

$$\widetilde{\rho} \to \rho \text{ in } L^2(\Omega). \tag{3.26}$$

We have proved the following result.

Theorem 3.1. *Let* $b_i = \frac{\partial\xi}{\partial x_i}$, $\xi \in W^{2,\infty}(\Omega)$, *$\xi$ small enough,* $\frac{\mathrm{d}p}{\mathrm{d}\rho} > 0$ *and $\mathcal{J}$ be the set of all solutions* $(\widetilde{\rho}, \widetilde{u})$ *of the problem* (1.2)–(1.13) *with the initial conditions* $v_0 \in W^{2,q}(\Omega) \cap W_0^{1,2}(\Omega)$, $\rho_0 \in C^1(\overline{\Omega})$. *Then the rest state* (ρ, v) *characterized by equations (1.1) is unconditionally asymptotically stable in the class $\mathcal{J}$ in the sense that there exists a subsequence* $\{t_n\}, t_n \to +\infty$ *such that*

$$\widetilde{\rho}(t_n) \to \rho \textit{ in } L^2(\Omega),$$
$$\widetilde{v}(t_n) \to 0 \quad \textit{in } L^2(\Omega),$$

and

$$\nabla p(\rho) = \rho b \textit{ in } \Omega.$$

Remark 3.1. The behaviour of incompressible Newtonian fluids under assumptions of sufficient regularity and $\rho > 0$ was studied by Salvi and Straškraba [25]. Semigroup approach we can find in the work of Neustupa [21].

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