

## A SPECTRAL STUDY OF AN INFINITE AXISYMMETRIC ELASTIC LAYER

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**Abstract.** We present here a theoretical study of eigenmodes in axisymmetric elastic layers. The mathematical modelling allows us to bring this problem to a spectral study of a sequence of unbounded self-adjoint operators  $A_n$ ,  $n \in \mathbb{N}$ , in a suitable Hilbert space. We show that the essential spectrum of  $A_n$  is an interval of type  $[\gamma, +\infty[$  and that, under certain conditions on the coefficients of the medium, the discrete spectrum is non empty.

**Résumé.** Nous présentons ici une étude théorique des modes propres dans une couche élastique axisymétrique. La modélisation mathématique permet de ramener ce problème à l'étude spectrale d'une suite d'opérateurs  $A_n$ ,  $n \in \mathbb{N}$ , non bornés et autoadjoints dans un espace de Hilbert adéquat. On montre que le spectre essentiel de  $A_n$  est un intervalle du type  $[\gamma, +\infty[$  et que, sous certaines conditions portant sur les coefficients du milieu, le spectre discret est non vide.

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## 1. THE PROBLEM SETTING

We consider an elastic layer occupying the open set  $\tilde{\Omega} = \{(x_1, x_2, x_3) \in \mathbb{R}^3; 0 < x_3 < h\}$ . We assume that the medium is axisymmetric, which means that the density  $\rho$  and the Lamé coefficients  $(\lambda, \mu)$  depend on  $(r, z)$  if  $r < R$ , and are constants if  $r > R$  (the triple  $(r, \theta, z)$  denote the cylindrical coordinates). We suppose that the surface  $z = 0$  is rigid and the one at  $z = h$  is stress free. We call an eigenmode a field of displacement  $U(x, t)$ , time-harmonic, travelling in the medium without source for  $t > 0$ , and with the amplitude  $|U| \in L^2(\tilde{\Omega})$  (*i.e.* the energy of the mode is localized in a neighbourhood of the axis of symmetry).

In this paper we consider the question of existence of such eigenmodes which amounts to studying the spectrum of a sequence of self-adjoint operators  $A_n$  derived from the linearized elasticity equations ( $n$  is the order of the harmonic). Generalized eigenfunction expansion of  $A_n$  is essential for the construction of the Green's function which enable us to resolve transient problems (with source). This problem appears, for example, in geophysics in the study of seismic waves generated in soil by a source placed in a borehole (with variable diameter and/or with casing) (*cf.* [3, 13]).

This paper is a generalisation of [4] where we studied the guided waves in an elastic space with a symmetry of revolution, in other words the coefficients depend only on the radial variable  $r$ . The spectral theory of self-adjoint operators and in particular the Min-Max principle are the main tools used (as in [1, 2, 5]).

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Our problem appears to be more difficult than those mentioned above. Indeed, in our case the free boundary does not permit us to extend directly the results of [1], moreover as the medium is heterogeneous with unbounded boundary, we cannot also extend the result of [2] and [5].

The main result of this paper is the determination of the essential spectrum of the operator  $A_n$ . Then we establish two existence results of eigenmodes for suitable Lamé coefficients.

Our paper is organized as follows. In Section 2, we obtain the equations for eigenmodes. Section 3 is devoted to the mathematical study of the problem. As we shall see, an eigenmode appears as an eigenfunction of an unbounded self-adjoint operator  $A_n$  (with two variables) defined in an adequate Hilbert space. In Section 4, we establish the regularity of  $u$ , the solution of  $A_n u = f$  in the exterior domain  $\Omega'_R = ]R, +\infty[ \times ]0, h[$ . In Section 5, we study  $\sigma_e(A_n)$  which is the essential spectrum of the operators  $A_n$ . This is a fundamental preliminary step in the analysis of their point spectrum. The value of the lower bound of  $\sigma_e(A_n)$  seems predictable but its proof is not so much trivial. Indeed, as the free surface is unbounded, this leads to significant difficulties in elasticity. To overcome these difficulties, we introduce weighted Sobolev spaces, which is not classic in the previous works on waveguides, to obtain a compactness result. In Section 6, we give sufficient existence conditions of eigenmodes for the operator  $A_n$ . Here we point out the influence of the perturbation.

## 2. THE EQUATIONS

Because the structure of the layer is cylindrical, we are looking for a particular solution  $U(x, t)$  of the form:

$$\begin{cases} U(r, \theta, z, t) = U_r e_r + U_\theta e_\theta + U_z e_z & \text{with} \\ (U_r, U_\theta, U_z) = (u_1(r, z), -i u_2(r, z), u_3(r, z)) \exp i(n\theta \pm \omega t), & n \in \mathbb{Z}, \end{cases} \tag{1}$$

where  $(e_r, e_\theta, e_z)$  is the local basis of the cylindrical coordinates  $(r, \theta, z)$ . The field  $U(r, \theta, z, t)$  must satisfy the elastodynamic equations (in cylindrical coordinates) [6]:

$$\begin{cases} \frac{\partial(r\sigma_{rr})}{\partial r} + \frac{\partial\sigma_{\theta r}}{\partial\theta} + r\frac{\partial\sigma_{zr}}{\partial z} - \sigma_{\theta\theta} = r\rho\frac{\partial^2 U_r}{\partial t^2} \\ \frac{1}{r}\frac{\partial(r^2\sigma_{r\theta})}{\partial r} + \frac{\partial\sigma_{\theta\theta}}{\partial\theta} + r\frac{\partial\sigma_{\theta z}}{\partial z} = r\rho\frac{\partial^2 U_\theta}{\partial t^2} \\ \frac{\partial(r\sigma_{rz})}{\partial r} + \frac{\partial\sigma_{\theta z}}{\partial\theta} + r\frac{\partial\sigma_{zz}}{\partial z} = r\rho\frac{\partial^2 U_z}{\partial t^2} \end{cases} \tag{2}$$

where  $\sigma$  is the stress tensor given by Hooke's law:

$$\begin{cases} \sigma_{rr} = \lambda \operatorname{div} U + 2\mu \frac{\partial U_r}{\partial r} & ; & \sigma_{r\theta} = \mu \left[ \frac{1}{r} \frac{\partial U_r}{\partial\theta} + r \frac{\partial}{\partial r} \left( \frac{U_\theta}{r} \right) \right] \\ \sigma_{rz} = \mu \left( \frac{\partial U_z}{\partial r} + \frac{\partial U_r}{\partial z} \right) & ; & \sigma_{\theta\theta} = \lambda \operatorname{div} U + \frac{2\mu}{r} \left( \frac{\partial U_\theta}{\partial\theta} + U_r \right) \\ \sigma_{\theta z} = \mu \left( \frac{\partial U_\theta}{\partial z} + \frac{1}{r} \frac{\partial U_z}{\partial\theta} \right) & ; & \sigma_{zz} = \lambda \operatorname{div} U + 2\mu \frac{\partial U_z}{\partial z} \end{cases} \tag{3}$$

with

$$\operatorname{div} U = \frac{1}{r} \left[ \frac{\partial(rU_r)}{\partial r} + \frac{\partial U_\theta}{\partial\theta} \right] + \frac{\partial U_z}{\partial z}$$

and the boundary conditions

$$U = 0, \text{ if } z = 0 \text{ and } \sigma_{rz}(U) = \sigma_{r\theta}(U) = \sigma_{zz}(U) = 0, \text{ if } z = h. \tag{4}$$

We denote by  $u(r, z) = (u_1, u_2, u_3)$  the field of amplitudes. Therefore, to find the eigenmodes we must solve, for each  $n \in \mathbb{Z}$ , the following two dimensional problem:

$$(\tilde{P}_n) \quad \begin{cases} \text{Find } u \in \mathcal{D}'(\Omega), u \neq 0, \text{ and } \omega > 0 \text{ such that :} \\ \sqrt{r}|u| \in L^2(\Omega) \text{ with } \Omega = \{(r, z) \in \mathbb{R}^2, r > 0, 0 < z < h\}; \\ \mathcal{A}_n u = \omega^2 u \text{ in } \Omega; \\ u(r, 0) = 0; \quad \sigma_{rz}^n(u)(r, h) = \sigma_{\theta z}^n(u)(r, h) = \sigma_{zz}^n(u)(r, h) = 0. \end{cases}$$

where  $\mathcal{A}_n$  is the partial differential operator defined formally by:

$$\mathcal{A}_n u = \frac{1}{\rho r} \left[ -\frac{\partial}{\partial r} \left( B \frac{\partial u}{\partial r} + B_2 u \right) + B_1 \frac{\partial u}{\partial r} + B_3 u - \frac{\partial}{\partial r} \left( B_6 \frac{\partial u}{\partial z} \right) - \frac{\partial}{\partial z} \left( B_7 \frac{\partial u}{\partial r} \right) - \frac{\partial}{\partial z} \left( B_8 \frac{\partial u}{\partial z} \right) + B_4 \frac{\partial u}{\partial z} - \frac{\partial}{\partial z} (B_5 u) \right] \tag{5}$$

and the matrices  $B$  and  $B_i, i = 1$  to  $8$ , are given by:

$$B = \begin{pmatrix} (\lambda + 2\mu)r & 0 & 0 \\ 0 & \mu r & 0 \\ 0 & 0 & \mu r \end{pmatrix} ; \quad B_1 = B_2^t = \begin{pmatrix} \lambda & -n\mu & 0 \\ n\lambda & -\mu & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B_3 = \begin{pmatrix} \frac{\lambda + (n^2 + 2)\mu}{r} & \frac{n(\lambda + 3\mu)}{r} & 0 \\ \frac{n(\lambda + 3\mu)}{r} & \frac{n^2\lambda + (2n^2 + 1)\mu}{r} & 0 \\ 0 & 0 & \frac{n^2\mu}{r} \end{pmatrix} ; \quad B_4 = B_5^t = \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & n\lambda \\ 0 & -n\mu & 0 \end{pmatrix}$$

$$B_6 = B_7^t = \begin{pmatrix} 0 & 0 & \lambda r \\ 0 & 0 & 0 \\ \mu r & 0 & 0 \end{pmatrix} ; \quad B_8 = \begin{pmatrix} \mu r & 0 & 0 \\ 0 & \mu r & 0 \\ 0 & 0 & (\lambda + 2\mu)r \end{pmatrix}.$$

The reduced stress vector  $\sigma_z^n(u)$  is given by:

$$\begin{cases} \sigma_{rz}^n(u) = \mu \left( \frac{\partial u_3}{\partial r} + \frac{\partial u_1}{\partial z} \right) \\ \sigma_{\theta z}^n(u) = \mu \left( \frac{\partial u_2}{\partial z} - \frac{n}{r} u_3 \right) \\ \sigma_{zz}^n(u) = (\lambda + 2\mu) \frac{\partial u_3}{\partial z} + \lambda \left( \frac{\partial u_1}{\partial r} + \frac{u_1 + nu_2}{r} \right) \end{cases} \tag{6}$$

**Remark 2.1.**

- 1) We have introduced  $(-i)$  in the solution (1) to deal with a system with real coefficients.
- 2) We point out that if the field  $u = (u_1, u_2, u_3)$  satisfies  $(\tilde{P}_n)$  then the field  $u = (u_1, -u_2, u_3)$  satisfies  $(\tilde{P}_{-n})$ . Therefore it is sufficient to study the problem  $(\tilde{P}_n)$  for  $n \in \mathbb{N}$ .
- 3) The system of solutions of the form (1) is complete since each solution of (2) can be expanded in series of solutions of type (1). This can be rigorously proved by using Fourier decomposition in  $\theta$  (see [10] for  $\tilde{\Omega}$  a bounded axisymmetric domain).

### 3. THE VARIATIONAL FORMULATION OF $\tilde{P}_n$

In the following we suppose that  $\lambda, \mu, \rho \in L^\infty(\Omega)$  and satisfy the assumptions:

- (i)  $\exists R > 0$  such that  $(\lambda(r, z), \mu(r, z), \rho(r, z)) = (\lambda_\infty, \mu_\infty, \rho_\infty)$  for all  $r > R$ ,
- (ii)  $\inf_\Omega \lambda(r, z) = \lambda_- > 0, \inf_\Omega \mu(r, z) = \mu_- > 0, \inf_\Omega \rho(r, z) = \rho_- > 0$ .

We also define the velocities of the  $P$  and  $S$  waves given respectively by

$$c_S = \left(\frac{\mu_\infty}{\rho_\infty}\right)^{\frac{1}{2}} \text{ and } c_P = \left(\frac{\lambda_\infty + 2\mu_\infty}{\rho_\infty}\right)^{\frac{1}{2}}.$$

We set  $\lambda_+ = \sup_\Omega \lambda(r, z), \mu_+ = \sup_\Omega \mu(r, z)$  and  $\rho_+ = \sup_\Omega \rho(r, z)$ .

We define the real Hilbert space:

$$H = \{u = (u_1, u_2, u_3) \in (L^2_{loc}(\Omega))^3; \sqrt{r}|u| \in L^2(\Omega)\} \tag{7}$$

with the inner product  $(u, v) = \iint_\Omega uv\rho r \, dr \, dz$ .

We then define a sequence of real Hilbert spaces:

$$V_n = \{u \in (H^1_{loc}(\Omega))^3 \text{ such that } \|u\|_{V_n} < +\infty \text{ and } u(r, 0) = 0\}, \quad n \in \mathbb{N}, \tag{8}$$

with the norms:

$$\|u\|_{V_n}^2 = \begin{cases} \iint_\Omega \left[ r(|u|^2 + |\nabla u|^2) + \frac{n^2 + 1}{r}(u_1^2 + u_2^2) + \frac{n^2 u_3^2}{r} \right] dr \, dz & \text{if } n \neq 1, \\ \iint_\Omega \left[ r(|u|^2 + |\nabla u|^2) + \frac{1}{r}(u_1 + u_2)^2 + \frac{u_3^2}{r} \right] dr \, dz & \text{if } n = 1. \end{cases} \tag{9}$$

Consider the following bilinear form on  $V_n \times V_n$  defined by:

$$a_n(u, v) = \iint_\Omega \left[ B \frac{\partial u}{\partial r} \cdot \frac{\partial v}{\partial r} + B_1 \frac{\partial u}{\partial r} \cdot v + u \cdot B_1 \frac{\partial v}{\partial r} + B_3 u \cdot v + \frac{\partial u}{\partial r} \cdot B_6 \frac{\partial v}{\partial z} + B_6 \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial r} + u \cdot B_4 \frac{\partial v}{\partial z} + B_4 \frac{\partial u}{\partial z} \cdot v + B_8 \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial z} \right] dr \, dz. \tag{10}$$

The variational formulation of  $(\tilde{P}_n)$  is:

$$(\mathcal{P}_n) \quad \text{Find } u \in V_n, u \neq 0, \text{ and } \omega > 0 \text{ such that: } a_n(u, v) = \omega^2(u, v), \quad \forall v \in V_n.$$

The bilinear form  $a_n(\cdot, \cdot)$  is obviously symmetric and continuous, we shall show that it is coercive and, consequently, it defines a self-adjoint operator  $A_n$  in  $H$ . Then the problem  $(\mathcal{P}_n)$  is equivalent to the spectral problem:

$$(P_n) \quad \text{Find } u \in D(A_n), u \neq 0, \text{ and } \omega > 0 \text{ such that: } A_n u = \omega^2 u.$$

#### The coerciveness of $a_n(\cdot, \cdot)$

We have the following decomposition:

$$a_n(u, u) = \iint_\Omega \lambda |\operatorname{div}^n u|^2 r \, dr \, dz + \iint_\Omega \mu \varepsilon_n(u, u) r \, dr \, dz, \tag{11}$$

where

$$\operatorname{div}^n u = \frac{\partial u_1}{\partial r} + \frac{\partial u_3}{\partial z} + \frac{u_1 + nu_2}{r} \quad , \quad \varepsilon_n(u, u) = 2 \sum_{\tau, \sigma} \varepsilon_{\tau\sigma}^n(u)^2 \tag{12}$$

with  $\varepsilon^n(u)$  the symmetric tensor

$$\begin{aligned} \varepsilon_{rr}^n(u) &= \frac{\partial u_1}{\partial r} \quad ; & \varepsilon_{zz}^n(u) &= \frac{\partial u_3}{\partial z} \quad ; & \varepsilon_{\theta\theta}^n(u) &= \frac{nu_2 + u_1}{r} \quad ; \\ 2\varepsilon_{rz}^n(u) &= \frac{\partial u_3}{\partial r} + \frac{\partial u_1}{\partial z} \quad ; & 2\varepsilon_{r\theta}^n(u) &= \frac{nu_1 + u_2}{r} - \frac{\partial u_2}{\partial r} \quad ; & 2\varepsilon_{\theta z}^n(u) &= \frac{\partial u_2}{\partial z} - \frac{nu_3}{r} \quad . \end{aligned}$$

To prove the coerciveness of  $a_n$  it suffices to establish the following inequality

$$\iint_{\Omega} \varepsilon_n(v, v) r \, dr \, dz \geq K \|v\|_{V_n(\Omega)}^2, \quad \forall v \in V_n \text{ with } K > 0. \tag{13}$$

For this we use Korn’s inequality in the half-space  $\Pi_h := \{x \in \mathbb{R}^3, x_3 < h\}$  (cf. [9]):

$$\iiint_{\Pi_h} \sum_{i,j=1}^3 |e_{ij}(\tilde{u})|^2 \, dx \geq C_1 \iiint_{\Pi_h} |\nabla \tilde{u}|^2 \, dx, \quad \text{for } \tilde{u} \in H^1(\Pi_h)^3 \text{ with } C_1 > 0. \tag{14}$$

The inequality (14) is also valid for  $\tilde{u} \in W_0^1(\tilde{\Omega}) := \{\tilde{v} \in H^1(\tilde{\Omega})^3; \tilde{v}|_{z=0} = 0\}$ .

We would like now to derive (13) from (14). We proceed in two steps.

**Step 1.** We recall some transformations between Cartesian and cylindrical coordinates (cf. [10]). Each vector field  $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$  is transformed into a vector field  $u = (u_r, u_\theta, u_z)$  on  $\tilde{\Omega}$  by

$$u_r = \tilde{u}_1 \cos \theta + \tilde{u}_2 \sin \theta, \quad u_\theta = -\tilde{u}_1 \sin \theta + \tilde{u}_2 \cos \theta, \quad u_z = \tilde{u}_3. \tag{15}$$

We get the following integrals

$$\begin{aligned} \iiint_{\tilde{\Omega}} |\nabla \tilde{u}|^2 \, dx &= \iiint_{\tilde{\Omega}} \left\{ \sum_{\tau=r,\theta,z} r \left( \left| \frac{\partial u_\tau}{\partial r} \right|^2 + \left| \frac{\partial u_\tau}{\partial z} \right|^2 \right) \right. \\ &\quad \left. + \frac{1}{r} \left[ \left( \frac{\partial u_r}{\partial \theta} - u_\theta \right)^2 + \left( \frac{\partial u_\theta}{\partial \theta} + u_r \right)^2 + \left( \frac{\partial u_z}{\partial \theta} \right)^2 \right] \right\} dr \, d\theta \, dz, \end{aligned} \tag{16}$$

$$\iiint_{\tilde{\Omega}} \sum_{i,j=1}^3 |e_{ij}(\tilde{u})|^2 \, dx = 2 \iiint_{\tilde{\Omega}} \sum_{\tau,\sigma} |\varepsilon_{\tau,\sigma}(u)|^2 r \, dr \, d\theta \, dz, \tag{17}$$

where

$$\begin{aligned} \varepsilon_{rr}(u) &= \frac{\partial u_r}{\partial r}, & \varepsilon_{\theta\theta}(u) &= \frac{1}{r} \left( u_r + \frac{\partial u_\theta}{\partial \theta} \right), & \varepsilon_{zz}(u) &= \frac{\partial u_z}{\partial z}, \\ 2\varepsilon_{rz}(u) &= \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, & 2\varepsilon_{\theta r}(u) &= \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}, & 2\varepsilon_{z\theta}(u) &= \frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta}. \end{aligned}$$

**Step 2.** We consider test functions  $u^{(n)}$ ,  $n \in \mathbb{N}$ , such that:

$$u^{(n)}(r, \theta, z) = (v_1(r, z) \cos n\theta, v_2(r, z) \sin n\theta, v_3(r, z) \cos n\theta) \tag{18}$$

where  $v = (v_1, v_2, v_3)$  is an arbitrary field in  $V_n$ . The vector field  $\tilde{u}^{(n)}$  associated to  $u^{(n)}$  by the inverse transformation (15) belongs to the space  $W_0^1(\tilde{\Omega})$ .

Inserting the test functions  $\tilde{u}^{(n)}$  in (14), and taking into consideration (16) and (17), we derive the estimate

$$\iint_{\Omega} \varepsilon_n(v, v)r \, dr \, dz \geq C_1 \iint_{\Omega} \left\{ \sum_{i=1}^3 r \left( \left| \frac{\partial v_i}{\partial r} \right|^2 + \left| \frac{\partial v_i}{\partial z} \right|^2 \right) + \frac{1}{r} [(n^2 + 1)(v_1^2 + v_2^2) + n^2 v_3^2 + 4nv_1 v_2] \right\} dr \, dz. \tag{19}$$

To conclude, we use the inequality

$$(n^2 + 1)(v_1^2 + v_2^2) + 4nv_1 v_2 \geq \begin{cases} (n - 1)^2(v_1^2 + v_2^2) & \text{if } n \neq 1 \\ 2(v_1 + v_2)^2 & \text{if } n = 1 \end{cases} \tag{20}$$

and the following Poincaré inequality

$$\iint_{\Omega} \left( \sum_{i=1}^3 \left| \frac{\partial v_i}{\partial z} \right|^2 \right) r \, dr \, dz \geq \frac{\pi^2}{4h^2} \iint_{\Omega} |v|^2 r \, dr \, dz, \text{ for } v \in V_n. \tag{21}$$

Hence, we have proved the following

**Proposition 3.1.** *The bilinear form  $a_n(u, v)$  is coercive i.e. there exists a constant  $K > 0$  (independent of  $n$ ) such that*

$$a_n(u, u) \geq K \|u\|_{V_n}^2, \quad \forall u \in V_n.$$

Applying the Kato first representation theorem (see [7]), we assert that  $a_n(., .)$  spans a self-adjoint operator  $A_n$  defined in  $H$  by:

$$\begin{cases} D(A_n) = \{u \in V_n ; \mathcal{A}_n u \in H \text{ and } \sigma_{rz}^n(u)(r, h) = \sigma_{\theta z}^n(u)(r, h) = \sigma_{zz}^n(u)(r, h) = 0\}, \\ \mathcal{A}_n u = A_n u \text{ for } u \in D(A_n). \end{cases}$$

Using the generalized Green’s formula, we obtain

$$a_n(u, v) = (A_n u, v) \quad \text{for } (u, v) \in D(A_n) \times V_n.$$

Notice that the traces  $\sigma_{\tau z}^n(u)(r, h)$ ,  $\tau = r, \theta, z$ , lie in the space  $H_{loc}^{-1/2}(\mathbb{R}^+)$ .

Now we are going to study the spectrum of the operator  $A_n$ , but in a preliminary step we study the essential spectrum. First of all we prove a result on the regularity of an element  $u \in D(A_n)$  in the exterior domain  $\Omega'_{R_1} = ]R_1, +\infty[ \times ]0, h[$  with  $R_1 > R$  (i.e. outside the perturbation). Such a result will be useful in Paragraph 5.1 to get a compactness result.

#### 4. A REGULARITY RESULT

We consider the equation:

$$A_n u = \tilde{f}, \quad (\tilde{f} \in H). \tag{22}$$

Due to the coerciveness of  $a_n(\cdot, \cdot)$ , the estimation  $\|u\|_{V_n} \leq C\|\tilde{f}\|_H$  follows. Our purpose now is to obtain a better estimation on  $\Omega'_{R_1}$ , namely:

$$\|\sqrt{r}u\|_{2,\Omega'_{R_1}} \leq C(\|\tilde{f}\|_H + \|u\|_{V_n}).$$

This inequality would be obvious if the domain was bounded. First of all we transform the equation (22). We set  $v = \sqrt{r}u$ , then  $v$  satisfies in  $\Omega'_R = ]R, +\infty[ \times ]0, h[$  the system:

$$\begin{cases} \mu_\infty \left( \frac{\partial^2 v_1}{\partial r^2} + \frac{\partial^2 v_1}{\partial z^2} \right) + (\lambda_\infty + \mu_\infty) \frac{\partial}{\partial r} \left( \frac{\partial v_1}{\partial r} + \frac{\partial v_3}{\partial z} \right) = f_1(v) \\ \frac{\partial^2 v_2}{\partial r^2} + \frac{\partial^2 v_2}{\partial z^2} = f_2(v) \\ \mu_\infty \left( \frac{\partial^2 v_3}{\partial r^2} + \frac{\partial^2 v_3}{\partial z^2} \right) + (\lambda_\infty + \mu_\infty) \frac{\partial}{\partial z} \left( \frac{\partial v_1}{\partial r} + \frac{\partial v_3}{\partial z} \right) = f_3(v) \end{cases}$$

where  $f(v) = (f_1(v), f_2(v), f_3(v))$  satisfies the inequality:

$$\|f(v)\|_{0,\Omega'_R} \leq c(\|\tilde{f}\|_H + \|u\|_{V_n}).$$

For  $\bar{v} = (v_1, v_3)$ , we consider the quantities:

$$\begin{cases} \sigma_{13}(\bar{v}) = \mu_\infty \left( \frac{\partial v_3}{\partial r} + \frac{\partial v_1}{\partial z} \right) \\ \sigma_{33}(\bar{v}) = (\lambda_\infty + 2\mu_\infty) \frac{\partial v_3}{\partial z} + \lambda_\infty \frac{\partial v_1}{\partial r}. \end{cases}$$

The condition of free surface  $\{\sigma_{rz}^n(u) = \sigma_{\theta z}^n(u) = \sigma_{zz}^n(u) = 0, \text{ for } z = h\}$  implies together with (6) that

$$\begin{cases} \sigma_{13}(\bar{v}) + \mu_\infty \frac{u_3}{2\sqrt{r}} = 0, & \text{for } z = h, \\ \sigma_{33}(\bar{v}) + \lambda_\infty \left( \frac{u_1}{2r} + \frac{u_1 + nu_2}{r\sqrt{r}} \right) = 0, & \text{for } z = h, \\ \frac{\partial v_2}{\partial z}(r, h) + \frac{n}{r\sqrt{r}} u_2(r, h) = 0. \end{cases}$$

Consequently, the traces  $\sigma_{13}(\bar{v})(r, h)$ ,  $\sigma_{33}(\bar{v})(r, h)$  and  $\frac{\partial v_2}{\partial z}(r, h)$  lie in the space  $H^{1/2}(R, +\infty)$ . Then the field  $\bar{v} \in [H^1(\Omega'_R)]^2$  satisfies the 2D elasticity problem:

$$(P_1) \quad \begin{cases} A\bar{v} = f & \text{in } \Omega'_R, \\ \bar{v}(r, 0) = 0, & \forall r > R, \\ \sigma_{13}(\bar{v})(r, h) = g_1(r), \quad \sigma_{33}(\bar{v})(r, h) = g_3(r), & \forall r > R, \end{cases}$$

with

$$A\bar{v} = \mu_\infty \Delta \bar{v} + (\lambda_\infty + \mu_\infty) \nabla(\nabla \cdot \bar{v}).$$

Moreover we have:

$$\|g_1\|_{\frac{1}{2}} + \|g_3\|_{\frac{1}{2}} \leq c\|u\|_{V_n} \quad (\text{with } \|g\|_s := \|g\|_{H^s}).$$

The function  $v_2 \in H^1(\Omega'_R)$  satisfies the scalar problem:

$$(P_2) \quad \begin{cases} \Delta v_2 = f_2, & \text{in } \Omega'_R, \\ v_2(r, 0) = 0, \\ \frac{\partial v_2}{\partial z}(r, h) = g_2(r), & \forall r > R, \end{cases}$$

with the estimation  $\|g_2\|_{\frac{1}{2}} \leq c\|u\|_{V_n}$ .

Now we look at the regularity of  $v$  in the unbounded domain  $\Omega'_{R_1}$ . Without loss of generality, we can suppose that  $R_1 = R + 2$ , then there exists a cut-off function  $\varphi \in D(]-1, 1[)$  such that:

$$\sum_{m \geq 0} \varphi(r - m) = 1, \quad \forall r > 0.$$

We let  $\varphi_m(r) = \varphi(r - m)$ ,  $I_m = ]m - 1, m + 1[$  and  $\Omega_m = I_m \times ]0, h[$ .

Setting  $v_m(r, z) = \varphi_m(r)v(r, z)$  where  $v$  is the solution of  $(P_1) - (P_2)$ , we see that  $\bar{v}_m = (v_{m1}, v_{m3})$  and  $v_{m2}$  satisfy regular elliptic boundary-value problems in the bounded domain  $\Omega_m$ . Hence  $v_m \in H^2(\Omega_m)^3$  and we have the following inequality:

$$\|v_m\|_{2, \Omega_m} \leq c(\|f\|_{0, \Omega} + \|g\|_{\frac{1}{2}}),$$

where the constant  $c$  is independent of  $m$  (the diameter of  $\Omega_m$  which is uniformly bounded). Since  $v = \sum_{m \geq m_0} v_m$

on  $\Omega'_{R_1}$  where  $m_0$  is the integer part of  $R_1$ , we obtain:

$$\|v\|_{2, \Omega} \leq c_1 \sum_{m \geq m_0} \|v_m\|_{2, \Omega_m} \leq c_2(\|f\|_{0, \Omega} + \|g\|_{\frac{1}{2}}).$$

Hence we have the result:

**Proposition 4.1.** *Let  $u \in V_n$  be a solution of (22), then  $u$  satisfies:*

- (1)  $\sqrt{r}u \in H^2(\Omega'_{R_1})^3$  with  $\Omega'_{R_1} = ]R_1, +\infty[ \times ]0, h[$  and  $R_1 > R$ ,
- (2)  $\|\sqrt{r}u\|_{2, \Omega'_{R_1}} \leq c(R_1)(\|\tilde{f}\|_H + \|u\|_{V_n})$ .

### 5. THE ESSENTIAL SPECTRUM OF $A_n$

The operator  $A_n$  is self-adjoint and non negative, then its spectrum is a subset of  $\mathbb{R}_+$ . Since the imbedding of  $V_n$  in  $H$  is not compact, the resolvent of  $A_n$  is not compact and its spectrum  $\sigma(A_n)$  is composed of a continuous part (the essential spectrum  $\sigma_e(A_n)$ ) and sometimes of a discrete part  $\sigma_d(A_n)$  corresponding to the eigenmodes. We recall (see [12]) that a real number  $\sigma \in \sigma_e(A_n)$  if and only if there exists a *singular sequence*  $u^p \in D(A_n)$  such that:

$$\|u^p\| = 1, \quad u^p \rightharpoonup 0 \text{ weakly in } H \quad \text{and} \quad A_n u^p - \sigma u^p \rightarrow 0 \text{ in } H. \tag{23}$$

Often the essential spectrum of the global operator  $A$ , defined on  $\tilde{\Omega}$  as Hilbertian sum of the operators  $A_n$ , coincides with the spectrum of the non-perturbed operator  $\bar{A}$  (corresponding to the homogeneous layer). The study of the reduced operators (obtained after performing Fourier transformation in the horizontal direction) enables us to predict the lower bound of  $\sigma(\bar{A})$ . Thus we should have  $\sigma_e(A_n) \subset \sigma(\bar{A})$ . But the initial hypothesis ( $\sigma_e(A) = \sigma(\bar{A})$ ) is not proved in this general case. We can prove this strictly (as in [5]) under a restrictive



hypothesis on Lamé coefficients, namely if  $\lambda$  and  $\mu$  are Lipschitz continuous functions, which restricts the generality. The aim of this section is to prove the inclusion  $\sigma_e(A_n) \subset \sigma(\bar{A})$  in the general case (*i.e.* when  $\lambda, \mu \in L^\infty(\Omega)$ ). First of all we establish a compactness result based on Proposition 4.1.

5.1. A compactness result

We recall some properties of weighted Sobolev spaces. To our knowledge these spaces have not been used before on the similar works in waveguides.

**Definitions.** For  $s, t \in \mathbb{R}$ , we define the Hilbert spaces:

$$L^{2,s}(\mathbb{R}) = \{u \in L^2_{loc}(\mathbb{R}) ; \rho^s u \in L^2(\mathbb{R})\}, \quad H^{s,t}(\mathbb{R}) = \{u \in S'(\mathbb{R}) ; F(\rho^t u) \in L^{2,s}(\mathbb{R})\}$$

with the norms  $\|u\|_{s,t} = \|F(\rho^t u)\|_{L^2_s}$  where  $F$  is the Fourier transform and  $\rho(x) = (1 + x^2)^{1/2}$ .

If  $I \subset \mathbb{R}$  is an unbounded interval we set  $H^{s,t}(I) = \{u = v|_I ; v \in H^{s,t}(\mathbb{R})\}$ .

We have the properties:

- 1) If  $s \geq s'$  and  $t \geq t'$  then the embedding  $H^{s,t}(\mathbb{R}) \subset H^{s',t'}(\mathbb{R})$  is continuous.
- 2) If  $s > s' \geq 0$  and  $t > t'$  then the embedding  $H^{s,t}(\mathbb{R}) \subset H^{s',t'}(\mathbb{R})$  is compact.
- 3) If  $s \geq 0$  and  $t \in \mathbb{R}$  then the dual  $(H^{s,t}(\mathbb{R}))' = H^{-s,-t}(\mathbb{R})$ .

**Proposition 5.1.** *Let  $(u^p)_{p \in \mathbb{N}}$  be a sequence of  $D(A_n)$  satisfying  $\|A_n u^p\|_H \leq c$  for all  $p$ . Then we can extract a subsequence, still denoted  $(u^p)$ , which converges weakly in  $V_n$  to an element  $u$  such that:*

- (i)  $\sqrt{r}u^p(\cdot, h) \rightarrow \sqrt{r}u(\cdot, h)$  strongly in  $Y^* = (H^{-\frac{1}{2},t}(\bar{R}, +\infty))^3$  with  $\bar{R} > R$  and  $t > 0$ ,
- (ii)  $\sqrt{r} \frac{\partial u^p}{\partial r}(\cdot, h) \rightarrow \sqrt{r} \frac{\partial u}{\partial r}(\cdot, h)$  weakly in  $Y = (H^{\frac{1}{2},-t}(\bar{R}, +\infty))^3$ ,
- (iii)  $u^p(\cdot, h) \rightarrow u(\cdot, h)$  strongly in  $(L^2(\bar{R}, +\infty))^3$ .

*Proof.* It follows from the coerciveness of  $a_n(\cdot, \cdot)$  that  $\|u^p\|_{V_n} \leq c$  for all  $p$ . Then there exists a subsequence which converges weakly in  $V_n$  to an element  $u$ . The proposition 4.1 shows that  $\forall p, \|\sqrt{r}u^p\|_{2,\Omega'_R} \leq c(\|A_n u^p\|_H + \|u^p\|_{V_n}) \leq c_1$ . This implies that the sequence of traces  $(\sqrt{r}u^p(\cdot, h))_p$  is bounded in the space  $H^{\frac{3}{2}}(\bar{R}, +\infty)^3$ . This last space is compactly embedded in the space  $Y = H^{\frac{1}{2},-t}(\bar{R}, +\infty)^3$  with  $t > 0$  arbitrary. Therefore  $\sqrt{r}u^p(\cdot, h)$  tends to  $\sqrt{r}u(\cdot, h)$  strongly in  $Y$ . Since the embedding  $Y \subset Y^* = H^{-\frac{1}{2},t}(\bar{R}, +\infty)^3$  is continuous, we obtain (i).

The sequence  $\sqrt{r} \frac{\partial u^p}{\partial r}(r, h)$  is bounded in  $H^{\frac{1}{2}}(\bar{R}, +\infty)^3$ , hence in  $Y$ , which proves the property (ii). The property (iii) follows from the fact that the sequence  $(u^p(\cdot, h))$  is bounded in  $H^{\frac{1}{2},\frac{1}{2}}(\bar{R}, +\infty)^3$  which is embedded compactly in  $L^2(\bar{R}, +\infty)^3$ . □

**Lemma 5.2.** *For every  $u \in D(A_n)$ :*

$$a_n(u, u) \geq \mu_\infty \iint_{\Omega'_R} |\nabla u|^2 r \, dr \, dz + \mu_\infty p_n(u, u), \quad (\bar{R} > R), \tag{24}$$

with

$$\begin{aligned} p_n(u, u) = & 2 \int_{\bar{R}}^{+\infty} \left[ r \frac{\partial u_3}{\partial r}(r, h) u_1(r, h) - n u_2(r, h) u_3(r, h) \right] dr \\ & + \int_0^h [u_1^2(\bar{R}, z) + u_2^2(\bar{R}, z) + 2n u_1(\bar{R}, z) u_2(\bar{R}, z)] dz \\ & - 2 \int_0^h \bar{R} \frac{\partial u_3}{\partial z}(\bar{R}, z) u_1(\bar{R}, z) dz \end{aligned}$$

*Proof.*  $a_n(.,.)$  can be written in the form:

$$a_n(u, u) = \iint_{\Omega} \lambda |\operatorname{div}^n u|^2 r \, dr \, dz + \iint_{\Omega} \mu \varepsilon_n(u, u) \, dr \, dz \tag{25}$$

with  $\operatorname{div}^n u$  given in (12) and

$$\begin{aligned} r\varepsilon_n(u, u) &= r|\operatorname{div}^n u|^2 + r|\nabla u|^2 + \frac{1}{r}((n^2 + 1)(u_1^2 + u_2^2) + n^2 u_3^2 + 4nu_1 u_2) \\ &\quad + 2r \left( \frac{\partial u_1}{\partial z} \frac{\partial u_3}{\partial r} - \frac{\partial u_1}{\partial r} \frac{\partial u_3}{\partial z} \right) - 2(nu_1 + u_2) \frac{\partial u_2}{\partial r} \\ &\quad - 2(u_1 + nu_2) \frac{\partial u_1}{\partial r} - 2n \frac{\partial}{\partial z} (u_2 u_3) - 2u_1 \frac{\partial u_3}{\partial z}. \end{aligned}$$

Using the inequality (20) the decomposition (25) gives

$$\begin{aligned} a_n(u, u) &\geq \mu_{\infty} \iint_{\Omega'_R} |\nabla u|^2 r \, dr \, dz \\ &\quad + 2\mu_{\infty} \iint_{\Omega'_R} \left[ r \left( \frac{\partial u_1}{\partial z} \frac{\partial u_3}{\partial r} - \frac{\partial u_1}{\partial r} \frac{\partial u_3}{\partial z} \right) - (nu_1 + u_2) \frac{\partial u_2}{\partial r} \right] \, dr \, dz \\ &\quad - 2\mu_{\infty} \iint_{\Omega'_R} \left[ (u_1 + nu_2) \frac{\partial u_1}{\partial r} + n \frac{\partial}{\partial z} (u_2 u_3) + u_1 \frac{\partial u_3}{\partial z} \right] \, dr \, dz. \end{aligned}$$

Since  $\sqrt{r}u \in H^2(\bar{\Omega}, +\infty[ \times ]0, h]^3)$  (see Prop. 4.1), we integrate twice by parts to get (24). □

The following proposition is the key result which will be useful in the next paragraph to determine the lower bound of the essential spectrum of  $A_n$ .

**Proposition 5.3.** *Let  $(u_p)$  be a singular sequence of  $D(A_n)$ . Then*

$$\lim_{p \rightarrow +\infty} p_n(u^p, u^p) = 0$$

*Proof.* It follows from the properties of singular sequence (see (23)) that  $u^p \rightharpoonup 0$  weakly in  $V_n$  and the sequence  $(A_n u^p)_p$  is bounded in  $H$ . The result follows from the regularity of  $u^p$  (see Prop. 4.1) and the compactness result (see Prop. 5.1) noting that when  $p \rightarrow +\infty$

$$\int_{\bar{R}}^{+\infty} u_1^p(r, h) \frac{\partial u_3^p}{\partial r}(r, h) r \, dr = \left\langle \sqrt{r} \frac{\partial u_3^p}{\partial r}(r, h), \sqrt{r} u_1^p(r, h) \right\rangle_{Y \times Y^*} \rightarrow 0.$$

□

### 5.2. The essential spectrum

The following lemma gives a lower bound for the essential spectrum of  $A_n$ .

**Lemma 5.4.** *For all  $n \in \mathbb{N}$  we have:*

$$\sigma_e(A_n) \subset [\omega_1^2, +\infty[ \quad \text{with } \omega_1^2 = \frac{\pi^2}{4h^2} c_S^2 \tag{26}$$

*Proof.* Let  $\omega^2 \in \sigma_e(A_n)$ . We suppose the contrary (*i.e.*  $\omega^2 < \omega_1^2$ ) and let  $(u_p)$  the associated singular sequence. The sequence  $(u^p)$  is bounded in  $V_n(\Omega_{\bar{R}}) = \{u = v|_{\Omega_{\bar{R}}}, v \in V_n\}$ ,  $\Omega_{\bar{R}} = ]0, \bar{R}[ \times ]0, h[$ , which is embedded compactly in  $L^2(\Omega_{\bar{R}})$ . Then we can extract a subsequence satisfying

$$\lim_{p \rightarrow \infty} \iint_{\Omega_{\bar{R}}} |u^p|^2 \rho r \, dr \, dz = 0. \tag{27}$$

Combining Lemma 5.2 and Poincaré inequality (21) we obtain the inequality

$$a_n(u^p, u^p) \geq \omega_1^2 \iint_{\Omega'_{\bar{R}}} |u^p|^2 \rho_{\infty} r \, dr \, dz + \mu_{\infty} p_n(u^p, u^p), \tag{28}$$

therefore

$$a_n(u^p, u^p) - \omega^2 \geq (\omega_1^2 - \omega^2) \iint_{\Omega'_{\bar{R}}} |u^p|^2 \rho_{\infty} r \, dr \, dz - \omega^2 \iint_{\Omega_{\bar{R}}} |u^p|^2 \rho r \, dr \, dz + \mu_{\infty} p_n(u^p, u^p). \tag{29}$$

As  $\lim_{p \rightarrow \infty} a_n(u^p, u^p) = \omega^2$ , the Proposition 5.3 together with (27) and (28) lead to

$$\lim_{p \rightarrow \infty} \iint_{\Omega'_{\bar{R}}} |u^p|^2 \rho_{\infty} r \, dr \, dz = 0,$$

which is in contradiction with  $\iint_{\Omega} |u|^2 \rho r \, dr \, dz = 1$ , which shows  $\omega^2 \geq \omega_1^2$ . □

**Theorem 5.5.** *For all  $n \in \mathbb{N}$  we have  $\sigma_e(A_n) = [\omega_1^2, +\infty[$ .*

*Proof.* We will now construct singular sequences to prove that the inclusion (26) is in fact an equality. For  $\sigma > \omega_1^2$ , we built a generalized eigenfunction  $u^\sigma$  of the non-perturbed operator  $\bar{A}_n$ , obtained from  $A_n$  by putting  $(\lambda, \mu, \rho) = (\lambda_{\infty}, \mu_{\infty}, \rho_{\infty})$ . To do this, let us consider the following problem:

$$\text{Find } u^\sigma \in D_{loc}(\bar{A}_n) \cap L^\infty(\Omega)^3 \text{ such that } \bar{A}_n u^\sigma = \sigma u^\sigma.$$

This problem can be solved explicitly by using the method of potentials (see for example [8]). The field  $u^\sigma$  is given by

$$\begin{cases} u_1^\sigma(r, z) = (J_{n-1}(kr) - \frac{n}{r} J_n(kr))(k g_1(z) + g_2'(z)) \\ u_2^\sigma(r, z) = -\frac{n}{r} J_n(kr)(g_1(z) + \frac{1}{k} g_2'(z)) \\ u_3^\sigma(r, z) = J_n(kr)(g_1(z) + k g_2'(z)) \end{cases}$$

where  $J_n(r)$  is the Bessel's function of the first kind and of order  $n$ , and

$$\begin{cases} g_1(z) = A(k) \cos(\alpha z) + B(k) \sin(\alpha z) \\ g_2(z) = C(k) \cos(\beta z) + D(k) \sin(\beta z) \end{cases}$$

with  $\alpha^2 = \frac{\sigma}{c_P^2} - k^2$ ,  $\beta^2 = \frac{\sigma}{c_S^2} - k^2$  and  $k = k_1(\sigma)$  is an analytic function defined implicitly by the following dispersion equation:

$$2k^2 \alpha \beta [\xi - \mu_{\infty}(k^2 - \beta^2)] + \alpha \beta [4k^2 \mu_{\infty} - \xi(k^2 - \beta^2)] \cos(\alpha h) \cos(\beta h) + k^2 [\xi(k^2 - \beta^2) - 4\mu_{\infty} \alpha^2 \beta^2] \sin(\alpha h) \sin(\beta h) = 0$$

with  $\xi = \lambda_\infty \frac{\sigma}{c_P^2} + 2\mu_\infty \alpha^2$ . The curve  $\sigma \rightarrow k_1(\sigma)$ ,  $\sigma > \omega_1^2$ , is plotted numerically, it is strictly increasing and has the properties:  $k_1(\sigma) > 0$  if  $\sigma > \omega_1^2$  and  $\lim_{\sigma \rightarrow \infty} \frac{\sigma}{k_1(\sigma)} = c_P^2$ .

The field  $u^\sigma$  is not in  $V_n$ . Consider now the sequence  $(u_p)$  in  $H$  defined as follows:

$$u_p(r, z) = \frac{1}{\sqrt{p}} \varphi_p(r) u^\sigma(r, z)$$

where  $(\varphi_p)$  is a sequence of cut-off functions defined by:

$$\begin{cases} \varphi_p(r) = \varphi\left(\frac{r}{p}\right), \\ \varphi \in C_0^\infty(]0, +\infty[), \varphi(r) = 0 \text{ if } r < R, \\ \int_0^{+\infty} |\varphi|^2 dr = 1. \end{cases}$$

The sequence  $(u^p)$  satisfies all the properties of a singular sequence. This proves that  $\sigma \in \sigma_e(A_n)$ . As  $\sigma_e(A_n)$  is a closed set, the theorem is proved. □

### 6. THE POINT SPECTRUM

We are interested now in the point spectrum of the operator  $A_n$ . The eigenvalues  $\omega^2$  such that  $\omega^2 < \omega_1^2$  form the discrete spectrum; they can be studied by means of the Min-Max principle (see [11]) as will be seen in the following. The study of the so-called *embedded eigenvalues* such that  $\omega^2 \geq \omega_1^2$  is much more complicated. Obviously, this discrete spectrum may be empty: it is the case for example if the layer is homogeneous. Therefore, our goal is to find out the types of perturbations for which eigenmodes exist.

**Hypothesis.** Suppose that there exists a rectangle  $G = ]a, b[ \times ]0, h[$ ,  $a \geq 0$ , such that:

$$\mu(r, z) = \mu_\infty \quad \text{and} \quad \rho(r, z) = \rho_\infty \quad \text{in } \Omega \setminus G. \tag{H1}$$

We put:  $\mu_0 = \sup_G \mu(r, z)$  and  $\lambda_0 = \sup_G \lambda(r, z)$ .

**Theorem 6.1.** *Suppose that the couple  $(\mu, \rho)$  satisfies hypothesis (H1), then there exists a number  $\delta > 0$  (which depends of  $(n, h, b - a, \mu_0, \lambda_0)$ ) such that the discrete spectrum of  $A_n$  is non empty if*

$$\mu_\infty - \mu_0 \geq \delta. \tag{H2}$$

*Proof.* We consider the Bessel's operator

$$\begin{cases} B_1 v = -\frac{1}{r} \frac{d}{dr} \left( r \frac{dv}{dr} \right) + \frac{v}{r^2} & \text{in } I = ]a, b[, \\ v(a) = v(b) = 0. \end{cases}$$

Denoting by  $v$  the eigenfunction of  $B_0$  associated to the first eigenvalue  $\alpha_1$ , we have:

$$\int_I \left( r \left| \frac{dv}{dr} \right|^2 + \frac{v^2}{r} \right) dr = \alpha_1 \quad \text{and} \quad \int_I |v|^2 r dr = 1.$$

Considering the function  $g(z) = \sqrt{\frac{2}{\rho_\infty h}} \sin(\frac{\omega_1}{c_S} z)$ , we can verify that:

$$\int_0^h |g|^2 \rho_\infty dz = 1 \quad \text{and} \quad \int_0^h |g'|^2 \mu_\infty dz = \omega_1^2.$$

Now we put  $u(r, z) = (w(r)g(z), 0, 0)$ , where  $w(r)$  is the extension by zero of  $v(r)$  outside the interval  $I$ . Then we have:

$$\begin{aligned} a_n(u, u) &= \iint_\Omega \left[ (\lambda + 2\mu)r \left| \frac{dw}{dr} \right|^2 |g|^2 + 2\lambda w \frac{dw}{dr} |g|^2 \right. \\ &\quad \left. + \frac{\lambda + (n^2 + 2)\mu}{r} |w|^2 |g|^2 + r\mu |w|^2 \left| \frac{dg}{dz} \right|^2 \right] dr dz \\ &= \iint_\Omega (\lambda + 2\mu)r \left( \frac{dw}{dr} + \frac{w}{r} \right)^2 |g|^2 dr dz - 4 \iint_\Omega \mu w \frac{dw}{dr} |g|^2 dr dz \\ &\quad + \iint_\Omega r\mu |w|^2 \left| \frac{dg}{dz} \right|^2 dr dz + n^2 \iint_\Omega \frac{\mu}{r} |w|^2 |g|^2 dr dz. \end{aligned}$$

On the other hand we have:

$$\begin{aligned} \iint_\Omega (\lambda + 2\mu)r \left( \frac{dw}{dr} + \frac{w}{r} \right)^2 |g|^2 dr dz &\leq \frac{\lambda_0 + 2\mu_0}{\rho_\infty} \int_I \left( \frac{dv}{dr} + \frac{v}{r} \right)^2 r dr \times \int_0^h |g|^2 \rho_\infty dz \\ &\leq \frac{\lambda_0 + 2\mu_0}{\rho_\infty} \alpha_1, \\ \iint_\Omega \frac{\mu}{r} |w|^2 |g|^2 dr dz &\leq \frac{\mu_0}{\rho_\infty} \int_I |v|^2 \frac{dr}{r} \leq \frac{\mu_0}{\rho_\infty} \alpha_1, \end{aligned}$$

and according to (H2) we have:

$$\iint_\Omega r\mu |w|^2 \left| \frac{dg}{dz} \right|^2 dr dz = \iint_G r\mu |v|^2 \left| \frac{dg}{dz} \right|^2 dr dz \leq \int_0^h (\mu_\infty - \delta) \left| \frac{dg}{dz} \right|^2 dz \leq \omega_1^2 - \delta \frac{\omega_1^2}{\mu_\infty}.$$

We also have

$$4 \iint_\Omega \mu |w| \left| \frac{dw}{dr} \right| |g|^2 dr dz \leq 4 \frac{\mu_0}{\rho_\infty} \int_I |v| \left| \frac{dv}{dr} \right| dr \leq 2 \frac{\mu_0}{\rho_\infty} \int_I \left( \frac{|v|^2}{r} + r \left| \frac{dv}{dr} \right|^2 \right) dr = 2\alpha_1 \frac{\mu_0}{\rho_\infty},$$

which gives

$$a_n(u, u) - \omega_1^2 \|u\|^2 \leq \alpha_1 \left( \frac{\lambda_0 + (n^2 + 4)\mu_0}{\rho_\infty} \right) - \delta \frac{\omega_1^2}{\mu_\infty}.$$

The left hand side is negative if  $\delta > \frac{4\alpha_1 h^2}{\pi^2} [\lambda_0 + (n^2 + 4)\mu_0]$ . The Min-Max principle proves that  $A_n$  has at least one eigenvalue less than  $\omega_1^2$ . □

**Remark 6.2.**

- 1) If  $a = 0$  then  $\alpha_1 = (\frac{\beta_1}{b})^2$  where  $\beta_1$  is the first root of Bessel function  $J_1(r)$ . More generally  $\alpha_1$  decreases when  $(b - a)$  grows.
- 2) Theorem 6.1 means that if in an elastic layer there is a region which is considerably softer than the rest then one would expect a "trapped mode" localised around the soft region, moreover it is a radial field. From the proof we can see that only a finite number of harmonics occur in the layer. This number increases with  $\delta_0 = \mu_\infty - \mu_0$ .

The following theorem complete the previous one.

**Theorem 6.3.** *If the following condition is satisfied:*

$$\iint_{\Omega_R} \left( \frac{\mu}{\rho} |g'|^2 - \omega_1^2 |g|^2 \right) \rho r \, dr \, dz < 0, \quad \text{with } g(z) = \sqrt{\frac{2}{\rho_\infty h}} \sin \left( \frac{\omega_1}{c_S} z \right), \tag{H3}$$

then the operator  $A_1$  has at least one discrete eigenvalue.

*Proof.* It suffices to find  $u \in V_1$  such that  $a_1(u, u) - \omega_1^2 \|u\|^2 < 0$ . For  $a > R$  we define  $\varphi_a(r)$  by:

$$\varphi_a(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq R, \\ \frac{\log \frac{a}{r}}{\log \frac{a}{R}} & \text{if } R \leq r \leq a, \\ 0 & \text{if } a \leq r. \end{cases}$$

We verify that:  $\sqrt{r}\varphi_a \in L^2(\mathbb{R}^+)$ ,  $\sqrt{r}\varphi'_a \in L^2(\mathbb{R}^+)$  and  $\lim_{a \rightarrow \infty} \int_0^\infty |\varphi'_a|^2 r \, dr = 0$ .

Putting  $u(r, z) = (\varphi_a(r)g(z), -\varphi_a(r)g(z), 0)$ , then  $u \in V_1$  and the decomposition (25) gives for  $n = 1$ :

$$a_1(u, u) - \omega_1^2 \|u\|^2 = b_1(u, u)$$

with

$$\begin{aligned} b_1(u, u) &= \iint_{\Omega} ((\lambda + 3\mu)|\varphi'_a|^2 |g|^2 + 2\mu|\varphi_a|^2 |g'|^2 - \omega_1^2 |g|^2 |\varphi_a|^2 \rho) r \, dr \, dz \\ &= (\lambda_\infty + 3\mu_\infty) \int_0^\infty |\varphi'_a|^2 r \, dr + 2 \iint_{\Omega_R} \left( \frac{\mu}{\rho} |g'|^2 - \omega_1 |g|^2 \right) \rho r \, dr \, dz. \end{aligned}$$

Under the hypothesis (H3), we can see that  $a_1(u, u) - \omega_1^2 \|u\|^2 < 0$  if  $a$  is sufficiently large and the result follows from the Min-Max Principle. □

**Remark 6.4.**

- 1) The condition (H3) is satisfied, for example, if  $\rho = \rho_\infty$  and  $\mu_\infty - \mu_0 > 0$ .
- 2) In Theorem 6.3 we consider the situation when  $\delta_0 = \mu_\infty - \mu_0$  is small. In this case there exists always a flexural mode (*i.e.* harmonic of order  $n = 1$ ).

## 7. CONCLUSION

We have studied in this article the spectrum of a self-adjoint operator which models the self-oscillations of an axisymmetric elastic layer considered as a locally perturbed homogeneous layer. In geophysics applications, the earth's layer is more often stratified, then it is important to extend the obtained results to perturbed stratified layer. Finally, we can think of the situation when the perturbation is not axisymmetric (the case where the harmonics in  $\theta$  are coupled).

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