

OBSTACLE PROBLEMS FOR SCALAR CONSERVATION LAWS

LAURENT LEVI¹

Abstract. In this paper we are interested in bilateral obstacle problems for quasilinear scalar conservation laws associated with Dirichlet boundary conditions. Firstly, we provide a suitable entropy formulation which ensures uniqueness. Then, we justify the existence of a solution through the method of penalization and by referring to the notion of entropy process solution due to specific properties of bounded sequences in L^∞ . Lastly, we study the behaviour of this solution and its stability properties with respect to the associated obstacle functions.

Résumé. Ce travail a pour objet l'étude de problèmes d'obstacles bilatéraux pour des lois de conservation scalaires quasi-linéaires du premier ordre associées à des conditions aux limites de Dirichlet. On donne d'abord une formulation entropique qui garantit l'unicité. On justifie alors l'existence d'une solution par utilisation de la méthode de pénalisation et au moyen de la notion de processus entropique solution due aux propriétés des suites bornées dans L^∞ . Enfin, on étudie le comportement de cette solution et ses propriétés de stabilité en fonction des contraintes d'obstacle associées.

Mathematics Subject Classification. 35L65, 35R35, 35L85.

Received: March, 2000. Revised: February 7, 2001.

1. INTRODUCTION AND MATHEMATICAL FRAMEWORK

1.1. Physical motivations

Obstacle problems for first-order hyperbolic operators have been introduced by Bensoussan and Lions [4] in 1973, as part of the study of cost-functions associated with deterministic processes. In this paper we are interested in non-homogeneous Dirichlet problems for general scalar conservation laws associated with a forced bilateral constraint. The physical motivations of such a study are diverse: for example, in the hydrological field we consider the simplified modelling of one phase saturating the subsoil and made of two components without any chemical interactions: water and the component c . Depending on the geological nature of the subsoil, the molecular diffusion-dispersion effects can be neglected in favor of the transport ones. Hence, by referring to [5], given the distribution of temperatures T and the pressure field P of the fluid phase, the transcription of the mass conservation law for c provides the equality ruling the mass fraction ω_c :

$$\partial_t \omega_c - \frac{k(x)}{\mu(\omega_c)} \nabla \omega_c (\nabla P - \rho(T, \omega_c) \vec{g}) = 0,$$

Keywords and phrases. Obstacle problem, conservation laws, entropy solution.

¹ University of Pau, CNRS, Laboratory of Applied Mathematics ERS 2055, I.P.R.A., Avenue de l'Université, 64000 Pau, France.
e-mail: laurent.levi@univ-pau.fr

where $k(x)$ denotes the absolute permeability at the point x , μ is the dynamic viscosity of the fluid phase and $\rho(T, \omega_c)$ its voluminal mass, defined by the composition ω_c at the temperature T . Lastly, \vec{g} is the gravity acceleration vector. Furthermore, ω_c must satisfy the bilateral obstacle condition:

$$\theta_{1,c}(T, P) \leq \omega_c \leq \theta_{2,c}(T, P),$$

where $\theta_{1,c}(T, P)$ and $\theta_{2,c}(T, P)$ are saturation thresholds. Beyond these values the appearance of a new phase, liquid or solid, for the same number of components changes the thermodynamical nature of the considered system and the latter cannot be described through a simplified balance equation.

In petroleum engineering one is led to examine the case of a unique oil phase, saturating the reservoir rock, which is made of two components: heavy and light oil. Within the context of isothermal flows, when pressure P is a determined sufficiently smooth function, writing the heavy oil component's mass conservation law leads to the first-order equality:

$$\partial_t C_o^h - \text{Div} \left[k(P) \frac{C_o^h}{\mu_o(C_o^h)} \left(\vec{\nabla} P - \rho_o(C_o^h) \vec{g} \right) \right] = 0.$$

Moreover, C_o^h has to fulfill the unilateral obstacle condition:

$$\theta(P) \leq C_o^h,$$

at pressure P . Indeed, beyond this value the appearance of a gaseous phase for the same number of components shows that the molar fraction of the pseudo heavy constituent is determined by the knowledge of the pressure; thus, the proposed model considers the dissolution of gas into oil, yet it excludes the gas-liberating phenomenon. A comprehensive treatment of those problems may be found in Gagneux and Madaune-Tort's book [10], particularly the *Black-Oil* model that we have referred to.

1.2. Mathematical setting and assumptions on data

The previous motivations lead us to introduce the first-order quasilinear hyperbolic operator including a reaction term

$$\mathbb{H}(t, x, \cdot) : u \rightarrow \partial_t u + \sum_{i=1}^p \partial_{x_i} f_i(t, x, u) + g(t, x, u),$$

and two measurable obstacle-functions θ_1 and θ_2 such that

$$\theta_1 \leq \theta_2 \text{ a.e. on } Q =]0, T[\times \Omega,$$

where T is a strictly positive and finite real and Ω is a subdomain of \mathbb{R}^p , $p \geq 1$, with a Lipschitz boundary $\Gamma = \partial\Omega$. Then, the purpose of this article is to study the free boundary problem:

$$\theta_1 \leq u \leq \theta_2 \text{ a.e. on } Q, \tag{1}$$

$$(u - \theta_1)\mathbb{H}(t, x, u) \leq 0, (u - \theta_2)\mathbb{H}(t, x, u) \leq 0, (u - \theta_1)(u - \theta_2)\mathbb{H}(t, x, u) = 0 \text{ on } Q, \tag{2}$$

$$u = u_\Gamma \text{ on (a part of) } \Sigma, u(0, \cdot) = u_0 \text{ on } \Omega, \tag{3}$$

where u_0 and u_Γ are two measurable and bounded functions respectively on Ω and $\Sigma =]0, T[\times \Gamma$ and such that:

$$\theta_1(0, \cdot) \leq u_0 \leq \theta_2(0, \cdot) \text{ a.e. on } \Omega,$$

$$\theta_1 \leq u_\Gamma \leq \theta_2 \text{ a.e. on } \Sigma.$$

What is more, in the rest of this paper we assume that the data satisfy the following assumptions:

- $f_i, i \in \{1, \dots, p\}$, (respect. g) is a C^1 -class function on $\bar{Q} \times \mathbb{R}$ (respect. a continuous function on $\bar{Q} \times \mathbb{R}$). In addition, $\partial_{x_i} f_i, i \in \{1, \dots, p\}$, and g are Lipschitzian with respect to the third variable, uniformly in (t, x) .
- θ_1 and θ_2 are elements of $W^{1,+\infty}(Q)$.

Those hypotheses guarantee the existence of the quantity $M(t)$ defined for any real t of $[0, T]$ by

$$M(t) = \max \left(\|u_0\|_{L^\infty(\Omega)} ; \|u_\Gamma\|_{L^\infty(\Sigma)} ; \|\theta_1\|_{L^\infty(Q)} \right) e^{C_1 t} + \frac{e^{C_1 t} - 1}{C_1} \max_{[0, T] \times \bar{\Omega}} |\mathbb{H}(t, x, 0)|, \tag{4}$$

where C_1 is the sum of the Lipschitz constants with respect to u of g and $\partial_{x_i} f_i, i \in \{1, \dots, p\}$.

- ν denotes the outer normal vector defined a.e. on Γ .

We first have to provide a mathematical formulation for obstacle problem (1), (2) and (3) by keeping in mind that on the one hand, for a general first-order quasilinear equation, it is classical to refer to an entropy criterium which warrants uniqueness. On the other hand, without any assumptions on the sign of the source term for \mathbb{H} , the introduction of obstacle constraints on the initial datum does not a priori pass on to the solution. Therefore we need an entropy formulation taking these constraints into account. Lastly, since u_0 and u_Γ are only bounded functions, we need to refer to the works of Otto in [14], Chapter 2, which introduce a new formulation of boundary conditions for quasilinear hyperbolic equations, since in the case of $L^\infty(Q)$ -solutions we cannot use the notion of a trace on Γ , available for a function of bounded variation on Q . With this view, in the rest of this paper, we denote $\mathbf{f} = (f_1, f_2, \dots, f_p)$ and

$$\begin{aligned} \mathbf{F}(u, v) &= \operatorname{sgn}(u - v) [\mathbf{f}(t, x, u) - \mathbf{f}(t, x, v)], \\ G(u, v) &= \operatorname{sgn}(u - v) [\nabla \cdot \mathbf{f}(t, x, v) + g(t, x, u) + \partial_t v], \\ \mathcal{L}(u, v, w) &= |u - v| \partial_t w + \mathbf{F}(u, v) \cdot \nabla w - G(u, v)w, \end{aligned}$$

the dependence on time and space variables of the non-linearities \mathbf{F} and G not being essential to comprehension. We thus say by denoting for any real k

$$K(t, x) = k(\theta_2(t, x) - \theta_1(t, x)) + \theta_1(t, x).$$

Definition 1.1. A measurable function u is called an entropy solution to (1), (2), (3) if it satisfies:

i) the bilateral constraint

$$\theta_1(t, x) \leq u(t, x) \leq \theta_2(t, x) \text{ for a.e. } (t, x) \text{ in } Q, \tag{5}$$

ii) the entropy condition, for all functions ξ of $\mathcal{D}_+(]0, T[\times \Omega)$ for any real k of $[0, 1]$,

$$\int_Q \mathcal{L}(u, K, \xi) \, dxdt \geq 0. \tag{6}$$

iii) the initial condition $u_0 \in L^\infty(\Omega)$ in the L^1 -sense:

$$\operatorname{ess} \lim_{t \rightarrow 0^+} \int_\Omega |u(t, x) - u_0(x)| \, dx = 0, \tag{7}$$

iv) the boundary condition $u_\Gamma \in L^\infty(\Sigma)$, in the weak sense:

$$\operatorname{ess} \lim_{\tau \rightarrow 0^-} \int_\Sigma \mathcal{F}(u(\sigma + \tau\nu), u_\Gamma, K) \cdot \nu \zeta \, d\sigma \geq 0, \tag{8}$$

for any real k of $[0, 1]$ and for all functions ζ of $L^1_+(\Sigma)$ where, for any real a, b and c ,

$$2\mathcal{F}(a, b, c) = \mathbf{F}(a, b) - \mathbf{F}(c, b) + \mathbf{F}(a, c).$$

2. THE UNIQUENESS PROOF

Let us verify that the weak entropy formulation proposed in Definition 1.1 ensures the uniqueness of a solution. With this view we refer to Otto’s works in [14], Chapter 2, relative to the study of Dirichlet problems for homogeneous scalar conservation laws and we adapt them to the present situation where \mathbb{H} involves a reaction term and K is a time-and-space-depending function. Thus, boundary conditions (8) allow us to provide an L^1 -continuity result with respect to boundary data thanks to the following relation:

Lemma 2.1. *Let u be an entropy solution to obstacle problem (1), (2), (3) in the sense of Definition 1.1. Then, for any real k of $[0, 1]$ and for any function ξ of $\mathcal{D}_+(]0, T[\times \mathbb{R}^p)$,*

$$-\int_Q \mathcal{L}(u, K, \xi) dx dt \leq \text{ess} \lim_{\tau \rightarrow 0^-} \int_{\Sigma} \mathbf{F}(u(\sigma + \tau \nu), u_{\Gamma}) \cdot \nu \xi d\sigma - \int_{\Sigma} \mathbf{F}(K, u_{\Gamma}) \cdot \nu \xi d\sigma. \tag{9}$$

Proof. By adapting Otto’s reasoning, one may be sure that since (6) holds, then for any function δ of $L^1_+(\Sigma)$,

$$\text{ess} \lim_{\tau \rightarrow 0^-} \int_{\Sigma} \mathbf{F}(u(\sigma + \tau \nu), w(\theta_2(\sigma + \tau \nu) - \theta_1(\sigma + \tau \nu)) + \theta_1(\sigma + \tau \nu)) \cdot \nu \delta d\sigma \text{ exists,}$$

for any w of $L^\infty(\Sigma)$ such that $0 \leq w \leq 1$ a.e. on Σ . So, due to the smoothness of \mathbf{f} and $\theta_i, i = 1, 2$ we infer:

$$\text{ess} \lim_{\tau \rightarrow 0^-} \int_{\Sigma} \mathbf{F}(u(\sigma + \tau \nu), w) \delta d\sigma \text{ exists,} \tag{10}$$

for any w of $L^\infty(\Sigma)$ such that $\theta_1 \leq w \leq \theta_2$ a.e. on Σ .

On the other hand, entropy relation (6) ensures that for any function ζ of $\mathcal{D}(]0, T[\times \mathbb{R}^p)$,

$$-\int_Q \mathcal{L}(u, K, \xi) dx dt \leq -\text{ess} \lim_{\tau \rightarrow 0^-} \inf \int_{\Sigma} \mathbf{F}(u(\sigma + \tau \nu), K(\sigma + \tau \nu)) \cdot \nu \xi d\sigma.$$

This inequality and boundary condition (8) give (9) thanks to the regularities of \mathbf{f} and θ_i . □

2.1. The Kruskov relation

The proof uses that one developed by Kruskov [12] by splitting the time and space variables into two. However, the introduction of a forced bilateral obstacle condition gives rise to the choice of an entropy family depending on (t, x) , through functions θ_1 and θ_2 . Even so, the next L^1 -stability result holds:

Theorem 2.1. *Let u and v be two entropy solutions of obstacle problem (1), (2), (3) in the sense of Definition 1.1 and corresponding to boundary conditions (u_0, u_{Γ}) and (v_0, v_{Γ}) . Then, for a.e. t in $]0, T[$,*

$$\int_{\Omega} |u(t, x) - v(t, x)| dx \leq \left(A \int_0^t \int_{\Gamma} |u_{\Gamma} - v_{\Gamma}| d\sigma + \int_{\Omega} |u_0 - v_0| dx \right) e^{M'_g t}, \tag{11}$$

where A and M'_g are respectively the Lipschitz constants of \mathbf{f} and g with respect to their third variable.

Proof. Let m be an element of \mathbb{N}^* and let ρ_m be the standard mollifier sequence in \mathbb{R}^m . For each $\epsilon > 0$, let $(\rho_{m,\epsilon})_{\epsilon>0}$ be the sequence defined by:

$$\rho_{m,\epsilon}(x) = \frac{1}{\epsilon^m} \rho_m\left(\frac{x}{\epsilon}\right)$$

Let p and \tilde{p} be elements of Q . To simplify the writing we add a “tilde” superscript to any function in “tilde” variables. Thus, we denote l and k the functions of $L^\infty(Q)$ such that:

$$\tilde{k} = \begin{cases} \frac{\tilde{V}}{\tilde{\Theta}} & \text{if } \theta_1(\tilde{p}) < \theta_2(\tilde{p}), \\ 0 & \text{else,} \end{cases} \quad \text{and } l = \begin{cases} \frac{U}{\Theta} & \text{if } \theta_1(p) < \theta_2(p), \\ 0 & \text{else,} \end{cases}$$

with $V = v - \theta_1$, $U = u - \theta_1$ and $\Theta = \theta_2 - \theta_1$; that way, $0 \leq k \leq 1$ and $0 \leq l \leq 1$ a.e. on Q . Then, we introduce the function Λ_ϵ defined on $Q \times Q$ through:

$$\Lambda_\epsilon(p, \tilde{p}) = \varphi\left(\frac{p + \tilde{p}}{2}\right) \rho_\epsilon(p - \tilde{p})$$

where φ is an element of $\mathcal{D}_+(\]0, T[\times \mathbb{R}^p)$ and $\rho_\epsilon(p - \tilde{p}) = \rho_{p,\epsilon}(x - \tilde{x})\rho_{1,\epsilon}(t - \tilde{t})$.

In entropy relation (9) satisfied by u one chooses $k = \tilde{k}$, the test-function being equal to $\Lambda(\cdot, \tilde{p})\tilde{\Theta}$ and we integrate with respect to $d\tilde{p}$ over Q . The same reasoning leads us to choose the test-function $\Lambda(p, \cdot)\Theta$ and $k = l$, in inequality (9) fulfilled by v and written in the variables \tilde{p} and $\tilde{\sigma}$. Then, we integrate over Q with respect to dp . The two expressions are added up and by taking into account the definition of Λ_ϵ it follows:

$$-I_{1,\epsilon} - I_{2,\epsilon} \leq I_{3,\epsilon} - I_{4,\epsilon} \text{ with:}$$

$$\begin{aligned} I_{1,\epsilon} &= \int_{Q \times Q} \left(|U\tilde{\Theta} - \tilde{V}\Theta| \partial_t \varphi - \text{sgn}(U\tilde{\Theta} - \tilde{V}\Theta) \{g(p, u)\tilde{\Theta} - g(\tilde{p}, \tilde{v})\Theta\} \varphi \right) \rho_\epsilon dp d\tilde{p} \\ &+ \int_{Q \times Q} \text{sgn}(U\tilde{\Theta} - \tilde{V}\Theta) \left\{ \left\{ \tilde{\Theta} \mathbf{f}(p, u) - \Theta \mathbf{f}(\tilde{p}, \tilde{v}) \right\} - \left\{ \tilde{\Theta} \mathbf{f}(p, \tilde{k}\Theta + \theta_1) - \Theta \mathbf{f}(\tilde{p}, l\tilde{\Theta} + \tilde{\theta}_1) \right\} \right\} \cdot \nabla \varphi \rho_\epsilon dp d\tilde{p} \\ &+ \int_{Q \times Q} \text{sgn}(U\tilde{\Theta} - \tilde{V}\Theta) \left\{ \tilde{\Theta} \left\{ \nabla \cdot \mathbf{f}(p, \tilde{k}\Theta + \theta_1) + \partial_t(\tilde{k}\Theta + \theta_1) \right\} - \Theta \left\{ \tilde{\nabla} \cdot \mathbf{f}(\tilde{p}, l\tilde{\Theta} + \tilde{\theta}_1) + \partial_t(l\tilde{\Theta} + \tilde{\theta}_1) \right\} \right\} \Lambda_\epsilon dp d\tilde{p}, \\ I_{2,\epsilon} &= \int_{Q \times Q} \text{sgn}(U\tilde{\Theta} - \tilde{V}\Theta) \left\{ \tilde{\Theta} \left\{ \mathbf{f}(p, u) - \mathbf{f}(p, \tilde{k}\Theta + \theta_1) \right\} + \Theta \left\{ \mathbf{f}(\tilde{p}, \tilde{v}) - \mathbf{f}(\tilde{p}, l\tilde{\Theta} + \tilde{\theta}_1) \right\} \right\} \cdot \nabla \rho_\epsilon \varphi dp d\tilde{p}, \\ I_{3,\epsilon} &= \text{ess} \lim_{\tau \rightarrow 0^-} \left(\int_Q \int_\Sigma \tilde{\Theta} \mathbf{F}(u(\sigma + \tau \nu, u_\Gamma)) \cdot \nu \Lambda_\epsilon(\sigma, \tilde{p}) d\sigma d\tilde{p} + \int_Q \int_\Sigma \Theta \tilde{\mathbf{F}}(v(\tilde{\sigma} + \tau \tilde{\nu}), \tilde{v}_\Gamma) \cdot \tilde{\nu} \Lambda_\epsilon(p, \tilde{\sigma}) d\tilde{\sigma} dp \right), \\ I_{4,\epsilon} &= \int_Q \int_\Sigma \tilde{\Theta} \mathbf{F}(\tilde{k}\Theta + \theta_1, u_\Gamma) \cdot \nu \Lambda_\epsilon(\sigma, \tilde{p}) d\sigma d\tilde{p} + \int_Q \int_\Sigma \Theta \tilde{\mathbf{F}}(l\tilde{\Theta} + \tilde{\theta}_1, \tilde{v}_\Gamma) \cdot \tilde{\nu} \Lambda_\epsilon(p, \tilde{\sigma}) d\tilde{\sigma} dp. \end{aligned}$$

We seek to calculate the limit of each integral $I_{i,\epsilon}$ when ϵ goes to 0^+ . For the first two ones the argumentation focuses on the notion of Lebesgue points for an integrable function and uses the Lipschitzian properties of the non-linearities in the same spirit as Kruskov in [12]. For example, let us describe the study of the second term in the first line of $I_{1,\epsilon}$ with typical arguments also used for the second and third lines. Clearly, owing to the

regularities of functions g and θ_i and to the Lipschitz property for g , the second term of the first line in $I_{1,\epsilon}$ has the same limit, when ϵ goes to 0^+ , as the integral:

$$J_\epsilon = \int_{Q \times Q} \operatorname{sgn}(U\tilde{\Theta} - \tilde{V}\Theta)\tilde{\Theta}\{g(p, u) - g(p, v)\}\Lambda_\epsilon dp d\tilde{p}.$$

Accordingly, by arguing whether or not $U\tilde{\Theta}$ is an element of the interval $\mathcal{I}(\Theta\tilde{V}; \tilde{\Theta}V)$, one proves the existence of a constant C , independent from ϵ such that:

$$|J_\epsilon - K_\epsilon| \leq C \int_{Q \times Q} \left\{ |v - \tilde{v}| + |\Theta - \tilde{\Theta}| \right\} \Lambda_\epsilon dp d\tilde{p}$$

where,

$$K_\epsilon = \int_{Q \times Q} \operatorname{sgn}(u - v)\tilde{\Theta}\{g(p, u) - g(p, v)\}\Lambda_\epsilon dp d\tilde{p}.$$

So J_ϵ has the same limits as K_ϵ , the latter being obtained thanks to the continuity of functions θ_i and φ .

This reasoning is still valid for the second line of $I_{1,\epsilon}$; but for the third line we also take advantage of the fact that since Θ is a non-negative function of $W^{1,+\infty}(Q)$, then for a.e. p_0 of Q such that $\Theta(p_0) = 0$, one has $\partial_t \Theta(p_0) = 0$ and $\nabla \Theta(p_0) = 0$. That way, a.e. on Q

$$\operatorname{sgn}(u - v)\partial_t \tilde{\Theta} = \operatorname{sgn}(U\tilde{\Theta} - \tilde{V}\tilde{\Theta})\partial_t \tilde{\Theta},$$

this equality staying true when $\partial_t \tilde{\Theta}$ is turned into $\nabla \tilde{\Theta}$. Consequently, it comes:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} I_{1,\epsilon} = \int_Q \left[|u - v| \partial_t (\Theta \varphi) + \Theta \mathbf{F}(u, v) \cdot \nabla \varphi - (\Theta \{g(p, u) \perp g(p, v)\}) \right. \\ \left. + \Theta \sum_{i=1}^p \{ \partial_{x_i} f_i(p, u) \perp \partial_{x_i} f_i(p, v) \} + \{ \omega(u) \perp \omega(v) \} \varphi \right] dp, \quad (12) \end{aligned}$$

with $h(a) \perp h(b) = \operatorname{sgn}(a - b)(h(a) - h(b))$ and $\omega(u) = \partial_u \mathbf{f}(p, u)(U\nabla \Theta + \Theta \nabla \theta_1)$.

To study $I_{2,\epsilon}$ we use the regularity of \mathbf{f} to compensate for the term $\nabla \rho_\epsilon$ since we must keep in mind that for any integrable function h on Q , in all the Lebesgue points p of h :

$$\lim_{\epsilon \rightarrow 0^+} \int_{Q \times Q} |h(p) - h(\tilde{p})| |p - \tilde{p}| |\nabla \rho_\epsilon(p - \tilde{p})| dp d\tilde{p} = 0, \quad (13)$$

this property being valid when the spatial derivatives are turned into the time one. This leads us to decompose $I_{2,\epsilon}$ into the next two integrals:

$$\begin{aligned} I_{21,\epsilon} &= \int_{Q \times Q} \operatorname{sgn}(U\tilde{\Theta} - \tilde{V}\Theta)[\tilde{\Theta} - \Theta] \left[\mathbf{f}(\tilde{p}, l\tilde{\Theta} + \tilde{\theta}_1) - \mathbf{f}(p, \tilde{k}\Theta + \theta_1) \right] \cdot \nabla \rho_\epsilon \varphi dp d\tilde{p}, \\ I_{22,\epsilon} &= \int_{Q \times Q} \operatorname{sgn}(U\tilde{\Theta} - \tilde{V}\Theta) \left\{ \tilde{\Theta} \left[\mathbf{f}(p, u) - \mathbf{f}(\tilde{p}, l\tilde{\Theta} + \tilde{\theta}_1) \right] + \Theta \left[\mathbf{f}(\tilde{p}, \tilde{v}) - \mathbf{f}(p, \tilde{k}\Theta + \theta_1) \right] \right\} \cdot \nabla \rho_\epsilon \varphi dp d\tilde{p}. \end{aligned}$$

Given that Θ is Lipschitzian on Q , (13) ensures that $I_{21,\epsilon}$ has the same limit as

$$\int_Q \operatorname{sgn}(u - v)[\tilde{\Theta} - \Theta] [\mathbf{f}(p, u) - \mathbf{f}(p, v)] \cdot \nabla \rho_\epsilon \varphi dp d\tilde{p},$$

which can be integrated by parts by considering the fact that $\nabla \rho_\epsilon(p - \tilde{p}) = -\nabla_{\tilde{x}} [\rho_\epsilon(p - \tilde{p})]$. Then, passing to the limit with ϵ provides:

$$\lim_{\epsilon \rightarrow 0^+} I_{21,\epsilon} = \int_Q \nabla \Theta \cdot \mathbf{F}(u, v) \varphi dp. \tag{14}$$

For $I_{22,\epsilon}$, let us grossly describe the development of the first term, the reasoning being the same for the second one. Thanks to the regularity of f_i one has a.e. on $Q \times Q$ and for any i in $\{1, \dots, p\}$,

$$f_i(p, u) - f_i(\tilde{p}, l\tilde{\Theta} + \tilde{\theta}_1) = \partial_u f_i(p, u)(u - l\tilde{\Theta} - \tilde{\theta}_1) + \frac{\partial_{uu}^2 f_i(p, \zeta)}{2}(u - l\tilde{\Theta} - \tilde{\theta}_1)^2 + \int_{\tilde{p}}^p \partial_1 f_i(\tau, l\tilde{\Theta} - \tilde{\theta}_1) d\tau,$$

where ζ belongs to $\mathcal{I}(u, l\tilde{\Theta} + \tilde{\theta}_1)$.

By noting that $u = l\Theta + \theta_1$ a.e. on Q we may refer to (13) and to the Lipschitz property of functions θ_i to pass to the limit in the integrals relative to the three previous terms. Therefore, the first line of $I_{22,\epsilon}$ has the same limit as:

$$\int_{Q \times Q} \operatorname{sgn}(u - v) \tilde{\Theta} \left[\partial_u f_i(p, u)(l(\Theta - \tilde{\Theta}) + \theta_1 - \tilde{\theta}_1) + \int_{\tilde{p}}^p \partial_1 f_i(\tau, u) d\tau \right] \partial_{x_i} \rho_\epsilon \varphi dp d\tilde{p}.$$

An identical procedure for the second term in $I_{22,\epsilon}$ proves that the latter has the same limit as:

$$\int_{Q \times Q} \operatorname{sgn}(u - v) \tilde{\Theta} \left[\partial_u f_i(p, v)(k(\tilde{\Theta} - \Theta) + \tilde{\theta}_1 - \theta_1) + \int_p^{\tilde{p}} \partial_1 f_i(\tau, v) d\tau \right] \partial_{x_i} \rho_\epsilon \varphi dp d\tilde{p}.$$

Consequently, by adding the two previous integrals, by integrating by parts with respect to \tilde{x}_i as for the study of $I_{21,\epsilon}$ and by taking the ϵ -limit, it comes:

$$\lim_{\epsilon \rightarrow 0^+} I_{22,\epsilon} = \int_Q \Theta \left[\sum_{i=1}^p \{ \partial_{x_i} f_i(p, u) \perp \partial_{x_i} f_i(p, v) \} + \{ \omega(u) \perp \omega(v) \} \right] \varphi dx dt. \tag{15}$$

So, to complete the proof of the Kruskov relation (11) we focus on the boundary terms $I_{3,\epsilon}$ and $I_{4,\epsilon}$. In fact, on account of the smoothness of Θ and \mathbf{f} , $I_{3,\epsilon}$ has the same limit as:

$$ess \lim_{\tau \rightarrow 0^-} \left(\int_Q \int_\Sigma \Theta \mathbf{F}(u(\sigma + \tau \nu), u_\Gamma) \cdot \nu \Lambda_\epsilon(\sigma, \tilde{p}) d\sigma d\tilde{p} + \int_Q \int_\Sigma \Theta \mathbf{F}(v(\tilde{\sigma} + \tau \tilde{\nu}), \tilde{v}_\Gamma) \cdot \tilde{\nu} \Lambda_\epsilon(p, \tilde{\sigma}) d\tilde{\sigma} d\tilde{p} \right),$$

and since Θ is a non-negative function of $W^{1,+\infty}(Q)$, relation (10) ensures that

$$ess \lim_{\tau \rightarrow 0^-} \int_\Sigma \mathbf{F}(u(\sigma + \tau \nu), \varphi) \Theta d\sigma \text{ exists,}$$

for any δ of $L^1_+(\Sigma)$. That is why, by referring to Otto's proof, there exist two functions γ_{u,u_Γ} and γ_{v,v_Γ} in $L^\infty(\Sigma)$ such that:

$$ess \lim_{\tau \rightarrow 0^-} \int_{\Sigma} \mathbf{F}(u(\sigma + \tau \nu), u_\Gamma) \Theta \delta d\sigma = \int_{\Sigma} \gamma_{u,u_\Gamma} \delta d\sigma \text{ and } ess \lim_{\tau \rightarrow 0^-} \int_{\Sigma} \mathbf{F}(v(\sigma + \tau \nu), v_\Gamma) \Theta \delta d\sigma = \int_{\Sigma} \gamma_{v,v_\Gamma} \delta d\sigma.$$

Moreover,

$$\lim_{\epsilon \rightarrow 0^+} \int_Q ess \lim_{\tau \rightarrow 0^+} \int_{\Sigma} \mathbf{F}(u(\sigma + \tau \nu), u_\Gamma) \Theta \Lambda_\epsilon(\sigma, \tilde{p}) d\sigma d\tilde{p} = \frac{1}{2} \int_{\Sigma} \gamma_{u,u_\Gamma} \Theta \varphi d\sigma$$

and as this relation holds with \tilde{v} , \tilde{v}_Γ and γ_{v,v_Γ} it comes:

$$\lim_{\epsilon \rightarrow 0^+} I_{3,\epsilon} = \frac{1}{2} \int_{\Sigma} \Theta \{ \gamma_{u,u_\Gamma} + \gamma_{v,v_\Gamma} \} \varphi d\sigma. \tag{16}$$

It is the same for $I_{4,\epsilon}$, since the regularities of \mathbf{f} and Θ ensure, for example, that the first term in $I_{4,\epsilon}$ has the same limit as the integrals

$$\int_Q \int_{\Sigma} \Theta \mathbf{F}(\tilde{v}, u_\Gamma) \cdot \nu \Lambda_\epsilon(\sigma, \tilde{p}) d\sigma d\tilde{p} + \int_Q \int_{\Sigma} \Theta \mathbf{F}(u, \tilde{v}_\Gamma) \cdot \tilde{\nu} \Lambda_\epsilon(\tilde{\sigma}, p) d\tilde{\sigma} dp.$$

That way, there exist two functions γ_{v,u_Γ} and γ_{u,v_Γ} in $L^\infty(\Sigma)$ such that:

$$\lim_{\epsilon \rightarrow 0^+} I_{4,\epsilon} = \frac{1}{2} \int_{\Sigma} \Theta \{ \gamma_{v,u_\Gamma} + \gamma_{u,v_\Gamma} \} \varphi d\sigma. \tag{17}$$

Finally, adding up limits (12),(14),(15),(16) and (17) results in:

$$- \int_Q (|u - v| \partial_t(\Theta \varphi) + \mathbf{F}(u, v) \cdot \nabla(\Theta \varphi) - \{g(p, u) \perp g(p, v)\} \theta \varphi) dp \leq \frac{1}{2} \int_{\Sigma} (\gamma_{u,u_\Gamma} - \gamma_{u,v_\Gamma} + \gamma_{v,v_\Gamma} - \gamma_{v,u_\Gamma}) \Theta \varphi d\sigma, \tag{18}$$

for any function φ of $\mathcal{D}_+(]0, T[\times \mathbb{R}^p)$.

By density this inequality is still fulfilled for all functions φ such that $\Theta \varphi$ is a positive element of $W^{1,1}(Q)$ with $\Theta(0, \cdot) \varphi(0, \cdot) = \Theta(T, \cdot) \varphi(T, \cdot) = 0$. So, one may consider the following test-function:

$$\varphi(t, x) = \varphi_\mu(t, x) = \begin{cases} \frac{\xi(t)}{\Theta(t, x)} & \text{if } \Theta(t, x) > \mu, \\ \frac{\xi(t)}{\mu} & \text{else,} \end{cases}$$

ξ being any element of $W^{1,1}_+(0, T)$ and μ a strictly positive real ($\mu < \|\Theta\|_\infty$). We thus split the integration field into $[0 < \Theta \leq \mu]$ and $[\mu < \Theta]$. So as to calculate the limit when μ goes to 0^+ for the left-hand member of (18) on $[0 < \Theta \leq \mu]$ we note that:

$$\frac{1}{\mu} |u - v| \leq 1 \text{ a.e. on } [\Theta \leq \mu], \text{ and } \lim_{\mu \rightarrow 0^+} \int_{[0 < \Theta \leq \mu]} \partial_t \Theta dp = 0, \quad \lim_{\mu \rightarrow 0^+} \int_{[0 < \Theta \leq \mu]} \nabla \Theta dp = 0.$$

That way,

$$\lim_{\mu \rightarrow 0^+} \frac{1}{\mu} \int_{[0 < \Theta \leq \mu]} (|u - v| \partial_t(\Theta \zeta) + \mathbf{F}(u, v) \cdot \nabla \Theta \zeta - \{g(p, u) \perp g(p, v)\} \Theta \zeta) dp = 0.$$

For the right-hand term of (18), we observe that for any real a, b, c, d and for any i of $\{1, \dots, p\}$,

$$|F_i(a, c) - F_i(a, d) + F_i(b, c) - F_i(b, d)| \leq 2A_i |c - d|,$$

where A_i is the Lipschitz constant of f_i with respect to its third variable, uniformly in (t, x) . Thus, when $a = u(\sigma + \tau \nu)$, $b = v(\sigma + \tau \nu)$, $c = u_\Gamma$ and $d = v_\Gamma$, the limit, when τ goes to 0^- outside of a set of measure zero, provides:

$$\int_\Sigma (\gamma_{u, u_\Gamma} - \gamma_{u, v_\Gamma} + \gamma_{v, v_\Gamma} - \gamma_{v, u_\Gamma}) \Theta \varphi d\sigma \leq 2A \int_\Sigma |u_\Gamma - v_\Gamma| \Theta \varphi d\sigma.$$

This inequality and the fact that $|u_\Gamma - v_\Gamma| \leq \mu$ a.e on Σ prove that the contribution on $[0 < \Theta \leq \mu]$ of the right-hand side term of (18) goes to zero with μ .

Eventually, since the Lebesgue-dominated convergence theorem gives the limit on the subdomain $[\mu < \Theta]$ when μ goes to 0^+ , one has:

$$- \int_Q |u - v| \partial_t \zeta dp \leq A \int_\Sigma |u_\Gamma - v_\Gamma| \zeta d\sigma + \int_Q \{g(p, u) \perp g(p, v)\} \zeta dp$$

for any function ζ of $W_+^{1,1}(0, T)$. Now the conclusion is classical: it uses the Lipschitz condition for g and a piecewise linear approximation of $\mathbb{I}_{]0, t[}$, t being given outside a of a set of measure zero. Thanks to the definition of initial condition (7) for u and v and to the Gronwall lemma we complete the proof of Theorem 2.1. \square

3. EXISTENCE VIA AN ENTROPY PROCESS SOLUTION

The method of penalization is applied with a view to obtaining an existence result. Hence, for each value of the strictly positive parameter η – intended to go to zero – the following regularized problem is introduced: find a measurable function u_η such that

$$\mathbb{H}(t, x, u_\eta) + \frac{1}{\eta} \beta_\eta(t, x, u_\eta) = 0 \text{ on } Q, \tag{19}$$

$$u_\eta = u_{\Gamma, \eta} \text{ on (a part of) } \Sigma, \quad u_\eta(0, \cdot) = u_{0, \eta} \text{ on } \Omega, \tag{20}$$

where $\beta_\eta(t, x, u) = -(u - \theta_{1, \eta}(t, x))^- + (u - \theta_{2, \eta}(t, x))^+$ and $u_{\Gamma, \eta}$ and $u_{0, \eta}$ are the standard regularizations of u_Γ and u_0 by means of mollifier sequences indexed on η . Moreover, $\theta_{i, \eta}$, $i = 1, 2$ is obtained through changing θ_i in its spatial regularization.

3.1. Study of the penalized problem

We need estimates of u_η that are independent from η . But as the penalizing operator β_η depends on time and space variables via obstacle functions θ_i one may even estimate the uniform bound of u_η . Indeed, the next statement holds:

Property 3.1. For any strictly positive real η :

$$\|u_\eta(t, \cdot)\|_{L^\infty(\Omega)} \leq M(t) \text{ for any } t \text{ of } [0, T], \tag{21}$$

$$\frac{1}{\eta} \|\beta_\eta(t, x, u_\eta)\|_{L^1(Q)} \leq C, \tag{22}$$

where M is given by (4) and C is a constant independent from any parameter.

Proof. According to the works of Bardos *et al.* [2], η being fixed, the penalized problem (19) – (20) has a unique weak entropy solution u_η in $BV(Q) \cap L^\infty(Q)$. In addition, when ϵ goes to 0^+ , u_η is the $L^q(Q)$ and $\mathcal{C}([0, T]; L^1(\Omega))$ -limit, $1 \leq q < +\infty$, of the sequence $(u_{\epsilon, \eta})_{\epsilon > 0}$ defined for any ϵ by: find $u_{\epsilon, \eta}$ in $H^2(Q) \cap L^\infty(Q)$, such that:

$$\begin{aligned} \mathbb{H}(t, x, u_{\epsilon, \eta}) + \frac{1}{\eta} \beta_\eta(t, x, u_{\epsilon, \eta}) &= \epsilon \Delta u_{\epsilon, \eta} \text{ a.e. on } Q, \\ u_{\epsilon, \eta} &= u_{\Gamma, \eta} \text{ a.e. on } \Sigma, \quad u_{\epsilon, \eta}(0, \cdot) = u_{0, \eta} \text{ a.e. on } \Omega. \end{aligned} \tag{23}$$

In fact, (21) is a consequence of the maximum principle for $u_{\epsilon, \eta}$ and of the monotony of the penalizing operator $w \rightarrow \beta_\eta(t, x, w)$. Nevertheless, we develop the demonstration with typical arguments used in the sequel. So, λ being any strictly positive real, we note sgn_λ the approximation of the function “sgn” defined, for any positive real x , through:

$$\text{sgn}_\lambda(x) = \min\left(\frac{x}{\lambda}, 1\right) \text{ and } \text{sgn}_\lambda(-x) = -\text{sgn}_\lambda(x).$$

Moreover, we denote

$$I_\lambda(x) = \int_0^x \text{sgn}_\lambda(\tau) \, d\tau.$$

Then, t being given in $]0, T]$, we consider the $L^2(]0, t[\times \Omega)$ -scalar product between the diffusion equation (23) and the test-function $\text{sgn}_\lambda(u_{\epsilon, \eta} - M(t))^+$. By taking into account the definition of $M(t)$ and the fact that $\beta_\eta(t, x, u_{\epsilon, \eta}) \text{sgn}_\lambda(u_{\epsilon, \eta} - M(t))^+ \geq \beta_\eta(t, x, M(t)) \text{sgn}_\lambda(u_{\epsilon, \eta} - M(t))^+ \geq 0$ a.e. on Q , some integrations by parts in time and space give the next inequality:

$$\begin{aligned} &\int_\Omega I_\lambda(u_{\epsilon, \eta}(t, x) - M(t))^+ \, dx + \int_0^t \int_\Omega \mathbb{H}(t, x, M) \text{sgn}_\lambda(u_{\epsilon, \eta} - M)^+ \, dx \, dt \\ &\leq \int_0^t \int_\Omega \left\{ \mathbf{f}(t, x, u_{\epsilon, \eta}) - \mathbf{f}(t, x, M) \right\} \cdot \nabla \text{sgn}_\lambda(u_{\epsilon, \eta} - M)^+ \, dx \, dt + M'_g \int_0^t \int_\Omega (u_{\epsilon, \eta} - M)^+ \, dx \, dt. \end{aligned}$$

The Sacks lemma shows that the first term of the right-hand member goes to zero when λ tends to 0^+ . Thus, as a result of the Gronwall lemma, one may conclude the proof of Property 3.1 ensuring that $\mathbb{H}(t, x, M)$ is non-negative a.e. on Q . For this purpose, we remark that:

$$M'(t) = C_1 M(t) + \max_{[0, T] \times \Omega} |\mathbb{H}(t, x, 0)|$$

and owing to the Lipschitz properties for g and \mathbf{f} ,

$$\mathbb{H}(t, x, M(t)) \geq -C_1 M(t) + \mathbb{H}(t, x, 0) + M'(t).$$

Accordingly, the λ -limit provides the majoration for $u_{\epsilon,\eta}$

$$u_{\epsilon,\eta}(t, x) \leq M(t) \text{ a.e. in } Q,$$

and the minoration is obtained through the same L^1 -cut off method. Estimate (21) follows by passing to the limit with respect to ϵ .

We now seek an estimate of the penalized term in (23). With this view we first consider firstly the $L^2(Q)$ -scalar product between (23) and the test-function $-\text{sgn}_\lambda(u_{\epsilon,\eta} - \theta_{1,\eta})^-$. The next inequality comes:

$$\begin{aligned} \frac{1}{\eta} \int_Q (u_{\epsilon,\eta} - \theta_{1,\eta})^- \text{sgn}_\lambda(u_{\epsilon,\eta} - \theta_{1,\eta})^- dxdt + \int_Q \left\{ \mathbf{f}(t, x, u_{\epsilon,\eta}) - \mathbf{f}(t, x, \theta_{1,\eta}) \right\} \cdot \nabla \text{sgn}_\lambda(u_{\epsilon,\eta} - \theta_{1,\eta})^- dxdt \\ \leq \int_Q \left\{ |\mathbb{H}(t, x, \theta_{1,\eta})| + \epsilon |\Delta \theta_{1,\eta}| + M'_g(u_{\epsilon,\eta} - \theta_{1,\eta})^- \right\} dxdt \end{aligned}$$

As previously, the Sacks lemma shows that the second term in the right-hand member goes to zero with λ and the left-hand one is bounded by a constant C independent from any parameter (due to the regularities of $\theta_{1,\eta}$ some majorations of the time and space derivatives of $\theta_{1,\eta}$, independent from η , hold). That way, the λ -limit and then, the ϵ -limit provide the existence of a constant C such that:

$$\frac{1}{\eta} \int_Q (u_\eta - \theta_{1,\eta})^- dxdt \leq C$$

Secondly, the $L^2(Q)$ -scalar product between (23) and the test-function $\text{sgn}_\lambda(u_{\epsilon,\eta} - \theta_{2,\eta})^+$ allows us to conclude the proof of Property 3.1. □

To study the behaviour of the sequence $(u_\eta)_{\eta>0}$ that is to pass to the η -limit in the nonlinear terms of \mathbb{H} we must refer to properties of bounded sequences in L^∞ .

3.2. Entropy process solution

Let \mathcal{O} be an open bounded subset of \mathbb{R}^q ($q \geq 1$) and let $(u_n)_{n>0}$ be a bounded sequence in $L^\infty(\mathcal{O})$. Clearly, for any continuous function h , there exists \bar{h} in $L^\infty(Q)$ such that, for a subsequence,

$$h(u_n) \rightharpoonup \bar{h} \text{ weakly in } L^\infty(\mathcal{O}).$$

Since the works of Tartar [17] and Ball [1], one has been able to describe the composite limit \bar{h} . Actually, thanks to the properties for weak-* topology on the space of Radon measures the next compacity result holds:

Property 3.2. *Let $(u_n)_{n>0}$ be a sequence of measurable functions on \mathcal{O} such that:*

$$\exists M > 0, \forall n > 0, \|u_n\|_{L^\infty(\mathcal{O})} \leq M.$$

Then, there exists a subsequence $(u_{\varphi(n)})_{n>0}$ and $(\nu_w)_{w \in \mathcal{O}}$ a family of probability measures on \mathbb{R} , with support in $[-M, M]$, such that, for all continuous bounded functions h on $\mathcal{O} \times]-M, M[$, the sequence $(h(\cdot, u_{\varphi(n)}))_{n>0}$ converges in $L^\infty(\mathcal{O})$ weak- towards the element:*

$$w \rightarrow \int_{\mathbb{R}} h(x, \lambda) d\nu_w(\lambda).$$

The mapping $\nu : w \rightarrow \nu_w$ is called "a Young measure associated with the sequence $(u_n)_{n>0}$ ".

Such a result has found its first application in [17] and [8] for the approximation through the artificial viscosity method of the Cauchy problem in \mathbb{R}^p for a scalar conservation law, as one can establish a uniform L^∞ -control of approximate solutions. It has also been applied to the numerical analysis of transport equations [6, 7, 9, 11, 19] since “Finite-Volume” schemes only give an L^∞ -estimate uniformly with respect to the mesh length of the numerical solution. Lastly, it has been extended to the Dirichlet problem for scalar conservation laws by introducing the notion of Young measure-trace [3, 15, 16, 18].

As a matter of fact, the properties for the generalized inverse of the distribution function linked to a probability measure permits to turn the integration with respect to the measure $d\nu_w$ into an integration with respect to the Lebesgue measure on $]0, 1[$, which can be permuted with the integration on \mathcal{O} . Indeed, owing to Eymard *et al.* [9], the next statement holds:

Property 3.3. *Let $w \rightarrow \nu_w$ be a Young measure with support in $[-M, M]$. There exists a function π in $L^\infty(]0, 1[\times\mathcal{O})$ such that for all continuous bounded functions h on $\mathcal{O}\times]-M, M[$:*

$$\int_{\mathbb{R}\times\mathcal{O}} h(x, \lambda)d\nu_w(\lambda)dx = \int_{]0, 1[\times\mathcal{O}} h\left(x, \pi(\alpha, w)\right)d\alpha dx \text{ for a.e. } w \text{ in } \mathcal{O}.$$

Here, we adapt this concept when the approximating sequence $(u_\eta)_{\eta>0}$ is the sequence of weak entropy solutions to penalized problems. As a result, *a priori* estimates of Property 3.1 allow us to announce the main result of this section:

Theorem 3.1. *There exists an entropy process solution to obstacle problem (1), (2), (3) in the sense that there exists a function π of $L^\infty(]0, 1[\times Q)$ such that:*

i) for a.e. (α, t, x) in $]0, 1[\times Q$

$$\theta_1(t, x) \leq \pi(\alpha, t, x) \leq \theta_2(t, x), \tag{24}$$

ii) for all functions ξ of $\mathcal{D}_+(]0, T[\times\Omega)$, for any real k of $[0, 1]$,

$$\int_{]0, 1[\times Q} \mathcal{L}(\pi, K, \xi)d\alpha dx dt \geq 0, \tag{25}$$

with $K = k(\theta_2 - \theta_1) + \theta_1$.

iii)

$$ess \lim_{t \rightarrow 0^+} \int_{]0, 1[\times\Omega} |\pi(\alpha, t, x) - u_0(x)|d\alpha dx = 0, \tag{26}$$

iv) for any real k of $[0, 1]$ and for all functions ζ of $L^1_+(\Sigma)$,

$$ess \lim_{\tau \rightarrow 0^-} \int_{]0, 1[\times\Sigma} \mathcal{F}(\pi(\alpha, \sigma + \tau\nu), u_\Gamma, K) \cdot \nu \zeta d\sigma \geq 0. \tag{27}$$

Proof. Further to (21) there exists a subsequence extracted from $(u_\eta)_{\eta>0}$ – still labeled $(u_\eta)_{\eta>0}$ – and a measurable and bounded function π on $]0, 1[\times Q$ such that for any continuous bounded function ψ on $Q\times]-M, M[$,

$$\lim_{\eta \rightarrow 0^+} \int_Q \psi(t, x, u_\eta)\varphi dx dt = \int_{]0, 1[\times Q} \psi(t, x, \pi(\alpha, t, x))\varphi d\alpha dx dt,$$

for any function φ of $L^1(Q)$.

So, (24) follows from (22) and in order to prove (25), (26), (27) we come back to the viscous equation (23). We introduce a boundary entropy-entropy flux pair in the sense of Otto [14], that means a pair of $(H, \mathbf{Q}) \in \mathcal{C}^2(\mathbb{R}^2)$ -class functions such that for any w of \mathbb{R} , $(H(\cdot, w), \mathbf{Q}(\cdot, w))$ is an entropy-entropy flux pair for \mathbb{H} :

$$z \rightarrow H(z, w) \text{ is convex and } \mathbf{Q}(z, w) = \int_w^z \partial_1 H(\tau, w) \partial_u \mathbf{f}(t, x, \tau) d\tau.$$

In addition, (H, \mathbf{Q}) satisfies

$$H(w, w) = 0, \mathbf{Q}(w, w) = 0, \partial_1 H(w, w) = 0,$$

where $\partial_1 H$ denotes the partial derivative of H with respect to its first variable.

To simplify the writing we drop the indexes ϵ and η temporarily. Thus, due to the convexity of $H(\cdot, w)$ for any w of \mathbb{R} and since $\theta_1 \leq K \leq \theta_2$ on Q ,

$$\partial_1 H(u, K) \Delta u \leq \nabla \cdot [\partial_1 H(u, K) \nabla u] - \partial_{21}^2 H(u, K) \nabla u \cdot \nabla K \text{ and } 0 \leq \partial_1 H(u, K) \beta(t, x, u) \text{ a.e. on } Q.$$

and if we transform the transport term into

$$\partial_1 H(u, K) \nabla \cdot \mathbf{f}(t, x, u) = \nabla \cdot \mathbf{Q}(u, K) + \int_K^u \left\{ \partial_u \mathbf{f}(t, x, \tau) \partial_{12}^2 H(\tau, K) \cdot \nabla K - \nabla \cdot \mathbf{f}(t, x, \tau) \partial_{11}^2 H(\tau, K) \right\} d\tau.$$

The multiplication of (23) by $\partial_1 H(u, K)$ ensures that a.e. on Q ,

$$\partial_t H(u, K) + \nabla \cdot \mathbf{Q}(u, K) + G(u, K) \leq \epsilon \left(\nabla \cdot [\partial_1 H(u, K) \nabla u] - \partial_{21}^2 H(u, K) \nabla u \cdot \nabla K \right), \tag{28}$$

where

$$G(u, K) = \int_K^u \left\{ \partial_u \mathbf{f}(t, x, \tau) \partial_{12}^2 H(\tau, K) \cdot \nabla K - \nabla \cdot \mathbf{f}(t, x, \tau) \partial_{11}^2 H(\tau, K) \right\} d\tau - \partial_2 H(u, K) \partial_t K + g(t, x, u) \partial_1 H(u, K).$$

Now let us consider the $L^2(Q)$ -scalar product between (28) and a test-function ξ of $\mathcal{D}_+(\cdot - \infty, T[\times \Omega])$. After integrating by parts one may have ϵ tend to 0^+ by referring to the convergence properties of the sequence $(u_{\epsilon, \eta})_{\epsilon > 0}$. It provides a regularized weak entropy formulation for the solution u_η of penalized problem (19)-(20), in which we can pass to the limit with η as all the non-linearities are continuous with respect to η -depending functions. Hence, it comes:

$$- \int_{]0, 1[\times Q} (H(\pi, K) \partial_t \xi + \mathbf{Q}(u, K) \cdot \nabla \xi - G(\pi, K) \xi) \, d\alpha \, dx \, dt \leq \int_{\Omega} H(u_0, K(0, x)) \xi(0, x) \, dx.$$

In the particular situation when, for any l of \mathbb{N} ,

$$H(z, w) = H_l(z, w) = \left((z - w)^2 + \left(\frac{1}{l}\right)^2 \right)^{1/2} - \frac{1}{l},$$

the limit when l goes to $+\infty$ provides:

$$- \int_{]0,1[\times Q} \mathcal{L}(\pi, K, \zeta) d\alpha dx dt \leq \int_{\Omega} |u_0 - K(0, x)| \xi(0, x) dx.$$

Thus, one gets (25). Moreover, Otto’s reasoning in [14] developed for an homogeneous scalar conservation law can be adapted to the present context where \mathbb{H} presents a reaction term and K is a time and space depending function. Hence, we may be sure that if the previous inequality holds then for any function φ of $L^\infty(\Omega)$, $0 \leq \varphi \leq 1$ a.e. on Ω ,

$$\begin{aligned} \text{ess lim}_{t \rightarrow 0^+} \sup \int_{]0,1[\times \Omega} |\pi(\alpha, t, x) - \varphi(x)(\theta_2(0, x) - \theta_1(0, x)) - \theta_1(0, x)| d\alpha dx \\ \leq \int_{\Omega} |u_0(x) - \varphi(x)(\theta_2(0, x) - \theta_1(0, x)) - \theta_1(0, x)| dx. \end{aligned}$$

So initial condition (26) is obtained by choosing:

$$\varphi(x) = \begin{cases} \frac{u_0(x)}{\theta_2(0, x) - \theta_1(0, x)} - \theta_1(0, x) & \text{if } \theta_1(0, x) < \theta_2(0, x), \\ 0 & \text{else.} \end{cases}$$

The demonstration of Dirichlet boundary condition (27) for π also refers to the Otto’s basic proof which, for any $\epsilon > 0$, introduces the function ζ defined by

$$\zeta(x) = 1 - \exp\left(-\frac{A + \epsilon L}{\epsilon} s(x)\right) \tag{29}$$

where A is the Lipschitz constant of \mathbf{f} with respect to its third variable, uniformly with respect to (t, x) and for any strictly positive parameter δ ,

$$s(x) = \begin{cases} \min(\text{dist}(x, \Gamma), \delta) & \text{for } x \in \Omega \\ -\min(\text{dist}(x, \Gamma), \delta) & \text{for } x \in \mathbb{R}^p \setminus \Omega, \end{cases}$$

with $L = \sup_{0 < s(x) < \delta} |\Delta s(x)|$. That way, for any φ of $W_+^{1,1}(\mathbb{R}^p)$,

$$A \int_{\Omega} |\nabla \zeta| \varphi dx \leq \epsilon \int_{\Omega} \nabla \zeta \cdot \nabla \varphi dx + (A + L\epsilon) \int_{\Gamma} \varphi d\lambda.$$

Consequently, we take the $L^2(Q)$ -scalar product between (28) and the test-function $\zeta \xi$, in which ξ is any element of $\mathcal{D}_+(]0, T[\times \mathbb{R}^p)$. Then, some integrations by parts give:

$$\begin{aligned} - \int_Q \left(H(u, K) \zeta \partial_t \xi + \mathbf{Q}(u, K) \cdot \nabla \xi \zeta - G(u, K) \zeta \xi \right) dx dt \\ \leq \int_Q \mathbf{Q}(u, K) \cdot \nabla \zeta \xi dx dt - \epsilon \int_Q \left\{ \partial_1 H(u, K) \nabla u \cdot \nabla (\xi \zeta) + \partial_{12}^2 H(u, K) \nabla u \cdot \nabla K \xi \zeta \right\} dx dt. \end{aligned}$$

In the right-hand member of the above inequality we consider the fact that

$$|\mathbf{Q}(u, K)| \leq AH(u, K) \text{ a.e. on } Q,$$

and we use the weak differential inequality for ζ , with $\varphi = H(u, K)\xi$. As ζ goes to 1 in $L^1(\Omega)$ and $\epsilon \nabla \zeta$ goes to 0 in $(L^1(\Omega))^p$ one may successively pass to the limit when ϵ and η tend to 0^+ . For any boundary entropy-entropy flux pair, it comes:

$$- \int_{]0,1[\times Q} \left\{ H(\pi, K) \partial_t \xi + \mathbf{Q}(\pi, K) \cdot \nabla \xi - G(\pi, K) \xi \right\} d\alpha dx dt \leq A \int_{\Sigma} H(u_{\Gamma}, K) \xi d\sigma.$$

Thus, when one refers to Otto's works and uses the smoothness of functions \mathbf{f} and θ_i , this inequality implies that for any φ of $L^{\infty}_+(\Sigma)$, $0 \leq \varphi \leq 1$ a.e. on Σ , and δ of $L^1_+(\Gamma)$,

$$\text{ess} \lim_{\tau \rightarrow 0^-} \int_{\Sigma} \mathbf{Q} \left(\pi(\alpha, \sigma + \tau \nu), \varphi(\theta_2 - \theta_1) + \theta_1 \right) \delta d\alpha d\sigma \geq -A \int_{\Sigma} H \left(u_{\Gamma}, \varphi(\theta_2 - \theta_1) + \theta_1 \right) \delta d\sigma.$$

To conclude, let us consider

$$\varphi(\sigma) = \begin{cases} \frac{u_{\Gamma}}{\theta_2 - \theta_1} - \theta_1 & \text{if } \theta_1(\sigma) < \theta_2(\sigma), \\ 0 & \text{else} \end{cases}$$

and for any l of \mathbb{N} and k of \mathbb{R} ,

$$H(z, w) = H_l(z, w) = \left(\left(\text{dist}(z, \mathcal{I}[w, k]) \right)^2 + \left(\frac{1}{l} \right)^2 \right)^{1/2} - \frac{1}{l},$$

that way, $(\mathbf{Q}_l)_{l \in \mathbb{N}}$ converges uniformly as l goes to $+\infty$, to $\mathcal{F}(z, w, k)$. Boundary condition (27) follows, which completes the proof of Theorem 3.1. □

3.3. Consequences

Let π and ω be two entropy process solutions of obstacle problem (1)-(3) in the sense of (24)-(27). Then, as Kruskov relation (11) obviously remains valid for in the context of entropy process, one has for a.e. t of $]0, T[$:

$$\int_0^t \int_{Q \times]0,1[^2} |\pi(\alpha, t, x) - \omega(\beta, t, x)| d\alpha d\beta dx dt = 0.$$

Therefore, as mentioned in [9], the two processes π and ω are equal, for a.e. (t, x) on Q , to a common value $u(t, x)$, which does not depend on α or β in $]0, 1[$. In addition, u is a measurable and bounded function on Q . It can also be understood that for a.e. (t, x) in Q , the probability measures $d\mu_{(t,x)}$ and $d\varpi_{(t,x)}$ associated with π and ω thanks to Property 3.2 have a punctual support and thus are equal to a Dirac mass centered on a point noted $u(t, x)$ (cf. [8]). Since relations (24),(25),(26) and (27) are now respectively written under the form (5),(6),(7) and (8), we are able to announce:

Theorem 3.2. *Obstacle problem (1), (2), (3) has a unique weak entropy solution characterized through the formulation of Definition 1.1.*

To conclude, let us keep in mind that this uniqueness property ensures that u is the limit in $L^r(Q)$, $1 \leq r < +\infty$, and a.e. on Q , of the whole sequence of approximate solutions (see for example [8]).

4. SOME PROPERTIES FOR THE BILATERAL OBSTACLE PROBLEM

We now take advantage of the strong convergence in L^r , $1 \leq r < +\infty$, of the penalized sequence $(u_\eta)_{\eta>0}$ toward the weak entropy solution u of (1),(2),(3) first to provide first some sensitivity properties of u with respect to the associated obstacle functions θ_i . These results are obtained thanks to the monotony of the penalizing operator $w \rightarrow \beta(t, x, w)$, which guarantees that (with the notations introduced in Th. 3.1) :

Lemma 4.1. *Let u and v be two weak entropy solutions to obstacle problem (1),(2),(3) associated respectively with boundary conditions (u_0, u_Γ) and (v_0, v_Γ) and corresponding to obstacle functions (θ_1, θ_2) and (ψ_1, ψ_2) . Let us assume that $\theta_1 \leq \psi_1 \leq \theta_2 \leq \psi_2$ a.e. on Q . Then, for a.e. t of $]0, T[$,*

$$\int_{\Omega} \left(u(t, x) - v(t, x) \right)^+ dx \leq \left(A \int_0^t \int_{\Gamma} (u_\Gamma - v_\Gamma)^+ d\sigma + \int_{\Omega} (u_0 - v_0)^+ dx \right) e^{M'_g t}. \tag{30}$$

Proof. By coming back to the viscous equation (23) for u and v , we obtain a.e. on Q (we have dropped indexes ϵ and η temporarily):

$$\mathbb{H}(t, x, u) - \mathbb{H}(t, x, v) + \frac{1}{\eta} \left(\beta(t, x, u) - \beta^*(t, x, v) \right) = \epsilon \Delta(u - v),$$

where $\beta^*(t, x, v) = -(v - \psi_1)^- + (v - \psi_2)^+$.

Given t in $]0, T[$, we take the $L^2(]0, t[\times \Omega)$ -scalar product with $\text{sgn}_\lambda(u - v)^+ \zeta$, the functions ζ and sgn_λ being respectively introduced in the proofs of Theorem 3.1 and Property 3.1. Thus, owing to the monotony of the penalizing operators $\beta(t, x, \cdot)$ and $\beta^*(t, x, \cdot)$ and to the definition of sgn_λ^+ , we make sure that for any t of $]0, T[$,

$$\left\{ \beta(t, x, u) - \beta^*(t, x, v) \right\} \text{sgn}_\lambda(u - v)^+ \geq 0 \text{ a.e. on }]0, t[\times \Omega.$$

Accordingly, some integrations by parts and the Lipschitz properties for \mathbf{f} and g provide the next inequality:

$$\begin{aligned} \int_{\Omega} I_\lambda(u - v)^+(t, \cdot) \zeta dx &\leq \int_{\Omega} I_\lambda(u_0 - v_0)^+ dx + M'_g \int_{Q_t} (u - v)^+ dx dt + A \int_{Q_t} (u - v)^+ |\nabla \text{sgn}_\lambda(u - v)^+| dx dt \\ &\quad - \epsilon \int_{Q_t} \text{sgn}_\lambda(u - v)^+ \nabla(u - v) \cdot \nabla \zeta dx dt + A \int_{Q_t} (u - v)^+ \text{sgn}_\lambda(u - v)^+ |\nabla \zeta| dx dt. \end{aligned}$$

In the right-hand member, for the second line, we use the differential inequality fulfilled by the function ζ with $\varphi = (u - v)^+ \text{sgn}_\lambda(u - v)^+$. Thus,

$$\begin{aligned} \int_{\Omega} I_\lambda(u - v)^+(t, \cdot) \zeta dx &\leq (A + L\epsilon) \int_{\Sigma_t} (u_\Gamma - v_\Gamma)^+ d\sigma + \int_{\Omega} I_\lambda(u_0 - v_0)^+ dx + M'_g \int_{Q_t} (u - v)^+ dx dt \\ &\quad + A \int_{Q_t} (u - v)^+ |\nabla \text{sgn}_\lambda(u - v)^+| dx dt + \epsilon \int_{Q_t} (u - v)^+ \nabla \zeta \cdot \nabla \text{sgn}_\lambda(u - v)^+ dx dt. \end{aligned}$$

Due to the Sacks lemma, ϵ being fixed, each term of the second line goes to 0 when λ goes to 0^+ . Then, the ϵ -limit and the Gronwall lemma give relation (30). □

The consequences of Lemma 4.1 are diverse. Of course, it supplies the monotonistic dependence on boundary data for the solution u to (1),(2),(3) but it also provides some approximation and stability results for u with respect to the associated obstacle functions. Indeed, the next statement holds:

Property 4.1. Let $(\theta_{i,n})_{n>0}$, $i = 1, 2$, be two sequences of $W^{1,+\infty}(Q)$ -functions such that:

$$\exists M_i > 0, \forall n > 0, \|\theta_{i,n}\|_{L^\infty(Q)} \leq M_i.$$

For any $n > 0$, we denote v_n the weak entropy solution to the following bilateral obstacle problem:

$$\begin{aligned} &\theta_{1,n} \leq v_n \leq \theta_{2,n} \text{ a.e. in } Q, \\ &(v_n - \theta_{1,n})\mathbb{H}(t, x, v_n) \leq 0, (v_n - \theta_{2,n})\mathbb{H}(t, x, v_n) \leq 0, (v_n - \theta_{1,n})(v_n - \theta_{2,n})\mathbb{H}(t, x, v_n) = 0 \text{ in } Q, \\ &v_n = u_\Gamma \text{ on (a part of) } \Sigma, v_n(0, \cdot) = u_0 \text{ in } \Omega. \end{aligned}$$

That way, if $(\theta_{i,n})_{n>0}$ converges toward θ_i in $W^{1,1}(Q)$ as n goes to $+\infty$, then $(v_n)_{n>0}$ strongly converges to u in $L^r(Q)$, $1 \leq r < +\infty$. In addition, if the sequences $(\theta_{i,n})_{n>0}$ are similarly monotonic, then so it is for $(v_n)_{n>n_0}$.

Proof. We first observe that on account of the uniform estimate for $(\theta_{i,n})_{n>0}$ the sequence $(v_n)_{n>0}$ is bounded in $L^\infty(Q)$. So, n being fixed, we refer to the arguments developed in the proof of Theorem 3.1 with $K = K_n$ now. Then the convergence hypothesis for the sequences $(\theta_{i,n})_{n>0}$ are sufficient to pass to the limit with n in the nonlinear terms $H(v_n, K_n)$, $\mathbf{Q}(v_n, K_n)$ and $G(v_n, K_n)$. The uniqueness of the entropy process solution constructed this way ensures that the whole sequence $(v_n)_n$ converges to the weak entropy solution u of obstacle problem (1), (2), (3). Moreover, let us suppose that sequences $(\theta_{i,n})_{n>0}$ are non-decreasing. Then, for n large enough ($n \geq n_0$), for $m > n$,

$$\theta_{1,n} \leq \theta_{1,m} \leq \theta_1 \leq \theta_{2,n} \leq \theta_{2,m} \leq \theta_2 \text{ a.e. on } Q.$$

Due to (30), $v_n \leq v_m \leq u$ a.e. on Q and the conclusion follows. □

We now examine sufficient conditions on data such that obstacle problem (1), (2), (3) is not a free boundary one. Namely, we look for relations between the weak entropy solution u of (1), (2), (3) and v , that of the unconstrained problem:

$$\begin{aligned} &\mathbb{H}(t, x, v) = 0 \text{ in } Q, \\ &v(t, \sigma) = u_\Gamma \text{ on (a part) of } \Sigma, v(0, \cdot) = u_0 \text{ in } \Omega, \end{aligned} \tag{31}$$

and the next statement holds:

Property 4.2.

- (i) If $\mathbb{H}(t, x, \theta_1) \leq 0$ and $\mathbb{H}(t, x, \theta_2) \geq 0$ a.e. on Q , then $u = v$.
- (ii) If v belongs to $W^{1,1}(Q) \cap L^\infty(Q)$, if $\mathbb{H}(t, x, \theta_1) \geq 0$ and $\mathbb{H}(t, x, \theta_2) \leq 0$ a.e. on Q , then:

$$u = \theta_1 + (v - \theta_1)^+ - (v - \theta_2)^+$$

Proof. The point (i) is a consequence of the general result, for any function θ of $W^{1,+\infty}(Q)$

$$M_L(t) \leq v(t, x) - \theta(t, x) \leq M_U(t) \text{ a.e. on } Q,$$

where

$$M_L(t) = -\max \left(\|(u_0 - \theta(0, \cdot))^- \|_\infty; \|(u_\Gamma - \theta_\Gamma)^- \|_\infty \right) e^{C_1^* t} - \frac{(e^{C_1^* t} - 1)}{C_1^*} \|\mathbb{H}(t, x, \theta)^+\|_{L^\infty(Q)}$$

and

$$M_U(t) = \max \left(\|(u_0 - \theta(0, \cdot))^+ \|_\infty; \|(u_\Gamma - \theta_\Gamma)^+ \|_\infty \right) e^{C_1^* t} + \frac{(e^{C_1^* t} - 1)}{C_1^*} \|\mathbb{H}(t, x, \theta)^-\|_{L^\infty(Q)},$$

with C_1^* depending on $\|\nabla\theta_1\|_\infty, M'_g, A$ and on the Lipschitz constants of $\partial_u \mathbf{f}$ with respect to its third variable.

Indeed, by using a cut-off method and resorting to the viscous problem the arguments developed in Property 3.1 prove that for any t of $]0, T[$ (we have dropped index ϵ),

$$\int_{\Omega} I_\lambda(v(t, \cdot) - M_L(t, \cdot) - \theta(t, \cdot))^- dx - \int_{Q_t} \mathbb{H}(t, x, \theta + M_L) \operatorname{sgn}_\lambda(v - M_L - \theta)^- dx dt$$

$$\leq M'_g \int_{Q_t} (v - M_L - \theta)^- dx dt + \epsilon \int_{Q_t} |\Delta\theta| dx dt,$$

The definition of M_L and the Lipschitz conditions for \mathbf{f} and g ensure that $\mathbb{H}(t, x, \theta + M_L)$ is nonpositive a.e. on Q . When ϵ and λ tend to 0^+ , the minoration for v follows from the Gronwall lemma. The majoration for v is based on the same principles.

Now, by taking account of the obstacle conditions for u_0 and u_Γ and the sign conditions for $\mathbb{H}(t, x, \theta_1)$ and $\mathbb{H}(t, x, \theta_2)$ we make sure that $\theta_1 \leq v \leq \theta_2$ a.e. in Q . So, to prove that $u = v$ we just have to ensure that v fulfills relations (6), (7) and (8) of Definition 1.1. In fact, the proof of Theorem 3.1 shows that (6), (7), (8) for v are obtained as soon as the viscous solution v_ϵ fulfills (28), which is obviously true.

(ii) Let us denote $w = \theta_1 + (v - \theta_1)^+ - (v - \theta_2)^+$. By construction, $\theta_1 \leq w \leq \theta_2$ a.e. on Q . Therefore, to prove that $w = u$ we have to establish that w fulfills relations (6), (7) and (8). To do so, we first remark that as k is fixed in $[0, 1]$ the sign conditions for $\mathbb{H}(t, x, \theta_1)$ and $\mathbb{H}(t, x, \theta_2)$ guarantee that a.e. on Q :

$$\left\{ \partial_t \theta_1 \operatorname{sgn}(v - \theta_1)^+ - \partial_t \theta_2 \operatorname{sgn}(v - \theta_2)^+ - \partial_t \theta_1 \right.$$

$$\left. + \{ \nabla \cdot \mathbf{f}(t, x, v) + g(t, x, v) \} \{ \operatorname{sgn}(v - \theta_1)^+ - \operatorname{sgn}(v - \theta_2)^+ \} \right\} \partial_1 H(w, K)$$

$$\geq \{ \nabla \cdot \mathbf{f}(t, x, w) + g(t, x, w) \} \partial_1 H(w, K),$$

where $K = k(\theta_2 - \theta_1) + \theta_1$ and (H, \mathbf{Q}) is a boundary entropy-entropy flux pair (see proof of Th. 3.1). In addition, owing to the regularity of v , (31) is fulfilled a.e. on Q . Hence, on the one hand one multiplies with $\operatorname{sgn}(v - \theta_1)^+ \partial_1 H(w, K)$ and on the other hand with $-\operatorname{sgn}(v - \theta_2)^+ \partial_1 H(w, K)$. By adding up and by considering the previous inequality it ensues:

$$\left\{ \partial_t w + \nabla \cdot \mathbf{f}(t, x, w) + g(t, x, w) \right\} \partial_1 H(w, K) \leq 0 \text{ a.e. on } Q.$$

As a consequence, as in Theorem 3.1 we prove that w fulfills relation (28) with $\epsilon = 0$. So, as mentioned above, one gets (6), (7), (8) for w , which completes the proof of Property 4.2. □

5. CONCLUSION

To conclude, we have proved the existence and uniqueness of the weak entropy solution for a nonlinear scalar conservation law associated with Dirichlet boundary data and a bilateral obstacle constraint in a bounded domain of $\mathbb{R}^p, p \geq 1$. Yet, the numerical approximation this solution remains to be proved. A first study has been achieved in [13] for a quasilinear first-order equation associated with a forced positiveness condition. Through the time-splitting method an L^1 -error bound in $\mathcal{O}(\sqrt{\Delta t})$ is established for the Cauchy problem and only an L^1 -convergence result has been demonstrated for the Dirichlet one so far.

REFERENCES

- [1] J.M. Ball, A version of the fundamental theorem for young measures, in *PDEs and continuum model of phase transition, Lect. Notes Phys.* **344**, Springer-Verlag, Berlin (1995) 241–259.
- [2] C. Bardos, A.Y. LeRoux and J.C. Nedelec, First-order quasilinear equations with boundary conditions. *Comm. Partial Differential Equations* **4** (1979) 1017–1034.
- [3] S. Benharbit, A. Chalabi and J.P. Vila, Numerical viscosity and convergence of finite volume methods for conservation laws with boundary conditions. *SIAM J. Numer. Anal.* **32** (1995) 775–796.
- [4] A. Bensoussan and J.L. Lions, Inéquations variationnelles non linéaires du premier et second ordre. *C. R. Acad. Sci. Paris, Sér. A* **276** (1973) 1411–1415.
- [5] P. Bia and M. Combarous, Les méthodes thermiques de production des hydrocarbures, Chap. 1 : Transfert de chaleur et de masse. *Revue de l'Institut français du pétrole* (1975) 359–394.
- [6] C. Chainais-Hillairet, Finite volume schemes for a nonlinear hyperbolic equation. Convergence toward the entropy solution and error estimate. *ESAIM: M2AN* **33** (1999) 129–156.
- [7] S. Champier, T. Gallouët and R. Herbin, Convergence of an upstream finite volume scheme for a nonlinear hyperbolic equation on triangular mesh. *Numer. Math.* **66** (1993) 139–157.
- [8] R.J. Diperna, Measure-valued solutions to conservation laws. *Arch. Rat. Mech. Anal.* **88** (1985) 223–270.
- [9] R. Eymard, T. Gallouët and R. Herbin, Existence and uniqueness of the entropy solution to a nonlinear hyperbolic equation. *Chin. Ann. Math.* **16B** (1995) 1–14.
- [10] G. Gagneux and M. Madaune-Tort, Analyse mathématique de modèles non linéaires de l'ingénierie pétrolière. *Math. Appl. SMAI* **22**, Springer-Verlag, Berlin (1996).
- [11] D. Kröner and M. Rokyta, Convergence of upwinding finite volume schemes for scalar conservation laws in two dimensions. *SIAM J. Numer. Anal.* **31** (1994) 324–343.
- [12] S.N. Kruskov, First-order quasilinear equations in several independent variables. *Math. USSR Sb.* **10** (1970) 217–243.
- [13] L. Lévi and F. Peyroutet, A time-fractional step method for conservation law related obstacle problem. (Preprint 99/37. Laboratory of Applied Math., ERS 2055, Pau University.), *Adv. Appl. Math.* (to appear).
- [14] F. Otto, Conservation laws in bounded domains, uniqueness and existence via parabolic approximation, in *Weak and measure-valued solutions to evolutionary PDE's*, J. Malek, J. Necas, M. Rokyta and M. Ruzicka Eds., Chapman & Hall, London (1996) 95–143.
- [15] A. Szepessy, Measure solutions to scalar conservation laws with boundary conditions. *Arch. Rat. Mech. Anal.* **107** (1989) 181–193.
- [16] A. Szepessy, Convergence of a streamline diffusion finite element method for scalar conservation laws with boundary condition. *ESAIM: M2AN* **25** (1991) 749–782.
- [17] L. Tartar, Compensated compactness and applications to partial differential equations, in *Nonlinear analysis and mechanics. Heriot-Watt Symposium*, R.J. Knops Ed., *Res. Notes Math.* **4**, Pitman Press, New-York (1979).
- [18] G. Vallet, Dirichlet problem for nonlinear conservation law. *Revista Matematica Complutense* **XIII** (2000) 1–20.
- [19] M.H. Vignal, Convergence of a finite volume scheme for elliptic-hyperbolic system. *RAIRO: Modél. Math. Anal. Numér.* **30** (1996) 841–872.