

ON THE ONE-DIMENSIONAL BOLTZMANN EQUATION FOR GRANULAR FLOWS

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Abstract. We consider a Boltzmann equation for inelastic particles on the line and prove existence and uniqueness for the solutions.

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1. INTRODUCTION

The simplest models of granular flows are one-dimensional inelastic point particle systems (see *e.g.* [2] and references quoted therein). If a pair of particles with precollisional velocities v and v_1 collides, the outgoing velocities v' and v'_1 are defined in such a way as to dissipate kinetic energy, preserving the total momentum. In other words the collision law is:

$$v' = \frac{1}{2}(v + v_1) + \frac{1}{2}(v - v_1)h; \quad v'_1 = \frac{1}{2}(v + v_1) - \frac{1}{2}(v - v_1)h, \quad (1)$$

where $h \in [0, 1]$ is the one-dimensional restitution coefficient and measures the dissipation of the relative velocity ($v' - v'_1 = h(v - v_1)$).

Note that for $h = 1$ the system becomes free while for $h = 0$ we have the so-called sticky particle model: after a collision the particles proceed with the same velocity given by the momentum conservation.

From (1) we also deduce:

$$v' = v - \varepsilon(v - v_1); \quad v'_1 = v_1 + \varepsilon(v - v_1), \quad (2)$$

where $\varepsilon = \frac{1}{2}(1 - h) \in [0, 1/2]$ is another parameter measuring the degree of inelasticity of the collision.

Although h (and hence ε) are usually assumed to be constant, one can also argue that, for some materials, when the relative velocity becomes very small, the collision is essentially elastic. Thus the restitution coefficient should be a suitable function of the relative velocity. In this paper, following [6] and references quoted therein, we assume the explicit phenomenological law:

$$h(|v - v_1|) = \frac{1}{1 + a|v - v_1|^\gamma}, \quad (3)$$

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where $a > 0$ and $\gamma \in (0, 1)$ are constants depending on the material. However all the considerations presented here can be easily extended to the case of any function h of the relative velocity behaving like (3) for small $|v - v_1|$.

We anticipate that in the present paper we do not consider the case $\gamma = 0$ (already considered in [3]).

In analogy with the standard (conservative) theory of rarefied gases, it is natural to derive a reduced description given in terms of a Boltzmann equation, which in our case reads as:

$$\partial_t f + v \partial_x f = Q_\gamma(f, f) \tag{4}$$

where

$$Q_\gamma(f, f)(v) = \int dv_1 |v - v_1| (J f^* f_1^* - f f_1). \tag{5}$$

Here $f = f(x, v, t)$ is the phase-space density function (as usual $f(x, v) dx dv$ denotes the number of particles in the cell $[x, x + dx] \times [v, v + dv]$ in the phase space), f, f_1, f^* and f_1^* denote f computed on the arguments v, v_1, v^*, v_1^* . Moreover, given the pair $(v, v_1), (v^*, v_1^*)$ denotes the incoming velocities:

$$v^* = \frac{1}{2}(v + v_1) + \frac{1}{2h^*}(v - v_1); \quad v_1^* = \frac{1}{2}(v + v_1) - \frac{1}{2h^*}(v - v_1), \tag{6}$$

where $h^* = h(|v^* - v_1^*|)$. Finally:

$$J = \frac{1}{h^*(h^* + (v^* - v_1^*)h'^*)} = \frac{1}{h^{*3}(1 + a(1 - \gamma)|v^* - v_1^*|^\gamma)},$$

for which

$$dv dv_1 J |v - v_1| = dv^* dv_1^* |v^* - v_1^*|. \tag{7}$$

In this paper we study the Cauchy problem for equation 4 and present a global existence theorem. The non-homogeneous problem is considerably more difficult than the homogeneous one with regards to the standard kinetic theory. The main difficulty, when an energy dissipation is present, is related to the absence of an *a priori* entropy control, due to the fact that the system has the tendency to cluster. In the present paper we shall consider only the case $\gamma > 0$ and this makes the clustering effect less effective. For the case $\gamma = 0$, considered in [3], we have only partial results and the problem of giving a global existence result (or evidence of collapses or blow-up) is still open for general initial conditions.

2. FORMAL PROPERTIES OF THE SOLUTION

In this section we deduce some properties of the solutions (if any) to equation 4, which will be crucial in the construction of the solution to the Cauchy problem. The main point is the control of the entropy of the system which, in our dissipative context, is not trivial. Then, following a well known classical argument, we can derive an uniform absolute continuity bound which allows us to pass from a short time to a globally in time result. We only sketch this last part in Section 4 for sake of completeness.

We underline once more that the entropy inequality we are able to obtain holds only for $\gamma > 0$.

We observe preliminary that:

$$\int dv \phi(v) Q_\gamma(f, f)(v) = \frac{1}{2} \int \int dv dv_1 |v - v_1| \{ \phi(v') + \phi(v_1') - \phi(v) - \phi(v_1) \} f(x, v, t) f(x, v_1, t). \tag{8}$$

Note that if ϕ is convex, then $\{ \phi(v') + \phi(v_1') - \phi(v) - \phi(v_1) \}$ in (8) is negative so that:

$$\frac{d}{dt} \int dv \int dx \phi(v) f(x, v, t) \leq 0 \tag{9}$$

and this expresses the tendency of the system to cluster. In particular the total kinetic energy is decreasing in time.

In constructing the solution it is crucial a control of the Entropy functional (which can increase in our context) to avoid concentration.

Entropy inequality

Defining

$$H(t) = \int dx \int dv f \log f(x, v, t) \tag{10}$$

we have

$$\dot{H}(t) \leq \frac{1}{2} \int dx \int dv \int dv_1 (J - 1) |v - v_1| f(x, v, t) f(x, v_1, t). \tag{11}$$

The proof of (11) is consequence of a direct computation using (8) and the inequality:

$$(\log h - \log k)k \leq \frac{k}{\xi h + (1 - \xi)k} (h - k) \leq (h - k) \tag{12}$$

which is valid for any $h, k \geq 0$ and some $\xi \in (0, 1)$.

Using the explicit form of J ,

$$J - 1 = a|w^*|^\gamma \frac{2 + \gamma + 3a|w^*|^\gamma + a^2|w^*|^{2\gamma}}{1 + a(1 - \gamma)|w^*|^\gamma} \leq c(|w^*|^\gamma + |w^*|^{2\gamma}),$$

where $w^* = v^* - v_1^*$ and c is a constant depending on a and γ . Being $|w^*| \leq |w|(1 + a|w|^\gamma)$ we obtain, as corollary of inequality (11),

$$\dot{H}(t) \leq C \int dx \int dv \int dv_1 (|v - v_1|^{1+\gamma} + |v - v_1|^{1+4\gamma}) f(x, v, t) f(x, v_1, t), \tag{14}$$

where C is a constant depending on a and γ .

Therefore to control H we must try and control the quantity

$$\mathcal{I}_\alpha(T) = \int_0^T dt \int dx \int dv \int dv_1 |v - v_1|^{1+\alpha} f(x, v, t) f(x, v_1, t) \tag{15}$$

for $\alpha \in (0, 1]$.

We estimate \mathcal{I}_α by extending an argument due to Bony [4].

Define, for any odd function ϕ :

$$I_\phi(t) = \int dx \int dy \int dv \int dw \mathcal{X}(x < y) \phi(v - w) f(x, v, t) f(y, w, t) \tag{16}$$

where \mathcal{X} is the characteristic function. Then

$$\begin{aligned} \dot{I}_\phi(t) = & - \int dx \int dv \int dw \phi(v - w) (v - w) f(x, v, t) f(x, w, t) \\ & + \frac{1}{2} \int dx \int dy \int dv \int dv_1 \int dw |v - v_1| (\mathcal{X}(x < y) - \mathcal{X}(y < x)) f(y, w, t) \\ & f(x, v, t) f(x, v_1, t) [\phi(v' - w) + \phi(v'_1 - w) - \phi(v - w) - \phi(v_1 - w)]. \end{aligned} \tag{17}$$

We shall use this identity for $\phi_\alpha = \text{sgn}(v)|v|^\alpha$.

Integrating with respect to time, we have:

$$\begin{aligned} \mathcal{I}_\alpha(T) &= I_{\phi_\alpha}(0) - I_{\phi_\alpha}(T) \\ &+ \frac{1}{2} \int_0^T \int dx \int dy \int dv \int dv_1 \int dw |v - v_1| (\mathcal{X}(x < y) - \mathcal{X}(y < x)) f(y, w, t) \\ &f(x, v, t) f(x, v_1, t) [\phi_\alpha(v' - w) + \phi_\alpha(v'_1 - w) - \phi_\alpha(v - w) - \phi_\alpha(v_1 - w)]. \end{aligned} \tag{18}$$

Note that, if $\alpha = 1$, using the momentum conservation, the last term in the right hand side of equation (18) vanishes and we have (see [4]):

$$\mathcal{I}_1(T) = I_{\phi_1}(0) - I_{\phi_1}(T). \tag{19}$$

For $\alpha < 1$ we need to estimate the last term in the right half side of equation (18). In doing this we arrive to an inequality (see Eq. (24) below) which involves $\mathcal{I}_{\alpha'}$ with $\alpha' > \alpha$. The idea is to iterate this relation to arrive to $\alpha = 1$ and hence to estimate \mathcal{I}_α in term of I_{ϕ_α} .

Namely for $\alpha < 1$, assuming $v' > v$ (the other case $v' \leq v$ can be handled analogously), we have:

$$\begin{aligned} \int dy \int dw f(y, w, t) [\phi_\alpha(v' - w) - \phi_\alpha(v - w)] &= \int dy \int_v^{v'} dw f(y, w, t) [\dots] + \int dy \int_{-\infty}^v dw f(y, w, t) [\dots] \\ &+ \int dy \int_{v'}^{+\infty} dw f(y, w, t) [\dots]. \end{aligned} \tag{20}$$

Moreover, if $w \in (v, v')$:

$$[\dots] = |v' - w|^\alpha + |v - w|^\alpha \leq 2|v' - v|^\alpha = 2\varepsilon^\alpha |v - v_1|^\alpha \leq 2^{1-\alpha} a^\alpha |v - v_1|^{(1+\gamma)\alpha} \tag{21}$$

If $w < v$:

$$\begin{aligned} [\dots] &= |v' - w|^\alpha - |v - w|^\alpha = |v - w - \varepsilon(v - v_1)|^\alpha - |v - w|^\alpha \\ &= \varepsilon^\alpha |v - v_1|^\alpha \left(\left| \frac{v - w}{\varepsilon(v - v_1)} - 1 \right|^\alpha - \left| \frac{v - w}{\varepsilon(v - v_1)} \right|^\alpha \right) \leq M 2^{-\alpha} a^\alpha |v - v_1|^{(1+\gamma)\alpha} \end{aligned} \tag{22}$$

where $M = \max_{\xi \in \mathbb{R}} (|\xi - 1|^\alpha - |\xi|^\alpha)$. The term $w > v'$ can be treated in the same way.

In conclusion:

$$\int dy \int dw f(y, w, t) [\phi_\alpha(v' - w) - \phi_\alpha(v - w)] \leq Cm |v - v_1|^{(1+\gamma)\alpha} \tag{23}$$

where C is a constant and $m = \int dx \int dv f(t)$ is the total mass of the system.

Inserting this estimate in the right hand side of equation (18) we readily arrive to the inequality:

$$\mathcal{I}_\alpha(T) \leq I_{\phi_\alpha}(0) - I_{\phi_\alpha}(T) + Cm \mathcal{I}_{(1+\gamma)\alpha}(T), \tag{24}$$

from which:

$$\mathcal{I}_\alpha(T) \leq \sum_{k=0}^{n-1} (Cm)^k (I_{\phi_{(1+\gamma)^k \alpha}}(0) - I_{\phi_{(1+\gamma)^k \alpha}}(T)) + (Cm)^n \mathcal{I}_{(1+\gamma)^n \alpha}. \tag{25}$$

Setting $\alpha_n = \frac{1}{(1+\gamma)^n}$, using (19) we obtain:

$$\mathcal{I}_{\alpha_n}(T) \leq \sum_{k=0}^n (Cm)^k (I_{\phi_{(1+\gamma)^{k-n}}}(0) - I_{\phi_{(1+\gamma)^{k-n}}}(T)). \tag{26}$$

Note now that

$$|I_{\alpha_n}(T)| \leq C \left(\int dx \int dv |v|^\alpha f(x, v, t) \right)^2 \leq C \left(\int dx \int dv (1 + |v|^2) f(x, v, t) \right)^2$$

and hence:

$$\mathcal{I}_{\alpha_n}(T) \leq C(m, E) \tag{27}$$

where $C(m, E)$ is a constant depending only on the mass m and the energy $E = \frac{1}{2} \int dx \int dv v^2 f_0(x, v)$ of the initial datum f_0 .

Finally, for any $\alpha \in (0, 1]$ we can find n such that $\alpha \in [\alpha_{n+1}, \alpha_n]$. Since

$$\mathcal{I}_\alpha \leq \mathcal{I}_{\alpha_n} + \mathcal{I}_{\alpha_{n+1}} \tag{28}$$

we conclude that:

$$\mathcal{I}_\alpha(T) \leq C(m, E). \tag{29}$$

Another property of the solution we are going to use is the following. Suppose that $v^2 > v_1^2$. Then

$$v'^2 = v^2[(1 - 2\varepsilon) + \varepsilon^2] + \varepsilon^2 v_1 + (2\varepsilon - 2\varepsilon^2)vv_1 \leq v^2[(1 - 2\varepsilon) + \varepsilon^2] + (2\varepsilon - 2\varepsilon^2)v^2 \leq v^2 \tag{30}$$

therefore the collision mechanism cannot increase the largest velocity. Hence if

$$f_0(x, v) = 0 \quad \text{when} \quad |v| > V \tag{31}$$

for a given $V > 0$ then:

$$f(x, v, t) = 0 \quad \text{when} \quad |v| > V \quad \text{for} \quad t > 0. \tag{32}$$

As consequence, under condition (31) we have, for $\alpha > \beta$:

$$\mathcal{I}_\alpha(T) \leq (1 + V^{\alpha-\beta})\mathcal{I}_\beta(T) \tag{33}$$

and by (14) and (29) we can control the entropy production. In conclusion:

Proposition 1. *Suppose $f = f(x, v, t)$ be a classical solution to equation (4) with initial datum f_0 satisfying (31) and such that $H(f_0) = \int dx \int dv f_0 \log f_0 < +\infty$. Then there exist a constant C depending on m, E and $H(f_0)$ for which $H(t) \equiv H(f(t)) \leq C$.*

3. THE CAUCHY PROBLEM FOR SMALL DATA

We shall now sketch a global existence and uniqueness result for the solution of equation (4) for small values of $\int dx dv f_0(x, v)$ (see [1] and [3]). In the next section we will use this result and the *a priori* estimate for the entropy production, in order to obtain the global existence and uniqueness theory for all initial data.

Theorem 1. *Let $f_0 \in \mathbf{L}_\infty(\mathbb{R})$, $f_0 \geq 0$. Then, if the total mass $\|f_0\|_1$ is small, there exists an unique mild, bounded solution in \mathbf{L}_∞ of (4) with initial datum given by f_0 .*

Proof. Let $f^\#(x, v, t) = f(x + vt, v, t)$. It is easy to realize that $f^\#$ satisfies

$$f^\#(x, v, t) \leq f_0(x, v) + \int_0^t ds \int dv_1 |v - v_1| J f^\#(x + (v - v^*)s, v^*, s) f^\#(x + (v - v_1^*)s, v_1^*, s). \tag{34}$$

Let

$$F(x, v) = \sup_{t \geq 0} f^\#(x, v, t). \tag{35}$$

From (34):

$$\|F\|_1 \leq \|f_0\|_1 + \int_0^\infty ds \int dx dv dv_1 |v - v_1| JF(x + (v - v^*)s, v^*)F(x + (v - v_1^*)s, v_1^*). \tag{36}$$

Coming back to the precollisional variables of integration, using (7) we have that:

$$\|F\|_1 \leq \|f_0\|_1 + \int_0^\infty ds \int dx dv dv_1 |v - v_1| F(x + (v' - v)s, v)F(x + (v' - v_1)s, v_1). \tag{37}$$

Putting $y = x + (v' - v)s$, and $z = x + (v' - v_1)s$, by elementary computations we obtain:

$$dx ds |v - v_1| = dy dz, \tag{38}$$

then

$$\|F\|_1 \leq \|f_0\|_1 + \|F\|_1^2, \tag{39}$$

and therefore

$$F \leq \frac{1}{2} \left(1 - \sqrt{1 - 4\|f_0\|_1} \right), \tag{40}$$

if

$$\|f_0\|_1 < \frac{1}{4}. \tag{41}$$

Now we are able to prove an L_∞ *a priori* estimate on $f(x, v, t)$.

Let

$$G = \sup_{x, v, t \geq 0} f^\#(x, v, t)$$

From (34):

$$G \leq \|f_0\|_\infty + G \int_0^\infty ds \int dv_1 |v - v_1| Jf^\#(x + (v^* - v_1^*)s, v_1^*, s). \tag{42}$$

We change the variables $(s, v_1) \rightarrow (y, v_1^*)$, with $y = x + (v - v_1^*)s$.

Taking into account that

$$\frac{\partial v_1^*}{\partial v_1} = \frac{2 + a(3 - \gamma)|w^*|^\gamma + a^2(1 - \gamma)|w^*|^{2\gamma}}{2(1 + a(1 - \gamma)|w^*|^\gamma)},$$

and that $|v - v_1^*| = \frac{1+h(|w^*|)}{2}|w^*|$, we obtain

$$J|v - v_1| ds dv_1 = \frac{2Jh(|w^*|)}{1 + h(|w^*|)} \frac{\partial v_1}{\partial v_1^*} dy dv_1^* = 4 \frac{(1+z)^2}{2+z} \frac{1}{2 + (3-\gamma)z + (1-\gamma)z^2} dy dv_1^*,$$

where $z = a|w^*|^\gamma$. By an explicit estimation of this expression, from (42) we conclude that

$$G \leq \|f_0\|_\infty + 2\|F\|_1 G.$$

Therefore G is bounded if

$$\|F\|_1 < \frac{1}{2}. \tag{43}$$

With this *a priori* bound the construction of the solutions is standard.

4. THE CAUCHY PROBLEM FOR ARBITRARY DATA

Theorem 4.1. *Let $f_0 \in \mathbf{L}_\infty(\mathbb{R})$, $f_0 \geq 0$, $f_0(x, v) = 0$ if $|v| \geq V$. Then there exists an unique mild, bounded solution in \mathbf{L}_∞ of (4) with initial datum given by f_0 .*

The proof of Theorem 4.1 follows from Theorem 3.1 by using the entropy bound in Proposition 1, and an argument due to Tartar [5] which we repeat here for completeness.

Proof. Being $f(x, v, t) = 0$ for $|v| > V$ and any time, the solution at time \bar{t} in \bar{x} depends only on $f_0(x, v)$ restricted to the interval $x \in (\bar{x} - V\bar{t}, \bar{x} + V\bar{t})$. We can choose \bar{t} such that

$$\sup_{\bar{x}} \int_{\bar{x}-V\bar{t}}^{\bar{x}+V\bar{t}} dx \int dv f_0(x, v)$$

is sufficiently small in order to verify the inequalities (40), (41) and (43). This gives a small time existence result for all initial data.

The entropy estimate in Proposition 1 allows us to choose \bar{t} independently from f_0 . Namely, let I be an interval of \mathbb{R} , with $\text{meas } I = |I| < 1$:

$$\begin{aligned} \int_I dx \int dv f(x, v, t) &= \int dv \int_{I \cap \{f \leq |I|^{-\frac{1}{2}}\}} dx f(x, v, t) + \int dv \int_{I \cap \{f > |I|^{-\frac{1}{2}}\}} dx f(x, v, t) \\ &\leq 2V|I|^{\frac{1}{2}} + \frac{2}{|\log |I||} \int dx dv f(x, v, t) |\log f(x, v, t)|. \end{aligned}$$

We conclude the proof by noticing that the last estimate depends only on the entropy and the total mass, and then the proof of the local in time existence can be iterated to reach any time.

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