

APPROXIMATION OF THE ARCH PROBLEM BY RESIDUAL-FREE BUBBLES

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Abstract. We consider a general loaded arch problem with a small thickness. To approximate the solution of this problem, a conforming mixed finite element method which takes into account an approximation of the middle line of the arch is given. But for a very small thickness such a method gives poor error bounds. The conforming Galerkin method is then enriched with residual-free bubble functions.

Résumé. On considère un problème de déplacement d'une arche chargée et de faible épaisseur. Pour approcher la solution de ce problème, on donne une méthode d'éléments finis Galerkin mixte conforme qui tient compte d'une approximation de la forme de l'arche. Cependant une application directe d'une telle méthode ne donne pas de résultat de convergence satisfaisant pour une faible épaisseur. On propose d'enrichir cette méthode par des fonctions bulles résiduelles.

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1. INTRODUCTION

We are interested with a numerical analysis of a general arch problem with a small thickness. The elastic, clamped and loaded arch is studied by following the Buidiansky-Sander's model in linearized elasticity. The variable considered herein is the displacement of the arch under the potential energy of exterior forces. The shape of (the middle line) of the arch is assumed to be represented by a smooth function Φ . The coefficients, in the state equation governing this problem, depend on Φ and on its third derivative. Following Lods [18] we give a mixed formulation that is equivalent to the state equation in which the primitive variables are the components of the displacement vector in a fixed basis, the rotation of the normal vector and the membrane energy. The coefficients of this reformulated problem only depend on Φ and on its first derivative. We explicate the dependence of these coefficients on a critical parameter that is the thickness of the arch. The resulting problem fit into a singularly perturbed one. For the discrete problem, following Bernadou-Ducatel [5], we approach the arch by straight beam elements of small length. But, when the thickness is very small, numerical results obtained from commonly used standard finite element method, deteriorate. Interesting study of this phenomenon has been performed by Habbal and Chenais in [17] where it is proved by a non-conforming finite elements that if the mesh size is taking in the order of the square of the thickness, the convergence is insured. We may also mention that, a precise mathematical meanings for locking and robustness phenomenon are giving by Babuska

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and Suri in [3]. To complete we notice that a study of the numerical behaviour is performed in [12] by Chenais and Paumier for a class of elliptic problem in which a small parameter is involved, the computation of elastic arches is analyzed using this mathematical framework and general conditions ensuring uniform convergence of a numerical scheme, with respect to the thickness, are given.

In this paper, we consider the finite element method where the finite-dimensional space used consists of piecewise polynomials enriched with bubble functions (see [4, 16, 21, 22]). More precisely our computed solution is the sum of a piecewise polynomial and a bubble function that vanishes on the boundary of each element. We shall concentrate on the variant of this method where the bubbles are residual-free (see [21, 22]). For piecewise linears considered the bubbles are exactly computed. Since these functions vanish on the boundary of each element, they can be eliminated or statically condensed. The discrete problem is finally written in terms of the classical polynomial-based test space.

The structure of the paper is as follows: in the next section we consider the continuous arch problem. We introduce a mixed formulation with Lagrange multipliers where the new unknowns are considered. Following Lods [18] we establish that this mixed formulation is equivalent to the usual state equation governing the arch problem. In the remainder of this section, we consider ε , the thickness of the arch, as a small parameter of perturbation. We explicitate the dependence of the coefficients of the variational formulation on ε , we then concentrate on the resulting perturbed problem for which the key conditions of Brezzi's theorem are given. In Section 3 we first consider the discrete problem obtained from a direct application of the Galerkin method and from an approximation by interpolation of the arch. Consistency error estimates are proved. The stability conditions, with associated constants depending on ε , are then established. Some error bounds are given. Thereafter we introduce the residual-free bubbles technique, we calculate the bubble functions associated to piecewise linears, then we use the static condensation procedure to obtain the new discrete problem. Finally, we report on some numerical simulations that illustrate the suitability of the method with bubbles presented herein.

2. THE CONTINUOUS PROBLEM

2.1. The state equation

The shape of the arch is represented by a function Φ of the space:

$$\Lambda_s = \{ \Phi \in W^{s+1, \infty}(I) \text{ such that } \Phi(0) = \Phi(l) = 0 \}$$

where l is the depth of the arch, $I = [0, l]$ and $s \geq 0$ an integer. If L is the length of the arch and e its thickness, the mean surface is:

$$\omega = \{ (x, y, z) \in \mathbb{R}^3, x \in I, y = \Phi(x) \text{ and } z \in]0, L[\}.$$

If $\vec{n}(m)$ denotes the unit normal to the surface ω at the point m , the arch is the domain:

$$\Omega = \left\{ m + x_3 \vec{n}(m), m \in \omega; -\frac{e}{2} < x_3 < \frac{e}{2} \right\}.$$

In the setting of Budiansky-Sanders model in linear elasticity theory (see [10]), the displacement at a point M of Ω is entirely determined by the displacement at a point m of the mean surface. Since the loading is invariant on the oz -axis, the displacement vector belongs to the (x, y) -plane. The problem is then reduced to a one dimensional problem. The displacement of a point $(x, \Phi(x))$ is marked by its pair of tangential and normal components in the local basis $(\vec{t}(\Phi), \vec{n}(\Phi))$:

$$u(x) = u_1(x) \vec{t}(\Phi)(x) + u_2(x) \vec{n}(\Phi)(x).$$

Where, with (\vec{e}_1, \vec{e}_2) the canonical basis of \mathbb{R}^2 , $\Phi' = \frac{d\Phi}{dx}$ and $S(\Phi) = \sqrt{1 + \Phi'^2}$

$$\vec{t}(\Phi) = \frac{1}{S(\Phi)} (\vec{e}_1 + \Phi' \vec{e}_2) \quad \vec{n}(\Phi) = \frac{1}{S(\Phi)} (-\Phi' \vec{e}_1 + \vec{e}_2).$$

The arch being clamped, the pair $u = (u_1, u_2)$ of components belongs to the admissible displacement space (see [18])

$$W = H_0^1(I) \times H_0^2(I) \tag{1}$$

where $H_0^1(I)$ and $H_0^2(I)$ are usual Sobolev spaces. Naturally the displacement u depends on the function Φ . From the virtual work principle u satisfies the state equation (see [5]):

$$u \in W \quad c(\Phi; u, v) = l(\Phi; v) \tag{2}$$

where the energy c of the arch is given by:

$$c(\Phi; u; v) = Eel \int_0^l \eta(\Phi; u) \eta(\Phi; v) s(\Phi) dx + \frac{Ee^3l}{12} \int_0^l k(\Phi; u) k(\Phi; v) s(\Phi) dx \tag{3}$$

with E : the Young modulus, η : the membrane energy and k : the bending energy, respectively given by:

$$\eta(\Phi; v) = \frac{1}{S(\Phi)} v_1' + \frac{1}{R(\Phi)} v_2; \quad k(\Phi; v) = \frac{(\theta(\Phi; v))'}{S(\Phi)} \tag{4}$$

the curvature $\frac{1}{R(\Phi)}$ of ω and the rotation of the normal vector $\theta(\Phi; v)$ are defined by:

$$\frac{1}{R(\Phi)} = -\frac{\Phi''}{S(\Phi)^3} \quad \text{and} \quad \theta(\Phi; v) = \frac{v_1}{R(\Phi)} - \frac{v_2'}{S(\Phi)}. \tag{5}$$

The virtual work of exterior load is(with ρ the density of the material):

$$l(\Phi, v) = - \int_0^l \rho e (\Phi' v_1 + v_2) dx. \tag{6}$$

2.2. The mixed continuous formulation

We fix notation and functional setting that shall be used in the analysis below.

Let (β, γ) denote the pair of components of the displacement vector in the fixed basis (e_1, e_2) . Designate by θ and η respectively the rotation of the normal vector and the membrane energy; the new unknown vector is: $(\beta, \gamma, \theta, \eta) \in (H_0^1(I))^3 \times L^2(I)$. Let $u = (\beta, \gamma, \theta, \eta)$ be the primal unknown and $v = (\underline{\beta}, \underline{\gamma}, \underline{\theta}, \underline{\eta})$ a generic variable corresponding to u . Denote by $p = (p_1, p_2)$ the unknown in the space of Lagrange multipliers and $q = (q_1, q_2)$ a generic variable associated. We employ the the conventional spaces $L^2(\Omega)$, $H^2(\Omega)$, $H_0^2(\Omega)$ and $W^{s,\infty}(\Omega)$ with their classical norms: $\|\cdot\|_{0,\Omega} = |\cdot|_{0,\Omega}$, $\|\cdot\|_{s,\Omega} = \sum_{k=0}^s |\cdot|_{k,\Omega}$ and $\|\cdot\|_{s,\infty,\Omega}$; as usual $(a, b)_\Omega = \int_\Omega ab dx$. For brevity we drop the subscript Ω when $\Omega = I$. Finally we introduce the spaces $V = (H_0^1(I))^3 \times L^2(I)$ and $\Sigma = L^2(I) \times L^2(I)$ equipped respectively of the norms:

$$\begin{aligned} \|v\|_V^2 &= \|\underline{\beta}\|_1^2 + \|\underline{\gamma}\|_1^2 + \|\underline{\theta}\|_1^2 + \|\underline{\eta}\|_0^2 \\ \|q\|_\Sigma^2 &= \|q_1\|_0^2 + \|q_2\|_0^2. \end{aligned}$$

Now, let us recall some results given by Lods in [18].

Proposition 1. *Assume that $\Phi \in \Lambda_s$. Let $v = (v_1, v_2)$ belong to the space W then the following equalities hold in $L^2(I)$:*

$$\beta' = \theta\Phi' + \eta \tag{7}$$

$$\gamma' = -\theta + \Phi'\gamma' \tag{8}$$

where $\beta\vec{e}_1 + \gamma\vec{e}_2 = v_1\vec{t}'(\Phi) + v_2\vec{n}(\Phi)$, $\theta = \theta(v; \Phi)$ and $\eta = \eta(v; \Phi)$.

Variational formulation. Define the subspace

$$K(\Phi) = \{v \in V \text{ such that } b(\Phi; v, \mu) = 0 \quad \forall \mu \in \Sigma\}$$

where the continuous bilinear form $b(\Phi; \cdot, \cdot)$ is given by:

$$b(v, \mu) = \int_0^l \mu_1 (\underline{\beta}' - \underline{\theta}\Phi' - \underline{\eta}) dx + \int_0^l \mu_2 (\underline{\gamma}' - \underline{\theta} - \Phi'\underline{\eta}) dx \quad \forall \mu = (\mu_1, \mu_2) \in \Sigma$$

then we have:

Proposition 2. *Assume that $\Phi \in \Lambda_s$, then the mapping:*

$$\begin{aligned} F^\Phi : W &\longrightarrow K(\Phi) \\ v = (v_1, v_2) &\longmapsto (\underline{\beta}, \underline{\gamma}, \underline{\theta}, \underline{\eta}) \end{aligned}$$

where $\underline{\beta} = \underline{\beta}(\Phi; v)$ and $\underline{\gamma} = \underline{\gamma}(\Phi, v)$ is an isomorphism.

By using the isomorphism F^Φ , the state equation (2) is equivalent to the following problem (see [18]): find $u = (\beta, \gamma, \theta, \eta) \in K(\Phi)$ such that:

$$a(\Phi; u, v) = l(\Phi; v), \quad \forall v \in K(\Phi) \tag{9}$$

where the bilinear form $a(\Phi; \cdot, \cdot)$ and the linear form $l(\Phi; \cdot)$ are respectively given by

$$a(u, v) = Eel \int_0^l \eta\underline{\eta}S(\Phi) dx + \frac{Ee^3l}{12} \int_0^l \theta'\underline{\theta}' \cdot \frac{1}{S(\Phi)} dx \tag{10}$$

$$l(v) = -\rho e \int_0^l \underline{\gamma}S(\Phi) dx. \tag{11}$$

By standard arguments, we can set the variational problem with Lagrange multipliers as: find $u = (\beta, \gamma, \theta, \eta) \in V$ and $p = (p_1, p_2) \in \Sigma$ such that:

$$(P) \quad \begin{cases} a(\Phi; u, v) + b(\Phi; v, p) = l(\Phi; v) & \forall v = (\underline{\beta}, \underline{\gamma}, \underline{\theta}, \underline{\eta}) \in V \\ b(\Phi; u, q) = 0 & \forall q = (q_1, q_2) \in \Sigma \end{cases}$$

where the bilinear form $b(\Phi; \cdot, \cdot)$ is given by

$$b(v, q) = \int_0^l q_1 (\underline{\beta}' - \underline{\theta}\Phi' - \underline{\eta}) dx + \int_0^l q_2 (\underline{\gamma}' - \underline{\theta} - \Phi'\underline{\eta}) dx.$$

Now, denoting by A and B the operators associated to the forms $a(.,.)$ and $b(.,.)$, we obtain from problem (P) the following Euler Lagrange equations:

$$Au + B^*p = f \tag{12}$$

$$Bu = 0 \tag{13}$$

with:

$$Au = \begin{pmatrix} 0 \\ 0 \\ -\frac{Ee^3l}{12} \left(\frac{\theta'}{s(\Phi)}\right)' \\ \frac{Eels(\Phi)\eta}{\eta} \end{pmatrix}; \quad B^*p = \begin{pmatrix} -p'_1 \\ -p'_2 \\ -p_1\Phi' + p_2 \\ -p_1 - \Phi'p_2 \end{pmatrix}; \quad f = \begin{pmatrix} 0 \\ e\rho s(\Phi) \\ 0 \\ 0 \end{pmatrix}$$

and

$$Bu = \begin{pmatrix} \beta' - \theta\Phi' - \eta \\ \gamma' + \theta - \Phi'\eta \end{pmatrix}.$$

The subsequent theorem establishes the equivalence between the mixed problem and the state equation (2), we have (see [18]).

Theorem 3. *Assume that $\Phi \in \Lambda_s$, then problem (P) admits a unique solution (u, p) and $\underline{u} = (F^{-\Phi})u$ is the solution of the state equation.*

2.3. The perturbed problem

Explicitating the dependence of the coefficients in equations of problem (P) on a small parameter $\varepsilon = \frac{\varepsilon}{l}$ (the rigidity), taking, for simplicity, $l = 1$ and dividing by E the two members of its first equation, problem (P) becomes:

$$(P_\varepsilon) \begin{cases} a_\varepsilon(w, \tau) + b(\tau, p) = l(\tau) & \forall \tau \in V \\ b(w, q) = 0 & \forall q \in \Sigma \end{cases}$$

where

$$a_\varepsilon(w, \tau) = \varepsilon \int_0^1 \eta \underline{\eta} S(\Phi) dt + \frac{\varepsilon^3}{12} \int_0^1 \theta' \underline{\theta}' \cdot \frac{1}{S(\Phi)} dt$$

$$b(\tau, q) = \frac{1}{El} \left\{ (\underline{\beta}' - \underline{\theta}\Phi' - \underline{\eta}, q_1)_0 + (\underline{\gamma}' + \underline{\theta} - \Phi'\underline{\eta}, q_2)_0 \right\}$$

and

$$l(\tau) = -\rho\varepsilon (\underline{\gamma}, S(\Phi))_{0,I}$$

with $I = (0, 1)$. To examine existence and uniqueness of solution for (P_ε) , the strategy is to fulfill the two key stability conditions: (K – ellipticity and LBB) (see [2] and [6]);

$$(S_1) \quad a(v, v) \geq C_1 \|v\|_V^2 \quad \forall v \in K$$

$$(S_2) \quad \sup_{\tau \in V} \frac{b(\tau, q)}{\|\tau\|_V} \geq C_2 \|q\|_\Sigma^2 \quad \forall q \in \Sigma$$

this has been done in the proof of the previous theorem. We observe that for $\varepsilon > 0$ the form $a(., .)$ is K -elliptic and the problem is well-posed. However for $\varepsilon = 0$ problem (P_0) is not solvable. From now on, we shall deal with problem (P_ε) for $0 < \varepsilon \ll 1$ (the numerically more delicate case) which can be written as:

$$\begin{cases} a_\varepsilon(w, \tau) + b_\varepsilon(\tau, p) = l(\tau) & \forall \tau \in V \\ b_\varepsilon(w, q) = 0 & \forall q \in \Sigma \end{cases} \tag{14}$$

where:

$$\begin{aligned} a_\varepsilon(w, \tau) &= \int_0^1 \eta \underline{\eta} s(\Phi) dt + \frac{1}{12} \varepsilon^2 \int_0^1 \theta' \underline{\theta}' \cdot \frac{1}{s(\theta)} dt \\ b_\varepsilon(\tau, q) &= \frac{1}{\varepsilon E} \{ (\underline{\beta}' - \underline{\theta} \Phi' - \underline{\eta}, q_1) + (\underline{\gamma}' + \underline{\theta} - \Phi' \underline{\eta}, q_2) \} \end{aligned}$$

and

$$l(\tau) = -\rho \int_0^1 \underline{\gamma} S(\Phi) dt.$$

We now only explicitate the dependence on ε of the constants C_1 and C_2 appearing in conditions (S_1) and (S_2) respectively, which is easy.

Lemma 4. *There exists a constant $C > 0$ which depends on Φ such that:*

$$a_\varepsilon(\tau, \tau) \geq C \varepsilon^2 |\tau|_V^2 \quad \forall \tau \in K(\Phi). \tag{15}$$

Lemma 5. *There exists a constant $C > 0$ independent of ε such that:*

$$\sup_{\tau \in V} \frac{b(\tau, q)}{\|\tau\|_V} \geq C \frac{1}{\varepsilon} \|q\|_\Sigma \quad \forall q \in \Sigma. \tag{16}$$

3. THE DISCRETE PROBLEM

Let us first fix notation and finite-dimensional spaces that shall be used throughout: consider T_h a regular partition (see Ciarlet [13]) of the closed interval $I = [0, 1]$ into elements $K_i = [x_i, x_{i+1}]$; $i = 0, N$ such that the mesh parameter, $h = \max_{1 \leq i \leq N+1} h_i$, with $h_i = (x_{i+1} - x_i)$ and $N \geq 1$ an integer. Let $P_r(K_i)$ be the space of polynomials of degree less or equal to r defined on an element K_i . Designate by $u_h = (\beta_h, \gamma_h, \theta_h, \eta_h)$ the primal discrete unknown and by $v_h = (\underline{\beta}_h, \underline{\gamma}_h, \underline{\theta}_h, \underline{\eta}_h)$ the corresponding variable. Denote by $p_h = (p_{1h}, p_{2h})$ the unknown in the Lagrange multipliers space and by $q_h = (q_{1h}, q_{2h})$ the associated variable. Let $M_h^l(I)$ be the space of C^{-1} piecewise polynomial interpolations of degree l , *i.e.*

$$M_h^l = \{ \lambda_h \in L^2(I), \lambda_h|_{K_i} \in P_l(K_i), i = 0, N \}.$$

Let $W_h^k(I)$ be the space of C^0 piecewise polynomial interpolations of degree k with zero value at $x = 0$ and $x = 1$, *i.e.*

$$W_h^k = \{ \mu_h \in C(I), \mu_h|_{K_i} \in P_k(K_i), i = 0, N \text{ and } \mu_h(0) = \mu_h(1) = 0 \}.$$

Consider, for $k, l \geq 1$, the finite-dimensional spaces:

$$V_h = V_h^k = W_h^k \times W_h^k \times W_h^k \times M_h^{k-1} \tag{17}$$

$$\Sigma_h = \Sigma_h^l = M_h^l \times M_h^l \tag{18}$$

as the discrete spaces of primal and Lagrange multipliers variables respectively.

3.1. The discrete formulation

The method consists to approach the shape function Φ by its interpolate function Φ_h on the space W_h^m , ($m \leq s$), and set the discrete problem (P_h) as follows: find $w_h \in V_h$ and $p_h \in \Sigma_h$ such that:

$$a_h(w_h, \tau_h) + b_h(\tau_h, p_h) = l_h(\tau_h) \quad \forall \tau \in V_h \tag{19}$$

$$b_h(w_h, q_h) = 0 \quad \forall q_h \in \Sigma_h. \tag{20}$$

Where:

$$a_h(w_h, \tau_h) = \int_0^1 \eta_h \underline{\eta}_h S(\Phi_h) dt + \frac{\varepsilon^2}{12} \int_0^1 \theta'_h \underline{\theta}'_h \cdot \frac{1}{S(\Phi_h)} dt$$

$$b_h(\tau_h, q_h) = \frac{1}{\varepsilon E} \left(\underline{\beta}'_h - \underline{\theta}_h \Phi'_h - \underline{\eta}_h, q_{1h} \right) + \frac{1}{\varepsilon E} \left(\underline{\gamma}'_h + \underline{\theta}_h - \Phi'_h \underline{\eta}_h, q_{2h} \right)$$

and

$$l_h(\tau_h) = -\rho \left(\underline{\gamma}_h, S(\Phi_h) \right)_{0,I}.$$

These approximations are conforming since we have:

$$W_h^m \subset W^{1,\infty}(I), \quad V_h \subset V \quad \text{and} \quad \Sigma_h \subset \Sigma.$$

We first recall from [18] the following technical result:

Lemma 6. Assume that Φ_h is an interpolant of $\Phi \in \Lambda_s$ on the space W_h^m , ($m \leq s$), then the subspace: $K_h(\Phi_h) = \left\{ v_h = \left(\underline{\beta}_h, \underline{\gamma}_h, \underline{\theta}_h, \underline{\eta}_h \right) \in V_h / b_h(v_h, q_h) = 0 \quad \forall q_h \in \Sigma_h \right\}$ is nothing but the space of functions v_h such that:

$$\underline{\beta}'_h = \underline{\theta}_h \Phi'_h + \underline{\eta}_h \quad \text{and} \quad \underline{\gamma}'_h = -\underline{\theta}_h + \Phi'_h \underline{\eta}_h \tag{21}$$

otherwise, one has: $K_h(\Phi_h) \subset K(\Phi_h)$.

Lemma 7. Under the same hypotheses as in Lemma 6, the two stability conditions also hold for the approximated bilinear forms $a(\Phi_h; \cdot, \cdot)$ and $b(\Phi_h; \cdot, \cdot)$, indeed there exist constants α_1 and α_2 independents of h and ε such that for $h \leq h_0$, we have

$$a_\varepsilon(\Phi_h; \tau_h, \tau_h) \geq \alpha_1 \varepsilon^2 \|\tau_h\|_V^2 \quad \forall \tau_h \in K_h(\Phi_h) \tag{22}$$

$$\sup_{\tau_h \in V_h} \frac{b(\Phi_h; \tau_h, q)}{\|\tau_h\|_V} \geq \alpha_2 \frac{1}{\varepsilon} \|q\|_\Sigma \quad \forall q_h \in \Sigma_h. \tag{23}$$

Proof. It suffices to remark that $S(\Phi_h) \geq 1$ and that $\frac{1}{S(\Phi_h)}$ is bounded below for $h \leq h_0$ with a constant independent of h . The proof of estimate (22) is then quite similar to this of estimate (15) in Lemma 4. For the estimate (23), denote for any $t_h \in \Sigma_h$, t_h^0 the mean value of t_h i.e. $t_h^0 = \int_0^1 t_h dx$. Now, for $k \geq l - 1$, select, for any $q_h \in \Sigma_h$, $\tilde{\tau} = (s_h, t_h, 0, 0)$ such that:

$$s_h(x) = \int_0^x (q_{1h}(\zeta) - q_{1h}^0) d\zeta \quad \text{and} \quad t_h(x) = \int_0^x (q_{2h}(\zeta) - q_{2h}^0) d\zeta$$

so that $b(\Phi_h; \tilde{\tau}, q_h) = \|q_h\|_\Sigma^2$ and hence

$$\varepsilon \sup_{\tau \in V_h} \frac{|b(\Phi_h; \tau_h, q_h)|}{\|\tau\|_V} \geq \varepsilon \frac{b(\Phi_h; \tilde{\tau}, q_h)}{\|\tau\|_V} \geq \frac{1}{2} \varepsilon \frac{\|q_h\|_\Sigma^2}{\|q_h\|_\Sigma} \geq \frac{1}{2} \varepsilon \|q_h\|_\Sigma$$

yielding $\alpha_2 = \frac{1}{2}$. □

Now, using Lemma 7 and Brezzi’s theorem (see [6]) the following result is immediate:

Theorem 8. *Assume that $\Phi \in \Lambda_s$ ($s \geq 1$), then there exists a real $h_0 > 0$ such that, for $h \leq h_0$, the discrete problem (P_h) admits unique solution (w_h, p_h) .*

Let us now establish some consistency error estimates:

3.2. Consistency error estimates

Assume that $\Phi \in \Lambda_s$. Let Φ_h be the interpolate function of Φ on the space $W_h^m(I)$, ($m \leq s$), such that the interpolation error satisfies:

$$|\Phi - \Phi_h|_1 \leq Ch^m |\Phi|_{m+1} \tag{24}$$

then, with C a generic constant independent of ε and h , the following results hold:

Lemma 9. *There exists a real $h_0^1 > 0$ such that, for $h \leq h_0^1$, the bilinear form $a_h(\cdot, \cdot)$ is well defined on $V \times V$, moreover one has:*

$$|a_\varepsilon(w_h, \tau_h) - a_h(w_h, \tau_h)| \leq Ch^m \|w_h\|_V \|\tau_h\|_V \quad \forall (w_h, \tau_h) \in V_h \times V_h. \tag{25}$$

Proof. On each K_i we have:

$$|a_\varepsilon(w_h, \tau_h) - a_h(w_h, \tau_h)| \leq \int_{K_i} |\eta_h \underline{\eta}_h| |S(\Phi) - S(\Phi_h)| dt + \frac{\varepsilon^2}{12} \int_{K_i} |\theta'_h \underline{\theta}'_h| \left| \frac{1}{S(\Phi)} - \frac{1}{S(\Phi_h)} \right| dt \tag{26}$$

we will give estimate for each term of the second member of (23). Since the function $x \rightarrow \sqrt{1+x^2}$ is globally Lipschitz on IR with constant 1 we have:

$$|S(\Phi) - S(\Phi_h)| \leq |\Phi' - \Phi'_h|$$

and

$$\left| \frac{1}{S(\Phi)} - \frac{1}{S(\Phi_h)} \right| \leq |S(\Phi) - S(\Phi_h)| \leq |\Phi' - \Phi'_h|.$$

By using the Schwarz inequality we get for the first term:

$$\int_{K_i} |\eta_h \underline{\eta}_h| |S(\Phi) - S(\Phi_h)| dt \leq |\Phi' - \Phi'_h|_{0,K_i} \|\eta_h\|_{0,K_i} \|\underline{\eta}_h\|_{0,K_i}.$$

Now, from the interpolation error we get, with C a constant independent of h :

$$\int_{K_i} |\eta_h \underline{\eta}_h| |S(\Phi) - S(\Phi_h)| dt \leq ch^m \|\eta_h\|_{0,K_i} \cdot \|\underline{\eta}_h\|_{0,K_i}.$$

Similarly, for the second term we obtain :

$$\frac{\varepsilon^2}{12} \int_{K_i} |\theta'_h \underline{\varrho}'_h| \left| \frac{1}{S(\Phi)} - \frac{1}{S(\Phi)_h} \right| dt \leq \frac{\varepsilon^2}{12} |\theta_h|_{1,K_i} |\underline{\varrho}_h|_{1,K_i} |\Phi - \Phi_h|_{0,K_i}.$$

So, from (24) we get:

$$\frac{\varepsilon^2}{12} \int_{K_i} |\theta'_h \underline{\varrho}'_h| \left| \frac{1}{S(\Phi)} - \frac{1}{S(\Phi)_h} \right| dt \leq ch^m \frac{\varepsilon^2}{12} |\theta_h|_{1,K_i} |\underline{\varrho}_h|_{1,K_i}.$$

Adding and using the Schwarz inequality, we obtain the estimate after summing the integrals over all elements K_i .

Lemma 10. *There exists a real $h_0^2 > 0$ such that, for $h \leq h_0^2$, the bilinear form $b_h(\cdot, \cdot)$ on $V \times \Sigma$ is well defined and:*

$$|b_h(\tau_h, q_h) - b(\tau_h, q_h)| \leq C \frac{h^m}{\varepsilon} \|\tau_h\|_V \|q_h\|_\Sigma \quad \forall \tau \in V_h; \quad \forall q \in \Sigma_h. \tag{27}$$

Proof. For any $\tau_h = (\beta_h, \gamma_h, \theta_h, \eta_h) \in V_h$ and any $q_h = (q_{1h}, q_{2h}) \in \Sigma_h$ we have:

$$|b(\tau_h, q_h) - b_h(\tau_h, q_h)| \leq \frac{C}{\varepsilon} \int_0^1 (|\theta_h q_{1h}| |\Phi' - \Phi'_h| + |\eta_h q_{2h}| |\Phi' - \Phi'_h|) dt.$$

From (24) and the Schwarz inequality yields:

$$|b(\tau_h, q_h) - b_h(\tau_h, q_h)| \leq C \frac{h^m}{\varepsilon} |\Phi|_{m+1,K} \left(\|\theta_h\|_{0,K} \|q_{1h}\|_{0,K} + \|\eta_h\|_{0,K} \|q_{2h}\|_{0,K} \right).$$

Thus, with C a constant which depends on Φ and doesn't depend on ε and h :

$$|b(\tau_h, q_h) - b_h(\tau_h, q_h)| \leq C \frac{h^m}{\varepsilon} (\|\theta_h\|_0 + \|\eta_h\|_0) (\|q_{1h}\|_0 + \|q_{2h}\|_0).$$

Lemma 11. *There exists a real $h_0^3 > 0$ such that, for $h \leq h_0^3$, the linear form $l_h(\cdot)$ on V_h is well defined and*

$$|l(\tau_h) - l_h(\tau_h)| \leq ch^m \|\tau_h\|_V \quad \forall \tau_h \in V_h. \tag{28}$$

Proof. For any $\tau_h = (\beta_h, \gamma_h, \theta_h, \eta_h) \in V_h$ we have:

$$|l(\tau_h) - l_h(\tau_h)| \leq \rho \left| \int_0^1 \gamma(S(\Phi) - S(\Phi_h)) dt \right|.$$

Using the Schwarz inequality and (24) we immediately obtain

$$|l(\tau_h) - l_h(\tau_h)| \leq Ch^m |\Phi|_{m+1} \|\gamma\|_0.$$

3.3. Accuracy error estimate

We can now prove the error estimates theorem. Designating by C a generic constant independent of ε and h , the following holds:

Theorem 12. *Let, for $\Phi \in \Lambda_s$ ($s \geq 1$), (u, p) be the solution of problem (P_ε) and (u_h, p_h) be the solution of problem (P_h) then we have:*

$$\|u - u_h\|_V \leq \frac{C}{\varepsilon^2} \left\{ \inf_{\tau_h \in V_h} \|u - \tau_h\|_V + \inf_{\tau_h \in V_h} \sup_{v_h \in V_h} \frac{|a_h(\tau_h, v_h) - a(\tau_h, v_h)|}{\|v_h\|_V} + \sup_{v_h \in V_h} \frac{|b_h(v_h, p) - b(v_h, p)|}{\|v_h\|_V} + \sup_{v_h \in V_h} \frac{|l(v_h) - l_h(v_h)|}{\|v_h\|_V} \right\}. \quad (29)$$

Assume that, $u \in \{H^{k+1}(I) \cap H_0^1(I)\}^3 \times H^k(I)$ and $p \in \{H^{l+1}(I) \times H^{l+1}(I)\}$. Let $\tilde{u}_h \in V_h$, $\tilde{p}_h \in \Sigma_h$ and $\Phi_h \in W_h$ be interpolants of u , p and Φ respectively. Assume that the interpolation errors $u - \tilde{u}_h$, $p - \tilde{p}_h$ and $\Phi - \Phi_h$ satisfy

$$\|u - \tilde{u}_h\|_V \leq Ch^k \quad (30)$$

$$\|p - \tilde{p}_h\|_\Sigma \leq Ch^{l+1} \quad (31)$$

$$\|\Phi - \Phi_h\|_1 \leq Ch^m \quad (32)$$

then, with $\nu = \min(k, l + 1, m)$:

$$\|u - u_h\|_V \leq C(p, u)\varepsilon^{-2}h^\nu. \quad (33)$$

In particular for $m = 1$ (when the arch is approximated by straight beams) we have, for any higher order elements

$$\|u - u_h\|_V \leq C(p, u)h\varepsilon^{-2}$$

and hence a locking phenomenon could appear when $(\varepsilon \prec \sqrt{h})$.

Proof. As the problem (P_h) satisfies the general hypotheses of Brezzi's theorem, the technique used to prove estimate (29) is classical (see [8], p. 67). For the usual term $\inf_{\tau_h \in V_h} \|u - \tau_h\|_V$ of the right hand side of inequality (30) the infimum is bounded from above taking $\tau_h = \tilde{\tau}_h$ the V_h -interpolate of τ . The additional tree terms which measure the consistency between the original forms and their approximated ones, are given by Lemmas 7 to 9. Indeed from Lemma 7 we have

$$|a(\tilde{u}_h, v_h) - a_h(\tilde{u}_h, v_h)| \leq Ch^m \|\tilde{u}_h\|_V \|v_h\|_V$$

so that

$$\sup_{v_h \in V_h} \frac{|a(\tilde{u}_h, v_h) - a_h(\tilde{u}_h, v_h)|}{\|v_h\|_V} \leq Ch^m \|\tilde{u}_h\|_V. \quad (34)$$

Similarly, by Lemma 8 we have, with \tilde{q}_h the Σ_h -interpolate of q

$$\sup_{v_h \in V_h} \frac{|b_h(v_h, \tilde{q}_h) - b(v_h, \tilde{q}_h)|}{\|v_h\|_V} \leq C \frac{h^m}{\varepsilon} \|\tilde{q}_h\|_\Sigma. \quad (35)$$

Since the operator of V_h -interpolation (resp. Σ_h -interpolation) satisfy (see [8])

$$\|\tilde{u}_h\|_V \leq C \|u\|_V \text{ and } \|\tilde{q}_h\|_\Sigma \leq C \|q\|_\Sigma \quad (36)$$

the error estimate (33) follows by combining (28) to (32) and (33) to (36).

4. THE RESIDUAL-FREE BUBBLES TECHNIQUE

It was established that standard Galerkin method do not be effective to give a suitably approximation of the solution of problem (P_ε) when $\varepsilon \ll h$. In order to achieve good approximation properties, we propose to enrich the Galerkin method by residual-free bubbles functions [22]. To be more precise, for the given grid T_h , we still employ piecewise linear elements to approach the geometry of the arch, but for approximation of the displacement, we add to the space V_h of polynomials a space spanned by bubbles functions to recover an augmented test space S_h . The bubble functions are assumed to satisfy the governing differential equations in each element interior subjected to the Dirichlet conditions on each element boundary. Denote:

$$V_l = W_h^1 \times W_h^1 \times W_h^1 \times W_h^1$$

$$\Sigma_l = \Sigma_h^1$$

where W_h^1 and Σ_l^1 are defined as in Section 3. Let V_b and Σ_b designate the spaces spanned by bubbles corresponding to V_l and Σ_l respectively. The new test spaces are

$$S_h = V_l \oplus V_b \quad \text{and} \quad \Sigma_h = \Sigma_l \oplus \Sigma_b.$$

Consequently each $w_h \in S_h$ and each p_h of Σ_h is the sum of a polynomial and a bubble component:

$$w_h = w_l + w_b ; \quad p_h = p_l + p_b.$$

Let us now shed some light on the effect of eliminating the bubble functions by the static condensation procedure. For simplicity writing, we shall drop the subscript ε in the forms $a_{\varepsilon h}(\cdot, \cdot)$ and $l_{\varepsilon h}(\cdot)$. Consider the problem: find $w_h \in S_h$ and $p_h \in \Sigma_h$ such that:

$$(P_h) : \begin{cases} a_h(w_h, \tau_h) + b_h(\tau_h, p_h) = l_h(\tau_h) & \forall \tau_h \in S_h \\ b_h(w_h, q_h) = 0 & \forall q_h \in \Sigma_h \end{cases}$$

where, with $w_l = (\beta_l, \gamma_l, \theta_l, \eta_l)$; $\tau_l = (\underline{\beta}_l, \underline{\gamma}_l, \underline{\theta}_l, \underline{\eta}_l)$; $w_b = (\beta_b, \gamma_b, \theta_b, \eta_b)$ and $\tau_b = (\underline{\beta}_b, \underline{\gamma}_b, \underline{\theta}_b, \underline{\eta}_b)$

$$a_h(w_h, \tau_h) = \varepsilon \int_0^1 \eta_h \underline{\eta}_h S(\Phi_h) dt + \frac{\varepsilon^3}{12} \int_0^1 \theta'_h \underline{\theta}'_h \frac{1}{S(\Phi_h)} dt$$

$$b_h(\tau_h, q_h) = \left(\underline{\beta}'_h - \underline{\theta}_h \Phi'_h - \underline{\eta}_h, q_{1h} \right) + \left(\gamma'_h + \underline{\theta}_h - \Phi'_h \underline{\eta}_h, q_{2,h} \right)$$

$$l_h(\tau_h) = -\varepsilon \int_0^1 \underline{\gamma}_h S(\Phi_h) dt.$$

Reporting in (P_h) yields:

$$a_h(w_l + w_b, \tau_l + \tau_b) + b_h(\tau_l + \tau_b, p_l + p_b) = l_h(\tau_l + \tau_b) \quad \forall \tau_l \in V_l \quad \forall \tau_b \in V_b \tag{34.a}$$

$$b_h(w_l + w_b, q_l + q_b) = 0 \quad \forall q_l \in \Sigma_l \quad \forall q_b \in \Sigma_b. \tag{34.b}$$

Now, if we take first $\tau_l = 0$ and $q_l = 0$ we get the formulation:

$$a_h(w_b, \tau_b) + b_h(\tau_b, p_b) = l_h(\tau_b) - a_h(w_l, \tau_b) - b_h(\tau_b, p_l) \quad \forall \tau_b \in V_b \tag{35.a}$$

$$b_h(w_b, q_b) = -b_h(w_l, q_b) \quad \forall q_b \in \Sigma_b. \tag{35.b}$$

Equivalently, with A and B the operators associated to the forms $a_\varepsilon(.,.)$ and $b(.,.)$

$$a_h(w_b, \tau_b) + b_h(\tau_b, p_b) = \langle f - Aw_l - B^*p_l, \tau_b \rangle \quad \forall \tau_b \in V_b \quad (37)$$

$$b_h(w_b, q_b) = -\langle Bw_l, q_b \rangle \quad \forall q_b \in \sum_b. \quad (38)$$

Consequently, for the given $(w_l, p_l) \in V_l \times \Sigma_l$, the pair of bubbles (w_b, p_b) satisfy:

$$Aw_b + B^*p_b = -Aw_l - B^*p_l - f \quad \text{in } K \quad (36.a)$$

$$Bw_b = -Bw_l \quad \text{in } K \quad (36.b)$$

$$w_b = p_b = 0 \quad \text{on } \partial K. \quad (36.c)$$

Selecting $\tau_b = q_b = 0$ in (34) yields:

$$a_h(w_b + w_l, \tau_l) + b_h(\tau_l, p_l + p_b) = l_h(\tau_l) \quad \forall \tau_l \in V_l \quad (39)$$

$$b_h(w_b + w_l, q_l) = 0 \quad \forall q_l \in \sum_l. \quad (40)$$

Which can be written as:

$$a_h(w_l, \tau_l) + \langle Aw_b + B^*p_b, \tau_l \rangle + b_h(\tau_l, p_l) = l_h(\tau_l) \quad \forall \tau_l \in V_l \quad (41)$$

$$b_h(w_l, q_l) = -b_h(w_b, q_l) \quad \forall q_l \in \sum_l. \quad (37)$$

Where the term $\langle Aw_b + B^*p_b, \tau_l \rangle$ gives the bubbles contribution to the reduced space formulation.

Determination of the bubble basis functions. Let, for $i = 1, n_K$, $\{\Psi_{li}\}$ and $\{p_{li}\}$ be the polynomial local basis corresponding to V_l and \sum_l respectively; $\{\Psi_{bi}, \Psi_f\}$ and $\{p_{bi}, p_f\}$ denote the local basis of bubble functions associated to V_b and \sum_b . Problem (36) splits in: find Ψ_{bi} and p_{bi} such that

$$A\Psi_{bi} + B^*p_{bi} = -A\Psi_{li} + B^*p_{li} \quad i = 1, n_K \quad \text{in } K \quad (38.a)$$

$$B\Psi_{bi} = -B\Psi_{li} \quad i = 1, n_K \quad \text{in } K \quad (38.b)$$

$$\Psi_{bi} = p_{bi} = 0 \quad i = 1, n_K \quad \text{on } \partial K \quad (38.c)$$

and, find Ψ_{bf} and p_f such that

$$A\Psi_{bf} = -B^*p_l - f \quad \text{in } K \quad (39.a)$$

$$B\Psi_{bf} = 0 \quad \text{in } K \quad (39.b)$$

$$\Psi_{bf} = 0 \quad \text{on } \partial K. \quad (39.c)$$

Since Φ_h is linear in each element K , for the sake of clarity, we set $\Phi'_h = a_K = a$, $S(\Phi_h) = \sqrt{1 + a_k^2} = S_K = S$ and $h_K = h$. Now, problems (38) and (39) can be solved for each Ψ_{li} and each p_{li} , we have:

Proposition 13. Assume that $\Phi \in \Lambda_s$ and problem (P_ε) admits a solution (w, p) such that $w \in (H_0^1(I) \cap H^2(I))^3 \times (H^2(I))$ and $p \in (H^2(I))^2$ then problem (39) admits the solution $\Psi_{bf} = (\beta_{bf}, \gamma_{bf}, \theta_{bf}, \eta_{bf})$ and $p_f = (p_{f1}, p_{f2})$ such that:

$$\begin{cases} \beta_{bf}(\xi) = -\frac{a(1+a^2)}{6\varepsilon^2} \left[\left(\frac{\xi^4}{4} - \frac{\xi^2}{2}h^2 \right) + \frac{1}{4}h\xi^2(3h-2\xi) \right] - a\frac{\xi^2}{2} + \frac{a}{2}h\xi \\ \gamma_{bf}(\xi) = \frac{a^2}{2}\xi(h-\xi) + \frac{1+a^2}{24\varepsilon^2}\xi^2(h-\xi)^2 \\ \theta_{bf}(\xi) = -\frac{(1+a^2)}{2\varepsilon^2} \left[\frac{1}{3}\xi(\xi^2-h^2) + \frac{1}{2}h^2\xi(h-\xi) \right] \\ \eta_{bf}(\xi) = a\left(\frac{h}{2}-\xi\right), \quad \xi \in]0, h[\text{ and } \eta_{bf}(0) = \eta_{bf}(1) = 0 \end{cases} \tag{40.a}$$

$$\begin{cases} p_{f1}(\xi) = 0 \\ p_{f2}(\xi) = -\varepsilon S\xi + \frac{S}{2}h\varepsilon. \end{cases} \tag{40.b}$$

Proof. $\Psi_b^i = (\beta_b^i, \gamma_b^i, \theta_b^i, \eta_b^i); \Psi_{bf} = (\beta_{bf}, \gamma_{bf}, \theta_{bf}, \eta_{bf}); p_f = (p_{f1}, p_{f2}); p_{li} = (p_{li}^i, p_{li}^i)$.

Now, we write (with the same coefficients $C_{i,K}$):

$$w_{l|K} = \sum_{i=1}^{nK} C_{i,K} \Psi_{li} \quad \text{and} \quad w_{b|K} = \sum_{i=1}^{nK} C_{i,K} \Psi_b^i + \Psi_{bf}.$$

Hence, equations (39) can be written as:

$$\begin{cases} (a) & p'_{f1} = 0 \\ (b) & p'_{f2} = -\varepsilon S(\Phi_h) \\ (c) & -\varepsilon^3 \frac{\theta''_{bf}}{S(\Phi_h)} = p_{f1}\Phi'_h - p_{f2} \\ (d) & \varepsilon S(\Phi_h)\eta_{bf} = p_{f1} + \Phi'_h p_{f2} \\ (e) & \beta'_{bf} = \theta_{bf}\Phi'_h + \eta_{bf} \\ (f) & \gamma'_{bf} = \Phi'_h \eta_{bf} - \theta_{bf}. \end{cases} \tag{41}$$

In (41), from equations (b) and (c) we get, respectively, after integrating once with respect to the local variable $\xi = x - x_i \in [0, h]$,

$$\begin{aligned} p_{f2}(\xi) &= -\varepsilon S\xi + C_1 \\ \theta''_{bf}(\xi) &= -\frac{S}{\varepsilon^3}(p_{f1}a + \varepsilon S\xi - C_1). \end{aligned} \tag{42}$$

Integrating twice with respect to ξ and using the conditions $\theta_{bf}(0) = \theta_{bf}(h) = 0$ yields

$$\theta_{bf}(\xi) = -\frac{S^2}{6\varepsilon^2}\xi(\xi^2-h^2) + \frac{S(C_1-p_{f1}a)}{2\varepsilon^3}\xi(\xi-h). \tag{43}$$

Using from (41) equation (d) we obtain for η_{bf} :

$$\eta_{bf} = \frac{1}{\varepsilon S}(p_{f1} + ap_{f2}).$$

Hence from (42), we get:

$$\eta_{bf}(\xi) = \frac{1}{\varepsilon S}(p_{f1} + aC_1 - a\varepsilon S\xi). \tag{44}$$

Similarly, we use from (41) equations (e) and (f), to have:

$$\beta_{bf}(\xi) = a \int_0^\xi \theta_{bf}(t) dt + \int_0^\xi \eta_{bf}(t) dt.$$

Reporting expressions of θ_{bf} and η_{bf} yields, with $d = S(C_1 - p_{f1}a)/\varepsilon^3$ and $f = -S^2/\varepsilon^2$:

$$\beta_{bf}(\xi) = \frac{f}{6}a \left(\frac{\xi^4}{4} - \frac{\xi^2}{2}h^2 \right) - a \frac{d}{12}\xi^2(3h - 2\xi) + \frac{1}{\varepsilon S} (p_{f1} + aC_1)\xi - a \frac{\xi^2}{2}. \quad (45)$$

Now, using the condition $\beta_{bf}(h) = 0$ and replacing f by its expression, we get with $\alpha = h/\varepsilon$:

$$C_1 = \frac{\frac{S^2}{2}h^2\alpha^2 + aSp_{f1}\alpha^3 - 6h^2}{S\alpha^3 - \frac{12}{S}\alpha} + \frac{12\frac{p_{f1}}{S}\alpha}{a(S\alpha^3 - \frac{12}{S}\alpha)}. \quad (46)$$

Otherwise, using from (41) the equation (f) and integrating once with respect to ξ yields

$$\gamma_{bf}(\xi) = a \int_0^\xi \eta_{bf}(t) dt + \int_0^\xi \theta_{bf}(t) dt.$$

So, substituting with the η_{bf} and θ_{bf} expressions we get:

$$\gamma_{bf}(\xi) = \frac{a}{\varepsilon S} (p_{f1} + aC_1)\xi - a^2 \frac{\xi^2}{2} + \frac{f}{6} \left(\frac{\xi^2}{2}h^2 - \frac{\xi^4}{4} \right) + \frac{S(C_1 - p_{f1}a)}{12\varepsilon^3} \xi^2(3h - 2\xi). \quad (47)$$

Hence, using (46) and (47) we obtain after brief calculation

$$C_1 = \frac{\sqrt{1+a^2}}{2}h\varepsilon; \quad p_{f1} = 0. \quad (48)$$

Now, recalling (48) and reporting in (43) and (44) we obtain:

$$\theta_{bf}(\xi) = -\frac{(1+a^2)}{2\varepsilon^2} \left[\frac{1}{3}\xi(\xi^2 - h^2) + \frac{1}{2}h^2\xi(h - \xi) \right] \quad (50)$$

and

$$\begin{cases} \eta_{bf}(\xi) = a(\frac{h}{2} - \xi), & \xi \in]0, h[\\ \eta_{bf}(0) = \eta_{bf}(h) = 0. \end{cases} \quad (51)$$

Similarly, replacing in (45) and (47) C_1 and p_{f1} by their expressions, we get

$$\beta_{bf}(\xi) = -\frac{a(1+a^2)}{24\varepsilon^2} \xi^2(\xi - h)^2 + \frac{a}{2}\xi(h - \xi) \quad (52)$$

and

$$\gamma_{bf}(\xi) = \frac{a^2}{2}\xi(h - \xi) + \frac{1+a^2}{24\varepsilon^2} \xi^2(h - \xi)^2. \quad (53)$$

Now, it remains to determine the local basis elements Ψ_b^i and p_b^i . Let us, for the sake of clarity, introduce the quantities:

$$\begin{cases} m_1^1 = \frac{12(a-1)\varepsilon^3}{S^3} - \frac{6\varepsilon^3}{S h^2} & m_1^2 = -\frac{6}{S} \left(\frac{4}{a} + h \right) \frac{\varepsilon^3}{h^3} \\ m_2^1 = \frac{a+1}{S} \frac{\varepsilon}{h} & m_2^2 = S \frac{\varepsilon}{h}. \end{cases} \tag{54}$$

With the notation above, the following holds:

Proposition 14. *Under the same hypotheses as in proposition 15, problem (39) admits the solutions $\Psi_b^i = (\beta_b^i, \gamma_b^i, \theta_b^i, \eta_b^i)$ and $p_b^i = (p_{b,1}^i, p_{b,2}^i)$; $i = 1, 2$ such that*

$$\begin{aligned} (a) \quad & \begin{cases} \beta_b^1(\xi) &= -\frac{aS}{6\varepsilon^3} m_1^1 \xi^3 - \left(\frac{aSm_1^1 h}{4\varepsilon^3} + \frac{a}{2h} \right) \xi^2 + \left(\frac{m_2^1}{\varepsilon S} - \frac{1}{h} \right) \xi \\ \gamma_b^1(\xi) &= \frac{S}{6\varepsilon^3} m_1^1 \xi^3 - \left(\frac{Sm_1^1 h}{4\varepsilon^3} + \frac{1}{2h} \right) \xi^2 + \left(\frac{am_2^1}{\varepsilon S} - \frac{1}{h} \right) \xi \\ \theta_b^1(\xi) &= \frac{Sm_1^1}{2\varepsilon^3} \xi(h - \xi) \\ \eta_b^1(\xi) &= -\frac{\xi}{h} + \frac{m_2^1}{\varepsilon S_K} \end{cases} \\ (b) \quad & \begin{cases} p_{b1}^1(\xi) &= -\frac{\xi}{h} + \frac{a+1}{S^3} \frac{\varepsilon}{h} - \frac{6a}{S^3} \frac{\varepsilon^3}{h^2} + \frac{12a(a-1)}{S^5} \frac{\varepsilon^3}{h^3} \\ p_{b2}^1(\xi) &= -\frac{\xi}{h} - \frac{(a+1)a}{S^3} \frac{\varepsilon}{h} - \frac{6}{S^3} \frac{\varepsilon^3}{h^2} + \frac{12(a-1)}{S^5} \frac{\varepsilon^3}{h^3} \end{cases} \end{aligned} \tag{55}$$

and

$$\begin{aligned} (a) \quad & \begin{cases} \beta_b^2(\xi) &= \left\{ -\frac{aS}{6\varepsilon^3} m_1^2 \xi^3 + \left(\frac{aSm_1^2 h}{4\varepsilon^3} - \frac{a}{2h} \right) \xi^2 + \left(\frac{m_2^2}{\varepsilon S} + \frac{1}{h} + a \right) \xi \right. \\ \gamma_b^2(\xi) &= \left\{ \frac{S}{6\varepsilon^3} m_1^2 \xi^3 - \left(\frac{Sm_1^2 h}{4\varepsilon^3} - \frac{1}{2h} \right) \xi^2 + \left(\frac{am_2^2}{\varepsilon S} + \frac{1}{h} - 1 \right) \xi \right. \\ \theta_b^2(\xi) &= \frac{Sm_1^2}{2\varepsilon^3} \xi(h - \xi) \\ \eta_b^2(\xi) &= -1 + \frac{\xi}{h} + \frac{m_2^2}{\varepsilon S_K} \end{cases} \\ (b) \quad & \begin{cases} p_{b1}^2(\xi) &= \frac{\xi}{h} - 1 - \frac{6}{S^3} (4 + ah) \frac{\varepsilon^3}{h^3} + \frac{1}{S} \frac{\varepsilon}{h} \\ p_{b2}^2(\xi) &= \frac{\xi}{h} - 1 - \frac{6}{S^3} \left(\frac{4}{a} + h \right) \frac{\varepsilon^3}{h^3} - \frac{a}{S} \frac{\varepsilon}{h}. \end{cases} \end{aligned} \tag{56}$$

Proof. Denote first, for $i = 1, n_K$, $\Psi_i^i = (\beta_i^i, \gamma_i^i, \theta_i^i, \eta_i^i)$ and let us ignore the subscript i in the proof. Using in (38) equations (a) and (b), we get for $(w_i, p_i) \in V_i \times \Sigma_i$:

$$\left\{ \begin{array}{l} (a) \quad p'_{b,1} \quad = \quad -p'_{l1} \\ (b) \quad p'_{b,2} \quad = \quad -p'_{l2} \\ (c) \quad -\varepsilon^3 \frac{\theta''_b}{S_K} \quad = \quad \varepsilon^3 \frac{\theta''_l}{S} + a(p_{l1} + p_{b1}) - p_{l2} - p_{b2} \\ (d) \quad \varepsilon S_K \eta_b \quad = \quad -\varepsilon S \eta_l + p_{l1} + p_{b1} + a(p_{l2} + p_{b2}) \\ (e) \quad \beta'_b \quad = \quad a(\theta_b + \theta_l) + (\eta_b + \eta_l) - \beta'_l \\ (f) \quad \gamma'_b \quad = \quad a(\eta_b + \eta_l) - (\theta_b + \theta_l) - \gamma'_l. \end{array} \right. \tag{57}$$

Hence, from equations (a) and (b) yields, with M_1 and M_2 generic constants:

$$p_{b1} = -p_{l1} + M_1; \quad p_{b2} = -p_{l2} + M_2.$$

Now, equation (d) gives:

$$\eta_b = -\eta_l + \frac{1}{\varepsilon S} [M_1 + aM_2].$$

Since θ_l is linear in K , we have $\theta''_l = 0$ and then from (c) yields:

$$\theta''_b = -\frac{S}{\varepsilon^3} (aM_1 - M_2).$$

As the elements $\theta_l, \beta_l, \gamma_l, p_{l1}$ and p_{l2} are linears in K , one has:

$$\left\{ \begin{array}{l} (i) \quad \theta_l = \theta_l^1 = \beta_l = \beta_l^1 = \gamma_l = \gamma_l^1 = p_{l1}^1 = p_{l2}^1 = \frac{\xi}{h} \\ (ii) \quad \theta_l = \theta_l^2 = \beta_l = \beta_l^2 = \gamma_l = \gamma_l^2 = p_{l1}^2 = p_{l2}^2 = 1 - \frac{\xi}{h}. \end{array} \right. \tag{58}$$

In the first case we get:

$$p_{b1}^1 = -\frac{\xi}{h} + M_1^1; \quad p_{b2}^1 = -\frac{\xi}{h} + M_2^1. \tag{59}$$

Setting: $m_1^1 = aM_1^1 - M_2^1$ and $m_2^1 = M_1^1 + aM_2^1$ yields:

$$\eta_b^1(\xi) = -\frac{\xi}{h} + \frac{m_2^1}{\varepsilon S}. \tag{60}$$

Integrating twice with respect to ξ and recalling $\theta_b(0) = \theta_b(h) = 0$, we obtain

$$\theta_b^1(\xi) = \frac{Sm_1^1}{2\varepsilon^3} \xi(h - \xi). \tag{61}$$

Otherwise, using from (57) equation (e) and integrating, we get:

$$\beta_b(\xi) = a \int_0^\xi (\theta_b + \theta_l)(t) dt + \int_0^\xi (\eta_b + \eta_l)(t) dt - \beta_l(\xi) + \beta_l(0)$$

so according to (60) and (61)

$$\beta_b^1(\xi) = -\frac{aS}{6\varepsilon^3}m_1^1\xi^3 - \left(\frac{aS m_1^1 h}{4\varepsilon^3} + \frac{a}{2h}\right)\xi^2 + \left(\frac{m_2^1}{\varepsilon S} - \frac{1}{h}\right)\xi. \tag{62}$$

Similarly, using equation (f) and integrating with respect to ξ we obtain:

$$\gamma_b(\xi) = a \int_0^\xi (\eta_b + \eta_l)(t) dt - \int_0^\xi (\theta_b + \theta_l)(t) dt - \gamma_l(\xi) + \gamma_l(0).$$

Hence, from (60) and (61):

$$\gamma_b^1(\xi) = \frac{S}{6\varepsilon^3}m_1^1\xi^3 - \left(\frac{S m_1^1 h}{4\varepsilon^3} + \frac{1}{2h}\right)\xi^2 + \left(\frac{a m_2^1}{\varepsilon S} - \frac{1}{h}\right)\xi. \tag{63}$$

In other words:

$$\gamma_b^1(\xi) = -\frac{\beta_b^1(\xi)}{a} + \left(-\frac{a+1}{ah} + \frac{a^2+1}{aS\varepsilon}m_2^1\right)\xi. \tag{64}$$

Finally it remains to determine the constants m_1^1 and m_2^1 . This can be done by the use of the conditions $\beta_b^1(h) = \gamma_b^1(h) = 0$ According to (63) we have:

$$m_2^1 = \frac{a+1}{S} \frac{\varepsilon}{h}. \tag{65}$$

On the other hand, using the expression of β_b^1 , we get after brief calculation

$$\frac{aS}{12}m_1^1\frac{h^2}{\varepsilon^3} + \frac{1}{\varepsilon S}m_2^1 = \frac{1}{h} - \frac{a}{2}.$$

Substituting m_2^1 by its expression we obtain:

$$m_1^1 = \frac{12(a-1)}{S^3} \frac{\varepsilon^3}{h^3} - \frac{6}{S} \frac{\varepsilon^3}{h^2}. \tag{66}$$

Back to M_1^1 and M_2^1 we get:

$$\begin{cases} M_1^1 &= \frac{a+1}{S^3} \frac{\varepsilon}{h} - \frac{6a}{S^3} \frac{\varepsilon^3}{h^2} + \frac{12a(a-1)}{S^5} \frac{\varepsilon^3}{h^3} \\ M_2^1 &= -\frac{(a+1)a}{S^3} \frac{\varepsilon}{h} - \frac{6}{S^3} \frac{\varepsilon^3}{h^2} + \frac{12(a-1)}{S^5} \frac{\varepsilon^3}{h^3}. \end{cases} \tag{67}$$

Finally, we obtain the exact expressions of bubble basis functions $p_b^1 = (p_{b1}^1, p_{b2}^1)$ and $\Psi_b^1 = (\beta_b^1, \gamma_b^1, \theta_b^1, \eta_b^1)$ by substituting in (60) to (63) the constants m_1^1, m_2^1, M_1^1 and M_2^1 by their values respectively. Similarly working as in case (i) we have for case (ii):

$$\begin{cases} \eta_b^2(\xi) &= -1 + \frac{\xi}{h} + \frac{m_2^2}{\varepsilon S} \\ \theta_b^2(\xi) &= \frac{S m_1^2}{2\varepsilon^3} \xi(h - \xi). \end{cases} \tag{68}$$

For the expressions of β_b^2 and γ_b^2 we get (respectively):

$$\beta_b^2(\xi) = -\frac{aS}{6\varepsilon^3}m_1^2\xi^3 + \left(\frac{aS m_1^2 h}{4\varepsilon^3} - \frac{a}{2h}\right)\xi^2 + \left(\frac{m_2^2}{\varepsilon S} + \frac{1}{h} + a\right)\xi \tag{69}$$

$$\gamma_b^2(\xi) = \frac{S}{6\varepsilon^3}m_1^2\xi^3 - \left(\frac{S m_1^2 h}{4\varepsilon^3} - \frac{1}{2h}\right)\xi^2 + \left(\frac{a m_2^2}{\varepsilon S} + \frac{1}{h} - 1\right)\xi \tag{70}$$

in other words, we have:

$$\gamma_b^2(\xi) = -\frac{\beta_b^2(\xi)}{a} + \left(\frac{a+1}{ah} + \frac{a+1}{aS\varepsilon}m_2^2\right)\xi. \tag{71}$$

We then determine the constants m_1^2 and m_2^2 by using the conditions $\beta_b^2(h) = \gamma_b^2(h) = 0$. Hence, according to (69) one has

$$m_2^2 = S\frac{\varepsilon}{h}. \tag{72}$$

Using the expression of $\beta_b^2(\xi)$ given by (69) and recalling $\beta_b^2(h) = 0$ yields:

$$\frac{aS}{12}m_1^2\frac{h^2}{\varepsilon^3} + \frac{1}{\varepsilon S}m_2^2 = -\frac{1}{h} - \frac{a}{2}.$$

Replacing m_2^2 by its value gives:

$$m_1^2 = -\frac{6}{S}\left(\frac{4}{a} + h\right)\frac{\varepsilon^3}{h^3}. \tag{73}$$

Consequently:

$$M_1^2 = -\frac{6}{S^3}(4 + ah)\frac{\varepsilon^3}{h^3} + \frac{1}{S}\frac{\varepsilon}{h}; \quad M_2^2 = -\frac{6}{S^3}\left(\frac{4}{a} + h\right)\frac{\varepsilon^3}{h^3} - \frac{a}{S}\frac{\varepsilon}{h}. \tag{74}$$

Finally, we obtain the expressions of $p_b^2 = (p_{b1}^2, p_{b2}^2)$ and $\Psi_b^2 = (\beta_b^2, \gamma_b^2, \theta_b^2, \eta_b^2)$ by replacing the constants m_1^2, m_2^2, M_1^2 and M_2^2 by their values respectively.

Application to the perturbed problem. Consider the problem (P_h) , if we select $\tau_h = \tau_l$ in the first equation and $q_h = q_l$ in the second one, we get:

$$\begin{cases} a_h(w_l, \tau_l) + b_h(\tau_l, p_l) + a_h(w_b, \tau_l) + b_h(\tau_l, p_b) = l_h(\tau_l) \\ b_h(w_l, q_l) + b_h(w_b, q_l) = 0. \end{cases} \tag{75}$$

Where:

$$\begin{aligned}
 a_h(w_l, \tau_l) &= \varepsilon \Sigma_K S_K \int_K \eta_l \underline{\eta}_l dx + \frac{\varepsilon^3}{12} \Sigma_K \frac{1}{S_K} \int_K \theta'_l \underline{\theta}'_l dx \\
 b_h(\tau_l, p_l) &= \Sigma_K \left\{ \left(\beta'_l - a_k \underline{\theta}_l - \underline{\eta}_l, p_{l1} \right)_K + \left(\gamma'_l + \underline{\theta}_l - a_K \underline{\eta}_l, p_{l2} \right)_K \right\} \\
 b_h(\tau_l, p_b) &= \Sigma_K \left\{ \left(\beta'_l - a_K \underline{\theta}_l - \underline{\eta}_l, p_{b1} \right)_K + \left(\gamma'_l + \underline{\theta}_l - a_K \underline{\eta}_l, p_{b2} \right)_K \right\} \\
 a_h(w_b, \tau_l) &= \varepsilon \Sigma_K S_K \int_K \eta_b \underline{\eta}_l dx \\
 b_h(w_b, q_l) &= \Sigma_K \left\{ \left(\beta'_b - a_K \theta_b - \eta_b, q_{l1} \right)_K + \left(\gamma'_b + \theta_b - a_K \eta_b, q_{l2} \right)_K \right\}.
 \end{aligned} \tag{76}$$

Where, integrating by parts we have used the equality:

$$\sum_K \frac{1}{S_K} \int_K \theta'_b \underline{\theta}'_l dx = \sum_K \frac{1}{S_K} \left[\left(\theta_b, \underline{\theta}'_l \right)_{\partial K} - \left(\theta_b, \underline{\theta}''_l \right)_K \right] = 0. \tag{78}$$

Now, writing:

$$w_b = \sum_{j=1}^{n_K} c_{j,K} \left(\beta_b^j, \gamma_b^j, \theta_b^j, \eta_b^j \right) + (\beta_{bf}, \gamma_{bf}, \theta_{bf}, \eta_{bf}); \quad (p_{b1}, p_{b2}) = \sum_{j=1}^{n_K} d_{j,K} \left(p_{b1}^j, p_{b2}^j \right) + (p_{f1}, p_{f2}).$$

If we substitute in (72) the expressions of $\beta_b^j, \gamma_b^j, \theta_b^j, \eta_b^j$ and in (75) those of (p_{b1}^j, p_{b2}^j) and (p_{f1}, p_{f2}) , we obtain a linear system with coefficients $c_{j,K}, d_{j,K}$ as unknowns.

4.1. Numerical experiments

Each of the standard Galerkin (G) and the Galerkin with bubbles ($G+BF$) leads to the matricial formulation:

$$(S) \begin{cases} A\underline{X} + B^T \underline{P} = \underline{F} \\ B\underline{X} = \underline{G} \end{cases}$$

where \underline{X} and \underline{P} are vectors which components are the nodal parameters of displacement and Lagrange multipliers respectively, A and B are the matrices corresponding to the bilinear forms $a(.,.)$ and $b(.,.)$, \underline{F} and \underline{G} the second members. If A_K and B_K denote the element matrices it will be interesting to see that, with piecewise linear elements, A_K is bloc diagonal where the diagonal terms are the matrices $A_\beta = A_{11}, A_\gamma = A_{22}, A_\theta = A_{33}$ and $A_\eta = A_{44}$ associated to the components displacement vector β, γ, θ and η respectively. Note that in the case of standard Galerkin (G), $A_\beta = A_\gamma = 0$ and:

$$A_\theta = \frac{1}{12(a^2 + 1)^{\frac{1}{2}}} \frac{\varepsilon^2}{h} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}; \quad A_\eta = \frac{(a^2 + 1)^{\frac{1}{2}}}{6} h \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

B is of the form, with $(B_{21} = B_{12} = 0, B_{13} = B_{24} = aB_{14})$ and with $B_{11} = B_{22} = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$;
 $B_{14} = -B_{23} = -\frac{h}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$$B = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \end{bmatrix}.$$

In the Galerkin with bubbles method the matrix A becomes $A + b(A)$, the the diagonal terms are now:

$$A_\beta + b(A_\beta), A_\gamma + b(A_\gamma), A_\theta + b(A_\theta) \text{ and } A_\eta + b(A_\eta)$$

where $b(A_\beta) = b(A_\gamma) = b(A_\theta) = 0$ and $b(A_\eta) = \begin{pmatrix} \frac{1-h}{2} & 0 \\ 0 & -\frac{h}{3} + \frac{1}{2} \frac{a+1}{a^2+1} \end{pmatrix}$ and similarly the matrix B becomes $B + b(B)$ with $b(B_{12}) = b(B_{21}) = 0$, $b(B_{14}) = -\frac{b(B_{13})}{a}$, $b(B_{24}) = -ab(B_{23})$ and:

$$b(B_{11}) = \begin{pmatrix} \frac{1}{2} - M_1^1 & 0 \\ 0 & \frac{1}{2} + M_1^2 \end{pmatrix}; \quad b(B_{13}) = \begin{pmatrix} ah \left(\frac{1}{3} - \frac{M_1^1}{2} \right) & 0 \\ 0 & ah \left(\frac{1}{3} - \frac{M_1^2}{2} \right) \end{pmatrix}$$

$$b(B_{22}) = \begin{pmatrix} -\frac{1}{2} + M_2^1 & 0 \\ 0 & \frac{1}{2} - M_2^2 \end{pmatrix}; \quad b(B_{23}) = \begin{pmatrix} -\frac{h}{3} + M_2^1 \frac{h}{2} & 0 \\ 0 & -\frac{h}{3} + M_2^2 \frac{h}{2} \end{pmatrix}.$$

To illustrate the results above, we test the methods on an arch of shape $\Phi(x) = x^2 - x$ subjected to uniform unit load.

4.2. Conclusion

A standard Galerkin mixed method (G) and the Galerkin method enriched with residual-free bubbles (RFB) for equal order linear elements are considered to approximate the displacement of a general arch problem. We show that the (RFB) method consists in fact of the standard Galerkin formulation plus mesh dependent terms. The bubble functions, assumed to satisfy the governing differential equations in each element interior of a regular grid, are determined exactly. Numerical experiments are carried out for an arch with a given smooth shape which supports a uniform unit load. The results are presented for the values $\varepsilon = 1$, $\varepsilon = 10^{-1}$ and $\varepsilon = 10^{-2}$ of the thickness. The graphics show that the approximated rotation of the normal vector by the Galerkin method has a similar behaviour to exact one (Figs. 1 and 2). On the other hand, the approached membrane energy with classical Galerkin method presents a pathological behaviour which consists of a boundary layer phenomena observed for $\frac{\varepsilon}{h} \leq 10^{-1}$. Experiments show that an improvement is brought to the behaviour of the membrane energy component by the use of the (RFB) method (Figs. 3 and 4).

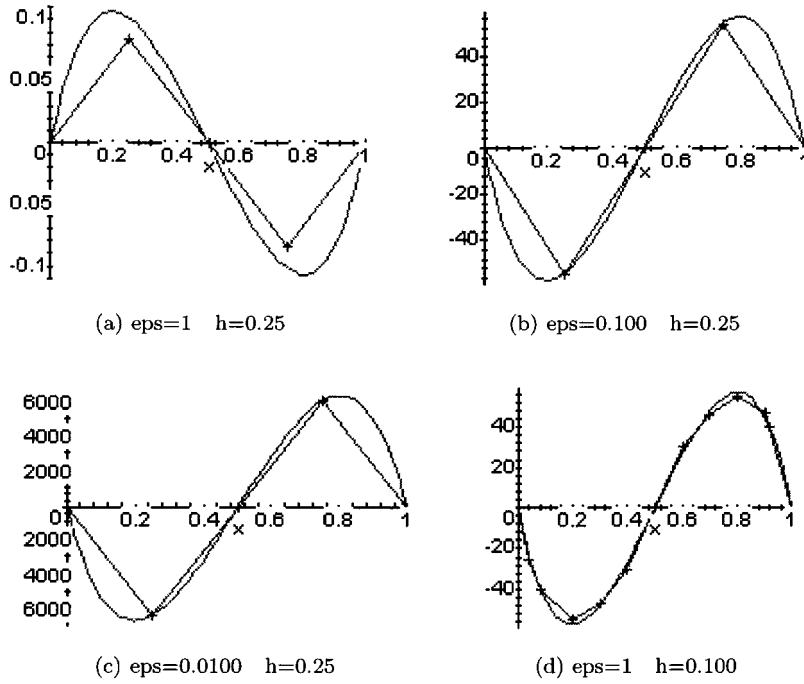


FIGURE 1. Standard Galerkin method with piecewise linear elements.

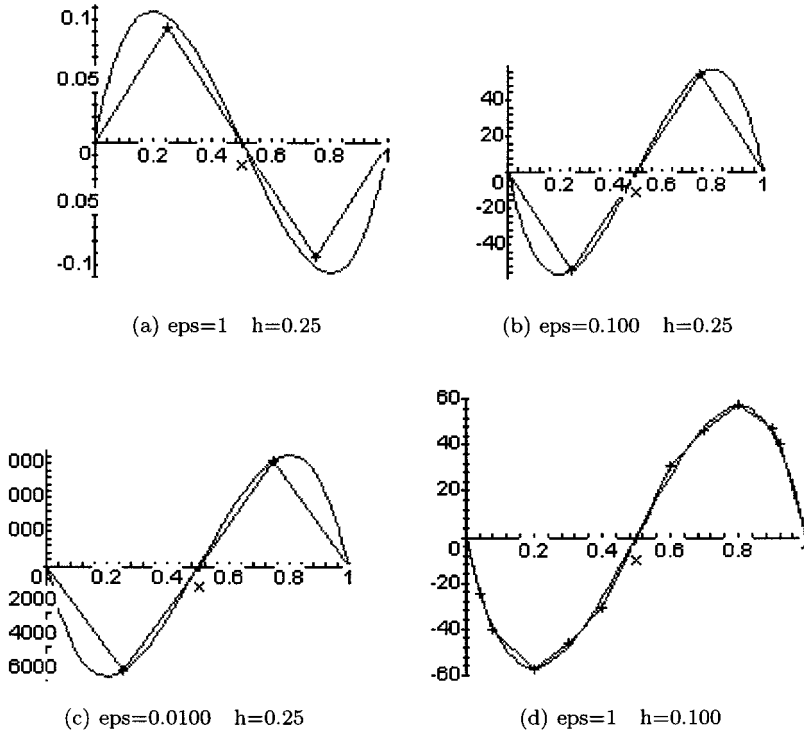


FIGURE 2. Galerkin+bubbles method.

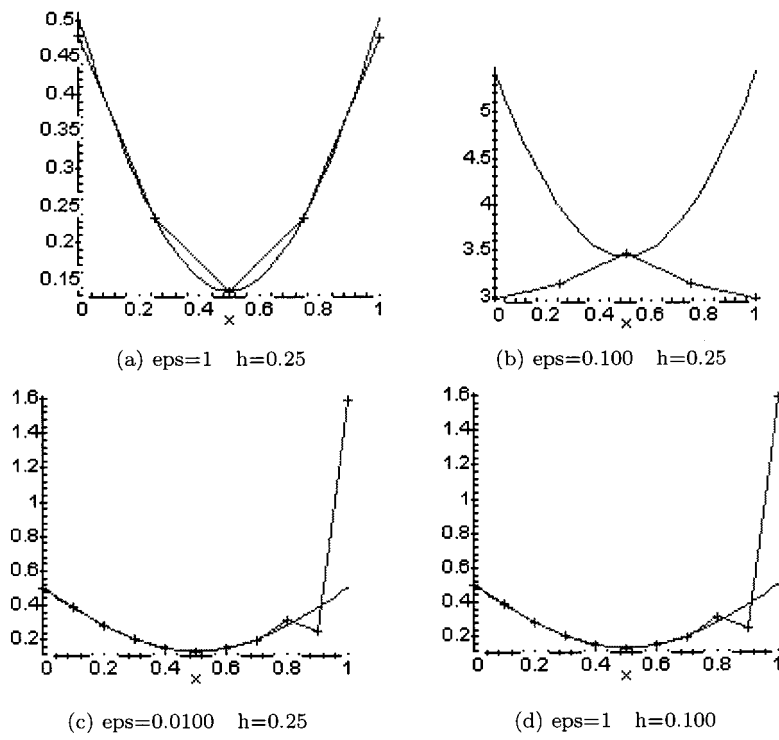


FIGURE 3. Standard Galerkin method with piecewise linear elements.

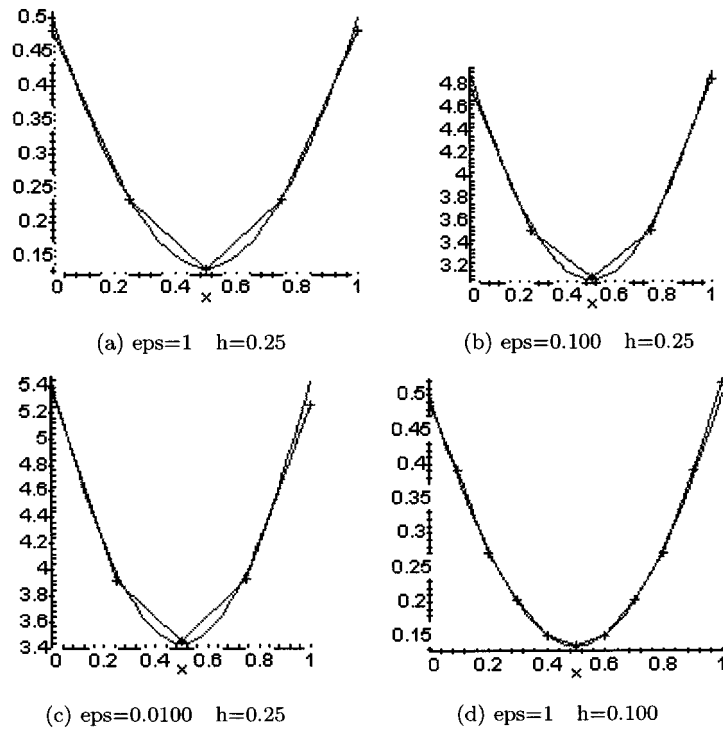


FIGURE 4. Galerkin+bubbles method.

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