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SOME MIXED FINITE ELEMENT METHODS ON ANISOTROPIC MESHES

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Abstract. The paper deals with some mixed finite element methods on a class of anisotropic meshes based on tetrahedra and prismatic (pentahedral) elements. Anisotropic local interpolation error estimates are derived in some anisotropic weighted Sobolev spaces. As particular applications, the numerical approximation by mixed methods of the Laplace equation in domains with edges is investigated where anisotropic finite element meshes are appropriate. Optimal error estimates are obtained using some anisotropic regularity results of the solutions.

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1. Introduction

Let us consider the Laplace equation and the Stokes system in a three-dimensional domain Ω. If Ω is smooth, then these solutions are smooth [15, 16], while if Ω is a polyhedral domain, then the solutions have in general singularities near the corners and the edges of Ω [4, 6, 11, 12]. Consequently if Ω is not convex, classical mixed finite element methods [13, 23, 24] on quasi-uniform meshes have a slow convergence rate.

For two-dimensional domains with corner singularities, the use of refined meshes in a neighbourhood of the singular corners allows to restore the optimal order of convergence [25, 26]. Our goal is then to extend the mesh refinement method in three-dimensional polyhedral domains in order to obtain an optimal order of convergence. For standard finite element method for the Laplace equation, it has been shown that anisotropic mesh grading (in the sense that elements in the refined region have an aspect ratio which grows to infinity as $h \to 0$, h being the global meshsize of the triangulation) is appropriate to compensate this effect and to obtain the optimal order of convergence [7, 17]. In [3, 4, 6] prismatic domains were considered. This restriction was made there because the authors wanted to focus on edge singularities, and such domains do not introduce additional corner singularities. The finite element meshes were graded perpendicularly to the edge and were quasi-uniform in the edge direction. In this paper, we extend these last results to some mixed FEM for the Laplace equation using Raviart-Thomas elements. The background is anisotropic regularity results of the solution of the Laplace

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equations on such prismatic domains obtained in [4]. The second step is to derive anisotropic local interpolation error estimates for functions from anisotropic weighted Sobolev spaces in the spirit of [1, 6, 20].

The outline of the paper is as follows. In Section 2 we describe two families of anisotropically graded finite element meshes based on prismatic elements and tetrahedral elements which turned out to be suited for the treatment of edge singularities [3, 4, 6]. We further introduce the Raviart-Thomas finite element spaces [23, 24] and show that the associated Raviart-Thomas interpolant is well-defined for functions in appropriate anisotropic weighted Sobolev spaces.

As usual our error analysis requires some local interpolation error estimates, which are investigated in Sections 3 and 4 for prismatic elements and tetrahedral elements respectively. Note that for tetrahedral elements, contrary to the habit, we use a composition of interpolation operators in order to avoid a geometrical obstacle.

In Section 5 we consider the mixed FEM of the Laplace equation. Using anisotropic regularity results from [4] and the results of Sections 3 and 4, we show that appropriate refined meshes lead to the optimal finite element error estimate

$$\|\mathbf{p} - \mathbf{p}_h\|_{0,\Omega} + \|u - u_h\|_{0,\Omega} \lesssim h|||f|||, \tag{1}$$

where $\|\cdot\|_{0,\Omega}$ is the standard norm of $L^2(\Omega)$, $\mathbf{p} = \nabla u$ and u is the solution of the Laplace equation, while $(\mathbf{p}_h, u_h)$ is the finite element solution approximating $(\mathbf{p}, u)$ by our mixed method. Finally $|||f|||$ is an appropriate norm of f (see the estimates (31) and (35) below). Hereabove and below, the notation $a \lesssim b$ means the existence of a positive constant C (which is independent of the meshsize h and of the function under consideration) such that $a \leq Cb$.

Let us notice that a similar analysis holds for the Stokes problem with Dirichlet boundary conditions using the regularity results from [6] and the interpolation error estimates from Sections 3 and 4 by introducing as new unknown the gradient of the velocity field.

2. Some finite element spaces

In the whole paper Ω is a prismatic domain of the form

$$\Omega = G \times Z, \tag{2}$$

where $G \subset \mathbb{R}^2$ is a bounded polygonal domain and $Z := (0, z_0) \subset \mathbb{R}$ is an interval. In this case the solutions of the Laplace equation have only edge singularities as we will show.

Without loss of generality, we may assume that the cross-section G has only one corner with interior angle $\omega \in (\pi, 2\pi)$ at the origin; thus Ω has only one "singular edge" which is part of the x_3-axis. The case of more than one singular edge introduces no additional difficulties because the edge singularities are of local nature. The properties of the solution will be described by using the anisotropic Sobolev spaces (of Kondratiev's type):

$$A^{1,p}_\beta(\Omega) := \{v \in \mathcal{D}'(\Omega) : \|v\|_{A^{1,p}_\beta(\Omega)} < \infty\}, \quad p \in (1, +\infty),\ \beta \in \mathbb{R}.$$

The norm is defined by

$$\|v\|^p_{A^{1,p}_\beta(\Omega)} := \|v\|^p_{0,p,\Omega} + \sum_{j=1,2} \|r^\beta \partial_j v\|^p_{0,p,\Omega} + \|\partial_3 v\|^p_{0,p,\Omega},$$

where $r(x) = (x_1^2 + x_2^2)^{1/2}$ is the distance between $x = (x_1, x_2, x_3)$ and the singular edge; the norm $\|\cdot\|_{0,p,\Omega}$ being the standard $L^p(\Omega)$-norm.

We now define families of meshes $\mathcal{Q}_h = \{Q\}$ and $\mathcal{T}_h = \{K\}$ by introducing in G the standard mesh grading for two-dimensional corner problems, see for example [21, 22]. Let $\{T\}$ be a regular triangulation of G in Ciarlet's

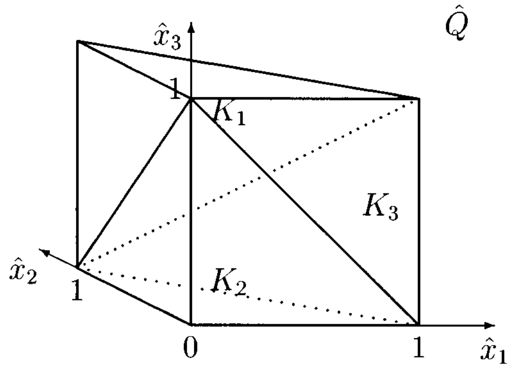

FIGURE 1. The basic reference prism and its subdivision into three tetrahedra.

sense [10, p. 124]; the elements are triangles. With h being the global mesh parameter, $\mu \in (0, 1]$ being the grading parameter, r_T being the distance of T to the corner,

$$r_T := \inf_{(x_1,x_2)\in T} (x_1^2 + x_2^2)^{1/2},$$

we assume that the element size $h_T := \operatorname{diam} T$ satisfies

$$h_T \sim \begin{cases} h^{1/\mu} & \text{for } r_T = 0, \\ h r_T^{1-\mu} & \text{for } r_T > 0. \end{cases}$$

This graded two-dimensional mesh is now extended in the third dimension using a uniform mesh size h. In this way we obtain a pentahedral (or prismatic) triangulation $\mathcal{Q}_h$ or, by dividing each pentahedron into three tetrahedra, a tetrahedral triangulation $\mathcal{T}_h$ of Ω, see Figures 1 and 2 for an illustration. Note that the number of elements is of the order h^{-3} for the full range of μ.

Let r_Q and r_K be the distance of an element Q or K to the edge (x_3-axis), respectively. Then the element sizes $h_{i,Q}$ (length of the projection of Q on the x_i-axis) satisfy

$$h_{3,Q} \sim h, \quad h_{1,Q} \sim h_{2,Q} \sim \begin{cases} h^{1/\mu} & \text{for } r_Q = 0, \\ h r_Q^{1-\mu} & \text{for } r_Q > 0. \end{cases} \tag{3}$$

The element sizes $h_{i,K}$ for tetrahedral elements K satisfy the same properties since $h_{i,K} = h_{i,Q}$ and $r_K = r_Q$ if $K \subset Q$.

On $\mathcal{T}_h$ we introduce the (Raviart-Thomas) finite element space X_h as follows:

$$X_h := \{\mathbf{p}_h \in H(\operatorname{div}, \Omega) : \mathbf{p}_h|_K \in RT_0(K), \forall K \in \mathcal{T}_h\}, \tag{4}$$

where the set $RT_0(K)$ is the Raviart-Thomas finite element defined by (see [23, 27])

$$RT_0(K) = \{\mathbf{p}(x) = \mathbf{a} + b\,\mathbf{x} : \mathbf{a} \in \mathbb{R}^3, b \in \mathbb{R}\},$$

where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

An appropriate choice of X_h for pentahedral meshes $\mathcal{Q}_h$ is (see Def. 4 of [19])

$$X_h := \{\mathbf{p}_h \in H(\operatorname{div}, \Omega) : \mathbf{p}_h|_Q \in \mathcal{D}_{1,0}(Q), \forall Q \in \mathcal{Q}_h\}, \tag{5}$$

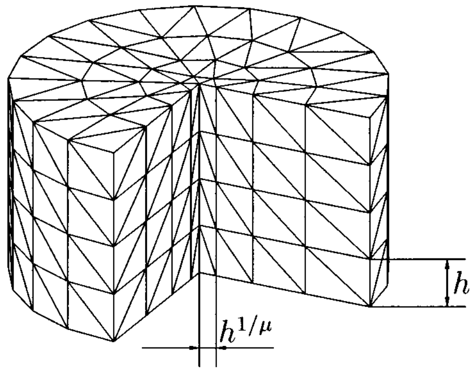

FIGURE 2. Example for an anisotropic mesh.

where the set $\mathcal{D}_{1,0}(Q)$ is defined by

$$\mathcal{D}_{1,0}(Q) = \{\mathbf{p}(x) = \begin{pmatrix} a_1 + bx_1 \\ a_2 + bx_2 \\ a_3 + cx_3 \end{pmatrix} : a_i, b, c \in \mathbb{R}, i = 1, 2, 3\}.$$

For the sake of shortness, we use the same symbol for the space defined by (4) on $\mathcal{T}_h$ and the space defined by (5) on $\mathcal{Q}_h$. The space X_h which will be used will be indicated by the context.

Our error estimates in the next sections will be derived using the standard interpolation operator Π_h^t defined, for tetrahedral triangulations, by

$$\int_F (\Pi_h^t \mathbf{p}|_K) \cdot \mathbf{n}_F \, \mathrm{d}\sigma = \int_F (\mathbf{p}|_K) \cdot \mathbf{n}_F \, \mathrm{d}\sigma, \forall F \in \mathcal{F}_K, \forall K \in \mathcal{T}_h, \tag{6}$$

where $\mathcal{F}_K$ denotes the set of faces of K and $\mathbf{n}_F$ the outward normal vector on F. For pentahedral triangulations, we simply replace K and $\mathcal{T}_h$ by Q and $\mathcal{Q}_h$ respectively and write this operator Π_h^q.

In view of the regularity results from Section 5, we need to show that these interpolation operators are well defined for any element from $A_\beta^{1,p}(\Omega)^3$.

Lemma 2.1. *Let $\beta \in [0, 1)$ and $p \geq 2$. Then any $\mathbf{p} \in A_\beta^{1,p}(\Omega)^3$ satisfies*

$$\mathbf{p} \cdot \mathbf{n}_F \in L^1(F), \forall F \in \mathcal{F}_K, K \in \mathcal{T}_h. \tag{7}$$

Proof. By Hölder's inequality, we have the embedding (which is meaningful since $\frac{2}{\beta+2/p} > 1$)

$$A_\beta^{1,p}(\Omega) \hookrightarrow W^{1,t}(\Omega), \forall t \in (1, \min\{p, \frac{2}{\beta + 2/p}\} = \frac{2}{\beta + 2/p}).$$

Therefore any $\mathbf{p} \in A_\beta^{1,p}(\Omega)^3$ satisfies $\mathbf{p} \in W^{1,t}(\Omega)^3$. A usual trace theorem leads to (since $t > 1$)

$$\mathbf{p}_{|F} \in W^{1-\frac{1}{t},t}(F)^3 \hookrightarrow L^1(F)^3.$$

$\square$

At the end, we define the finite dimensional subspace M_h of $L^2(\Omega)$ corresponding to $\mathcal{T}_h$ as follows:

$$M_h = \{v_h \in L^2(\Omega) : v_{h|K} \in \mathcal{P}_0(K), \forall K \in \mathcal{T}_h\}. \tag{8}$$

We similarly define M_h for the prismatic triangulation $\mathcal{Q}_h$ by simply replacing $\mathcal{T}_h$ by $\mathcal{Q}_h$.

3. Local and global interpolation error estimates for prismatic triangulations

We start with technical estimates which will be useful later on.

Lemma 3.1. *Let $Q \in \mathcal{Q}_h$, $\beta \in [0, 1)$ and $p \geq 2$. Then for any $\mathbf{p} \in A^{1,p}_\beta(\Omega)^3$ such that $\operatorname{div} \mathbf{p} \in L^p(\Omega)$, it holds*

$$\|\partial_i \Pi_h^q \mathbf{p}\|_{0,p,Q} \leq \frac{1}{2}\|\operatorname{div} \mathbf{p}\|_{0,p,Q} + \|\partial_3 p_3\|_{0,p,Q}, \forall i = 1, 2, 3. \tag{9}$$

Proof. We start with the case $i = 3$. In that case, by definition (6) and Green's formula, we obtain

$$\partial_3 (\Pi_h^q \mathbf{p})_{3|Q} = \frac{1}{|Q|}\int_Q \partial_3 p_3 \, dx.$$

This identity and Hölder's inequality yield (9) for $i = 3$ since $\partial_3(\Pi_h^q \mathbf{p})_1 = \partial_3(\Pi_h^q \mathbf{p})_2 = 0$.

For $i = 1$ or 2, it suffices to estimate $\|\operatorname{div} \Pi_h^q \mathbf{p}\|_{0,p,Q}$. As before Green's formula implies that

$$\operatorname{div} \Pi_h^q \mathbf{p} = \frac{1}{|Q|}\int_Q \operatorname{div} \mathbf{p} \, dx,$$

and therefore

$$\|\operatorname{div} \Pi_h^q \mathbf{p}\|_{0,p,Q} \leq \|\operatorname{div} \mathbf{p}\|_{0,p,Q}. \tag{10}$$

$\square$

We are now ready to establish our local interpolation error estimates:

Theorem 3.2. *Let $\beta \in [0, 2/3)$ and $p \geq 2$ such that $\beta \leq 1 - \mu$. Then for any $\mathbf{p} \in A^{1,p}_\beta(\Omega)^3$ such that $\operatorname{div} \mathbf{p} \in L^p(\Omega)$, we have*

$$\|\mathbf{p} - \Pi_h^q \mathbf{p}\|_{0,p,Q} \lesssim \sum_{k=1}^{3} h_{k,Q}^{1-\beta_k} \|r^{\beta_k} \partial_k \mathbf{p}\|_{0,p,Q} + h\|\operatorname{div} \mathbf{p}\|_{0,p,Q}, \tag{11}$$

with $\beta_1 = \beta_2 = \beta$, $\beta_3 = 0$ if Q is along the singular edge and $\beta_1 = \beta_2 = \beta_3 = 0$ else.

Proof. Fix two lateral faces F_i, $i = 1, 2$ of Q and denote by $F_3 = F_t$ the top face of Q and let $\mathbf{n}^{(i)}$ be the normal vector on F_i. Then (6) implies that $(\mathbf{p} - \Pi_h^q \mathbf{p}) \cdot \mathbf{n}^{(i)}$ has a vanishing average on F_i. Consequently by Lemma 4.2 of [20] if $r_Q = 0$ and Corollary 4.3 of [20] if $r_Q > 0$, we get

$$\|(\mathbf{p} - \Pi_h^q \mathbf{p}) \cdot \mathbf{n}^{(i)}\|_{0,p,Q} \lesssim \sum_{k=1}^{3} h_{k,Q}^{1-\beta_k} \|r^{\beta_k} \partial_k (\mathbf{p} - \Pi_h^q \mathbf{p})\|_{0,p,Q}.$$

Since the condition $\beta \leq 1 - \mu$ and the refinement rules imply that $h_{k,Q}^{1-\beta_k} \lesssim h$, for $k = 1, 2$, and $\beta_k \geq 0$, by Lemma 3.1, we obtain

$$\|(\mathbf{p} - \Pi_h^q \mathbf{p}) \cdot \mathbf{n}^{(i)}\|_{0,p,Q} \lesssim \sum_{k=1}^{3} h_{k,Q}^{1-\beta_k} \|r^{\beta_k} \partial_k \mathbf{p}\|_{0,p,Q} + h\|\operatorname{div} \mathbf{p}\|_{0,p,Q}.$$

To conclude, we notice that

$$\begin{pmatrix} (\mathbf{p} - \Pi_h^q \mathbf{p})_1 \\ (\mathbf{p} - \Pi_h^q \mathbf{p})_2 \\ (\mathbf{p} - \Pi_h^q \mathbf{p})_3 \end{pmatrix} = N^{-1} \begin{pmatrix} (\mathbf{p} - \Pi_h^q \mathbf{p}) \cdot \mathbf{n}^{(1)} \\ (\mathbf{p} - \Pi_h^q \mathbf{p}) \cdot \mathbf{n}^{(2)} \\ (\mathbf{p} - \Pi_h^q \mathbf{p}) \cdot \mathbf{n}^{(3)} \end{pmatrix},$$

when $N = \begin{pmatrix} \mathbf{n}^{(1)} \\ \mathbf{n}^{(2)} \\ \mathbf{n}^{(3)} \end{pmatrix}$ is the 3×3 matrix made with $\mathbf{n}^{(1)}$, $\mathbf{n}^{(2)}$ and $\mathbf{n}^{(3)}$ as its rows. Since $\mathbf{n}^{(3)} = (0, 0, 1)$, we readily check that

$$\|N^{-1}\| \lesssim \frac{1}{\sin \theta_0},$$

where θ_0 is the minimal angle of the projection of Q on the x_1, x_2-plane, $\| \cdot \|$ being the Euclidean matrix norm. We conclude because the triangulation on the basis G is regular. □

We now deduce the global interpolation error estimate in X_h.

Theorem 3.3. *Let $\beta \in [0, 2/3)$ satisfy $\beta \leq 1 - \mu$. Then for any $\mathbf{p} \in A_\beta^{1,2}(\Omega)^3$ such that $\operatorname{div} \mathbf{p} \in L^2(\Omega)$, we have*

$$\|\mathbf{p} - \Pi_h^q \mathbf{p}\|_{0,\Omega} \lesssim h\{\|\mathbf{p}\|_{A_\beta^{1,2}(\Omega)^3} + \|\operatorname{div} \mathbf{p}\|_{0,\Omega}\}. \tag{12}$$

Proof. The estimation of the global error is reduced to the evaluation of the local errors where we distinguish between the elements away from the singular edge ($r_Q > 0$) and the elements touching the edge ($r_Q = 0$).

For all elements Q with $r_Q > 0$, estimate (11) (with $\beta_k = 0$) implies that

$$\|\mathbf{p} - \Pi_h^q \mathbf{p}\|_{0,Q} \lesssim \sum_{k=1}^{3} h_{k,Q} \|\partial_k \mathbf{p}\|_{0,Q} + h\|\operatorname{div} \mathbf{p}\|_{0,Q}.$$

Since for $\beta \geq 0$, $r^{-\beta} \leq r_Q^{-\beta}$ in Q, the above estimate becomes

$$\|\mathbf{p} - \Pi_h^q \mathbf{p}\|_{0,Q} \lesssim \sum_{k=1}^{3} h_{k,Q} r_Q^{-\beta_k} \|r^{\beta_k} \partial_k \mathbf{p}\|_{0,Q} + h\|\operatorname{div} \mathbf{p}\|_{0,Q},$$

with $\beta_1 = \beta_2 = \beta$ and $\beta_3 = 0$. Since refinement rules (3) and the condition $\beta \leq 1 - \mu$ imply that $h_{k,Q} r_Q^{-\beta_k} \lesssim h$ for $k = 1, 2, 3$, we get

$$\|\mathbf{p} - \Pi_h^q \mathbf{p}\|_{0,Q} \lesssim h\{\sum_{k=1}^{3} \|r^{\beta_k} \partial_k \mathbf{p}\|_{0,Q} + \|\operatorname{div} \mathbf{p}\|_{0,Q}\}. \tag{13}$$

Consider now the elements Q with $r_Q = 0$. Then estimate (11) with $\beta_1 = \beta_2 = \beta$ and $\beta_3 = 0$ yield

$$\|\mathbf{p} - \Pi_h^q \mathbf{p}\|_{0,Q} \lesssim \sum_{k=1}^{3} h_{k,Q}^{1-\beta_k} \|r^{\beta_k} \partial_k \mathbf{p}\|_{0,Q} + h\|\operatorname{div} \mathbf{p}\|_{0,Q}.$$

Again refinement rules (3) and the condition $\beta \leq 1 - \mu$ imply that $h_{k,Q}^{1-\beta_k} \lesssim h$ for $k = 1, 2, 3$, consequently the above estimate implies that (13) still holds in this case.

We have just shown that (13) holds for all elements $Q \in \mathcal{Q}_h$. The sum of the square of this estimate on all $Q \in \mathcal{Q}_h$ yields (12). □

To finish this section, we state the global interpolation error estimate in M_h.

Theorem 3.4. *Let $\beta \in [0, 2/3)$ satisfy $\beta \leq 1-\mu$, $u \in A^{1,2}_\beta(\Omega)$ and denote by $\rho_h u$ the orthogonal projection of u on M_h (with respect to the $L^2(\Omega)$-inner product, which is meaningful). Then one has*

$$\|u - \rho_h u\|_{0,\Omega} \lesssim h\|u\|_{A^{1,2}_\beta(\Omega)}. \tag{14}$$

Proof. We first remark that the condition $\beta \in [0, 2/3)$ guarantees the compact embedding

$$A^{1,2}_\beta(\Omega) \hookrightarrow_c L^2(\Omega).$$

This gives a meaning to $\rho_h u$ for $u \in A^{1,2}_\beta(\Omega)$ as well as the estimate (as in Th. 3.2)

$$\|u - \rho_h u\|_{0,Q} \lesssim \sum_{k=1}^{3} h^{1-\beta_k}_{k,Q} \|r^{\beta_k} \partial_k u\|_{0,Q},$$

with β_k defined as in Theorem 3.2. The refinement rules then yield as above

$$\|u - \rho_h u\|_{0,Q} \lesssim h\{\sum_{k=1,2} \|r^\beta \partial_k u\|_{0,Q} + \|\partial_3 u\|_{0,Q}\}.$$

The sum of the square of this estimate on $Q \in \mathcal{Q}_h$ leads to the conclusion. □

4. Local and global interpolation error estimates for tetrahedral triangulations

For the tetrahedral triangulations introduced in Section 2, the arguments of Theorem 3.2 partially fail. Indeed some tetrahedra of the triangulation $\mathcal{T}_h$ do not have faces such that the third component-of the unit normal vector is uniformly bounded from below (this is the case of K_3 in Fig. 1), actually such tetrahedra do not satisfy the regular vertex property from [1] due to the anisotropy (see Rem. 6 in [1]). Therefore we need to modify the above arguments. We first show the following technical result.

Lemma 4.1. *Let $K \in \mathcal{T}_h$ and Q be the unique prism of $\mathcal{Q}_h$ such that $K \subset Q$. Then for all $\mathbf{p} \in \mathcal{D}_{1,0}(Q)$, it holds*

$$\|\mathbf{p} - \Pi^t_h \mathbf{p}\|_{0,p,K} \lesssim h\{\|\partial_3 p_3\|_{0,p,K} + \|\operatorname{div} \mathbf{p}\|_{0,p,K}\}, \tag{15}$$

for all $p > 1$.

Proof. Since $Q = T \times I$, where T is a triangle on the basis G, we make the change of variables $\mathbf{x} = B_Q \hat{\mathbf{x}} + b_Q$, from $\hat{Q} = \hat{T} \times (0,1)$ onto Q where the 3×3 matrix B_Q has the form

$$B_Q = \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{B_T}} & 0 \\ & & 0 \\ 0 & 0 & h_{3,Q} \end{pmatrix},$$

where B_T is a 2×2 matrix which satisfies (since the triangulation on G is regular)

$$\|B_T\| \sim h_T \sim h_{1,Q} \sim h_{2,Q}.$$

Setting $\hat{\mathbf{p}}(\hat{x}) = B_Q^{-1}\mathbf{p}(B_Q\hat{\mathbf{x}} + b_Q)$ we know that (see *e.g.* [18, Sects. 1.1 and 1.3] and [19, Sects. 2.1 and 2.3]) $\hat{\mathbf{p}}$ belongs to $\mathcal{D}_{1,0}(\hat{Q})$ and

$$\widehat{\Pi_h^t \mathbf{p}} = \hat{\Pi}^t \hat{\mathbf{p}}, \operatorname{div} \mathbf{p} = \operatorname{div} \hat{\mathbf{p}},\ \partial_3 p_3 = \partial_3 \hat{p}_3, \tag{16}$$

where $\hat{\Pi}^t$ is the Raviart-Thomas interpolation operator on $\hat{K}$.

On $\hat{K}$, $\hat{\mathbf{p}}$ may be written

$$\hat{\mathbf{p}}(\hat{x}) = \mathbf{a} + \begin{pmatrix} b\,\hat{x}_1 \\ b\,\hat{x}_2 \\ c\,\hat{x}_3 \end{pmatrix} = \hat{\mathbf{p}}_1(\hat{x}) + (c-b)\begin{pmatrix} 0 \\ 0 \\ \hat{x}_3 \end{pmatrix},$$

for $\mathbf{a} \in \mathbb{R}^3$ and $b, c \in \mathbb{R}$ and $\hat{\mathbf{p}}_1 \in RT_0(\hat{K})$. Therefore, we get

$$\hat{\mathbf{p}} - \hat{\Pi}^t \hat{\mathbf{p}} = (c-b)(I - \hat{\Pi}^t)\begin{pmatrix} 0 \\ 0 \\ \hat{x}_3 \end{pmatrix}.$$

This identity directly leads to the estimate

$$\|\hat{\mathbf{p}} - \hat{\Pi}^t \hat{\mathbf{p}}\|_{0,p,\hat{K}} \lesssim |c-b|.$$

Since $\operatorname{div} \hat{\mathbf{p}} = 2b + c$ and $\partial_3 \hat{p}_3 = c$, by the triangular inequality, we arrive at

$$\|\hat{\mathbf{p}} - \hat{\Pi}^t \hat{\mathbf{p}}\|_{0,p,\hat{K}} \lesssim |\operatorname{div} \hat{\mathbf{p}}| + |\partial_3 \hat{p}_3|. \tag{17}$$

The conclusion follows from identity (16), the above change of variables and the fact that $\operatorname{div} \mathbf{p}$ and $\partial_3 p_3$ are constant. $\square$

Let us now pass to the local interpolation error estimates:

Theorem 4.2. *Let $K \in \mathcal{T}_h$ and Q be the unique prism of $\mathcal{Q}_h$ such that $K \subset Q$. Let $\beta \in [0, 2/3)$ and $p \geq 2$ such that $\beta \leq 1 - \mu$. Then for any $\mathbf{p} \in A_\beta^{1,p}(\Omega)^3$ such that $\operatorname{div} \mathbf{p} \in L^p(\Omega)$, we have*

$$\|\mathbf{p} - \Pi_h^t \Pi_h^q \mathbf{p}\|_{0,p,K} \lesssim \sum_{k=1}^{3} h_{k,Q}^{1-\beta_k} \|r^{\beta_k} \partial_k \mathbf{p}\|_{0,p,Q} + h\|\operatorname{div} \mathbf{p}\|_{0,p,Q}, \tag{18}$$

with $\beta_1 = \beta_2 = \beta$, $\beta_3 = 0$ if Q is along the singular edge and $\beta_1 = \beta_2 = \beta_3 = 0$ else.

Proof. We first use the identity

$$\mathbf{p} - \Pi_h^t \Pi_h^q \mathbf{p} = \mathbf{p} - \Pi_h^q \mathbf{p} + (I - \Pi_h^t)\Pi_h^q \mathbf{p}.$$

By the triangular inequality and estimates (11) and (15), we obtain

$$\begin{aligned} \|\mathbf{p} - \Pi_h^t \Pi_h^q \mathbf{p}\|_{0,p,K} \quad \lesssim \quad & \sum_{k=1}^{3} h_{k,Q}^{1-\beta_k} \|r^{\beta_k} \partial_k \mathbf{p}\|_{0,p,Q} + h\|\operatorname{div} \mathbf{p}\|_{0,p,Q} \\ & + h\{\|\partial_3 (\Pi_h^q \mathbf{p})_3\|_{0,p,K} + \|\operatorname{div} \Pi_h^q \mathbf{p}\|_{0,p,K}\}. \end{aligned} \tag{19}$$

Estimates (9) and (10) into the above one yield (18). $\square$

Remark 4.3. Note that the use of the interpolation operator $\Pi_h^t \Pi_h^q$ instead of Π_h^t comes from the fact that one can show that there exist tetrahedral elements $K \in \mathcal{T}_h$ (like K_3 in Fig. 1) with $r_K = 0$ and some polynomial vector fields $\mathbf{p} \in \mathcal{P}_1(K)^3$ for which the estimate

$$\|\mathbf{p} - \Pi_h^t \mathbf{p}\|_{0,p,K} \lesssim \sum_{k=1}^{3} h_{k,Q}^{1-\beta_k} \|r^{\beta_k} \partial_k \mathbf{p}\|_{0,p,K} + h\|\operatorname{div} \mathbf{p}\|_{0,p,K},$$

with $\beta_1 = \beta_2 = \beta$, $\beta_3 = 0$, fails. Therefore such an estimate cannot hold for all $\mathbf{p} \in A_\beta^{1,p}(\Omega)^3$ such that $\operatorname{div} \mathbf{p} \in L^p(\Omega)$. $\square$

As in the previous section, estimates (18) and refinement rules (3) lead to the global interpolation error estimate in X_h.

Theorem 4.4. *Let $\beta \in [0, 2/3)$ satisfy $\beta \leq 1 - \mu$. Then for any $\mathbf{p} \in A_\beta^{1,2}(\Omega)^3$ such that $\operatorname{div} \mathbf{p} \in L^2(\Omega)$, we have*

$$\|\mathbf{p} - \Pi_h^t \Pi_h^q \mathbf{p}\|_{0,\Omega} \lesssim h\{\|\mathbf{p}\|_{A_\beta^{1,2}(\Omega)^3} + \|\operatorname{div} \mathbf{p}\|_{0,\Omega}\}. \tag{20}$$

Note that for $\mathbf{p} \in A_\beta^{1,2}(\Omega)^3$, with $\beta < 1$, $\Pi_h^t \Pi_h^q \mathbf{p}$ belongs to X_h since $\Pi_h^q \mathbf{p}$ belongs to $H(\operatorname{div}, \Omega)$ and is smooth enough.

5. Mixed formulation of the Laplace equation

Let $f \in L^2(\Omega)$ and let $u \in \mathring{H}^1(\Omega)$ be the unique solution of the Dirichlet problem for the Laplace equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma = \partial\Omega. \end{cases} \tag{21}$$

For our future uses, we recall the next regularity result which follows directly from Corollary 2.7 of [4].

Theorem 5.1. *The solution $u \in \mathring{H}^1(\Omega)$ of problem (21) with $f \in L^2(\Omega)$ satisfies*

$$\partial_3 u \in H^1(\Omega), \partial_k u \in A_\beta^{1,2}(\Omega), k = 1, 2, \tag{22}$$

for all $1 > \beta > 1 - \lambda$, with $\lambda := \frac{\pi}{\omega}$. Moreover we have the estimate

$$\|\partial_3 u\|_{1,\Omega} + \sum_{k=1,2} \|\partial_k u\|_{A_\beta^{1,2}(\Omega)} \lesssim \|f\|_{0,\Omega}. \tag{23}$$

The mixed formulation of (21) is well known (see *e.g.* [23, 24, 27]) and consists in finding $(\mathbf{p}, u)$ in $X \times M$ solution of

$$\begin{cases} \int_\Omega \mathbf{p} \cdot \mathbf{q} \, \mathrm{d}x + \int_\Omega u \operatorname{div} \mathbf{q} \, \mathrm{d}x = 0, \forall \mathbf{q} \in X, \\ \\ \int_\Omega v \operatorname{div} \mathbf{p} \, \mathrm{d}x = -\int_\Omega f v \, \mathrm{d}x, \quad \forall v \in M, \end{cases} \tag{24}$$

where $X = H(\operatorname{div}, \Omega)$ and $M = L^2(\Omega)$. Since this problem has at most one solution [27, p. 16], the unique solution $(\mathbf{p}, u)$ is given by $\mathbf{p} = \nabla u$, when u is the unique solution of (21).

The discrete problem associated with (24) is to find $(\mathbf{p}_h, u_h) \in X_h \times M_h$ such that

$$\begin{cases} \int_\Omega \mathbf{p}_h \cdot \mathbf{q}_h \, \mathrm{d}x + \int_\Omega u_h \operatorname{div} \mathbf{q}_h \, \mathrm{d}x = 0, \forall \mathbf{q}_h \in X_h, \\ \\ \int_\Omega v_h \operatorname{div} \mathbf{p}_h \, \mathrm{d}x = -\int_\Omega f v_h \, \mathrm{d}x, \quad \forall v_h \in M_h. \end{cases} \tag{25}$$

Existence and uniqueness of this problem (25) are based on the following surjectivity result (compare with Th. 4 of [23]), which also guarantees the uniform discrete inf-sup condition.

Lemma 5.2. *Let $v_h \in M_h$. Then there exists $\mathbf{q}_h \in X_h$ such that*

$$\operatorname{div} \mathbf{q}_h = v_h \text{ in } \Omega. \tag{26}$$

Assume further that the refinement parameter μ satisfies $\mu < \lambda$ and $\mu \geq \frac{1}{2}$. Then it holds

$$\|\mathbf{q}_h\|_{0,\Omega} \lesssim \|v_h\|_{0,\Omega}. \tag{27}$$

Proof. We start with the case of prismatic triangulations. Let $\psi \in \overset{\circ}{H}{}^1(\Omega)$ be the unique solution of $\Delta\psi = v_h$. As $v_h \in L^2(\Omega)$, by Theorem 5.1, $\mathbf{q} = \nabla\psi$ has the regularities (22). By Lemma 2.1, we may take

$$\mathbf{q}_h = \Pi_h^q \mathbf{q}.$$

By Green's formula, for all $Q \in \mathcal{Q}_h$, it satisfies

$$\operatorname{div} \mathbf{q}_{h|Q} = |Q|^{-1} \int_Q \operatorname{div} \mathbf{q}_h \, \mathrm{d}x = |Q|^{-1} \int_Q \operatorname{div} \mathbf{q} \, \mathrm{d}x = |Q|^{-1} \int_Q v_h \, \mathrm{d}x = v_{h|Q}.$$

This proves (26) since $\mathbf{q}_h$ belongs to X.

Furthermore by Theorem 3.3 with $1 - \lambda < \beta \leq 1 - \mu$, we have

$$\|\mathbf{q} - \mathbf{q}_h\|_{0,\Omega} \lesssim h\{\|\mathbf{q}\|_{A_\beta^{1,2}(\Omega)^3} + \|\operatorname{div} \mathbf{q}\|_{0,\Omega}\}.$$

Theorem 5.1 then yields

$$\|\mathbf{q} - \mathbf{q}_h\|_{0,\Omega} \lesssim h\|v_h\|_{0,\Omega}.$$

By the triangular inequality and Poincaré's estimate

$$\|\mathbf{q}\|_{0,\Omega} = \|\nabla\psi\|_{0,\Omega} \lesssim \|v_h\|_{0,\Omega},$$

we obtain (27).

For tetrahedral triangulations, we also need to take

$$\mathbf{q}_h = \Pi_h^t \nabla\psi,$$

with ψ as before. This guarantees (26) by Green's formula. To get (27), we cannot refer to Theorem 4.2 since $\Pi_h^t \Pi_h^q \mathbf{q}$ was used there to have an optimal error estimate. The use of $\Pi_h^t \mathbf{q}$ yields the non optimal error estimate

$$\|\mathbf{q} - \mathbf{q}_h\|_{0,\Omega} \lesssim h^{2-\frac{1}{\mu}} \|\mathbf{q}\|_{A_\beta^{1,2}(\Omega)^3}, \tag{28}$$

for $1 - \lambda < \beta \leq 1 - \mu$. This leads to the estimate

$$\|\mathbf{q} - \mathbf{q}_h\|_{0,\Omega} \lesssim \|\mathbf{q}\|_{A_\beta^{1,2}(\Omega)^3},$$

since $2 - \frac{1}{\mu} \geq 0$ by the assumption on μ. As before this estimate, the triangular inequality and Poincaré's estimate lead to (27).

It remains to establish (28). This is proved by standard arguments using affine transformations. Let us first consider $K \in \mathcal{T}_h$ such that $r_K = 0$. As in Theorem 3.4, using the transformation $\hat{\mathbf{q}}(\hat{\mathbf{x}}) = B_K^{-1}\mathbf{q}(\mathbf{x})$, we have

$$\|\mathbf{q} - \mathbf{q}_h\|_{0,K} \lesssim \|B_K\|\|B_K^{-1}\| \sum_{k=1}^{3} h_{k,K}^{1-\beta_k}\|r^{\beta_k}\partial_k \mathbf{q}\|_{0,K},$$

with $\beta_1 = \beta_2 = \beta$ and $\beta_3 = 0$ and $B_K = B_Q$ was defined in Lemma 4.1. Since $\|B_K\| \lesssim h$ and $\|B_K^{-1}\| \lesssim h_{1,K}^{-1} \sim h^{-\frac{1}{\mu}}$, the above estimate becomes

$$\|\mathbf{q} - \mathbf{q}_h\|_{0,K} \lesssim h^{1-\frac{\beta}{\mu}} \sum_{k=1,2} \|r^{\beta}\partial_k \mathbf{q}\|_{0,K} + h^{2-\frac{1}{\mu}}\|\partial_3 \mathbf{q}\|_{0,K}.$$

Since the condition $\beta \leq 1 - \mu$ implies that $h^{1-\frac{\beta}{\mu}} \lesssim h^{2-\frac{1}{\mu}}$, we arrive at

$$\|\mathbf{q} - \mathbf{q}_h\|_{0,K} \lesssim h^{2-\frac{1}{\mu}}\{\sum_{k=1,2} \|r^{\beta}\partial_k \mathbf{q}\|_{0,K} + \|\partial_3 \mathbf{q}\|_{0,K}\}. \tag{29}$$

If $K \in \mathcal{T}_h$ is such that $r_K > 0$, then standard arguments yield as previously

$$\|\mathbf{q} - \mathbf{q}_h\|_{0,K} \lesssim \|B_K\|\|B_K^{-1}\| \sum_{k=1}^{3} h_{k,K}\|\partial_k \mathbf{q}\|_{0,K}.$$

Therefore we get

$$\|\mathbf{q} - \mathbf{q}_h\|_{0,K} \lesssim h h_{1,K}^{-1}\{\sum_{k=1,2} h_{k,K} r_K^{-\beta}\|r^{\beta}\partial_k \mathbf{q}\|_{0,K} + h\|\partial_3 \mathbf{q}\|_{0,K}\}.$$

The condition $\beta \leq 1 - \mu$ and refinement rules (3) imply that

$$h r_K^{-\beta} \lesssim h^{2-\frac{1}{\mu}}, h^2 h_{1,K}^{-1} \lesssim h^{2-\frac{1}{\mu}}.$$

Therefore the above estimate leads to (29).

The sum of the square of estimates (29) gives (28). □

Corollary 5.3. *Assume that the refinement parameter μ satisfies $\mu \in [\frac{1}{2}, \lambda)$. Then there exists a constant $\beta^* > 0$ independent of h such that for every $v_h \in M_h$*

$$\sup_{\mathbf{q}_h \in X_h} \frac{\int_\Omega v_h \operatorname{div} \mathbf{q}_h \, \mathrm{d}x}{\|\mathbf{q}_h\|_{H(\operatorname{div},\Omega)}} \geq \beta^* \|v_h\|_{0,\Omega}. \tag{30}$$

Proof. For a fixed $v_h \in M_h$, it suffices to take $\mathbf{q}_h$ from Lemma 5.2, which clearly satisfies

$$\frac{\int_\Omega v_h \operatorname{div} \mathbf{q}_h \, \mathrm{d}x}{\|\mathbf{q}_h\|_{H(\operatorname{div},\Omega)}} \geq \beta^* \|v_h\|_{0,\Omega}.$$

□

Corollary 5.4. *Problem (25) has a unique solution $(\mathbf{p}_h, u_h) \in X_h \times M_h$.*

We now establish some error estimates between the exact solution and the discrete one. We start with the case of prismatic triangulations.

Theorem 5.5. *Assume that we use the prismatic family of meshes $\{\mathcal{Q}_h\}_{h>0}$ defined in Section 2 and that the refinement parameter μ satisfies $\mu \in [\frac{1}{2}, \lambda)$, then we have*

$$\|\mathbf{p} - \mathbf{p}_h\|_{0,\Omega} + \|u - u_h\|_{0,\Omega} \lesssim h\|f\|_{0,\Omega}. \tag{31}$$

Proof. Proposition II.2.4 of [9] implies that

$$\|\mathbf{p} - \mathbf{p}_h\|_{0,\Omega} \leq 2\|\mathbf{p} - \Pi_h^q \mathbf{p}\|_{0,\Omega},$$

since by Green's formula, $\operatorname{div} \Pi_h^q \mathbf{p} = \operatorname{div} \mathbf{p}_h = -\rho_h f$. By Theorem 3.3, we obtain

$$\|\mathbf{p} - \mathbf{p}_h\|_{0,\Omega} \lesssim h\{\|\mathbf{p}\|_{A_\beta^{1,2}(\Omega)^3} + \|\operatorname{div} \mathbf{p}\|_{0,\Omega}\},$$

with $1 - \lambda < \beta \leq 1 - \mu$, which is meaningful due to Theorem 5.1. By estimate (23), we get

$$\|\mathbf{p} - \mathbf{p}_h\|_{0,\Omega} \lesssim h\|f\|_{0,\Omega}. \tag{32}$$

For the estimation of $\|u - u_h\|_{0,\Omega}$, we use the arguments of Proposition II.2.7 of [9] to get

$$\|u - u_h\|_{0,\Omega} \lesssim \inf_{v_h \in M_h} \|u - v_h\|_{0,\Omega} + \|\mathbf{p} - \mathbf{p}_h\|_{0,\Omega}.$$

By Theorem 3.4 (with $\beta = 0$) and estimate (32), we arrive at

$$\|u - u_h\|_{0,\Omega} \lesssim h\|f\|_{0,\Omega}.$$

□

In the case of tetrahedral meshes the above arguments fail since $\Pi_h^t \Pi_h^q \mathbf{p}$ (necessary for the use of Th. 4.4) no more satisfies $\operatorname{div}(\Pi_h^t \Pi_h^q \mathbf{p}) = \operatorname{div} \mathbf{p}_h$. Therefore as in [23], we use the following general result on the approximation of mixed problems (see for instance Th. II.1.1 of [15] or Prop. II.2.6 and II.2.7 of [9]): the uniform inf-sup condition (30) and the coercivity property (directly satisfied here)

$$\int_\Omega |\mathbf{p}(x)|^2 \,\mathrm{d}x \geq \alpha \|\mathbf{p}\|^2_{H(\operatorname{div},\Omega)}, \forall \mathbf{p} \in V,$$

for some $\alpha > 0$, where $V = \{\mathbf{p} \in X : \operatorname{div} \mathbf{p} = 0\}$, imply that the next error estimate holds

$$\begin{aligned} \|\mathbf{p} - \mathbf{p}_h\|_{H(\operatorname{div},\Omega)} + \|u - u_h\|_{0,\Omega} \quad &\lesssim \quad \inf_{\mathbf{q}_h \in X_h} \|\mathbf{p} - \mathbf{q}_h\|_{H(\operatorname{div},\Omega)} \\ &\quad + \inf_{\mu_h \in M_h} \|u - \mu_h\|_{0,\Omega}. \end{aligned} \tag{33}$$

The term $\inf_{\mu_h \in M_h} \|u - \mu_h\|_{0,\Omega}$ is easily estimated with the help of Theorem 3.4. For the other term, we may take $\mathbf{q}_h = \Pi_h^t \Pi_h^q \mathbf{p}$ and write

$$\inf_{\mathbf{q}_h \in X_h} \|\mathbf{p} - \mathbf{q}_h\|_{H(\operatorname{div},\Omega)} \leq \|\mathbf{p} - \Pi_h^t \Pi_h^q \mathbf{p}\|_{H(\operatorname{div},\Omega)}.$$

Since the term $\|\mathbf{p} - \Pi_h^t \Pi_h^q \mathbf{p}\|_{0,\Omega}$ may be estimated by Theorem 4.4, it remains the term $\|\operatorname{div}(\mathbf{p} - \Pi_h^t \Pi_h^q \mathbf{p})\|_{0,\Omega}$ that we treat as follows using more regularity on the datum f (compare with [23]):

Lemma 5.6. *Assume that the refinement parameter μ satisfies $\mu \in [\frac{1}{2}, \lambda)$ and that $f \in A_\beta^{1,2}(\Omega)$ for $1 - \lambda < \beta \leq 1 - \mu$. Then we have*

$$\|\operatorname{div}(\mathbf{p} - \Pi_h^t \Pi_h^q \mathbf{p})\|_{0,\Omega} \lesssim h\|f\|_{A_\beta^{1,2}(\Omega)}. \tag{34}$$

Proof. We notice that

$$(\operatorname{div}\Pi_h^t \mathbf{p})_{|K} = |K|^{-1}\int_K \operatorname{div}\mathbf{p}\,\mathrm{d}x, \forall K \in \mathcal{T}_h,$$

in other words, $\operatorname{div}\Pi_h^t\mathbf{p} = \rho_h^t \operatorname{div}\mathbf{p}$, where ρ_h^t means the L^2 projection on M_h based on the triangulation $\mathcal{T}_h$. By the triangular inequality, we may write

$$\begin{aligned}\|\operatorname{div}(\mathbf{p} - \Pi_h^t\Pi_h^q\mathbf{p})\|_{0,\Omega} &\le \|\operatorname{div}(\mathbf{p} - \Pi_h^t\mathbf{p})\|_{0,\Omega} + \|\operatorname{div}\Pi_h^t(I - \Pi_h^q)\mathbf{p}\|_{0,\Omega}\\ &\le \|(I - \rho_h^t)(\operatorname{div}\mathbf{p})\|_{0,\Omega} + \|\rho_h^t\operatorname{div}(I - \Pi_h^q)\mathbf{p}\|_{0,\Omega}.\end{aligned}$$

Since ρ_h^t is a projection, we obtain

$$\|\operatorname{div}(\mathbf{p} - \Pi_h^t\Pi_h^q\mathbf{p})\|_{0,\Omega} \le \|(I - \rho_h^t)(\operatorname{div}\mathbf{p})\|_{0,\Omega} + \|\operatorname{div}(I - \Pi_h^q)\mathbf{p}\|_{0,\Omega}.$$

Using the same property on $\mathcal{Q}_h$, we get

$$\|\operatorname{div}(\mathbf{p} - \Pi_h^t\Pi_h^q\mathbf{p})\|_{0,\Omega} \le \|(I - \rho_h^t)(\operatorname{div}\mathbf{p})\|_{0,\Omega} + \|(I - \rho_h^q)(\operatorname{div}\mathbf{p})\|_{0,\Omega},$$

where ρ_h^q means the L^2 projection on M_h based on the triangulation $\mathcal{Q}_h$. As $\operatorname{div}\mathbf{p} = -f$, we arrive at

$$\|\operatorname{div}(\mathbf{p} - \Pi_h^t\Pi_h^q\mathbf{p})\|_{0,\Omega} \le \|(I - \rho_h^t)f\|_{0,\Omega} + \|(I - \rho_h^q)f\|_{0,\Omega}.$$

The conclusion follows from Theorem 3.4. □

The above arguments show the

Theorem 5.7. *Assume that we use the tetrahedral family of meshes $\{\mathcal{T}_h\}_{h>0}$ defined in Section 2, that the refinement parameter μ satisfies $\mu \in [\frac{1}{2}, \lambda)$ and that $f \in A_\beta^{1,2}(\Omega)$ for $1 - \lambda < \beta \le 1 - \mu$. Then we have*

$$\|\mathbf{p} - \mathbf{p}_h\|_{0,\Omega} + \|u - u_h\|_{0,\Omega} \lesssim h\|f\|_{A_\beta^{1,2}(\Omega)}. \tag{35}$$

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