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ON THE ASYMPTOTIC ANALYSIS OF A NON-SYMMETRIC BAR

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Abstract. We study the 3-D elasticity problem in the case of a non-symmetric heterogeneous rod. The asymptotic expansion of the solution is constructed. The coercitivity of the homogenized equation is proved. Estimates are derived for the difference between the truncated series and the exact solution.

Résumé. Nous construisons un développement asymptotique de la solution du problème d'élasticité dans une barre non symétrique. Nous obtenons des équations homogénéisées unidimensionnelles dont nous montrons la positivité. Des estimations d'erreur permettent la justification de l'aspect asymptotique de l'approximation.

Mathematics Subject Classification. 65N99.

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1. Introduction

The main objective of the paper is to construct and justify an asymptotic expansion of a solution of the 3-D elasticity problem for a non-symmetric heterogeneous rod. The symmetric case has been considered in [9], and it was proved that the limit problem when the ratio of the diameter of a rod to its length tends to zero is represented by a diagonal matrix of the ordinary differential operators. Thus the construction of the homogenized equation and the proofs of existence and uniqueness of this solution are substantially simplified. The non-symmetric bar is considered below. We construct below an asymptotic expansion of a solution of the problem. The limit operator is presented by a full symmetric matrix, this expresses the coupling between the components of displacement (tension, bending, and torsional component). The existence and uniqueness of its solution are the consequence of the positivity of elastic energy. We give *a priori* estimates on the error. Similar questions for anisotropic beams are studied in [11] and for bars in [12]. The phenomenon of coupling appeared implicitly in the case of plates and explicitly for beams. A result of convergence was obtained in [5] for the thin cylinder. The equations of the homogenized problem are unidimensional, but their coefficients are averages of the solutions of auxilliary problems in three-dimensional cells. Except in particular cases where one can explicitly calculate these coefficients, the resolution of these equations requires a numerical approximation.

Keywords and phrases. Elastic bar homogenization.

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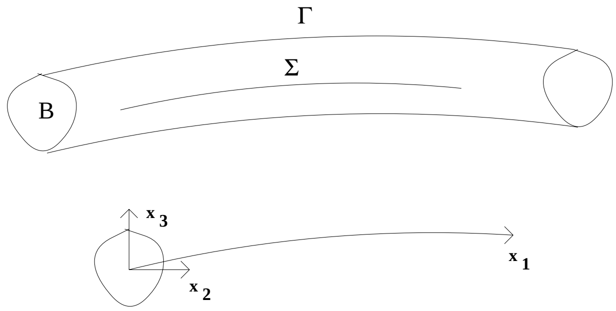

FIGURE 1. An example of a non-symmetric rod.

2. Formulation of the problem

Let B be a two-dimensional bounded domain in $\mathbb{R}^2$. Assume that its boundary ∂B is smooth. Set $\hat{x} = (x_2, x_3)$. Let ε be a small parameter ($\varepsilon \ll 1$) and let Ω_ε be a rod, Γ_ε denotes its lateral boundary (see Fig. 1).

$$\Omega_\varepsilon = \{x \in \mathbb{R}^3; x_1 \in [0,1], \frac{\hat{x}}{\varepsilon} \in B\}.$$
$$\Gamma_\varepsilon = \{x \in \mathbb{R}^3;\ x_1 \in [0,1], \frac{\hat{x}}{\varepsilon} \in \partial B\}.$$

The 3×3 matrices A_{ij} are 1-periodic functions in ξ_1. Let a_{ijkl} denote the element of A_{ij} in link k and column l. The functions $a_{ijkl}(\xi)$ are piecewise smooth: they are infinitely differentiable except on some smooth surfaces Σ_i, these surfaces do not intersect the boundary of Ω_ε. Denote $\Sigma = \cup \Sigma_i$. These coefficients satisfy the following symmetry and positivity relations:

$$\begin{cases} a_{kijl} \ = a_{iklj} \ = a_{ljki} \text{ for all } k,i,l,j, \\ \exists\, \kappa > 0 \mid \ a_{kilj}(\xi)\eta_{lj}\eta_{ki} \geq \kappa\ \eta_{lj}\eta_{lj}, \quad \text{for each symmetric constant matrix } \eta_{lj}. \end{cases}$$

Throughout the paper the convention of summation over repeated indices is used. Introduce the notations

$$\begin{aligned} L_\varepsilon u(x) &= -\frac{\partial}{\partial x_i}\left(A_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial u(x)}{\partial x_j}\right), \\ \frac{\partial u(x)}{\partial \nu} &= -\nu_i A_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial u(x)}{\partial x_j}, \end{aligned}$$

were ν_i are the components of the unit normal vector to the boundary Γ_ε. We consider the elasticity equations

$$\begin{cases} L_\varepsilon u(x) = f(x), & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u(x)}{\partial \nu} = 0, & \text{on } \Gamma_\varepsilon, \\ u(x) \text{ is 1-periodic} & \text{in } x_1, \end{cases} \tag{2.1}$$

with the natural conjugation conditions on the interface of discontinuity of the coefficients

$$\begin{cases} \left[\dfrac{\partial u(x)}{\partial \nu}\right]_{|\Sigma} = 0, \\ [u(x)]_{|\Sigma} = 0. \end{cases} \tag{2.2}$$

[.] denotes the jump of a function on the interface of discontinuity. The right-hand side $f(x)$ and the unknown $u(x)$ are 3-dimensional vector functions. The Problem (2.1)–(2.2) simulates the stressed state of a non-homogeneous rod. The elements a_{ijkl} are the elasticity coefficients. The right-hand side $f(x)$ is the density of mass forces. We study the asymptotic behavior of the solution of (2.1)–(2.2) in the case were the bar is non-symmetric. We construct a formal asymptotic solution (FAS) of the Problem (2.1)–(2.2). We establish the positivity of the limit problem. We prove that the FAS is an asymptotic expansion of an exact solution of the problem and establish error estimates for the partial sum of the series of FAS's. If the ends of the rod are fixed we must construct the boundary layer corrector. Full asymptotic expansion of the solution to (2.1) with mixed conditions were constructed in [9, 10]. In [6], the error estimates were proved for other norms.

3. A particular example

3.1. Asymptotic expansion

For the first time, we study the particular case were the right-hand side $f(x)$ depends only on the first component x_1, so the density of mass forces are unvarying on the cross-section, the torsional part of displacement is zero. Let us present the right-hand side $f(x_1)$ as the form:

$$f^\varepsilon(x) = \Lambda^{-1} f(x_1), \qquad \text{were} \quad \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon^{-1} & 0 \\ 0 & 0 & \varepsilon^{-1} \end{pmatrix}.$$

We seek a formal asymptotic expansion as

$$u^{(\infty)}(x) = \sum_{l=0}^{+\infty} \varepsilon^l N_l(\xi) \frac{\mathrm{d}^l \Lambda v(x_1)}{\mathrm{d}x_1^l}, \quad \xi = (x_1, \hat{x}/\varepsilon), \tag{3.1}$$

were $N_l(\xi)$ are 3×3 matrix functions 1-periodic in ξ_1, and $v(x_1)$ is a 3-vector function 1-periodic. Substituting (3.1) into equations of the Problem (2.1)–(2.2), we obtain

$$L_\varepsilon u^{(\infty)}(x) - \Lambda^{-1} f(x_1) = \sum_{l=0}^{+\infty} \varepsilon^{l-2} K_l(\xi) \frac{\mathrm{d}^l \Lambda v(x_1)}{\mathrm{d}x_1^l} - \Lambda^{-1} f(x_1).$$

On the lateral boundary we have

$$\frac{\partial u^{(\infty)}(x)}{\partial \nu} = \sum_{l=0}^{+\infty} \varepsilon^{l-1} G_l(\xi) \frac{\mathrm{d}^l \Lambda v(x_1)}{\mathrm{d}x_1^l} = 0.$$

The conjugation conditions on Σ land

$$\begin{aligned} \left[\frac{\partial u^{(\infty)}(x)}{\partial \nu}\right]_{|\Sigma} &= \sum_{l=0}^{+\infty} \varepsilon^{l-1} \left[G_l(\xi)\right]_{|\Sigma} \frac{\mathrm{d}^l \Lambda v(x_1)}{\mathrm{d}x_1^l} = 0, \\ \sum_{l=0}^{+\infty} \varepsilon^l \left[N_l(\xi)\right]_{|\Sigma} &= 0, \end{aligned}$$

where

$$\begin{aligned}
K_l(\xi) &= L_{\xi\xi} N_l(\xi) + T_l(\xi), \\
L_{\xi\xi} &= \frac{\partial}{\partial \xi_i}\left(A_{ij}(\xi)\frac{\partial}{\partial \xi_j}\right), \\
T_l(\xi) &= A_{1j}(\xi)\frac{\partial N_{l-1}(\xi)}{\partial \xi_j} + A_{11}(\xi)N_{l-2}(\xi) + \frac{\partial}{\partial \xi_i}\left(A_{i1}(\xi)N_{l-1}(\xi)\right), \\
G_l(\xi) &= \nu_i A_{ij}(\xi)\frac{\partial N_l(\xi)}{\partial \xi_j} + \nu_i A_{i1}(\xi)N_{l-1}(\xi).
\end{aligned}$$

Let us require that

$$\begin{cases} K_l(\xi) = k_l, \quad G_l(\xi) = 0, \\ [N_l(\xi)]_{|\Sigma} = 0, \quad [G_l(\xi)]_{|\Sigma} = 0, \end{cases}$$

where $k_l = const$. We obtain the following recurrent chain of problem for $N_l(\xi)$.

$$\begin{cases} L_{\xi\xi} N_l(\xi) = k_l - T_l(\xi), \quad \xi \in \mathbb{R} \times B, \\ \dfrac{\partial N_l(\xi)}{\partial \nu} = -\nu_i A_{i1}(\xi)N_{l-1}(\xi), \quad \text{on } \Gamma, \\ [N_l(\xi)]_{|\Sigma} = 0, \; \left[\dfrac{\partial N_l(\xi)}{\partial \nu}\right]_{|\Sigma} = [\nu_i A_{i1}(\xi)N_{l-1}(\xi)]_{|\Sigma}, \\ N_l(\xi) \text{ is 1-periodic in } \xi_1. \end{cases} \tag{3.2}$$

Here, k_l are chosen from the solvability conditions for (3.2):

$$k_l = \left\langle A_{1j}(\xi)\frac{\partial N_{l-1}(\xi)}{\partial \xi_j} + A_{11}(\xi)N_{l-2}(\xi)\right\rangle.$$

Thus, the algorithm for constructing the matrices $N_l(\xi)$ is inductive. Suppose that $N_l(\xi) = 0$ for each $l < 0$ and $N_0(\xi) = I_3$. If $l \geq 1$ then $N_l(\xi)$ is the solution of the Problem (3.2), note that the right-hand side is defined by the functions $N_m(\xi)$ with priority $m < l$. This condition may be used for defining $N_l(\xi)$ inductively in terms of the functions M_m of lower priority. We have

$$k_0 = 0, \quad k_1 = 0 \text{ and } k_2 = \left\langle A_{1j}(\xi)\frac{\partial N_1(\xi)}{\partial \xi_j} + A_{11}(\xi)\right\rangle$$

and the following assertion is valid:

Lemma 3.1. *The element k_2^{11} of the matrix k_2 is positive, all the other elements are zero.*

Proof. By substitution into (3.2), we verify that the second (the third) column of the matrix $N_1(\xi)$ satisfies

$$N_1^{(s)} = \begin{pmatrix} -\xi_s \\ 0 \\ 0 \end{pmatrix} \quad for\ s = 2, 3.$$

For $r,\ s \in \{1,\ 2,\ 3\ \}$, we have

$$k_l^{rs} = \left\langle a_{1j}^{rp}(\xi)\frac{\partial N_{l-1}^{ps}(\xi)}{\partial \xi_j} + a_{11}^{rp}(\xi)N_{l-2}^{ps}(\xi)\right\rangle$$

for s = 2, 3, we have

$$
\begin{aligned}
k_2^{1s} &= \left\langle a_{1j}^{1p}(\xi)\frac{\partial N_1^{ps}(\xi)}{\partial \xi_j} + a_{11}^{1p}(\xi) I_3^{ps} \right\rangle \\
&= \left\langle -a_{1s}^{11}(\xi) + a_{11}^{1s}(\xi) \right\rangle = 0.
\end{aligned}
$$

Now let us prove $k_2^{s1} = 0$, we have

$$
\begin{aligned}
k_2^{s1} &= \left\langle a_{1j}^{sp}(\xi)\frac{\partial N_1^{p1}(\xi)}{\partial \xi_j} + a_{11}^{sp}(\xi) I_3^{p1} \right\rangle \\
&= \left\langle a_{sj}^{1p}(\xi)\frac{\partial N_1^{p1}(\xi)}{\partial \xi_j} + a_{s1}^{11}(\xi) \right\rangle \\
&= \left\langle a_{sj}^{1p}(\xi)\frac{\partial}{\partial \xi_j}\left(N_1^{p1} + \xi_1 I^{p1}\right) \right\rangle \\
&= \left\langle a_{ij}^{1p}(\xi)\frac{\partial}{\partial \xi_j}\left(N_1^{p1} + \xi_1 I_3^{p1}\right)\frac{\partial}{\partial \xi_i}\xi_s \right\rangle.
\end{aligned}
$$

Applying the integral identity for Problem (3.2) for $N_1(\xi)$, with the function $w(\xi) = \xi_s$, which is 1-periodic in ξ_1. This proves the lemma. To prove that $k_2^{11} > 0$ is similar to the proof of Theorem 1 of Ref. [9]. □

Lemma 3.2. *In the matrix* k_3, $k_3^{22} = k_3^{23} = k_3^{32} = k_3^{33} = 0$. *In the matrix* k_4, $k_4^{22} \neq 0$, $k_4^{33} \neq 0$.

Proof. Let us prove that $k_3^{22} = 0$,

$$
\begin{aligned}
k_3^{22} &= \left\langle a_{1j}^{2p}(\xi)\frac{\partial N_2^{p2}(\xi)}{\partial \xi_j} + a_{11}^{2p}(\xi) N_1^{p2}(\xi) \right\rangle \\
&= \left\langle a_{2j}^{1p}(\xi)\frac{\partial N_2^{p2}(\xi)}{\partial \xi_j} + a_{21}^{1p}(\xi) N_l^{p2}(\xi) \right\rangle \\
&= \left\langle \left(a_{ij}^{1p}(\xi)\frac{\partial N_2^{p2}(\xi)}{\partial \xi_j} + a_{i1}^{1p}(\xi) N_l^{p2}(\xi) \right)\frac{\partial \xi_2}{\partial \xi_i} \right\rangle.
\end{aligned}
$$

Applying the integral identity for Problem (3.2) for $l = 2$, we obtain

$$
\begin{aligned}
k_3^{22} &= -\left\langle \frac{\partial}{\partial \xi_i}\left(a_{ij}^{1p}(\xi)\frac{\partial N_2^{p2}(\xi)}{\partial \xi_j} + a_{i1}^{1p}(\xi) N_l^{p2}(\xi) \right)\xi_2 \right\rangle \\
&= -\left\langle \left(k_2^{12} - a_{1j}^{1p}(\xi)\frac{\partial N_1^{p2}(\xi)}{\partial \xi_j} - a_{11}^{1p}(\xi) I_3^{p2}(\xi) \right)\xi_2 \right\rangle \\
\text{or } k_2^{12} = 0 \quad k_3^{22} &= \left\langle \left(a_{1j}^{1p}(\xi)\frac{\partial N_1^{p2}(\xi)}{\partial \xi_j} + a_{11}^{1p}(\xi) I_3^{p2}(\xi) \right)\xi_2 \right\rangle \\
&= \left\langle \left(a_{12}^{11}(\xi) - a_{11}^{12}(\xi) \right)\xi_2 \right\rangle = 0.
\end{aligned}
$$

Similarly, we prove that $k_3^{23} = 0$, $k_3^{32} = 0$ and $k_3^{33} = 0$. For k_4, we use the proof of Lemma 5 of Ref. [9]. □

3.2. Homogenized problems

Let us rewrite the equation for the function $v(x_1)$ and its derivatives

$$\sum_{l=0}^{+\infty} \varepsilon^{l-2}\, k_l\, \frac{\mathrm{d}^l \Lambda v(x_1)}{\mathrm{d}x_1^l} = \Lambda^{-1} f(x_1), \quad x_1 \in (0,1). \tag{3.3}$$

Let us multiply (3.3) by Λ

$$\sum_{l=0}^{+\infty} \varepsilon^{l-2}\, \Lambda k_l\, \frac{\mathrm{d}^l \Lambda v(x_1)}{\mathrm{d}x_1^l} = f(x_1).$$

According to the previous lemmas, one can write the first terms of (3.3)

$$\begin{pmatrix} k_2^{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} L_2^\varepsilon + \varepsilon \begin{pmatrix} k_3^{11} & k_3^{12} & k_3^{13} \\ \varepsilon^{-1} k_3^{21} & 0 & 0 \\ \varepsilon^{-1} k_3^{32} & 0 & 0 \end{pmatrix} L_3^\varepsilon + \varepsilon^2 \Lambda\, k_4 L_4^\varepsilon + \sum_{l=5}^{+\infty} \varepsilon^{l-2}\, \Lambda k_l\, \frac{\mathrm{d}^l \Lambda v(x_1)}{\mathrm{d}x_1^l} = f(x_1).$$

$$\text{where } L_2^\varepsilon = \begin{pmatrix} \dfrac{\mathrm{d}^2}{\mathrm{d}x_1^2} v^1(x_1) \\ \varepsilon^{-1} \dfrac{\mathrm{d}^2}{\mathrm{d}x_1^2} v^2(x_1) \\ \varepsilon^{-1} \dfrac{\mathrm{d}^2}{\mathrm{d}x_1^2} v^3(x_1) \end{pmatrix}, \quad L_3^\varepsilon = \begin{pmatrix} \dfrac{\mathrm{d}^3}{\mathrm{d}x_1^3} v^1(x_1) \\ \varepsilon^{-1} \dfrac{\mathrm{d}^3}{\mathrm{d}x_1^3} v^2(x_1) \\ \varepsilon^{-1} \dfrac{\mathrm{d}^3}{\mathrm{d}x_1^3} v^3(x_1) \end{pmatrix}, \quad L_4^\varepsilon = \begin{pmatrix} \dfrac{\mathrm{d}^4}{\mathrm{d}x_1^4} v^1(x_1) \\ \varepsilon^{-1} \dfrac{\mathrm{d}^4}{\mathrm{d}x_1^4} v^2(x_1) \\ \varepsilon^{-1} \dfrac{\mathrm{d}^4}{\mathrm{d}x_1^4} v^3(x_1) \end{pmatrix},$$

The $v^s(x_1)$ (s = 1, 2, 3) are the components of the vector $v(x_1)$. Collecting similar terms of ε, we obtain

$$\sum_{l=0}^{+\infty} \varepsilon^l L_l v(x_1) = f(x_1), \quad x_1 \in (0,1). \tag{3.4}$$

Denote $\dfrac{\mathrm{d}^s}{\mathrm{d}x^s}$ by d^s, thus

$$L_0 = \begin{pmatrix} k_2^{11}\mathrm{d}^2 & k_3^{12}\mathrm{d}^3 & k_3^{13}\mathrm{d}^3 \\ k_3^{21}\mathrm{d}^3 & k_4^{22}\mathrm{d}^4 & k_4^{23}\mathrm{d}^4 \\ k_3^{31}\mathrm{d}^3 & k_4^{32}\mathrm{d}^4 & k_4^{33}\mathrm{d}^4 \end{pmatrix}.$$

We seek the function $v(x_1)$, solution of (3.4) as

$$v(x_1) = \sum_{t=0}^{+\infty} \varepsilon^t v_t(x_1). \tag{3.5}$$

The functions $v_t(x_1)$ are periodic and independent of ε. After substitution (3.5) into (3.4), we obtain

$$\sum_{l=0}^{+\infty} \varepsilon^l L_l \sum_{t=0}^{+\infty} v_t(x_1) = \sum_{t=0,l=0}^{+\infty} \varepsilon^{l+t} L_l v_t(x_1) = f(x_1).$$

After the following changes of index: $l' = l + t$, we obtain

$$\sum_{l'=0}^{+\infty} \varepsilon^{l'} \left(\sum_{l=0}^{l=l'} L_l v_{l'-l}(x_1) - \delta_{l'0} f(x_1) \right) = 0.$$

By identification of the coefficient of $\varepsilon^{l'}$ to zero, we have for $l' \geq 0$

$$\sum_{l=0}^{l=l'} L_l v_{l'-l}(x_1) - \delta_{l'0} f(x_1) = 0, \quad x_1 \in [0,1].$$

Thus for $l' = 0$, we have

$$L_0 v_0(x_1) = f(x_1), \quad x_1 \in [0,1],$$

for $l' \geq 1$,

$$L_0 v_{l'}(x_1) = F_{l'}(x_1), \quad x_1 \in [0,1], \tag{3.6}$$

where

$$F_{l'}(x_1) = \sum_{l=1}^{l=l'} L_l v_{l'-l}(x_1).$$

The conditions for periodicity of $v_{l'}(x_1)$ and its derivatives added to equations (3.6) define the homogenized problem of $v_t(x_1)$. The right-hand side $F_{l'}(x_1)$ of (3.6) contains the derivatives of functions $v_t(x_1)$, with $t < l'$, thus we define $v_l(x_1)$ inductively. Now let us prove that this equation has one solution. The proof is based on the Lax-Milgram lemma. Denote $\mathcal{V}$ to be the space of vector functions of $H^1(\mathbb{R})^3$ 1-periodic. We define in $\mathcal{V} \times \mathcal{V}$ the bilinear functional

$$a(\psi, \phi) = \int_0^1 (L_0 \psi(x_1), \phi(x_1)) \mathrm{d}x_1.$$

For $l \geq 0$, H_l is the linear functional defined in $\mathcal{V}$ by

$$H_l(\phi(x_1)) = \int_0^1 F_l(x_1) \phi(x) \mathrm{d}x_1.$$

The Problem (3.6) is equivalent to a variational problem

$$(\mathcal{P}_l) \begin{cases} \text{find } \psi \in \mathcal{V} \\ a(\psi, \phi) = H_l(\phi) \qquad \forall \phi \in \mathcal{V}. \end{cases}$$

Lemma 3.3. *There exists a positive constant $\mathcal{C}$ such that, for each function $\psi(x_1) \in \mathcal{V}$, we have*

$$a(\psi, \psi) \geq \mathcal{C} \|\psi(x_1)\|^2_{H^1([0,1])}.$$

Proof. Let the elasticity operator

$$\begin{cases} -\dfrac{\partial}{\partial x_i} \left(A_{ij} \left(\dfrac{x}{\varepsilon}\right) \dfrac{\partial u(x)}{\partial x_j} \right), & \text{in } \Omega_\varepsilon, \\ -\nu_i A_{ij} \left(\dfrac{x}{\varepsilon}\right) \dfrac{\partial u(x)}{\partial x_j}, & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{3.7}$$

Let us make the following changes of variables

$$\begin{cases} x_1 = \xi_1, \\ x_2 = \varepsilon \xi_2, \\ x_3 = \varepsilon \xi_3, \end{cases}$$

we define $w(\xi)$

$$\begin{cases} w_1(\xi) = u_1(x), \\ w_2(\xi) = \varepsilon u_2(x), \\ w_3(\xi) = \varepsilon u_3(x). \end{cases}$$

Denote Q_0 the bar $Q_0 = [0,1] \times B$. Let us multiply the second and the third component of the first equation of (3.7) by ε^{-1} and multiply components of the second equation by ε^{-2} and multiply the first component of the second equation by ε^{-1}. Then the equations (3.7) take the form

$$\begin{cases} -\dfrac{\partial}{\partial \xi_i}\left(B_{ij}(\xi)\dfrac{\partial w(\xi)}{\partial \xi_j}\right), & \text{in } Q_0, \\ -\nu_i B_{ij}(\xi)\dfrac{\partial w(\xi)}{\partial \xi_j}, & \text{on } \partial B, \end{cases} \tag{3.8}$$

$b_{ij}^{kl}(\xi) = \left(\frac{1}{\varepsilon}\right)^{\delta_{ijkl}} a_{ij}^{kl}\left(\dfrac{\xi_1}{\varepsilon}, \xi_2, \xi_3\right)$, where $\delta_{ijkl} = \delta_{2i} + \delta_{3i} + \delta_{2j} + \delta_{3j} + \delta_{2l} + \delta_{3l} + \delta_{2k} + \delta_{3k}$.
The matrices B_{ij} satisfy conditions of symmetry and positivity. The following inequality holds for any function from $\mathcal{V}$ that satisfies relation $\displaystyle\int_{Q_0} w(\xi)\mathrm{d}\xi = 0$

$$\int_{Q_0}\left(B_{ij}(\xi)\frac{\partial w(\xi)}{\partial \xi_j}, \frac{\partial w(\xi)}{\partial \xi_i}\right)\mathrm{d}\xi \geq C\|\omega(\xi)\|^2,$$

$C > 0$, where C is the constant form of Korn's inequality for Q_0. Since, we have

$$\int_{Q_0}\left(B_{ij}(\xi)\frac{\partial w(\xi)}{\partial \xi_j}, \frac{\partial w(\xi)}{\partial \xi_i}\right)\mathrm{d}\xi = \varepsilon^{-2}\int_{\Omega_\varepsilon}\left(A_{ij}(x)\frac{\partial u(x)}{\partial x_j}, \frac{\partial u(x)}{\partial x_i}\right)\mathrm{d}x,$$

thus

$$\varepsilon^{-2}\int_{\Omega_\varepsilon}\left(A_{ij}(x)\frac{\partial u(x)}{\partial x_j}, \frac{\partial u(x)}{\partial x_i}\right)\mathrm{d}x \geq C\|\omega(\xi)\|^2.$$

Applying integral identity, we obtain

$$\begin{aligned} I = \int_{\Omega_\varepsilon}\left(A_{ij}(x)\frac{\partial u(x)}{\partial x_j}, \frac{\partial u(x)}{\partial x_i}\right)\mathrm{d}x &= -\int_{\Omega_\varepsilon}\left(\frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial u(x)}{\partial x_j}\right), u(x)\right)\mathrm{d}x \\ &\quad - \int_{\partial\Omega_\varepsilon}\left(\nu_i A_{ij}(x)\frac{\partial u(x)}{\partial x_j}, u(x)\right)\mathrm{d}s - \int_{\Sigma}\left(\nu_i\left[A_{ij}(x)\frac{\partial u(x)}{\partial x_j}\right], u(x)\right)\mathrm{d}s. \end{aligned}$$

Let the function

$$u(x) = \sum_{l=0}^{5}\varepsilon^l N_l(\xi)\frac{\mathrm{d}^l \Lambda\psi(x_1)}{\mathrm{d}x_1^l},$$

where $N_l(\xi)$ are the solutions of Problem (3.2). We can verify by the direct substitution that

$$\begin{aligned} I &= \int_{\Omega_\varepsilon}\left(\sum_{l=2}^{4}\varepsilon^{l-2}\Lambda h_l\frac{\mathrm{d}^l\Lambda\psi(x_1)}{\mathrm{d}x_1^l},\psi(x_1)\right)\mathrm{d}x+\mathcal{O}(\varepsilon^3)\\ &= \varepsilon^2\mathrm{mes}(B)\int_0^1\left(\sum_{l=2}^{4}\varepsilon^{l-2}\Lambda h_l\frac{\mathrm{d}^l\Lambda\psi(x_1)}{\mathrm{d}x_1^l},\psi(x_1)\right)\mathrm{d}x_1+\mathcal{O}(\varepsilon^3)\\ &= \varepsilon^2\mathrm{mes}(B)\int_0^1(L_0\psi(x_1),\psi(x_1))\,\mathrm{d}x_1+\mathcal{O}(\varepsilon^3). \end{aligned}$$

Finally

$$\mathrm{mes}(B)\int_0^1(L_0\psi(\xi_1),\psi(\xi_1))\,d\xi_1+\mathcal{O}(\varepsilon)\geq C\|\omega(\xi)\|^2.$$

On the other hand, we have

$$\|\omega(\xi)\|^2=\|\psi(\xi_1)\|^2_{\mathcal{V}}+\mathcal{O}(\varepsilon).$$

We obtain

$$\mathrm{mes}(B)a(\psi,\psi)\geq C\|\psi(\xi_1)\|^2_{\mathcal{V}}.$$

□

The bilinear functional $a(\psi,\phi)$ is positive, H_l is continuous in the norm of $\mathcal{V}$. By Lax-Milgram lemma, we prove that Problem $(\mathcal{P}_l)$ has one and only one solution.

3.3. An error estimation

In order to justify the expansion, we replace the formal asymptotic expansion obtained above by partial sums of $u^{(\infty)}(x)$ and de $v(x_1)$. We substitute the partial sums into the variational formulation of Problem (2.1), and derive *a priori* estimates on the errors due to these truncations. So we prove that the formal asymptotic expansion built above is the asymptotic expansion of the exact solution. The following theorem gives *a priori* error estimation.

Theorem 3.1. *For each integer K, there exists a constant M independent of ε such that*

$$\|\Lambda^{-1}\left(u^\varepsilon(x)-u^{(K)}(x)\right)\|_{H^1(\mathcal{C}_\varepsilon)}\leq M\varepsilon^{K+1}.$$

Proof. We prove this estimation in the general case. For $K=0$, we have:

Corollary 3.1. *There exists a constant M independent of ε such that*

$$\begin{aligned} &\|\Lambda^{-1}u^\varepsilon(x)-v_0(x_1)-\varepsilon N_1(\frac{x}{\varepsilon})\frac{\mathrm{d}}{\mathrm{d}x_1}v_0(x_1)\|_{H^1(\Omega_\varepsilon)}\leq M\varepsilon.\\ &\|u^1_\varepsilon(x)-v^1_0(x_1)\|_{L^2(\Omega_\varepsilon)}\leq M\varepsilon.\\ &\|\varepsilon u^2_\varepsilon(x)-v^2_0(x_1)\|_{L^2(\Omega_\varepsilon)}\leq M\varepsilon.\\ &\|\varepsilon u^3_\varepsilon(x)-v^3_0(x_1)\|_{L^2(\Omega_\varepsilon)}\leq M\varepsilon. \end{aligned}$$

4. A case with torsion

4.1. Formal expansion

Now let us suppose that the right-hand side f^ε depends on all components of x and has the following structure:

$$f^\varepsilon(x) = \Phi\left(\frac{\hat{x}}{\varepsilon}\right)\mathcal{K}^{-1}\hat{\Theta}\left(\frac{x}{\varepsilon}\right)\Psi(x_1).$$

In the above notation $\Phi(\hat{\xi})$ is the rigid displacement matrix

$$\Phi(\hat{\xi}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -a\xi_3 \\ 0 & 0 & 1 & a\xi_2 \end{pmatrix}$$

where the constant a is defined by

$$a = \left(\frac{1}{\mathrm{mes}(B)}\int_B (y_2^2 + y_3^2)\mathrm{d}y_2\,\mathrm{d}y_3\right)^{-1/2}$$

and $\mathcal{K}$ is defined by

$$\mathcal{K} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \varepsilon^{-1} & 0 & 0 \\ 0 & 0 & \varepsilon^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We assume that the 4×4 matrix function $\hat{\Theta}(\xi)$ is 1-periodic in ξ_1, $\Psi(x_1)$ is a 4-dimensional vector and $\Psi \in \mathcal{C}^\infty(\mathbb{R})$. Let us rewrite $f(x)$ as the sum

$$f^\varepsilon(x) = \Phi(\hat{\xi})\mathcal{K}^{-1}\bar{\Theta}\Psi(x_1) + \Phi(\hat{\xi})\mathcal{K}^{-1}\hat{\Theta}(\xi)\Psi(x_1).$$

where

$$\begin{cases} \bar{\Theta} = \dfrac{1}{\mathrm{mes}(B)}\displaystyle\int_Y \Phi^T(\hat{\xi})\Phi(\hat{\xi})\hat{\Theta}(\xi)\mathrm{d}\xi, \\ \dfrac{1}{\mathrm{mes}(B)}\displaystyle\int_Y \Phi^T(\hat{\xi})\Phi(\hat{\xi})\Theta(\xi)\mathrm{d}\xi = 0, \end{cases}$$

and $Y = [0,1] \times B$. We seek a formal asymptotic expansion as

$$u^{(\infty)}(x) = \sum_{l=0}^{+\infty} \varepsilon^l N_l(\xi)\frac{\mathrm{d}^l\mathcal{K}v(x_1)}{\mathrm{d}x_1^l} + \sum_{l=0}^{+\infty} \varepsilon^{l+2} M_l(\xi)\frac{\mathrm{d}^l\Psi(x_1)}{\mathrm{d}x_1^l},\ \xi = (x_1, \hat{x}/\varepsilon) \tag{4.1}$$

where $N_l(\xi)$, $M_l(\xi)$ are matrix functions 1-periodic in ξ_1, and $v(x_1)$ is a 3-vector function. As in the first case, we obtain

$$\begin{aligned} L_\varepsilon u^{(\infty)}(x) \quad - \quad f(x) &= \sum_{l=0}^{+\infty} \varepsilon^{l-2} H_l^N(\xi)\frac{\mathrm{d}^l\mathcal{K}v(x_1)}{\mathrm{d}x_1^l} + \sum_{l=0}^{+\infty} \varepsilon^l H_l^M(\xi)\frac{\mathrm{d}^l\Psi(x_1)}{\mathrm{d}x_1^l} \\ &\quad -\Phi(\hat{\xi})\mathcal{K}^{-1}\bar{\Theta}(\xi)\Psi(x_1) - \Phi(\hat{\xi})\mathcal{K}^{-1}\hat{\Theta}(\xi)\Psi(x_1), \end{aligned} \tag{4.2}$$

$$\frac{\partial u^{(\infty)}(x)}{\partial \nu} \quad = \quad \sum_{l=0}^{+\infty} \varepsilon^{l-1} G_l^N(\xi)\frac{\mathrm{d}^l\mathcal{K}v(x_1)}{\mathrm{d}x_1^l} + \sum_{l=0}^{+\infty} \varepsilon^{l+1} G_l^M(\xi)\frac{\mathrm{d}^l\Psi(x_1)}{\mathrm{d}x_1^l}. \tag{4.3}$$

H_l^M, T_l^M, G_l^M are obtained by replacing $N(\xi)$ with $M(\xi)$. Here $N_l(\xi) = M_l(\xi) = 0$ whenever $l < 0$. As in the procedure from the previous case, we require that

$$\begin{aligned}&\text{for } l \geq 0,\ H_l^N(\xi) = \Phi(\hat{\xi})h_l^N, \quad \text{for } l > 0,\ H_l^M(\xi) = \Phi(\hat{\xi})h_l^M,\ H_0^M(\xi) = \Phi(\hat{\xi})\mathcal{K}^{-1}\Theta(\xi),\\ &G_l^N(\xi) = G_l^M(\xi) = 0\ [N_l(\xi)] = [M_l(\xi)] = 0,\ \left[G_l^N(\xi)\right] = \left[G_l^M(\xi)\right] = 0,\end{aligned}$$

where h_l^N and h_l^M are constant 4×4 matrices. We obtain the following recurrent chain of problems for $N_l(\xi)$, $M_l(\xi)$

$$\begin{cases} L_{\xi\xi}N_l(\xi) = \Phi(\hat{\xi})h_l^N - T_l^N(\xi), & \xi \in (0,1) \times B. \\ \dfrac{\partial N(\xi)}{\partial \nu} = -\nu_i A_{i1}(\xi)N_{l-1}(\xi), & \text{on } \Gamma. \\ [N_l(\xi)]\ |_\Sigma = 0,\ \left[\dfrac{\partial N_l(\xi)}{\partial \nu}\right] = [\nu_i A_{i1}(\xi)N_{l-1}(\xi)]\,. \\ N_l \text{ is 1-periodic in } \xi_1. \end{cases} \tag{4.4}$$

Here, as previously, h_l^N are chosen from the solvability conditions for the Problem (4.4):

$$h_l^N = \left\langle \Phi^T(\hat{\xi}) \left(A_{1j}(\xi)\frac{\partial N_{l-1}(\xi)}{\partial \xi_j} + A_{11}(\xi)N_{l-2}(\xi) \right) \right\rangle.$$

$M_l(\xi)$ are the solution of the same problems with N_l replaced by M_l. However $M_0(\xi)$ is the solution of the problem

$$\begin{cases} L_{\xi\xi}M_0(\xi) = \Phi(\hat{\xi})\mathcal{K}^{-1}\Theta(\xi), & \xi \in (0,1) \times B. \\ \dfrac{\partial}{\partial \nu}M_0(\xi) = 0, & \hat{\xi} \in \partial B. \\ [M_0(\xi)]\ |_\Sigma = 0,\ \left[\dfrac{\partial M_0(\xi)}{\partial \nu}\right]\ |_\Sigma = 0. \end{cases} \tag{4.5}$$

Thus, the algorithm for constructing the matrices N_l and M_l is inductive. Suppose that $N_l = 0$, $M_l = 0$ for $l < 0$, $N_0(\xi) = \Phi(\hat{\xi})$, and M_0 is the solution of (4.5). If $l > 0$, then N_l and M_l are solution of Problem (4.4). The right-hand side of these problems contains N_m and M_m with indices $m < l$, which permits us to define them successively. We have

$$h_0^M = 0, \quad h_0^N = 0, \quad h_1^N = 0, \quad h_2^N = \left\langle \Phi^T(\hat{\xi}) \left(A_{1j}(\xi)\frac{\partial N_1(\xi)}{\partial \xi_j} + A_{11}(\xi)\Phi(\hat{\xi}) \right) \right\rangle.$$

For matrices h_l^N (l= 2, 3, 4), as in the first case, we prove

Lemma 4.1. *In the matrix h_2^N non-zero elements can occupy only the following positions marked by stars: we have*

$$h_2^N = \begin{pmatrix} * & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & 0 & 0 & * \end{pmatrix}$$

$h_2^{11} \neq 0$, $h_2^{44} > 0$. *In the matrix* h_3^N, $h_3^{22} = h_3^{23} = h_3^{33} = h_3^{32} = 0$. *In the matrix* h_4^N, $h_4^{22} \neq 0$ *and* $h_4^{33} \neq 0$.

4.2. **Homogenized problems**

After the construction of h_l^N and h_l^M, one can write the equation of $v(x_1)$

$$\sum_{l=0}^{+\infty} \varepsilon^{l-2}\mathcal{K}h_l^N \frac{\mathrm{d}^l \mathcal{K}v(x_1)}{\mathrm{d}x_1^l} + \sum_{l=1}^{+\infty} \varepsilon^{l}\mathcal{K}h_l^M \frac{\mathrm{d}^l \Psi(x_1)}{\mathrm{d}x_1^l} = \bar{\Theta}\Psi(x_1), \quad x_1 \in (0,1). \tag{4.6}$$

We seek $v(x_1)$, solution of (4.6), as

$$v(x_1) = \sum_{t=0}^{+\infty} \varepsilon^t v_t(x_1).$$

The $v_t(x_1)$ are periodic 4-dimensional vector functions. As in the first case, we obtain the homogenized problems, for $l' \geq 0$, we have

$$\sum_{l=0}^{l=l'} L_l v_{l'-l}(x_1) - \delta_{l'0}\bar{\Theta}\Psi(x_1) + \mathcal{K}h_{l'}^M \frac{\mathrm{d}^{l'}\Psi(x_1)}{\mathrm{d}x_1^{l'}} = 0, \quad x_1 \in [0,1].$$

For $l' = 0$, we obtain

$$L_0 v_0(x_1) = \bar{\Theta}\Psi(x_1), \quad x_1 \in [0,1].$$

For $l' \geq 1$,

$$L_0 v_{l'}(x_1) = F_{l'}(x_1), \quad x_1 \in [0,1]. \tag{4.7}$$

where

$$F_l(x_1) = \sum_{p=1}^{p=l} L_l v_{l-p}(x_1) + \mathcal{K}h_l^M \frac{\mathrm{d}^l \Psi(x_1)}{\mathrm{d}x_1^l}.$$

The algorithm for constructing $v_t(x_1)$, solution of the Problem (4.7) is inductive. As in the Paragraph (3.2), we write

$$L_0 = \begin{pmatrix} h_2^{11}\mathrm{d}^2 & h_3^{12}\mathrm{d}^3 & h_3^{13}\mathrm{d}^3 & h_{14}^{2}\mathrm{d}^2 \\ h_3^{21}\mathrm{d}^3 & h_4^{22}\mathrm{d}^4 & h_4^{23}\mathrm{d}^4 & h_{24}^{3}\mathrm{d}^3 \\ h_3^{31}\mathrm{d}^3 & h_4^{32}\mathrm{d}^4 & h_4^{33}\mathrm{d}^4 & h_{34}^{3}\mathrm{d}^3 \\ h_2^{41}\mathrm{d}^2 & h_3^{42}\mathrm{d}^3 & h_3^{43}\mathrm{d}^3 & h_{44}^{2}\mathrm{d}^2 \end{pmatrix}.$$

In order to prove that Problem (4.7) has one solution, let us add some conditions. Denote $\mathcal{M}$ the matrix defined by

$$\mathcal{M} = \frac{1}{\mathrm{mes}(B)} \int_B \Phi^T(y_2,y_3)\Phi(y_2,y_3)\mathrm{d}y_2\,\mathrm{d}y_3.$$

The domain B is non-symmetric, so $\mathcal{M}$ is different to the identity matrix I_4. We must change the center of reference to have $\mathcal{M} = I_4$, then

$$\int_B \xi_2\,\mathrm{d}\xi_2 = 0, \text{ et } \int_B \xi_3\,\mathrm{d}\xi_3 = 0.$$

The Problem (4.7) is equivalent to a variational problem

$$(\mathcal{P}_l)\begin{cases} \text{find } \psi \in \mathcal{V} \quad \text{such that} \\ a(\psi,\phi) = H_l(\phi) \qquad \forall \phi \in \mathcal{V}. \end{cases}$$

Let us prove that the Problem $(\mathcal{P}_l)$ has one and only one solution. Let K be an integer and define the function $u^K(x)$

$$u^K(x) = \sum_{l=0}^{K+1} \varepsilon^l N_l(\xi)\frac{\mathrm{d}^l \mathcal{K}\psi(x_1)}{\mathrm{d}x_1^l} + \sum_{l=0}^{K+1} \varepsilon^{l+2} M_l(\xi)\frac{\mathrm{d}^l \Psi(x_1)}{\mathrm{d}x_1^l},\ \xi = (x_1, \hat{x}/\varepsilon).$$

The functions $N_l(\xi)$ and $M_l(\xi)$ are solutions of Problems (4.4), as in the Paragraph (3.2), there exists a positive constant C such that

$$(L_0\psi, \psi)_{\mathcal{V}} \geq C\|\omega\|^2_{H^1(Q_0)}$$

where

$$\omega(\xi) = \begin{pmatrix} \psi^1(\xi_1) \\ \psi^2(\xi_1) - \varepsilon a \xi_3 \psi^4(\xi_1) \\ \psi^3(\xi_1) + \varepsilon a \xi_2 \psi^4(\xi_1) \end{pmatrix}$$

in fact, according to the Korn's inequality, we have

$$\varepsilon^{-2} \int_{\Omega_\varepsilon} \left(A_{ij}(x) \frac{\partial u(x)}{\partial x_j}, \frac{\partial u(x)}{\partial x_i} \right) \mathrm{d}x \geq C\|\omega\|^2_{H^1(Q_0)}.$$

Applying the integral identity for Problem (2.1), we have

$$\begin{aligned} I = \int_{\Omega_\varepsilon} \left(A_{ij}(x) \frac{\partial u(x)}{\partial x_j}, \frac{\partial u(x)}{\partial x_i} \right) \mathrm{d}x &= - \int_{\Omega_\varepsilon} \left(\frac{\partial}{\partial x_i} \left(A_{ij}(x) \frac{\partial u(x)}{\partial x_j} \right), u(x) \right) \mathrm{d}x \\ &\quad - \int_{\partial\Omega_\varepsilon} \left(\nu_i A_{ij}(x) \frac{\partial u(x)}{\partial x_j}, u(x) \right) \mathrm{d}s - \int_{\Sigma} \left(\nu_i \left[A_{ij}(x) \frac{\partial u(x)}{\partial x_j} \right], u(x) \right) \mathrm{d}s. \end{aligned}$$

As in Paragraph (3.2), we obtain

$$\mathrm{mes}(B) \int_0^1 (L_0\psi(\xi_1), \psi(\xi_1))\, \mathrm{d}\xi_1 + \mathcal{O}(\varepsilon) \geq C\|\omega(\xi)\|^2.$$

Taking into account the previous conditions, we have for $s = 1, 2,$ $\int_{Q_0} \xi_s \psi^4(\xi_1) \mathrm{d}\xi = 0$, then

$$\|\psi^2(\xi_1) - \varepsilon a \xi_3 \psi^4(\xi_1)\|^2_{H^1(Q_0)} \geq \|\psi^2(\xi_1)\|^2_{H^1(Q_0)} + \varepsilon^2 a^2 \|\xi_2 \psi^4(\xi_1)\|^2_{H^1(Q_0)}$$

$$\|\psi^3(\xi_1) + \varepsilon a \xi_2 \psi^4(\xi_1)\|^2_{H^1(Q_0)} \geq \|\psi^3(\xi_1)\|^2_{H^1(Q_0)} + \varepsilon^2 a^2 \|\xi_3 \psi^4(\xi_1)\|^2_{H^1(Q_0)}.$$

On the other hand,

$$\int_{\Omega_\varepsilon} \left(A_{ij}(x) \frac{\partial u(x)}{\partial x_j}, \frac{\partial u(x)}{\partial x_i} \right) \mathrm{d}x \geq C \left(\varepsilon^4 \|\nabla u\|^2_{L^2(C_\varepsilon)} + \varepsilon^2 \|u\|^2_{L^2(\Omega_\varepsilon)} \right).$$

Finally, we obtain

$$(L_0\psi, \psi)_{Q_0} + \mathcal{O}(\varepsilon) \geq \varepsilon^2 \|\nabla u\|^2_{L^2(C_\varepsilon)} + \|u\|^2_{L^2(C_\varepsilon)}.$$

Then

$$(L_0\psi, \psi) \geq \|\psi^4(\xi_1)\|^2_{H^1(Q_0)}.$$

There exists a positive constant C such that

$$(L_0\psi, \psi)_{Q_0} \geq C\|\psi\|^2_{H^1(Q_0)}.$$

The bilinear functional $a(\psi, \phi) = (L_0\psi, \phi)$ defined in $\mathcal{V}$ is positive. $H_l(\phi) = \int_0^1 (F_l(x_1), \phi(x_1)) \mathrm{d}x_1$ is continuous in $\mathcal{V}$. By Lax-Milgram lemma, we prove that for each l the Problem $(\mathcal{P}_l)$ has one and only one solution in $\mathcal{V}$.

4.3. Justification of the asymptotic

In order to justify the expansion, we replace the formal asymptotic expansions obtained above by partial sums of series u^ε and v. First, we prove some lemmas on the properties of the functions $v_t(x_1)$, $N_l(\xi)$ and $M_l(\xi)$.

Lemma 4.2. *For each integer k, there exists a constant C dependent on t but independent of ε such that*

$$\|v_t(x_1)\|_{\mathcal{C}^k([0,1])} \leq C.$$

Proof. All functions $v_t(x_1)$ are solutions of the ordinary Equations (4.7) with mixed boundary conditions. We can prove by induction using the form of the right hand-side functions that these solution are infinitely smooth. □

Lemma 4.3. *The functions $N_l(\xi)$, solutions of Problems (4.4), satisfy the following inequalities:*

$$\int_{\Omega_\varepsilon} \left(\left| \nabla_\xi N_l \left(\frac{x}{\varepsilon} \right) \right|^2 + \left| N_l \left(\frac{x}{\varepsilon} \right) \right|^2 \right) \mathrm{d}x \leq M_l,$$

where M_l is a positive constant independent of ε.

Proof. Is analogous to that of Lemma 3.2 of [3]. □

For functions $M_l(\xi)$, a similar lemma holds. Consider the partial sum $u^{(K)}(x)$ of the expansion (4.1)

$$u^{(K)}(x) = \sum_{l=0}^{K+1} \varepsilon^l N_l(\xi) \frac{\mathrm{d}^l \mathcal{K} v^{(K)}(x_1)}{\mathrm{d}x_1^l} + \sum_{l=0}^{K+1} \varepsilon^{l+2} M_l(\xi) \frac{\mathrm{d}^l \Psi(x_1)}{\mathrm{d}x_1^l},\ \xi\,(x_1, \hat{x}/\varepsilon) \tag{4.8}$$

where

$$v^{(K)}(x_1) = \sum_{t=0}^{K} \varepsilon^t v_t(x_1).$$

Lemma 4.4. *The function $u^{(K)}(x)$ is the solution of the following problem*

$$\begin{cases} L_\varepsilon(u^{(K)}(x)) = f(x) + \varepsilon^{K+1} F^0(x) + \varepsilon^{K+1} \dfrac{\partial}{\partial x_m} F^m(x) \quad \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u^{(K)}(x)}{\partial \nu} = \nu_m \varepsilon^{K+1} F^m(x) \quad \text{on } \Gamma, \\ \left[\dfrac{\partial u^{(K)}(x)}{\partial \nu} \right]_{|\Sigma} = 0, \\ \left[u^{(K)}(x) \right]_{|\Sigma} = 0, \\ u^{(K)} \text{ is 1-periodic in } x_1, \end{cases} \tag{4.9}$$

where $\|F^m\|_{L_2(\Omega_\varepsilon)} \leq M$ and M is a constant independent of ε.

Proof. The substitution of (4.8) in (2.1) leads to

$$\begin{aligned}
L_\varepsilon(u^{(K)}(x)) = &\sum_{l=0}^{K+1} \varepsilon^{l-2} h_l^N \frac{\mathrm{d}^l}{\mathrm{d}x_1^l}\mathcal{K}v^{(K)}(x_1) + \varepsilon^{K+1} A_{11}(\xi) N_{K+1}(\xi) \frac{\mathrm{d}^{K+3}\mathcal{K}v^{(K)}(x_1)}{\mathrm{d}x_1^{K+3}} \\
&+ \varepsilon^K \left(\frac{\partial}{\partial \xi_i} A_{i1}(\xi) N_{K+1}(\xi) + A_{11}(\xi) N_K(\xi) + A_{1j}(\xi) \frac{\partial}{\partial \xi_j} N_{K+1}(\xi) \right) \frac{\mathrm{d}^{K+2}\mathcal{K}v^{(K)}(x_1)}{\mathrm{d}x_1^{K+2}} \\
&+ \sum_{l=0}^{K+1} \varepsilon^l h_l^M \frac{\mathrm{d}^l}{\mathrm{d}x_1^l} \Psi(x_1) + \varepsilon^{K+1} A_{11}(\xi) M_{K+1}(\xi) \frac{\mathrm{d}^{K+3}\Psi(x_1)}{\mathrm{d}x_1^{K+3}} \\
&+ \varepsilon^K \left(\frac{\partial}{\partial \xi_i} A_{i1}(\xi) N_{K+1}(\xi) + A_{11}(\xi) M_K(\xi) + A_{1j}(\xi) \frac{\partial}{\partial \xi_j} M_{K+1}(\xi) \right) \frac{\mathrm{d}^{K+2}\Psi(x_1)}{\mathrm{d}x_1^{K+2}} \,.
\end{aligned}$$

We replace the terms between parenthesis respectively by $h_{K+2}^N - L_{\xi\xi} N_{K+2}(\xi)$ and $h_{K+2}^M - L_{\xi\xi} M_{K+2}(\xi)$, we obtain

$$\begin{aligned}
L_\varepsilon(u^{(K)}(x)) \;&=\; \sum_{l=0}^{K+2} \varepsilon^{l-2} h_l^N \frac{\mathrm{d}^l}{\mathrm{d}x_1^l}\mathcal{K}v^{(K)}(x_1) + \sum_{l=0}^{K+2} \varepsilon^l h_l^M \frac{\mathrm{d}^l}{\mathrm{d}x_1^l}\Psi(x_1) + \beta_\varepsilon, \qquad (4.10)\\
\text{where} \quad \beta_\varepsilon \;&=\; -\varepsilon^{K+1} L_{\xi\xi} N_{K+2} \frac{\mathrm{d}^{K+2}\mathcal{K}v^{(K)}(x_1)}{\mathrm{d}^{K+2}} + \varepsilon^{K+1} A_{11}(\xi) N_{K+1}(\xi) \frac{\mathrm{d}^{K+3}\mathcal{K}v^{(K)}(x_1)}{\mathrm{d}x_1^{K+3}} \\
&\quad - \varepsilon^{K+3} L_{\xi\xi} M_{K+2} \frac{\mathrm{d}^{K+2}\Psi(x_1)}{\mathrm{d}^{K+2}} + \varepsilon^{K+3} A_{11}(\xi) M_{K+1}(\xi) \frac{\mathrm{d}^{K+3}\Psi}{\mathrm{d}x_1^{K+3}} \cdot
\end{aligned}$$

In (4.10), we set

$$\begin{cases} \bar{h}_l^N = h_l^N & \text{for } l \le K+2, \quad = 0 \quad \text{if } l \ge K+3, \\ \bar{h}_l^M = h_l^M & \text{for } l \le K+2, \quad = 0 \quad \text{if } l \ge K+3. \end{cases}$$

then

$$\begin{aligned}
L_\varepsilon(u^{(K)}(x)) \;&=\; \sum_{l=0}^{K+2} \varepsilon^{l-2} h_l^N \frac{\mathrm{d}^l}{\mathrm{d}x_1^l} \sum_{t=0}^{K} \mathcal{K}v_t(x_1) + \sum_{l=0}^{K+2} \varepsilon^l h_l^M \frac{\mathrm{d}^l}{\mathrm{d}x_1^l}\Psi(x_1) + \beta_\varepsilon \\
&=\; \sum_{l'=0}^{2K+2} \varepsilon^{l'-2} \sum_{l=0}^{l'} \left(\bar{h}_l \frac{\mathrm{d}^l}{\mathrm{d}x_1^l} \mathcal{K}v_{l'-l}(x_1) + \bar{h}_l^M \frac{\mathrm{d}^l}{\mathrm{d}x_1^l}\Psi(x_1) \right) + \beta_\varepsilon.
\end{aligned}$$

As in the construction of the homogenized equation, we obtain

$$L_\varepsilon(u^{(K)}(x)) \;=\; f(x_1) + \sum_{l'=1}^{K} \varepsilon^{l'} \left(L_0 v_l'(x_1) + \sum_{l=1}^{l'} L_l v_{l'-l}(x_1) \right) + \varepsilon^{K+1} F^0(x) + \varepsilon^{K+1} \frac{\partial}{\partial \xi_m} F^m(x),$$

where

$$\begin{aligned}
\|F^m(x)\|_{L^2(\Omega_\varepsilon)} &\le \left\| A_{mj}(\xi) \frac{\partial N_{K+2}(\xi)}{\partial \xi_j} \frac{\mathrm{d}^{K+2} v^{(K)}(x_1)}{\mathrm{d}^{K+2} x_1} + A_{mj}(\xi) \frac{\partial M_{K+2}(\xi)}{\partial \xi_j} \frac{\mathrm{d}^{K+2}\Psi(x_1)}{\mathrm{d}^{K+2} x_1} \right\|_{L^2(\Omega_\varepsilon)}, \\
\|F^0(x)\|_{L^2(\Omega_\varepsilon)} &\le \left\| A_{11}(\xi) N_{K+1}(\xi) \frac{\mathrm{d}^{K+3} v^{(K)}(x_1)}{\mathrm{d}x_1^{K+3}} + A_{11}(\xi) M_{K+1}(\xi) \frac{\mathrm{d}^{K+3}\Psi(x_1)}{\mathrm{d}x_1^{K+3}} \right\|_{L^2(\Omega_\varepsilon)}.
\end{aligned}$$

Analogously we have

$$\frac{\partial u^{(K)}(x)}{\partial\nu} = \sum_{l=0}^{K+1}\varepsilon^{l-1}G_l^N(\xi)\frac{\mathrm{d}^l\mathcal{K}v^{(K)}(x_1)}{\mathrm{d}x_1^l} + \sum_{l=0}^{K+1}\varepsilon^{l+1}G_l^M(\xi)\frac{\mathrm{d}^l\Psi(x_1)}{\mathrm{d}x_1^l}$$
$$+\nu_m\varepsilon^{K+1}A_{m1}(\xi)N_{K+1}(\xi)\frac{\mathrm{d}^{K+2}\mathcal{K}v^{(K)}(x_1)}{\mathrm{d}x_1^{K+2}} + \nu_m\varepsilon^{K+3}A_{m1}(\xi)M_{K+1}(\xi)\frac{\mathrm{d}^{K+2}\Psi(x_1)}{\mathrm{d}x_1^{K+2}}\cdot$$

In the other hand, we have

$$G_{K+2}^N(\xi) = \nu_m A_{mj}(\xi)\frac{\partial N_{K+2}(\xi)}{\partial\xi_j} + \nu_m A_{m1}(\xi)N_{K+1}(\xi),$$

$$G_{K+2}^M(\xi) = \nu_m A_{mj}(\xi)\frac{\partial M_{K+2}(\xi)}{\partial\xi_j} + \nu_m A_{m1}(\xi)M_{K+1}(\xi)$$

then

$$\frac{\partial u^{(K)}(x)}{\partial\nu} = \sum_{l=0}^{K+2}\varepsilon^{l-1}G_l^N(\xi)\frac{\mathrm{d}^l\mathcal{K}v^{(K)}(x_1)}{\mathrm{d}x_1^l} - \nu_m\varepsilon^{K+1}A_{mj}(\xi)\frac{\partial N_{K+2}(\xi)}{\partial\xi_j}\frac{\mathrm{d}^{K+2}\mathcal{K}v^{(K)}(x_1)}{\mathrm{d}x_1^{K+2}}$$
$$+\sum_{l=0}^{K+2}\varepsilon^{l+1}G_l^M(\xi)\frac{\mathrm{d}^l\Psi(x_1)}{\mathrm{d}x_1^l} - \nu_m\varepsilon^{K+3}A_{mj}(\xi)\frac{\partial M_{K+2}(\xi)}{\partial\xi_j}\frac{\mathrm{d}^{K+2}\Psi(x_1)}{\mathrm{d}x_1^{K+2}}\cdot$$

The matrices $G_l^N(\xi)$ and $G_l^M(\xi)$ are zero, we obtain

$$\frac{\partial u^{(K)}(x)}{\partial\nu} = \nu_m\varepsilon^{K+1}F^m(x) \quad \text{on } \Gamma.$$

Taking into account inequalities of functions $N_l(\xi)$, $M_l(\xi)$ and $v_t(x_1)$, we prove the inequality announced for $F^m(x)$. □

Introduce

$$\tilde{u}^{(K)}(x) = u^\varepsilon(x) - u^{(K)}(x)$$

where $u^\varepsilon(x)$ is the solution of (2.1)–(2.2). The function $\tilde{u}^{(K)}(x)$ is the solution of the following problem

$$\begin{cases} L_\varepsilon(\tilde{u}^{(K)}(x)) = \varepsilon^{K+1}F^0(x) + \varepsilon^{K+1}\dfrac{\partial}{\partial x_m}F^m(x) \quad \text{in } \Omega_\varepsilon, \\ \dfrac{\partial\tilde{u}^{(K)}(x)}{\partial\nu} = \nu_m\varepsilon^{K+1}F^m(x,\varepsilon) \quad \text{on } \Gamma, \\ \left[\dfrac{\partial\tilde{u}^{(K)}(x)}{\partial\nu}\right]_{|\Sigma} = 0, \\ \left[\tilde{u}^{(K)}(x)\right]_{|\Sigma} = 0, \\ \tilde{u}^{(K)} \text{ is 1-periodic in } x_1. \end{cases} \tag{4.11}$$

where $\|F^m\|_{L_2(\Omega_\varepsilon)} \le M$ and M is a constant independent of ε.
Using Theorem (1.6) of [3] we prove the following theorem:

Theorem 4.1. *For each integer K, there exists a positive constant $\mathcal{C}$ independent of ε such that*

$$\left\|\left(u^\varepsilon(x) - u^{(K)}(x)\right)\right\|_{H^1(\Omega_\varepsilon)} \le \mathcal{C}\varepsilon^{K+1}.$$

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