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REGULARIZATION OF AN UNILATERAL OBSTACLE PROBLEM

Ahmed Addou[1], E. Bekkaye Mermri[1] and Jamal Zahi[1]

Abstract. The aim of this article is to give a regularization method for an unilateral obstacle problem with obstacle ψ and second member f, which generalizes the one established by the authors of [4] in case of null obstacle and a second member is equal to constant 1.

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1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $g \in H^{1/2}(\partial\Omega)$ and $\psi \in H^1(\Omega)$. We consider the variational inequality problem - called unilateral obstacle problem -:
Find

$$u \in K = \{v \in H^1(\Omega); v \geq \psi \text{ a.e. in } \Omega,\ v = g \text{ on } \partial\Omega\} \tag{1}$$

such that

$$\int_\Omega \nabla u \nabla(v-u)\mathrm{d}x + \langle f, v-u\rangle \geq 0 \quad \forall v \in K, \tag{2}$$

where $f \in H^{-1}(\Omega)$. It is well known that Problem (1-2) admits a unique solution (see [5]).

The aim of this article is to develop a regularization method for solving a non differentiable minimization problem which is equivalent to Problem (1-2). The idea of the regularization method is to approximate the non differentiable term by a sequence of differentiable ones depending on $\varepsilon > 0$, $\varepsilon \to 0$. To establish this regularization we give a new formulation of the obstacle problem, which is the subject of Theorem 1. We give three forms of regularization for which we establish the convergence result and *a priori* error estimates. Next by the duality method by conjugate functions (see [2]) we provide *a posteriori* error estimates which is desired for practical implementation for the regularization method.

This study is a generalization of an other one established by the authors of [4], where the obstacle ψ is equal to zero and the second member f is taken equal to the constant 1.

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[1] Department of Mathematics, Faculty of Sciences, University Mohammed I, Oujda, Morocco.
e-mail: `zahi-j@sciences.univ-oujda.ac.ma`

2. Formulation and regularization of the problem

Let Ω be a bounded domain of $\mathbb{R}^n$, with smooth boundary $\partial\Omega$ and $g \in H^{1/2}(\partial\Omega)$, we denote by

$$H_g^1(\Omega) = \{v \in H^1(\Omega); v = g \text{ on } \partial\Omega\}.$$

For ψ an element of $H^1(\Omega)$ with $\psi \leq g$ on $\partial\Omega$, we set

$$K_\psi = \{v \in H_g^1(\Omega) : v \geq \psi \text{ a.e. on } \Omega\}.$$

Let $f \in H^{-1}(\Omega)$, we assume that f and ψ verify the following hypothesis:

$$f - \Delta\psi = F \in L^2(\Omega).$$

We denote by $\langle .,. \rangle$ the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, and $(.,.)$ the inner product of $L^2(\Omega)$. Consider the following variational inequality problem:

$$(P_\psi) \begin{cases} \text{Find } u \in K_\psi \\ a(u, v-u) + \langle f, v-u \rangle \geq 0 \quad \text{for all } v \in K_\psi, \end{cases}$$

where $a(.,.)$ is defined by

$$a(u,v) = \int_\Omega \nabla u . \nabla v, \qquad u, v \in H^1(\Omega).$$

It is well-known that Problem (P_ψ) admits a unique solution.
For all element $z \in L^2(\Omega)$ we denote

$$z^+ = \max\{z, o\} \text{ and } z^- = \min\{z, o\}\cdot$$

If $v \in H^1(\Omega)$, then we have $v^+, v^- \in H^1(\Omega)$ and

$$a(v^+, v^-) = 0. \tag{3}$$

In the sequel we use the same notation g to designate an element of $H^{1/2}(\partial\Omega)$ and an element of $H^1(\Omega)$ which its trace on $\partial\Omega$ is g. We write the obstacle problem (P_ψ) on a new form.

Theorem 1. *u is solution of Problem (P_ψ) if and only if $w = u - g$ is solution of the following problem:*

$$(P) \begin{cases} \textit{Find } w \in H_0^1(\Omega), \\ a(w + g - \psi, v - w) + \varphi(v) - \varphi(w) + (F^-, v - w) \geq 0 \;\; \textit{for all } v \in H_0^1(\Omega), \end{cases}$$

where φ is the functional defined by

$$\varphi(v) = (F^+, \phi(v + g - \psi)), \quad v \in H_0^1(\Omega),$$

with

$$\phi(t) = t^+, \quad t \in \mathbb{R}.$$

Proof. From the general theory of variational inequalities (see [3]), Problem (P) admits a unique solution, so it is sufficient to show that $w + g$ is a element of K_ψ, where w is the solution of Problem (P). Indeed, for $v = (w + g - \psi)^+ - g + \psi \in H_0^1(\Omega)$, the inequality of (P) becomes

$$-a((w + g - \psi)^-, (w + g - \psi)^-) + \varphi(w) - \varphi(w) - (F^-, (v + g - \psi)^-) \geq 0.$$

Hence, from (3) we obtain

$$a((w+g-\psi)^-,(w+g-\psi)^-)=0,$$

we deduce that $w+g-\psi\geq 0$, consequently $w+g\in K_\psi$.
It is easy to see that $u=w+g$ verify the inequality of Problem (P), hence the proof is complete. □

The functional φ being non differentiable on $H_0^1(\Omega)$, we approximate it by a sequence of differentiable functionals, $\varphi_\varepsilon(v)=\int_\Omega F^+\phi_\varepsilon(v+g-\psi)\mathrm{d}x$, ($\varepsilon>0$, tends to 0). The regularized problem is

$$(P_\varepsilon)\begin{cases}\text{Find } w_\varepsilon\in H_0^1(\Omega),\\ a(w_\varepsilon,v-w_\varepsilon)+\varphi_\varepsilon(v)-\varphi_\varepsilon(w_\varepsilon)+\langle l,v-w_\varepsilon\rangle\geq 0 \ \text{ for all } v\in H_0^1(\Omega),\end{cases}$$

where

$$\langle l,v\rangle=a(g-\psi,v)+(F^-,v).$$

Problems (P) and (P_ε) are, respectively, equivalent to

$$u\in H_g^1(\Omega)\ :\ a(u,v-u)+\tilde\varphi(v)-\tilde\varphi(u)+\int_\Omega(F^-+\Delta\psi)(v-u)\mathrm{d}x\geq 0\quad\forall v\in H_g^1(\Omega),\tag{4}$$

with

$$\tilde\varphi(v)=\int_\Omega\phi(v-\psi)\mathrm{d}x,\quad v\in H_g^1(\Omega),$$

and

$$u_\varepsilon\in H_g^1(\Omega):\ a(u_\varepsilon,v-u_\varepsilon)+\tilde\varphi_\varepsilon(v)-\tilde\varphi_\varepsilon(u_\varepsilon)+\int_\Omega(F^-+\Delta\psi)(v-u_\varepsilon)\mathrm{d}x\geq 0\quad\forall v\in H_g^1(\Omega)\tag{5}$$

with

$$\tilde\varphi_\varepsilon(v)=\int_\Omega F^+\phi_\varepsilon(v-\psi)\mathrm{d}x,\quad v\in H_g^1(\Omega).$$

There are many methods to construct sequences of differentiable approximations. In this article we take the sequence ϕ_ε verifying one of the following choices:

$$\text{c1}:\qquad \phi_\varepsilon^1(t)=\begin{cases}t-\frac{\varepsilon}{2} & \text{if } t\geq\varepsilon\\ \frac{t^2}{2\varepsilon} & \text{if } 0\leq t\leq\varepsilon\\ 0 & \text{if } t\leq 0.\end{cases}$$

$$\text{c2}:\qquad \phi_\varepsilon^2(t)=\begin{cases}t & \text{if } t\geq\varepsilon\\ \frac{1}{2}(\frac{t^2}{\varepsilon}+\varepsilon) & \text{if } 0\leq t\leq\varepsilon\\ \frac{\varepsilon}{2} & \text{if } t\leq 0.\end{cases}$$

$$\text{c3}:\qquad \phi_\varepsilon^3(t)=\begin{cases}\sqrt{t^2+\varepsilon^2} & \text{if } t\geq 0\\ \varepsilon & \text{if } t\leq 0.\end{cases}$$

With these choices Problem (P_ε) admits a unique solution. To establish the convergence of Sequence u_ε we need the following results (see [3]).

Lemma 1. *Let V be a Hilbert space, $a:V\times V\to\mathbb{R}$ a continuous, V-elliptic bilinear form, $j:V\to\mathbb{R}$ proper, non negative, convex, weakly continuous function and f is a linear continuous form on V. Assume that $j_\varepsilon:V\to\mathbb{R}$, $(\varepsilon>0)$, is a family of non negative convex weakly lower semi-continuous (l.s.c.) functions verifying*

$$j_\varepsilon(v)\to j(v)\quad\forall v\in V,\tag{6}$$

$$\textit{If } u_\varepsilon \to u \textit{ weakly in } V \textit{ then we have } j(u) \le \lim_{\varepsilon\to 0}\inf j_\varepsilon(u_\varepsilon). \tag{7}$$

Let u, $u_\varepsilon \in V$ be the solutions of the following variational inequalities:

$$a(u, v-u) + j(v) - j(u) + \langle f, v-u\rangle \ge 0, \quad \forall v \in V,$$
$$a(u_\varepsilon, v-u_\varepsilon) + j_\varepsilon(v) - j_\varepsilon(u_\varepsilon) + \langle f, v-u_\varepsilon\rangle \ge 0, \quad \forall v \in V,$$

respectively. Then we have $u_\varepsilon \to u$ in V when $\varepsilon \to 0$.

Lemma 2. *Assume that*

$$j(v) = \int_\Omega \phi(v)\mathrm{d}x, \;\; j_\varepsilon(v) = \int_\Omega \phi_\varepsilon(v)\mathrm{d}x$$

and j is weakly l.s.c. If

$$\phi_\varepsilon(t) \to \phi(t) \textit{ uniformly in } t, \;\; \textit{as } \varepsilon \to 0, \tag{8}$$

then (6) and (7) are verified.

We notice that if

$$|\phi_\varepsilon(t) - \phi(t)| \le c\varepsilon \quad \forall t \in \mathbb{R}, \tag{9}$$

then (8) is verified. Since the functions ϕ_ε^j, $j = 1, 2, 3$ verify the inequality (9), then we have the convergence $w_\varepsilon \to w$ in $H_0^1(\Omega)$.
Taking $v = w_\varepsilon$(resp. $v = w$) in the inequality of Problem (P) (resp. (P_ε)), we obtain

$$a(w - w_\varepsilon, w - w_\varepsilon) \le \varphi(w_\varepsilon) - \varphi_\varepsilon(w_\varepsilon) + \varphi_\varepsilon(w) - \varphi(w).$$

Consequently, we obtain the following *a priori* estimate

$$\| w - w_\varepsilon \|_{H_0^1(\Omega)} \le (2c\int_\Omega F^+)^{\frac{1}{2}}\sqrt{\varepsilon}.$$

3. *A-posteriori* error estimates

In this section we use the duality method by conjugate functions in order to derive the *a posteriori* error estimates of solutions of approximate problems. We need the following preliminary results (see [2])

Let V and V^* (resp. Y and Y^*) two topological vector spaces and $\langle .,.\rangle_V$ (resp. $\langle .,.\rangle_Y$) denotes the duality pairing between V^* and V (resp. Y^* and Y). Let φ be a function from V to $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, its conjugate function is defined by

$$\varphi(v^*) = \sup_{v\in V}\langle v^*, v\rangle_V - \varphi(v), \quad v^* \in V^*.$$

Assume there exists a continuous linear operator L from V to Y, $L \in \mathcal{L}(V, Y)$, with transpose $L^* \in \mathcal{L}(Y^*, V^*)$. Let J be a function from $V \times Y$ to $\overline{\mathbb{R}}$. We consider the following minimization problem:

$$u \in V, \;\; J(u, Lu) = \inf_{v\in V} J(v, Lv). \tag{10}$$

The conjugate function of J is given by

$$J^*(y^*, v^*) = \sup_{v\in V, y\in Y}\{\langle v^*, v\rangle_V + \langle y^*, y\rangle_Y - J(v, y)\}$$

Theorem 2. *Assume that V is a reflexive Banach space and Y a normed vector space. Let $J : V \times Y \to \overline{\mathbb{R}}$ be a proper l.s.c. strictly convex function verifying:*
(i) $\exists\, u_0 \in V$, such that $J(u_0, Lu_0) < \infty$ and $y \to J(u_0, y)$ is continuous at Lu_0.
(ii) $J(v, Lv) \to +\infty$, as $\| v \|_V \to +\infty$, $v \in V$.
Then Problem (10) *admits a unique solution and*

$$J(u, Lu) = \inf_{v\in V} J(v, Lv) = -\sup_{y^*\in Y^*} J^*(-y^*, L^*y^*).$$

Let Ω be an open subset of $\mathbb{R}^N$, $g : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be the Carathéodory function *i.e.*, $\forall s \in \mathbb{R}^n$, $x \to g(x, s)$ is a measurable function and for almost all $x \in \Omega$, the function $s \to g(x, s)$ is continuous. Then the conjugate function of

$$G(v) = \int_\Omega g(x, v(x))\mathrm{d}x$$

(assuming G is well defined over some a function space V) is

$$G^*(v^*) = \int_\Omega g^*(x, v^*(x))\mathrm{d}x, \quad \forall v^* \in V^*,$$

where

$$g^*(x, y) = \sup_{s\in\mathbb{R}^N} \{ys - g(x, s)\}\cdot$$

For Problem (P) we take

$$\begin{array}{rcl} V &=& H^1(\Omega), \quad Y = Y^* = (L^2(\Omega))^n \times L^2(\Omega) \\ Lv &=& (\nabla v, v) \\ J(v, Lv) &=& H(v) + G(Lv) \\ H(v) &=& \left\{ \begin{array}{ll} 0 & \text{if } v = g \text{ on } \partial\Omega \\ +\infty & \text{otherwise} \end{array} \right. \\ G(y) &=& \int_\Omega \frac{1}{2}|y_1|^2 + F^+(y_2 - \psi)^+ + F^- y_2 + \Delta\psi y_2 \\ && (\text{ Furthermore we assume that } \psi \geq 0) \end{array}$$

where $y = (y_1, y_2)$ with $y_1 \in (L^2(\Omega))^n$ and $y_2 \in L^2(\Omega)$. A similar notation is used for $y^* \in Y^*$. So the obstacle problem (P) can be rewritten in the form (10). To apply Theorem 2, we compute the conjugate of the functional J.

Lemma 3. *Let h be a function defined by*

$$\begin{array}{rccl} h: & \mathbb{R} & \to & \mathbb{R} \\ & s & \mapsto & as + b(s - t)^+ \end{array}$$

where a, b and t are constants with $b \geq 0$, then the conjugate function h^* of h is given by

$$h^*(s^*) = \left\{ \begin{array}{ll} t(s^* - a) & \text{if } a \leq s^* \leq a + b, \\ +\infty & \text{otherwise.} \end{array} \right.$$

If $t \geq 0$ then we have

$$0 \leq h^*(s^*) \leq tb.$$

Proof. We have

$$\begin{aligned} h^*(s^*) &= \sup_{s\in\mathbb{R}}\{ss^* - h(s)\} \\ &= \sup_{s\in\mathbb{R}}\{ss^* - as - b(s-t)^+\} \\ &= \max\{\sup_{s\geq t}\{ss^* - as - bs + bt\}, \sup_{s\leq t}\{ss^* - as\}\} \\ &= \max\{\sup_{s\geq t}\{s(s^* - a - b) + bt\}, \sup_{s\leq t}\{s(s^* - a)\}\}\cdot \end{aligned}$$

It is easy to check that

$$h^*(s^*) = \begin{cases} t(s^* - a) & \text{si } a \leq s^* \leq a + b \\ +\infty & \text{otherwise.} \end{cases}$$

Hence the proof is complete. □

We have

$$J^*(-y^*, L^*y^*) = H^*(L^*y^*) + G^*(-y^*)$$

where

$$\begin{aligned} H^*(L^*y^*) &= \sup_{v\in H^1(\Omega)}\{\langle Lv, y\rangle - H(v)\} \\ &= \sup_{v\in H^1_g(\Omega)} \int_\Omega (\nabla v y_1^* + v y_2^*)\mathrm{d}x \\ &= \int_\Omega (\nabla g y_1^* + g y_2^*)\mathrm{d}x + \sup_{v\in H^1_0(\Omega)} \int_\Omega (\nabla v y_1^* + v y_2^*)\mathrm{d}x \\ &= \begin{cases} \displaystyle\int_\Omega (\nabla g y_1^* + g y_2^*)\mathrm{d}x & \text{if } -\operatorname{div} y_1^* + y_2^* = 0 \text{ dans } \Omega \\ \infty & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} G^*(-y^*) &= \sup_{y\in Y}\{\langle -y^*, y\rangle - G(y)\} \\ &= \sup_{y\in Y}\int_\Omega (-y_1^* y_1 - y_2^* y_2 - \frac{1}{2}|y_1|^2 - F^+(y_2 - \psi)^+ - F^- y_2 - \Delta\psi y_2)\mathrm{d}x \end{aligned}$$

and from Lemma 3 we obtain

$$G^*(-y^*) = \begin{cases} \leq \displaystyle\int_\Omega (\frac{1}{2}|y_1^*|^2 + F^+\psi)\mathrm{d}x & \text{if } -f \leq y_2^* \leq F^+ - f, \\ \infty & \text{otherwise.} \end{cases}$$

Hence

$$J^*(-y^*, L^*y^*) = \begin{cases} \leq \displaystyle\int_\Omega (\nabla g y_1^* + g y_2^* + \frac{1}{2}|y_1^*|^2 + F^+\psi)\mathrm{d}x & \text{if } \operatorname{div} y_1^* + y_2^* = 0 \\ & \text{and } -f \leq y_2^* \leq F^+ - f, \\ \infty & \text{otherwise.} \end{cases} \tag{11}$$

We have

$$\begin{aligned} J(u_\varepsilon, Lu_\varepsilon) - J(u, Lu) &= \int_\Omega \frac{1}{2}|\nabla u_\varepsilon|^2 - \frac{1}{2}|\nabla u|^2 + F^+(u_\varepsilon - \psi)^+ - F^+(u - \psi)^+ \\ &\quad + F^- u_\varepsilon - F^- u + \Delta\psi u_\varepsilon - \Delta\psi u. \end{aligned}$$

Using (4), with $v = u_\varepsilon$, we obtain

$$J(u_\varepsilon, Lu_\varepsilon) - J(u, Lu) \geq \frac{1}{2} \parallel \nabla(u_\varepsilon - u) \parallel^2_{L^2(\Omega)} .$$

Applying Theorem 2 and using (11), we have

$$\begin{aligned} J(u_\varepsilon, Lu_\varepsilon) - J(u, Lu) \leq & \int_\Omega (\frac{1}{2}|\nabla u_\varepsilon|^2 + F^+(u_\varepsilon - \psi)^+ + F^- u_\varepsilon + \Delta\psi u_\varepsilon \\ & + \nabla g y_1^* + g y_2^* + \tfrac{1}{2}|y_1^*|^2 + F^+\psi)\mathrm{d}x \end{aligned}$$

$\forall y^* = (y_1^*, y_2^*) \in Q^*$, with $-\operatorname{div} y_1^* + y_2^* = 0$ and $-f \leq y_2^* \leq F^+ - f$ *a.e.* in Ω.
Since ϕ_ε is differentiable the inequality (5) is equivalent to

$$u_\varepsilon \in H_g^1(\Omega) : a(u_\varepsilon, v) + \int_\Omega (F^+\phi_\varepsilon'(u_\varepsilon - \psi) + F^- + \Delta\psi)v\mathrm{d}x = 0. \quad \forall v \in H_0^1(\Omega). \tag{12}$$

Hence u_ε verifies the following Dirichlet problem:

$$\begin{cases} -\Delta u_\varepsilon + F^+\phi_\varepsilon'(u_\varepsilon - \psi) + F^- + \Delta\psi = 0 & \text{in } \Omega. \\ u_\varepsilon = g & \text{on } \partial\Omega. \end{cases}$$

If we take

$$y_1^* = -\nabla u_\varepsilon \quad \text{and} \quad y_2^* = -(F^+\phi_\varepsilon'(u_\varepsilon - \psi) + F^- + \Delta\psi).$$

Then we have

$$-\operatorname{div} y_1^* + y_2^* = 0 \text{ and } -f \leq y_2^* \leq F^+ - f.$$

Therefore, we have the *a posteriori* estimate

$$\begin{aligned} \tfrac{1}{2} \| \nabla(u_\varepsilon - u) \|_{L^2(\Omega)}^2 \leq & \int_\Omega (\nabla u_\varepsilon \nabla(u_\varepsilon - g) + F^+(u_\varepsilon - \psi)^+ + F^- u_\varepsilon + \Delta\psi u_\varepsilon \\ & - g(F^+\phi_\varepsilon'(u_\varepsilon - \psi) + F^- + \Delta\psi) + F^+\psi)\mathrm{d}x. \end{aligned} \tag{13}$$

Taking $v = u_\varepsilon - g \in H_0^1(\Omega)$ in (12), we obtain

$$\int_\Omega \nabla u_\varepsilon \nabla(u_\varepsilon - g)\mathrm{d}x + \int_\Omega (F^+\phi_\varepsilon'(u_\varepsilon - \psi) + F^- + \Delta\psi)(u_\varepsilon - g)\mathrm{d}x = 0.$$

The estimate (13) becomes

$$\frac{1}{2} \| \nabla(u_\varepsilon - u) \|_{L^2(\Omega)}^2 \leq \int_\Omega (F^+(u_\varepsilon - \psi)^+ + F^- u_\varepsilon + \Delta\psi u_\varepsilon - (F^+\phi_\varepsilon'(u_\varepsilon - \psi) + F^- + \Delta\psi)u_\varepsilon + F^+\psi)\mathrm{d}x.$$

Hence we obtain the *a posteriori* error estimates.
For choices c1 and c2, we have

$$\phi_\varepsilon'(t) = \begin{cases} 1 & \text{if } t \geq \varepsilon, \\ \frac{t}{\varepsilon} & \text{if } 0 \leq t \leq \varepsilon, \\ 0 & \text{if } t \leq 0. \end{cases}$$

The *a posteriori* error estimate is

$$\frac{1}{2} \| \nabla(u_\varepsilon - u) \|_{L^2(\Omega)}^2 \leq \int_{[0 \leq u_\varepsilon - \psi \leq \varepsilon]} F^+ u_\varepsilon (1 - \frac{u_\varepsilon - \psi}{\varepsilon})\mathrm{d}x + \int_{[u_\varepsilon - \psi < 0]} F^+\psi\mathrm{d}x.$$

For choice c3, we have

$$\phi_\varepsilon'(t) = \begin{cases} \frac{t}{\sqrt{t^2+\varepsilon^2}} & \text{if } t \geq 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

The *a posteriori* error estimate is

$$\frac{1}{2} \parallel \nabla(u_\varepsilon - u) \parallel^2_{L^2(\Omega)} \leq \int_{[u_\varepsilon - \psi \geq 0]} F^+ u_\varepsilon (1 - \frac{u_\varepsilon - \psi}{\sqrt{(u_\varepsilon - \psi)^2 + \varepsilon^2}}) \mathrm{d}x + \int_{[u_\varepsilon - \psi < 0]} F^+ \psi \mathrm{d}x.$$

In particular, when $\psi = 0$ we find

$$\frac{1}{2} \parallel \nabla(u_\varepsilon - u) \parallel^2_{L^2(\Omega)} \leq \int_{[0 \leq u_\varepsilon \leq \varepsilon]} f^+ u_\varepsilon (1 - \frac{u_\varepsilon}{\varepsilon}) \mathrm{d}x,$$
$$\frac{1}{2} \parallel \nabla(u_\varepsilon - u) \parallel^2_{L^2(\Omega)} \leq \int_{[u_\varepsilon \geq 0]} f^+ u_\varepsilon (1 - \frac{u_\varepsilon}{\sqrt{u_\varepsilon^2 + \varepsilon^2}}) \mathrm{d}x,$$

respectively.

4. *A-posteriori* error estimates for regularized discrete problem

Let V_h be a finite element space approximating $H^1(\Omega)$, let V_{0h} be the finite element subspace of V_h consisting of all element of V_h which are zero on the boundary of the domain. We have $V_{0h} \subset H^1_0(\Omega)$. Assume the boundary function g can be represented exactly by a function from V_h. Then, a finite element solution $u_h \in V_h$ for the obstacle problem (P) is determined from the following problem:

$$(P_h) \begin{cases} u_h \in V_h, \ u_h = g \ \text{ on } \partial\Omega \\ a(u_h, v_h - u_h) + (F^+, (u_h - \psi)^+ - (v_h - \psi)^+) + (F^- - \Delta\psi, v_h - u_h) \geq 0 \\ \forall v_h \in V_h, \ v_h = g \text{ on } \partial\Omega. \end{cases}$$

If we set $u_{0h} = u_h - g$, then u_{0h} is the solution of the problem

$$(P_{0h}) \begin{cases} u_{0h} \in V_{0h} \\ a(u_{0h}, v_h - u_{0h}) + \varphi(v_h) - \varphi(u_{0h}) + \langle l, v_h - u_{0h} \rangle \geq 0 \quad \forall v_h \in V_{0h}. \end{cases}$$

We can proceed similarly as in [3] to prove the convergence of the finite element approximations and to have *a priori* error estimates.
The regularized problem of (P_{0h}) is

$$(P_{0h,\varepsilon}) \begin{cases} u_{0h,\varepsilon} \in V_{0h,\varepsilon} \\ a(u_{0h,\varepsilon}, v_h - u_{0h,\varepsilon}) + \varphi_\varepsilon(v_h) - \varphi_\varepsilon(u_{0h,\varepsilon}) + \langle l, v_h - u_{0h,\varepsilon} \rangle \geq 0 \quad \forall v_h \in V_{0h}. \end{cases}$$

We can similarly prove that $(P_{0h,\varepsilon})$ have unique solutions and their solution converge to corresponding solution of Problem (P_{0h}). By the duality theory on the discrete problems we prove the following *a posteriori* error estimates.
For choices c1 and c2, the *a posteriori* error estimate is

$$\frac{1}{2} \parallel \nabla(u_{h,\varepsilon} - u_h) \parallel^2_{L^2(\Omega)} \leq \int_{[0 \leq u_{h,\varepsilon} - \psi \leq \varepsilon]} F^+ u_{h,\varepsilon} (1 - \frac{u_{h,\varepsilon} - \psi}{\varepsilon}) \mathrm{d}x + \int_{[u_{h,\varepsilon} - \psi < 0]} F^+ \psi \mathrm{d}x.$$

For choice c3, the *a posteriori* error estimate is

$$\frac{1}{2} \parallel \nabla(u_{h,\varepsilon} - u) \parallel^2_{L^2(\Omega)} \leq \int_{[u_{h,\varepsilon} - \psi \geq 0]} F^+ u_{\varepsilon,h} (1 - \frac{u_{h,\varepsilon} - \psi}{\sqrt{(u_{h,\varepsilon} - \psi)^2 + \varepsilon^2}}) \mathrm{d}x + \int_{[u_{h,\varepsilon} - \psi < 0]} F^+ \psi \mathrm{d}x.$$

In particular, when $\psi = 0$ we find

$$\frac{1}{2} \parallel \nabla(u_{h,\varepsilon} - u) \parallel^2_{L^2(\Omega)} \leq \int_{[0 \leq u_{h,\varepsilon} \leq \varepsilon]} f^+ u_{h,\varepsilon}(1 - \frac{u_{h,\varepsilon}}{\varepsilon})\mathrm{d}x,$$
$$\frac{1}{2} \parallel \nabla(u_{h,\varepsilon} - u) \parallel^2_{L^2(\Omega)} \leq \int_{[u_{h,\varepsilon} \geq 0]} f^+ u_{h,\varepsilon}(1 - \frac{u_{h,\varepsilon}}{\sqrt{u^2_{h,\varepsilon} + \varepsilon^2}})\mathrm{d}x,$$

respectively.

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