ON BLOW-UP OF SOLUTION FOR EULER EQUATIONS

ERIC BEHR[1], JINDŘICH NEČAS[1] AND HONGYOU WU[1]

Abstract. We present numerical evidence for the blow-up of solution for the Euler equations. Our approximate solutions are Taylor polynomials in the time variable of an exact solution, and we believe that in terms of the exact solution, the blow-up will be rigorously proved.

Résumé. Nous présentons une solution numérique des équations d'Euler montrant la solution non-bornée : l'approximation de la solution est donnée par une série de Taylor dans la variable de temps de la solution exacte, et il est probable que cet exemple fournira le résultat.

1. INTRODUCTION

The question about the blow-up of regular solution for the Euler equations or the Navier-Stokes equations emerged with the first fundamental papers on the topic. Here we restrict ourselves to quoting the famous paper of Leray [7] from 1934, in which he proposed to show the blow-up of solution for the Cauchy problem of the Navier-Stokes equations using backward self-similar solutions. Only recently, in papers [11,12] and [9], it was proven that such a weak solution to the problem must be identically zero. The authors of this paper together with several collaborators tried, without success, to construct a non-zero backward self-similar solution for the boundary value problem on a half-space.

There is a series of papers showing the blow-up by numerical simulations; see, for example [3,6,13,14] and [4].

An eventual blow-up of solution for the Navier-Stokes equations is also a key for the proof of non-uniqueness. All these results will create the need for a more adequate theory of incompressible fluids, where essential role will be played by the theory of multipolar fluids (see [10]) as well as by the theory of fluids of higher grade (see [1]).

This paper, in the authors' belief, may be the basis for the proof of the blow-up of solution for the Euler equations in the space-periodic case. For a special non-zero spatial vorticity we conjecture some estimates implying blow-up. These estimates follow from a large computation of the solution as a Taylor series in time, with the series computed up to the term of degree 35, with about 11 million of the coefficients of the series being calculated. As the theoretic preparation for this numerical computation, some theorems about the analyticity

in time of the solution with values in L^2 will be mentioned in the next section. These results are very close to those of Delort [2]. We also prove there that any L^2-solution that is analytic in time and any W^{3,2}-solution of the same initial-value problem coincide on their common time interval.

It is known [8] (pp. 151–152) that for each $p \in (1, +\infty)$, there are no uniform W^{1,p}-estimates, even for a very short time. However, in this paper we deal with solutions that are analytic in the spatial variables.

2. ANALYTICITY OF SOLUTIONS IN TIME

We look for the space-periodic solutions $\boldsymbol{u} = (u_1, u_2, u_3)$ with period 2π in the x_1-, x_2- and x_3-directions to the Euler equations

$$\frac{\partial u_j}{\partial t} + \sum_{m=1}^{3} u_m \frac{\partial u_j}{\partial x_m} + \frac{\partial p}{\partial x_j} = 0 \quad \text{in } (0, T) \times (-\pi, \pi)^3 \text{ for each } j,$$

$$\sum_{m=1}^{3} \frac{\partial u_m}{\partial x_m} = 0 \quad \text{in } (0, T) \times (-\pi, \pi)^3, \tag{2.1}$$

where $T > 0$ is a constant.

The second equation of (2.1) implies that for each $j \in \{1, 2, 3\}$,

$$\sum_{m=1}^{3} u_m \frac{\partial u_j}{\partial x_m} = \sum_{m=1}^{3} \frac{\partial}{\partial x_m}(u_m u_j). \tag{2.2}$$

So, from the first equation of (2.1) and the space-periodicity of $\boldsymbol{u}$ we deduce that the condition

$$\int_{(-\pi, \pi)^3} \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} = 0 \tag{2.3}$$

is satisfied for all time if it is so at $t = 0$. Therefore, we will always assume (2.3).

Let $\boldsymbol{u}^{(0)}$ be a real-valued, space-periodic, divergence-free and real analytic function. Note that $\boldsymbol{u}^{(0)}$ is real analytic if and only if there is a constant C such that

$$\left\| \mathrm{D}^{\boldsymbol{l}} \boldsymbol{u}^{(0)} \right\|_{\mathrm{L}^2} \le C^{|\boldsymbol{l}|} \, \boldsymbol{l}! \, \left\| \boldsymbol{u}^{(0)} \right\|_{\mathrm{L}^2} \tag{2.4}$$

with $\boldsymbol{l}! = l_1! \, l_2! \, l_3!$ for all $\boldsymbol{l} = (l_1, l_2, l_3) \in \mathbb{N}_0^3$, where $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$. We shall look for a function $\boldsymbol{u}$ with values in L$^2\big((-\pi, \pi)^3\big)$, real analytic in t on $(-T, T)$, and satisfying

$$\boldsymbol{u}(0, \boldsymbol{x}) = \boldsymbol{u}^{(0)}(\boldsymbol{x}) \qquad \text{in } (-\pi, \pi)^3 \tag{2.5}$$

as well as (2.1) with p real analytic in t on $(-T, T)$. It follows from (2.1) that

$$-\sum_{j,m=1}^{3} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_m}(u_m u_j) = \triangle p. \tag{2.6}$$

Let

$$\boldsymbol{u}(t, \boldsymbol{x}) = \sum_{\boldsymbol{l} \in \mathbb{Z}^3} \boldsymbol{u}_{\boldsymbol{l}}(t) \mathrm{e}^{\mathrm{i}\boldsymbol{l} \cdot \boldsymbol{x}} = \sum_{\boldsymbol{l} \in \mathbb{Z}^3} \big(u_{1,\boldsymbol{l}}(t), u_{2,\boldsymbol{l}}(t), u_{3,\boldsymbol{l}}(t)\big) \mathrm{e}^{\mathrm{i}\boldsymbol{l} \cdot \boldsymbol{x}}, \tag{2.7}$$

then it follows (see [5]) that as an L^2-valued function, $\boldsymbol{u}$ is real analytic in t on $(-T, T)$ if and only if each $\boldsymbol{u_l}$ is so and the series (2.7) converges in $\mathrm{L}^2\big((-\pi, \pi)^3\big)$. We also put

$$p(t, \boldsymbol{x}) = \sum_{\boldsymbol{l} \in \mathbb{Z}^3} p_{\boldsymbol{l}}(t) \mathrm{e}^{\mathrm{i}\boldsymbol{l} \cdot \boldsymbol{x}}, \tag{2.8}$$

where the sum is taken in the sense of distributions, then we have a similar claim relating p and $p_{\boldsymbol{l}}$. Since

$$\frac{\partial}{\partial x_m}\big(u_m u_j\big) = \mathrm{i} \sum_{\boldsymbol{l} \in \mathbb{Z}^3} \sum_{\boldsymbol{s} \in \mathbb{Z}^3} u_{m, \boldsymbol{s}} u_{j, \boldsymbol{l}-\boldsymbol{s}} l_m \mathrm{e}^{\mathrm{i}\boldsymbol{l} \cdot \boldsymbol{x}} \tag{2.9}$$

in the sense of distributions, (2.6) yields that

$$\mathrm{i} \sum_{j=1}^{3} C_{j,\boldsymbol{l}} l_j = (l_1^2 + l_2^2 + l_3^2) p_{\boldsymbol{l}} \qquad \text{for each } \boldsymbol{l} \in \mathbb{Z}^3, \tag{2.10}$$

where

$$C_{j,\boldsymbol{l}}(t) = \mathrm{i} \sum_{m=1}^{3} \sum_{\boldsymbol{s} \in \mathbb{Z}^3} u_{m, \boldsymbol{s}}(t) u_{j, \boldsymbol{l}-\boldsymbol{s}}(t) l_m. \tag{2.11}$$

Thus, the first equation of (2.1) together with (2.10) imply that for each $j \in \{1, 2, 3\}$ and $\boldsymbol{l} \in \mathbb{Z}^3$ satisfying $\boldsymbol{l} \neq \boldsymbol{0}$,

$$u'_{j,\boldsymbol{l}}(t) + C_{j,\boldsymbol{l}}(t) - \frac{l_j}{l_1^2 + l_2^2 + l_3^2} \sum_{m=1}^{3} C_{m,\boldsymbol{l}}(t) l_m = 0. \tag{2.12}$$

For each $j \in \{1, 2, 3\}$, we set

$$u_j(t, \boldsymbol{x}) = \sum_{k=0}^{+\infty} \sum_{\boldsymbol{l} \in \mathbb{Z}^3} u_{j,k,\boldsymbol{l}} t^k \mathrm{e}^{\mathrm{i}\boldsymbol{l} \cdot \boldsymbol{x}}, \tag{2.13}$$

then (2.12) can be rewritten as

$$(k+1) u_{j,k+1,\boldsymbol{l}} + C_{j,k,\boldsymbol{l}} - \frac{l_j}{l_1^2 + l_2^2 + l_3^2} \sum_{m=1}^{3} C_{m,k,\boldsymbol{l}} l_m = 0 \tag{2.14}$$

for all $j \in \{1, 2, 3\}$, $k \in \mathbb{N}_0$ and $\boldsymbol{l} \in \mathbb{Z}^3$ satisfying $\boldsymbol{l} \neq \boldsymbol{0}$, where

$$C_{j,k,\boldsymbol{l}} = \mathrm{i} \sum_{r=0}^{k} \sum_{\boldsymbol{s} \in \mathbb{Z}^3} u_{j,k-r,\boldsymbol{l}-\boldsymbol{s}} \sum_{m=1}^{3} l_m u_{m,r,\boldsymbol{s}}. \tag{2.15}$$

Note also that now the reality of $\boldsymbol{u}$ is equivalent to

$$\overline{u_{j,k,\boldsymbol{l}}} = u_{j,k,-\boldsymbol{l}} \tag{2.16}$$

for each $j \in \{1, 2, 3\}$, $k \in \mathbb{N}_0$ and $\boldsymbol{l} \in \mathbb{Z}^3$, and that $\boldsymbol{u}$ is divergence-free if and only if

$$\sum_{m=1}^{3} u_{m,k,\boldsymbol{l}} l_m = 0 \tag{2.17}$$

for each $k \in \mathbb{N}_0$ and $\boldsymbol{l} \in \mathbb{Z}^3$.

By [2], there is a unique solution to the problem consisting of (2.1) and (2.5) with values in the real analytic functions and real analytic in time. Let $T' > 0$ be the constant such that $(-T', T')$ is the maximal open interval of time on which this real analytic solution is defined. The existence of the real analytic solution implies that there is a unique solution to the problem with values in $L^2\big((-\pi, \pi)^3\big)$ and real analytic in time. We shall denote by T_0 this L^2-solution's radius of convergence at $t = 0$.

Proposition 2.18. *Let $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x})$ be the L^2-solution to the problem consisting of (2.1) and (2.5) that is real analytic in time. Then,*

$$\int_{(-\pi,\pi)^3} |\boldsymbol{u}(t, \boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x} = \int_{(-\pi,\pi)^3} |\boldsymbol{u}(0, \boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x} \qquad \text{for } t \in (-T_0, T_0). \tag{2.19}$$

Proof. For $t \in (-T', T')$, the equality follows from classical estimates of the kinetic energy. Since the left hand side of (2.19) is a real analytic function in t, (2.19) holds. $\qquad\square$

For each $n \in \mathbb{N}$, let $\boldsymbol{u}^{(n)} = \big(u_1^{(n)}, u_2^{(n)}, u_3^{(n)}\big)$ be the Taylor expansion of $\boldsymbol{u}$ in t up to the power t^n. Then, $\boldsymbol{u}^{(n)}$ is divergence-free and space-periodic. This notation will be used in the proof of the following uniqueness result.

Lemma 2.20. *Let $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x})$ be the L^2-solution to the problem consisting of (2.1) and (2.5) that is real analytic in time, and $\boldsymbol{w} = \boldsymbol{w}(t, \boldsymbol{x})$ any solution to the problem in $\mathrm{C}\big([0, T), W^{3,2}\big) \cap \mathrm{C}^1\big([0, T), \mathrm{W}^{2,2}\big)$. Then, $\boldsymbol{u} = \boldsymbol{w}$ for $t \in [0, T_0) \cap [0, T)$.*

Proof. Since each $\boldsymbol{u}^{(n)}$ is divergence-free, we have that for $t \in [0, T_0)$,

$$\lim_{n\to+\infty} \sum_{j=1}^3 \int_{(-\pi,\pi)^3} \left(\frac{\partial u_j^{(n)}}{\partial t} + \sum_{m=1}^3 u_m^{(n)} \frac{\partial u_j^{(n)}}{\partial x_m} \right) u_j^{(n)} \mathrm{d}\boldsymbol{x} = \lim_{n\to+\infty} \frac{1}{2} \int_{(-\pi,\pi)^3} \frac{\mathrm{d}}{\mathrm{d}t} \left| \boldsymbol{u}^{(n)} \right|^2 \mathrm{d}\boldsymbol{x}$$

$$= \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{(-\pi,\pi)^3} |\boldsymbol{u}|^2 \, \mathrm{d}\boldsymbol{x} = 0. \tag{2.21}$$

Let $T_* = \min\{T_0, T\}$. For any $\epsilon > 0$, there is a constant $C(\epsilon)$ such that

$$\left\| \frac{\partial w_j}{\partial x_m} \right\|_{L^\infty} \leq C(\epsilon) \quad \text{for } t \in [0, T_* - \epsilon]. \tag{2.22}$$

Thus, from

$$\sum_{j=1}^3 \int_{(-\pi,\pi)^3} \left[\frac{\partial w_j}{\partial t} - \frac{\partial u_j^{(n)}}{\partial t} + \sum_{m=1}^3 \left(w_m \frac{\partial w_j}{\partial x_m} - u_m^{(n)} \frac{\partial u_j^{(n)}}{\partial x_m} \right) \right] \left(w_j - u_j^{(n)} \right) \mathrm{d}\boldsymbol{x}$$

$$= \sum_{j=1}^3 \int_{(-\pi,\pi)^3} \left[\frac{\partial w_j}{\partial t} - \frac{\partial u_j^{(n)}}{\partial t} + \sum_{m=1}^3 \left(w_m \frac{\partial w_j}{\partial x_m} - u_m^{(n)} \frac{\partial w_j}{\partial x_m} \right. \right.$$

$$\left. \left. + u_m^{(n)} \frac{\partial w_j}{\partial x_m} - u_m^{(n)} \frac{\partial u_j^{(n)}}{\partial x_m} \right) \right] \left(w_j - u_j^{(n)} \right) \mathrm{d}\boldsymbol{x} \tag{2.23}$$

$$= \frac{1}{2} \int_{(-\pi,\pi)^3} \frac{\mathrm{d}}{\mathrm{d}t} \left| \boldsymbol{w} - \boldsymbol{u}^{(n)} \right|^2 \mathrm{d}\boldsymbol{x}$$

$$+ \sum_{j,m=1}^3 \int_{(-\pi,\pi)^3} \left(w_m - u_m^{(n)} \right) \left(w_j - u_j^{(n)} \right) \frac{\partial w_j}{\partial x_m} \, \mathrm{d}\boldsymbol{x}$$

and (2.21) we obtain that when $t \in [0, T_* - \epsilon]$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{(-\pi,\pi)^3} \left| \boldsymbol{w} - \boldsymbol{u}^{(n)} \right|^2 \mathrm{d}\boldsymbol{x} \leq D(\epsilon) \int_{(-\pi,\pi)^3} \left| \boldsymbol{w} - \boldsymbol{u}^{(n)} \right|^2 \mathrm{d}\boldsymbol{x} + E(\epsilon, n) \tag{2.24}$$

for some constants $D(\epsilon)$ and $E(\epsilon, n)$ satisfying $\lim_{n \to +\infty} E(\epsilon, n) = 0$ and, hence,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{(-\pi,\pi)^3} |\boldsymbol{w} - \boldsymbol{u}|^2 \, \mathrm{d}\boldsymbol{x} \leq D(\epsilon) \int_{(-\pi,\pi)^3} |\boldsymbol{w} - \boldsymbol{u}|^2 \, \mathrm{d}\boldsymbol{x}. \tag{2.25}$$

So, $\boldsymbol{u} = \boldsymbol{w}$ for $t \in [0, T_* - \epsilon]$, since $\boldsymbol{u}(0, \cdot) = \boldsymbol{w}(0, \cdot)$ by our assumption. $\square$

Remark 2.26. Note that the proof above only uses the divergence-freeness and space-periodicity conditions. So, it works for many other equations. In particular, it implies a similar result for the Navier-Stokes equations.

Proposition 2.27. *Let* $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x})$ *be the* L^2-*solution to the problem consisting of (2.1) and (2.5) that is real analytic in time, assume that*

$$\sum_{\boldsymbol{l} \in \mathbb{Z}^3} \left\| \boldsymbol{u_l}(T_*, \cdot) \right\|_{\mathrm{L}^2}^2 (l_1^2 + l_2^2 + l_3^2)^2 = +\infty, \tag{2.28}$$

for some $T_* \in (0, T_0)$, *and denote by* T_c *the largest value of* $T > 0$ *such that there is a solution to the problem in* $\mathrm{C}\big([0, T), rmW^{3,2}\big) \cap \mathrm{C}^1\big([0, T), rmW^{2,2}\big)$. *Then,* $T_c \leq T_*$ *and, hence, the solution to the problem in* $\mathrm{C}\big([0, T_c), rmW^{3,2}\big) \cap \mathrm{C}^1\big([0, T_c), rmW^{2,2}\big)$ *blows up at* T_c, *i.e.,*

$$\varlimsup_{t \to T_c^-} \left\| \boldsymbol{u}(t, \cdot) \right\|_{rmW^{3,2}} = +\infty. \tag{2.29}$$

Proof. We can assume that T_* is the smallest number satisfying (2.28). To reach a contradiction, suppose that $T_* < T_c$. Then, $T_* < T =: \min\{T_0, T_c\}$. By Lemma 2.20, $\boldsymbol{u}$ is equal to the solution to the problem in $\mathrm{C}\big([0, T), rmW^{3,2}\big) \cap \mathrm{C}^1\big([0, T), rmW^{2,2}\big)$ for $t \in [0, T)$. This is impossible, since (2.28) implies

$$\varlimsup_{t \to T_*^-} \left\| \boldsymbol{u}(t, \cdot) \right\|_{rmW^{3,2}} = +\infty. \tag{2.30}$$

Therefore, we must have $T_c \leq T_*$. $\square$

3. NUMERICAL APPROXIMATIONS TO A SOLUTION AND BLOW-UP OF SOLUTION

In this section, we discuss how the Taylor polynomial (in time) approximations to a specially chosen solution can be computed and then present some numerical results so obtained, plus our comments on blow-up.

Take the initial condition

$$\begin{aligned}
\boldsymbol{u}(0, \boldsymbol{x}) &= (1, -1, 0)\Big(\mathrm{e}^{\mathrm{i}(x_1 + x_2)} + \mathrm{e}^{-\mathrm{i}(x_1 + x_2)}\Big) + (1, 0, -1)\Big(\mathrm{e}^{\mathrm{i}(x_1 + x_3)} + \mathrm{e}^{-\mathrm{i}(x_1 + x_3)}\Big) \\
&\quad + (0, 1, -1)\Big(\mathrm{e}^{\mathrm{i}(x_2 + x_3)} + \mathrm{e}^{-\mathrm{i}(x_2 + x_3)}\Big).
\end{aligned} \tag{3.1}$$

Then, using (2.14) and (2.15) one can prove by induction that for every $j \in \{1, 2, 3\}$ and $k \in \mathbb{N}_0$,

$$C_{j,k,\boldsymbol{l}} = u_{j,k+1,\boldsymbol{l}} = 0 \quad \text{if } l_m < -k - 2 \text{ or } l_m > k + 2 \text{ for some } m \tag{3.2}$$

and when $\boldsymbol{l}$ satisfies $-k - 2 \leq l_m \leq k + 2$ for each m, we have

$$C_{j,k,\boldsymbol{l}} = \mathrm{i} \sum_{r=0}^{k} \sum_{s_1=a_1}^{b_1} \sum_{s_2=a_2}^{b_2} \sum_{s_3=a_3}^{b_3} u_{j,k-r,\boldsymbol{l}-\boldsymbol{s}} \sum_{m=1}^{3} l_m u_{m,r,\boldsymbol{s}} \tag{3.3}$$

with

$$a_n = \max\{-r, l_n - k + r\} - 1 \quad \text{and} \quad b_n = \min\{r, l_n + k - r\} + 1 \tag{3.4}$$

for each $n \in \{1, 2, 3\}$. Thus, for each $N \in \mathbb{N}_0$, the N-th Taylor polynomial approximation

$$\boldsymbol{u}^{(N)} = \sum_{k=0}^{N} \sum_{\boldsymbol{l} \in \mathbb{Z}^3} \big(u_{1,k,\boldsymbol{l}}, u_{2,k,\boldsymbol{l}}, u_{3,k,\boldsymbol{l}}\big) \mathrm{e}^{\mathrm{i}\boldsymbol{l}\cdot\boldsymbol{x}} t^k \tag{3.5}$$

of the solution $\boldsymbol{u}$ is actually a Fourier polynomial in the space variables. Moreover (3.2) together with the symmetries of the Euler equations (2.1) and the initial condition (3.1) imply that for each $N \in \mathbb{N}_0$, the N-th Taylor polynomial approximation $\boldsymbol{u}^{(N)}$ satisfies

$$(2\pi)^3 \big\|\boldsymbol{u}^{(N)}\big\|_{\mathrm{L}^2}^2 = 3 \sum_{k=0}^{2N} \sum_{m=a}^{b} \sum_{|l_1|,|l_2|,|l_3| \leq L} u_{1,k-m,\boldsymbol{l}} \overline{u_{1,m,\boldsymbol{l}}} \, t^k,$$

$$(2\pi)^3 \big\|\boldsymbol{u}^{(N)}\big\|_{\mathrm{H}^3}^2 = 3 \sum_{k=0}^{2N} \sum_{m=a}^{b} \sum_{|l_1|,|l_2|,|l_3| \leq L} (l_1^2 + l_2^2 + l_3^2)^3 u_{1,k-m,\boldsymbol{l}} \overline{u_{1,m,\boldsymbol{l}}} \, t^k, \tag{3.6}$$

where

$$a = \max\{0, k - N\}, \quad b = \min\{N, k\}, \quad L = \min\{k - m, m\} + 1. \tag{3.7}$$

It is straightforward to compute $\boldsymbol{u}^{(N)}$ (with exact values of the coefficients) using *Mathematica* or similar software for small N. We were able to compute $\boldsymbol{u}^{(N)}$ using *Mathematica* for $N = 1, 2, 3, ..., 10$.

Using any code directly implementing (3.3, 2.14), and (3.6), one can also approximate $\boldsymbol{u}^{(N)}$ for relatively large, but not too large, N. Note also that the reality condition (2.16) can be used to save almost half of the computations. We approximated $\boldsymbol{u}^{(N)}$ for $N = 11, 12, 13, ..., 35$ with a C++ program. Here we summarize our numerical results so obtained:

i) Among the $3(5^3 + 7^3 + \cdots + 73^3) = 11,240,775$ coefficients that have to be approximated, about $976,455$ are non-zero, and the largest absolute value of them is about 6.21×10^{12}.

ii) At $t = 0.32$, the values of $L(N) = \frac{(2\pi)^3}{12}\big\|\boldsymbol{u}^{(N)}\big\|_{\mathrm{L}^2}^2$ are illustrated by the following graph, with $L(35) \approx 1.00000$:
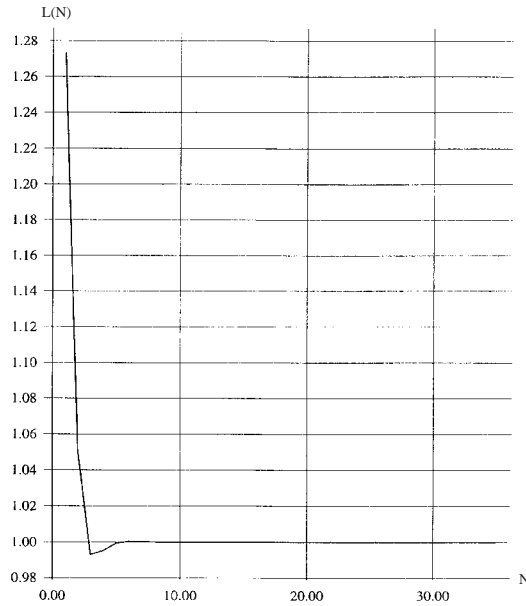
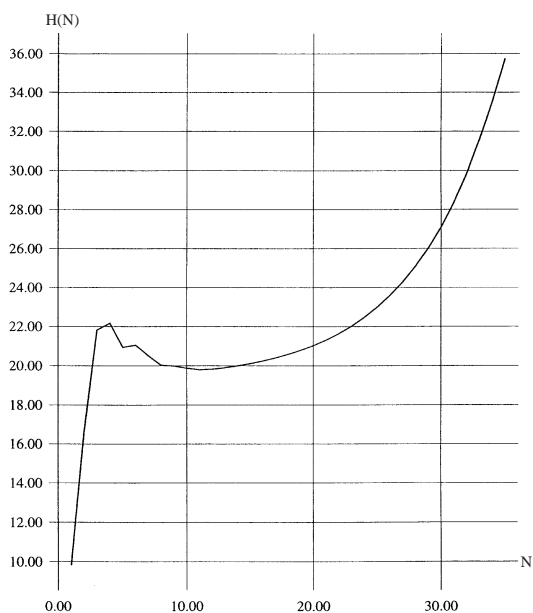while the values of $H(N) = \frac{(2\pi)^3}{96}\big\|\boldsymbol{u}^{(N)}\big\|_{\mathrm{H}^3}^2$ are shown in the following graph, with $H(35) \approx 13.85000$:
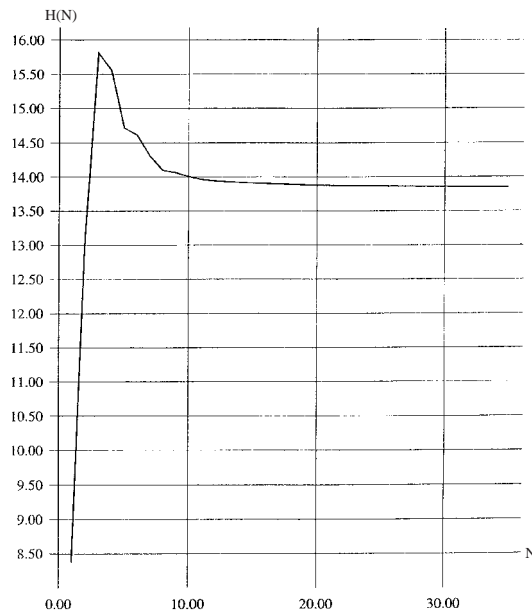
We believe that this strongly indicates that both norms stay finite as $N \to +\infty$.

iii) At $t = 0.35$, the values of $L(N) = \frac{(2\pi)^3}{12}\left\|\boldsymbol{u}^{(N)}\right\|^2_{L^2}$ are illustrated by the following graph, with $L(35) \approx 1.00001$:
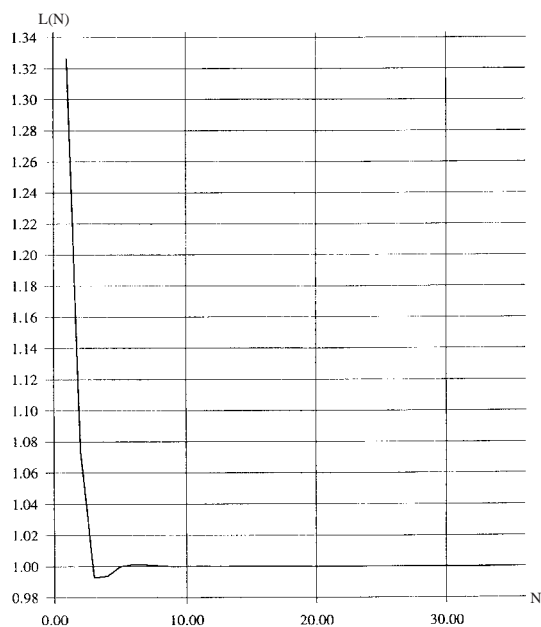
while the values of $H(N) = \frac{(2\pi)^3}{96}\left\|\boldsymbol{u}^{(N)}\right\|^2_{H^3}$ are shown in the following graph, with $H(35) \approx 35.7467$:
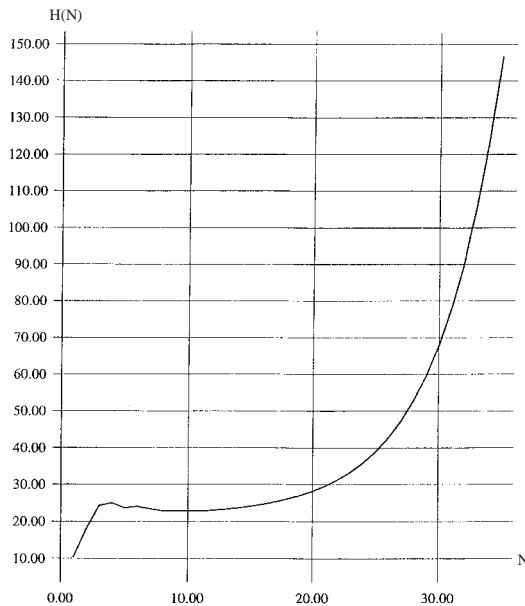

We believe that this gives strong indications that, as $N \to +\infty$, the L^2-norm approaches 1 and the H^3-norm approaches $+\infty$, *i.e.*, blows up.

iv) At $t = 0.36$, the values of $L(N) = \frac{(2\pi)^3}{12}\big\|\boldsymbol{u}^{(N)}\big\|_{\mathrm{L}^2}^2$ are illustrated by the following graph, with $L(35) \approx 1.00007$:
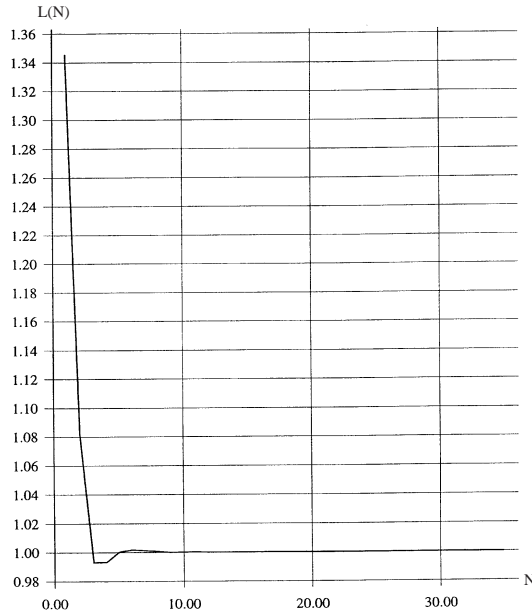
while the values of $H(N) = \frac{(2\pi)^3}{96}\big\|\boldsymbol{u}^{(N)}\big\|_{\mathrm{H}^3}^2$ are shown in the following graph, with $H(35) \approx 146.777$:

We believe that this gives even stronger evidence that there is a T_* at which, as $N \to +\infty$, the L^2-norm approaches 1 and the H^3-norm blows up. We also believe that here we begin to see some rounding error in computations.

v) Our numerical results also indicate that even though the H^3-norm blows up at some point between 0.32 and 0.35, the convergence radius of the L^2-solution is between 0.38 and 0.42.

REFERENCES

[1] H. Bellout, J. Nečas and K.R. Rajagopal, On the existence and uniqueness of flows of multipolar fluids of grade 3 and their stability. *Internat. J. Engrg. Sci.* **37** (1999) 75–96.

[2] J.-M. Delort, Estimations fines pour des opérateurs pseudo-différentiels analytiques sur un ouvert à bord de $\mathbb{R}^n$ application aux equations d'Euler. *Comm. Partial Differential Equations* **10** (1985) 1465–1525.

[3] R. Grauer and T. Sideris, Numerical computation of three dimensional incompressible ideal fluids with swirl. *Phys. Rev. Lett.* **67** (1991) 3511.

[4] R. Grauer and T. Sideris, Finite time singularities in ideal fluids with swirl. *Phys. D* **88** (1995) 116–132.

[5] E. Hille and R.S. Phillips, *Functional analysis and semi-Groups*. Amer. Math. Soc., Providence, R.I. (1957).

[6] R. Kerr, Evidence for a singularity of the three-dimensional incompressible Euler equations. *Phys. Fluids* **A5** (1993) 1725–1746.

[7] J. Leray, Sur le mouvement d'un liquide visqueux remplissant l'espace. *Acta Math.* **63** (1934) 193–248.

[8] P.-L. Lions, *Mathematical topics in fluid mechanics, Vol. 1. Incompressible models*. Oxford University Press, New York (1996).

[9] J. Málek, J. Nečas, M. Pokorný and M. Schonbek, On possible singular solutions to the Navier-Stokes equations. *Math. Nachr.* **199** (1999) 97–114.

[10] J. Nečas, *Theory of multipolar fluids*. Problems and methods in mathematical physics (Chemnitz, 1993) 111–119. Teubner, Stuttgart, *Teubner-Texte Math.* **134** (1994).

[11] J. Nečas, M. Růžička and V. Šverák, Sur une remarque de J. Leray concernant la construction de solutions singulières des équations de Navier-Stokes. *C. R. Acad. Sci. Paris Sér. I Math.* **323** (1996) 245–249.

[12] J. Nečas, M. Růžička and V. Šverák, On Leray's self-similar solutions of the Navier-Stokes equations. *Acta Math.* **176** (1996) 283–294.

[13] A. Pumir and E. Siggia, Collapsing solutions to the 3-D Euler equations. *Phys. Fluids* **A2** (1990) 220–241.

[14] A. Pumir and E. Siggia, Development of singular solutions to the axisymmetric Euler equations. *Phys. Fluids* **A4** (1992) 1472–1491.