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ISOPARAMETRIC MIXED FINITE ELEMENT APPROXIMATION OF EIGENVALUES AND EIGENVECTORS OF 4TH ORDER EIGENVALUE PROBLEMS WITH VARIABLE COEFFICIENTS

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Abstract. Estimates for the combined effect of boundary approximation and numerical integration on the approximation of (simple) eigenvalues and eigenvectors of 4th order eigenvalue problems with variable/constant coefficients in convex domains with curved boundary by an isoparametric mixed finite element method, which, in the particular case of bending problems of aniso-/ortho-/isotropic plates with variable/constant thickness, gives a simultaneous approximation to bending moment tensor field $\Psi = (\psi_{ij})_{1\leq i,j\leq 2}$ and displacement field 'u', have been developed.

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1. Introduction

In *all* papers [14, 22, 23, 29] on mixed finite element analysis of 4th order eigenvalue problems, it has been assumed that *neither* any numerical integration is essential *nor* any approximation of the boundary is necessary (since the boundary of the *convex* domain is a *polygonal* one in all the cases, the convexity of the domain being a requirement for the regularity of the solution [18, 21, 24]). But in many situations,we are to consider convex domains with *curved* boundary Γ. Then an approximation of the curved boundary and possibly numerical evaluation of integrals will be essential, but convergence analysis becomes much more complex and complicated. Even for *classical, standard* finite element analysis of second order self-adjoint eigenvalue problems in domains with curved boundary we find the situation as stated in ([40], p. 254): "$\cdots$ In contrast to finite element analysis of boundary value problems, in the finite element analysis of eigenvalue problems, there does *not* exist any abstract error estimate consisting of the sum of *three* terms (error of interpolation, error of approximation of the boundary and error of numerical integration) $\cdots$". Hence, in such a situation error analysis for each specific problem can be attempted at and the proofs involved in finding the estimates will be quite complex and too *technical* in nature due to these additional complicacies introduced by the boundary approximation and obligatory use (for example, in the isoparametric case) of numerical integration. In fact, we find only two papers [25, 40], in which this combined effect of boundary approximation and numerical integration on

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second order self-adjoint eigenvalue approximations using *classical* isoparametric finite element methods has been estimated, [5] and [6] being the papers which deal with the effect of *only* numerical integration on eigenvalue approximations. But the situation is still worse in the case of *isoparametric mixed* finite element analysis of eigenvalue problems, for which error estimates are to be developed again for a specific mixed method formulation (since abstract results for the isoparametric case do **not** exist even for source problems) and the proofs for the estimates will be much more complex and much more technical in nature. Indeed *to our knowledge*, [8] is probably the *first* publication on the estimates for the combined effect of boundary approximation and numerical integration on the *mixed finite element* approximation of (*simple*) eigenvalues and eigenvectors of 4th order self-adjoint eigenvalue problems with variable/constant coefficients, many proofs in which, as stated earlier, have remained quite technical in spite of the best efforts of the authors to avoid these technical aspects in some proofs. The present paper, the results of which were announced in [8] (see also [31]), relies heavily on [10] for the corresponding source problem ([9] contains error estimates due to *polygonal* approximation of the curved boundary along with numerical integration for the same source problem) and also on the results of [4] on the mixed method scheme (see also [33, 36]) for polygonal domains. For other interesting references on eigenvalue approximations, we refer to [2, 15]. Finally, the present paper also contains interesting results of numerical experiments on some problems of practical importance and research interest.

2. The continuous mixed variational eigenvalue problem

Consider the eigenvalue problem: Find $\lambda \in \mathbb{R}$ for which $\exists$ non-null u such that

$$(\mathbf{P^E}): \quad \Lambda u \;=\; \lambda\, u \;\text{ in }\; \Omega,\;\; u|_\Gamma \;=\; (\frac{\partial u}{\partial n})|_\Gamma \;=\; 0, \tag{2.1}$$

$$\text{where } (\Lambda u)(x) \equiv \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2} \frac{\partial^2}{\partial x_k \partial x_l}(a_{ijkl}\frac{\partial^2 u}{\partial x_i \partial x_j})(x) \equiv (a_{ijkl}u_{,ij})_{,kl}(x) \quad \forall x=(x_1,x_2)\in\Omega. \tag{2.2}$$

(In (2.2) and also in the sequel, Einstein's summation convention with respect to twice repeated indices $i, j, k, l = 1, 2$ has been followed unless stated otherwise).

(**A1**): Ω is a bounded, open, convex domain with Lipschitz continuous boundary Γ which is piecewise of C^k- class, $k \geq 3$ [1, 21, 35, 41];

coefficients $a_{ijkl} = a_{ijkl}(x) \;\; \forall x = (x_1, x_2) \in \bar{\widetilde{\Omega}} \subset \mathbb{R}^2$,

(**A2**): $\widetilde{\Omega}$ being a bounded, open set with boundary $\widetilde{\Gamma}$, which is piecewise of C^k-class, $k \geq 3$, such that $\bar{\Omega} = \Omega \cup \Gamma \subset \widetilde{\Omega}$

satisfy the following assumptions: $\forall i, j, k, l = 1, 2$,

(**A3**): $a_{ijkl} \in W^{2,\infty}(\widetilde{\Omega}) \hookrightarrow C^1(\bar{\widetilde{\Omega}})$; $\;a_{ijkl} \geq 0$; $a_{ijkl}(x) = a_{klij}(x) = a_{lkij}(x) = a_{lkji}(x) \;\; \forall x \in \bar{\widetilde{\Omega}}$;

(**A4**): $\exists \alpha > 0$ such that $\forall \underline{\xi} = (\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}) \in \mathbb{R}^4$ with $\xi_{21} = \xi_{12}, a_{ijkl}(x)\xi_{ij}\xi_{kl} \geq \alpha \|\underline{\xi}\|^2_{\mathbb{R}^4} \;\; \forall x \in \bar{\widetilde{\Omega}}$.

Then, the corresponding **Galerkin Variational Eigenvalue Problem** ($\mathbf{P^E_G}$) is defined by:
Find $\lambda \in \mathbb{R}$ for which $\exists$ non-null $u \in H^2_0(\Omega)$ [1, 28] such that

$$(\mathbf{P^E_G}): \quad a(u, v) = \lambda \langle u, v\rangle_{0,\Omega} \quad \forall v \in H^2_0(\Omega) \tag{2.3}$$

where $a(u,v) = \langle \Lambda u, v\rangle_{0,\Omega} = \int_\Omega a_{ijkl} u_{,ij} v_{,kl} \,\mathrm{d}\Omega = a(v,u) \;\; \forall u,\; v \in H_0^2(\Omega)$;
$\exists \alpha_0 > 0$ such that $a(v,v) \geq \alpha_0 \|v\|^2_{2,\Omega} \;\; \forall v \in H_0^2(\Omega)$ [3, 20] $\quad \langle u, v\rangle_{0,\Omega} = \int_\Omega uv \,\mathrm{d}x$.

Since $a(\cdot,\cdot)$ is continuous and $H_0^2(\Omega)$-elliptic, the associated **Galerkin Variational (Source) Problem** $(\mathbf{P_G})$ defined by: For given $f \in L^2(\Omega)$, find $u \in H_0^2(\Omega)$ such that

$$(\mathbf{P_G}): \quad a(u,v) = \langle f, v\rangle_{0,\Omega} \quad \forall v \in H_0^2(\Omega), \tag{2.4}$$

has a unique solution by Lax-Milgram lemma, and we have:

Theorem 2.1 ([8]). $(\mathbf{P^E_G})$ *has a countable non-decreasing system of strictly positive eigenvalues with possibly finite multiplicities and accumulation point at ∞: i.e.* $0 < \widehat{\lambda}_1 \leq \widehat{\lambda}_2 \leq \cdots \leq \widehat{\lambda}_m \leq \cdots \uparrow \infty$, *and $\exists$ a system of eigenpairs $(\widehat{\lambda}_m, \widehat{v}_m)_{m=1}^\infty$ such that the eigensystem $(\widehat{v}_m)_{m=1}^\infty$ is a Hilbert basis in $(H_0^2(\Omega); \langle\langle\cdot,\cdot\rangle\rangle_{a(\cdot,\cdot)})$ with $\langle\langle \widehat{v}_m, \widehat{v}_n\rangle\rangle_{a(\cdot,\cdot)} = a(\widehat{v}_m, \widehat{v}_n) = \delta_{mn}$. Moreover, $(\sqrt{\widehat{\lambda}_m}\widehat{v}_m)_{m=1}^\infty$ is a Hilbert basis in $L^2(\Omega)$.*

Now, defining Hilbert space $\mathbf{H}$ of symmetric tensor-valued functions in Ω by:

$$\mathbf{H} = \{\Phi : \Phi = (\phi_{ij})_{1\leq i,j\leq 2} \text{ with } \phi_{ij} = \phi_{ji} \in L^2(\Omega)\} \text{ with } \|\Phi\|^2_{\mathbf{H}} = \|\Phi\|^2_{0,\Omega} = \sum_{i,j=1}^{2} \int_\Omega |\phi_{ij}(x)|^2 \,\mathrm{d}x$$

and new coefficients $A_{ijkl} = A_{ijkl}(x) \;\; \forall x \in \bar{\tilde{\Omega}}$ in terms of coefficients a_{ijkl}, the algorithm for which is given in [4], satisfying the following properties: $\forall i,j,k,l = 1,2$

$$\bullet\; \forall x \in \bar{\tilde{\Omega}},\;\; A_{ijkl}(x) = A_{klij}(x) = A_{lkij}(x) = A_{lkji}(x); \tag{2.5}$$

$$\bullet \exists \alpha > 0 \;\text{ such that } A_{ijkl}(x)\xi_{ij}\xi_{kl} \geq \alpha \|\underline{\xi}\|^2_{\mathbb{R}^4} \;\forall x \in \bar{\tilde{\Omega}},\;\; \forall \underline{\xi} = (\xi_{ij})_{i,j=1,2} \in \mathbb{R}^4 \text{ with } \xi_{12} = \xi_{21}; \tag{2.6}$$

$$\begin{aligned} &\bullet\; \forall x \in \bar{\tilde{\Omega}},\;\; \forall \underline{\xi} = (\xi_{ij})_{i,j=1,2} \in \mathbb{R}^4 \text{ with } \xi_{21} = \xi_{12},\; \forall \underline{\zeta} = (\zeta_{ij})_{i,j=1,2} \in \mathbb{R}^4 \text{ with } \zeta_{21} = \zeta_{12}, \\ &\qquad A_{ijkl}(x)a_{ijmn}(x)\xi_{mn}\zeta_{kl} = \xi_{ij}\zeta_{ij};\;\; A_{ijkl}(x)a_{ijmn}(x)\xi_{mn} = \xi_{kl}, \text{ and} \end{aligned} \tag{2.7}$$

$(\mathbf{A5})$: $A_{ijkl} \in W^{2,\infty}(\tilde{\Omega}) \hookrightarrow C^1(\bar{\tilde{\Omega}})$,

we construct an **Auxiliary Continuous Mixed Variational Eigenvalue Problem** $(\mathbf{Q^E_{AUX}})$ as follows: Find $\lambda \in \mathbb{R}$ for which $\exists$ non-null $(\Psi, u) \in \mathbf{H} \times H_0^2(\Omega)$ (*i.e.* $\Psi \neq \mathbf{0}$, $u \neq 0$) such that

$$(\mathbf{Q^E_{AUX}}): \;A_0(\Psi,\Phi) + b_0(\Phi,u) = 0 \quad \forall \Phi \in \mathbf{H}; \qquad -b_0(\Psi,v) = \lambda\langle u,v\rangle_{0,\Omega} \;\; \forall v \in H_0^2(\Omega). \tag{2.8}$$

The associated **Source Problem** $(\mathbf{Q_{AUX}})$ in **Continuous Mixed Variational Formulation** is defined by: For given $f \in L^2(\Omega)$, find $(\Psi, u) \in \mathbf{H} \times H_0^2(\Omega)$ such that:

$$(\mathbf{Q_{AUX}}): \;A_0(\Psi,\Phi) + b_0(\Phi,u) = 0 \;\; \forall \Phi \in \mathbf{H}; \qquad -b_0(\Psi,v) = \langle f,v\rangle_{0,\Omega} \;\; \forall v \in H_0^2(\Omega), \tag{2.9}$$

where $A_0(\cdot,\cdot)$ and $b_0(\cdot,\cdot)$ are continuous bilinear forms defined by:

$$A_0(\Psi,\Phi) = \int_\Omega A_{ijkl}\psi_{ij}\phi_{kl}\,\mathrm{d}x = A_0(\Phi,\Psi) \;\text{ with } |A_0(\Psi,\Phi)| \leq \bar{M}_0 \|\Psi\|_{\mathbf{H}}\|\Phi\|_{\mathbf{H}} \text{ for some } \bar{M}_0 > 0,$$

$$A_0(\Phi,\Phi) \geq \alpha \|\Phi\|^2_{\mathbf{H}} \;\; \forall \Phi \in \mathbf{H} \text{ and for some } \alpha > 0; \tag{2.10}$$

$$b_0(\Phi, v) = -\int_\Omega \phi_{ij} v_{,ij}\, \mathrm{d}\,\Omega \;\; \forall\Phi \in \mathbf{H}\, \forall v \in H^2_0(\Omega) \text{ with } |b_0(\Phi,v)| \leq \bar{m}_0 \|\Phi\|_{\mathbf{H}} \|v\|_{2,\Omega} \text{ for some } \bar{m}_0 > 0,$$

$$\sup_{\Phi\in\mathbf{H}-\{0\}} \frac{|b_0(\Phi,v)|}{\|\Phi\|_{\mathbf{H}}} \geq \beta_0 \|v\|_{2,\Omega}\; \forall v \in H^2_0(\Omega) \;\text{ for some }\; \beta_0 > 0. \tag{2.11}$$

As a consequence of (2.10) and (2.11), $(\mathbf{Q_{AUX}})$ has a unique solution $(\Psi, u) \in \mathbf{H} \times H^2_0(\Omega)$ [2, 11, 12], and we define $\vec{\mathbf{T}}_{\mathbf{0}} : f \in L^2(\Omega) \mapsto \vec{\mathbf{T}}_{\mathbf{0}} f = (S_0\, f, T_0\, f) = (\Psi, u) \in \mathbf{H} \times H^2_0(\Omega)$ such that

$$A_0(S_0\, f, \Phi) + b_0(\Phi, T_0\, f) = 0 \;\; \forall \Phi \in \mathbf{H}; \qquad -b_0(S_0\, f, v) = \langle f, v\rangle_{0,\Omega} \;\; \forall v \in H^2_0(\Omega), \tag{2.12}$$

where $S_0 \in \mathcal{L}(L^2(\Omega); \mathbf{H}),\ T_0 \in \mathcal{L}(L^2(\Omega); H^2_0(\Omega))$ with

$$\|S_0\, f\|_{0,\Omega} + \|T_0\, f\|_{2,\Omega} \leq C\|f\|_{0,\Omega} \;\; \forall f \in L^2(\Omega),\; S_0 f = \Psi, T_0 f = u;\; S_0(\cdot) = ((a_{ijkl} T_0(\cdot)_{,kl})_{1\leq i,j\leq 2}. \tag{2.13}$$

Then, $\hookrightarrow \cdot T_0 = T_0 \in \mathcal{L}(L^2(\Omega); L^2(\Omega)) \equiv \mathcal{L}(L^2(\Omega))$ with $\hookrightarrow : H^2_0(\Omega) \longrightarrow L^2(\Omega)$ is a *compact, positive, symmetric, linear* operator and the eigenvalue problem of $T_0 \in \mathcal{L}(L^2(\Omega))$: $T_0 u = \mu\, u$ is "equivalent" to the eigenvalue problem $(\mathbf{Q^E_{AUX}})$ with $\mu = 1/\lambda > 0$. Hence, we have:

Theorem 2.2. $(\mathbf{Q^E_{AUX}})$ *has a countable system of strictly positive, non-decreasing system of eigenvalues with possibly finite multiplicities and accumulation point at* ∞*:*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots \uparrow \infty \text{ with } \mu_m = 1/\lambda_m \;\; \forall m \in \mathbb{N}, \tag{2.14}$$

and $\exists$ *eigenpairs* $(\lambda_m; (\Psi_m, u_m))_{m=1}^\infty$ *of* $(\mathbf{Q^E_{AUX}}) : \forall m \in \mathbb{N}$,

$$A_0(\Psi_m, \Phi) + b_0(\Phi, u_m) = 0 \qquad \forall \Phi \in \mathbf{H};\;\; -b_0(\Psi_m, v) = \lambda_m \langle u_m, v\rangle_{0,\Omega} \;\; \forall v \in H^2_0(\Omega), \tag{2.15}$$

$(u_m)_{m=1}^\infty$ *being a Hilbert basis in* $L^2(\Omega)$ *with* $\Psi_m = S_0(\lambda_m u_m) = (a_{ijkl} u_{m,kl})_{i,j=1,2}$ $\forall m \in \mathbb{N}$ *(see (2.13)). Moreover,* $\left(\dfrac{\Psi_m}{\sqrt{\lambda_m}}\right)_{m=1}^\infty$ *is an orthonormal system in* $(\mathbf{H}, [\cdot,\cdot]_{A_0(\cdot,\cdot)})$ *with*
$\left[\dfrac{\Psi_m}{\sqrt{\lambda_m}}, \dfrac{\Psi_n}{\sqrt{\lambda_n}}\right]_{A_0(\cdot,\cdot)} = A_0\left(\dfrac{\Psi_m}{\sqrt{\lambda_m}}, \dfrac{\Psi_n}{\sqrt{\lambda_n}}\right) = \delta_{mn} \;\; \forall m, n \in \mathbb{N}.$

As a consequence of (2.10) and (2.11), $\forall$ **fixed** $v \in H^2_0(\Omega)$, *there exists* a unique $\underline{\sigma} \in \mathbf{H}$ such that $A_0(\underline{\sigma}, \Phi) + b_0(\Phi, v) = 0 \;\; \forall \Phi \in \mathbf{H}$ by virtue of Lax-Milgram lemma and this correspondence defines $\mathcal{I} : v \in H^2_0(\Omega) \mapsto \mathcal{I}v = \underline{\sigma} \in \mathbf{H}$ and we set

$$\mathcal{E} = \mathcal{I}(H^2_0(\Omega)) = \{\underline{\sigma} : \underline{\sigma} \in \mathbf{H} \text{ for which } \exists v \in H^2_0(\Omega) \text{ such that } \mathcal{I}v = \underline{\sigma}\} \subset \mathbf{H}. \tag{2.16}$$

Proposition 2.1. *(i)* $(\mathcal{E}; [\cdot,\cdot]_{A_0(\cdot,\cdot)})$ *equipped with inner product* $[\underline{\sigma}, \underline{\omega}]_{A_0(\cdot,\cdot)} = A_0(\underline{\sigma}, \underline{\omega})$ $\forall \underline{\sigma}, \underline{\omega} \in \mathcal{E}$ *is a Hilbert space and* $\forall$ *eigenpair* $(\lambda_m; (\Psi_m, u_m))$ *of* $(\mathbf{Q^E_{AUX}})$, $\Psi_m = \mathcal{I}u_m = S_0(\lambda_m u_m)$, $m \in \mathbb{N}$.
(ii) $\mathcal{I} : (H^2_0(\Omega), \langle\langle\cdot,\cdot\rangle\rangle_{a(\cdot,\cdot)}) \longrightarrow (\mathcal{E}; [\cdot,\cdot]_{A_0(\cdot,\cdot)})$ *is a linear, continuous bijection with*
$\langle\langle v, w\rangle\rangle_{a(\cdot,\cdot)} = [\mathcal{I}v, \mathcal{I}w]_{A_0(\cdot,\cdot)} = [\underline{\sigma}, \underline{\omega}]_{A_0(\cdot,\cdot)} = A_0(\underline{\sigma}, \underline{\omega})$, $(\underline{\sigma}, v)$, $(\underline{\omega}, w)$ *being the* ***linked pairs*** *in* $\mathcal{E} \times H^2_0(\Omega)$.

$$\text{Set } \mathcal{M} = \mathcal{E} \times H^2_0(\Omega) = \text{ product space of linked pairs } (\underline{\sigma}, v) \text{ with } \underline{\sigma} = \mathcal{I}v \;\; \forall v \in H^2_0(\Omega). \tag{2.17}$$

Rayleigh quotient characterization of eigenpairs.
To $(\mathbf{P_G^E})$ we can associate Rayleigh coefficient $R(v) = \dfrac{a(v,v)}{\langle v,v\rangle_{0,\Omega}}$ $\forall v \in H^2(\Omega) - \{0\}\cdot$
But $a(v,v) = \langle\langle v,v\rangle\rangle_{a(\cdot,\cdot)} = A_0(\underline{\sigma},\underline{\sigma})$ with $\underline{\sigma} = \mathcal{I}v \in \mathcal{E}$. So $\forall v \in H_0^2(\Omega)$ with $\underline{\sigma} = \mathcal{I}v \in \mathcal{E},\ R(v) = \dfrac{A_0(\underline{\sigma},\underline{\sigma})}{\langle v,v\rangle_{0,\Omega}}\cdot$
i.e. $R(v)$ is expressed through a **linked pair** $(\underline{\sigma}, v) = (\mathcal{I}v, v) \in \mathcal{E} \times H_0^2(\Omega)$. Hence, it suggests to define a new Rayleigh quotient $\Re(\cdot,\cdot)$ on $\mathcal{M} \equiv \mathcal{E} \times H_0^2(\Omega)$ by (see also [14]): $\Re(\underline{\sigma}, v) = \dfrac{A_0(\underline{\sigma},\underline{\sigma})}{\langle v,v\rangle_{0,\Omega}}$ $\forall$ linked pair $(\underline{\sigma}, v) = (\mathcal{I}v, v) \in \mathcal{M}$ such that $\Re(\underline{\sigma}, v) \equiv R(v) = \dfrac{a(v,v)}{\langle v,v\rangle_{0,\Omega}}$ $\forall v \in H_0^2(\Omega) - \{0\}$, for which we can apply various extrema [2,14,35,37].
Define a p-dimensional subspace $\mathcal{M}_p$ (resp. U_p) of $\mathcal{M}$ (resp. $H_0^2(\Omega)$) by: $\mathcal{M}_p = \mathrm{Span}\{(\Psi_m, u_m)_{m=1}^p\}$; $U_p = \mathrm{Span}\{(u_m)_{m=1}^p\},\ (\lambda_m; (\Psi_m, u_m)) \in \mathbb{R}^+ \times (\mathcal{E} \times H_0^2(\Omega)),\ \ 1 \le m \le p$, being the first '$p$' eigenpairs of $(\mathbf{Q_{AUX}^E})$ with $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_p$.

Theorem 2.3 (Min-Max Principle, [2,37])**.**

(*i*) *Eigensolutions of* $(\mathbf{Q_{AUX}^E})$ *are the stationary points of* $\Re(\cdot,\cdot)$ *on* $\mathcal{M}$, *the corresponding eigenvalues being the values of* $\Re(\cdot,\cdot)$ *at these stationary points*;

(*ii*) $\forall p \in \mathbb{N},\ \ \lambda_p = \min\limits_{\substack{\mathcal{S}_p \subset \mathcal{M} \\ \dim(\mathcal{S}_p) = p}} \max\limits_{(\underline{\sigma},v) \in \mathcal{S}_p} \Re(\underline{\sigma}, v) = \max\limits_{(\underline{\sigma},v) \in \mathcal{M}_p} \Re(\underline{\sigma}, v) = \Re(\Psi_p, u_p).$ (2.18)

We will need another Rayleigh quotient $Q(v) = \dfrac{\langle T_0 v, v\rangle_{0,\Omega}}{\langle v,v\rangle_{0,\Omega}}$ $\forall v \in H_0^2(\Omega)$, where $T_0 \in \mathcal{L}(L^2(\Omega))$ is compact, positive and symmetric. Hence,

Theorem 2.4 (Max-Min Principle, [2,37])**.** $\forall p \in \mathbb{N}$,

$$\mu_p = \max_{\substack{S_p^* \subset L^2(\Omega) \\ \dim S_p^* = p}} \min_{v \in S_p^*} Q(v) = \min_{\substack{v \in U_p \\ U_p = \mathrm{Span}\{(u_m)_{m=1}^p\}}} Q(v) = Q(u_p), \tag{2.19}$$

where $(\mu_m, u_m)_{m=1}^p$ *are the first 'p' eigenpairs of* T_0 *corresponding to the first 'p' eigenvalues* $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_p$ *of* T_0, u_m *being the m-th eigenvector of* T_0 $(\mu_m = 1/\lambda_m,\ 1 \le m \le p)$.

But $(\mathbf{Q_{AUX}^E})$ is **not** suitable for finite element approximation, since C^1-elements are to be used for construction of finite element subspaces of $H_0^2(\Omega)$. Hence, we construct a new **Continuous Mixed Variational Eigenvalue Problem** $(\mathbf{Q^E})$, which will be eminently suitable for finite element approximation using C^0-elements as follows:
Find $\lambda \in \mathbb{R}$ for which $\exists$ non-null $(\Psi, u) \in \mathbf{V} \times W$ such that

$$(\mathbf{Q^E}):\ A(\Psi,\Phi) + b(\Phi,u) = 0\ \ \forall \Phi \in \mathbf{V},\quad -b(\Psi,v) = \lambda\langle u,v\rangle_{0,\Omega}\ \ \forall v \in W, \tag{2.20}$$

where $\mathbf{V} = \{\Phi : \Phi = (\phi_{ij})_{i,j=1,2} \in \mathbf{H},\ \phi_{ij} \in H^1(\Omega)\ \forall i,j = 1,2\}$ with $\|\Phi\|_{\mathbf{V}}^2 = \|\Phi\|_{1,\Omega}^2 = \sum\limits_{i=1}^2 \sum\limits_{j=1}^2 \|\phi_{ij}\|_{1,\Omega}^2$
$W \equiv H_0^1(\Omega) = \{v : v \in H^1(\Omega),\ v|_\Gamma = 0\}$ with $\|v\|_W = \|v\|_{1,\Omega}$;
$A(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous bilinear forms defined by [4]:

$$A(\Psi,\Phi) = A_0(\Psi,\Phi)\ \ \forall \Psi,\Phi \in \mathbf{V} \subset \mathbf{H} \text{ such that}$$

$$|A(\Psi,\Phi)| \le M\|\Psi\|_{\mathbf{V}}\|\Phi\|_{\mathbf{V}} \text{ for some } M > 0;\ \ A(\Phi,\Phi) \ge \alpha\|\Phi\|_{\mathbf{H}}^2\ \ \forall \Phi \in \mathbf{V},\ \text{for some } \alpha > 0\ \ [4], \tag{2.21}$$

$$b(\Phi, v) = \int_\Omega \phi_{ij,j} v_{,i} \, d\Omega \;\; \forall \Phi \in \mathbf{V} \;\; \forall v \in W \text{ with } b(\Phi, v) = b_0(\Phi, v) \; \forall \Phi \in \mathbf{V} \; \forall v \in H_0^2(\Omega) \tag{2.22}$$

such that $|b(\Phi, v)| \leq m\|\Phi\|_{\mathbf{V}}\|v\|_W, \;\; \forall \Phi \in \mathbf{V}, \forall v \in W$ and some $m > 0$; and $\exists \beta > 0$ such that $\sup_{\Phi \in \mathbf{V} - \{\mathbf{0}\}} \frac{|b(\Phi, v)|}{\|\Phi\|_{\mathbf{V}}} \geq \beta\|v\|_W \;\; \forall v \in W$ ([4]).
If $(\lambda; (\Psi, u)) \in \mathbb{R} \times (\mathbf{V} \times W)$ be an eigenpair of $(\mathbf{Q^E})$, then $\lambda \in \mathbb{R}^+$.
Then, the corresponding continuous **Mixed Variational Source Problem** $(\mathbf{Q})$ [4] is defined by:
For given $f \in L^2(\Omega)$, find $(\Psi, u) \in \mathbf{V} \times W$ such that

$$(\mathbf{Q}): \; A(\Psi, \Phi) + b(\Phi, u) = 0 \;\; \forall \Phi \in \mathbf{V}, \qquad -b(\Psi, v) = \langle f, v \rangle_{0,\Omega} \;\; \forall v \in W. \tag{2.23}$$

Since $A(\cdot, \cdot)$ is **not** $\mathbf{V}$-elliptic, $(\mathbf{Q})$ is not well-posed *a priori*. But we have:

Theorem 2.5 ([4]). *Let* $(\mathbf{A1}\text{–}\mathbf{A5})$ *hold. If* $u \in H^3(\Omega) \cap H_0^2(\Omega)$ *be the solution of the Galerkin Variational Source Problem* $(\mathbf{P_G})$ *with* $\psi_{ij} = a_{ijkl}u_{,kl} \in H^1(\Omega) \;\; \forall i, j = 1, 2$ *and* $\Psi = (\psi_{ij})_{1 \leq i,j \leq 2}$*, then* $(\Psi, u) \in \mathbf{V} \times W$ *is the unique solution of* $(\mathbf{Q})$*. Conversely, let* $(\Psi, u) \in \mathbf{V} \times W$ *be the solution of* $(\mathbf{Q})$*. Then,* $u \in H_0^2(\Omega)$ *and is the unique solution of* $(\mathbf{P_G})$ *and* $\psi_{ij} = a_{ijkl}u_{,kl} \;\; \forall i, j = 1, 2;\; u_{,kl} = A_{ijkl}\psi_{ij} \;\; \forall k, l = 1, 2;\;\; \Psi = (\psi_{ij})_{1 \leq i,j \leq 2}$.

Hence, under the assumption that the solution $u \in H_0^2(\Omega)$ of Galerkin Variational Source Problem $(\mathbf{P_G})$ in (2.4) has the additional regularity [18, 21, 24]:

$$(\mathbf{A6}): \; u \in H^3(\Omega) \cap H_0^2(\Omega) \text{ with } \|u\|_{3,\Omega} \leq C\|f\|_{0,\Omega} \text{ for some } C > 0, \tag{2.24}$$

the correspondence $f \in L^2(\Omega) \mapsto (\Psi, u) \in \mathbf{V} \times W$ with $u \in H^3(\Omega) \cap H_0^2(\Omega)$ defines an operator $\vec{\mathbf{T}}f = (Sf, Tf) = (\Psi, u) \in \mathbf{V} \times W$ with

$$A(Sf, \Phi) + b(\Phi, Tf) = 0 \;\; \forall \Phi \in \mathbf{V}, \qquad -b(Sf, v) = \langle f, v \rangle_{0,\Omega} \;\; \forall v \in W, \tag{2.25}$$

$T: f \in L^2(\Omega) \mapsto Tf = u \in H^3(\Omega) \cap H_0^2(\Omega),\; S: f \in L^2(\Omega) \mapsto Sf = \Psi = (a_{ijkl}(Tf)_{,kl})_{i,j=1,2} \in \mathbf{V}$, being the solution component operators with $S(\cdot) = (a_{ijkl}(T(\cdot))_{,kl})_{1 \leq i,j \leq 2}$ and $\|Tf\|_{1,\Omega} \leq C\|f\|_{0,\Omega}$; $\|Sf\|_{1,\Omega} \leq C\|f\|_{0,\Omega}$; $\|Sf\|_{0,\Omega} + \|Tf\|_{1,\Omega} \leq C\|f\|_{0,\Omega}$.

Theorem 2.6 ([8, 31]). *Under* $(\mathbf{A6})$*, the source problems* $(\mathbf{Q})$ *and* $(\mathbf{Q_{AUX}})$ *are "equivalent" in the sense that these have the same solution* $(\Psi, u) \in \mathbf{V} \times W$ *with* $u \in H^3(\Omega) \cap H_0^2(\Omega)$,
$\psi_{ij} = a_{ijkl}u_{,kl} \in H^1(\Omega) \; \forall i, j = 1, 2,\; \Psi = (\psi_{ij})_{1 \leq i,j \leq 2} \in \mathbf{V} \subset \mathbf{H}$.

Hence under $(\mathbf{A6})$, $\forall f \in L^2(\Omega),\; Sf = S_0 f = \Psi \in \mathbf{V} \subset \mathbf{H},\; Tf = T_0 f = u \in H^3(\Omega) \cap H_0^2(\Omega) \subset W \subset L^2(\Omega)$ and all the results associated with $T_0 \in \mathcal{L}(L^2(\Omega))$ and $S_0 \in \mathcal{L}(L^2(\Omega))$ will hold for $T \in \mathcal{L}(L^2(\Omega))$ and $S \in \mathcal{L}(L^2(\Omega); \mathbf{V})$. Hence, we have the important result:

Theorem 2.7 ([8]). *Under* $(\mathbf{A6})$*, mixed variational eigenvalue problems* $(\mathbf{Q^E})$ *and* $(\mathbf{Q^E_{AUX}})$ *are equivalent in the sense that both of these eigenvalue problems have the same strictly positive eigenvalues* $(\lambda_m)_{m=1}^\infty$ *and the same eigenpairs* $(\lambda_m; (\Psi_m, u_m)) \in \mathbb{R}^+ \times (\mathbf{V} \times W)$ *with* $u_m \in H^3(\Omega) \cap H_0^2(\Omega)$, $(u_m)_{m=1}^\infty$ *being a Hilbert basis in* $L^2(\Omega)$ *and* $(\Psi_m/\sqrt{\lambda_m})_{m=1}^\infty$ *being an orthonormal system in* $(\mathbf{H}, A(\cdot, \cdot))$.

Define a linked pair $(\underline{\sigma}_p, \chi_p) = \sum_{m=1}^{p} c_m(\Psi_m, u_m) \in \mathcal{M}_p$ with $\underline{\sigma}_p = \sum_{m=1}^{p} c_m \Psi_m \in \mathbf{V}$,
$\chi_p = \sum_{m=1}^{p} c_m u_m \in H^3(\Omega) \cap H_0^2(\Omega) \; c_m \in \mathbb{R} \;\; \forall m = 1, 2, \cdots, p$, where $(\Psi_m, u_m) \in \mathbf{V} \times W$ with $u_m \in H^3(\Omega) \cap H_0^2(\Omega)$ is an eigenelement of $(\mathbf{Q^E})$ corresponding to the eigenvalue λ_m, $0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_p$. In

particular, for $c_m = 0\ \forall\, m \neq p,\ c_p = 1,\ \underline{\sigma}_p = \Psi_p,\ \chi_p = u_p$. Then, $\forall(\underline{\sigma}_p, \chi_p) \in \mathcal{M}_p$ with $\chi_p \in H^3(\Omega) \cap H_0^2(\Omega)$ and $\underline{\sigma}_p \in \mathbf{V}$, $\exists$ a unique linked pair $(\underline{\sigma}_p^*, \chi_p^*) \in \mathcal{M}_p$ with

$$\chi_p^* = \sum_{m=1}^{p} \lambda_m c_m u_m \text{ and } \underline{\sigma}_p^* = \mathcal{I}\chi_p^* = \sum_{m=1}^{p} \lambda_m c_m \Psi_m \tag{2.26}$$

such that $\vec{\mathbf{T}}\chi_p^* = (\underline{\sigma}_p, \chi_p)$, $\vec{\mathbf{T}}$ being the linear operator defined in (2.25).

Examples [8]: **I. Biharmonic Eigenvalue Problem** is obtained from (2.2) a_{ijkl} defined by: $a_{iiii} = 1$; $a_{1212} = a_{2121} = a_{2112} = a_{1221} = 1/2$; $a_{ijkl} = 0$ otherwise, which satisfy (**A3**–**A4**) [3], in $\widetilde{\Omega}$. Then, we have $\Lambda \equiv \Delta\Delta$, for which (**A6**) holds [4]. ($\mathbf{Q^E}$) corresponds to **H-H-M** (Hellan-Hermann-Miyoshi) mixed method scheme for biharmonic eigenvalue problem [4,13,30]. $\forall m \in \mathbb{N}\ \ (\lambda_m; (\Psi_m, u_m)) \in \mathbb{R}^+ \times (\mathbf{V} \times W)$ with $u_m \in H^3(\Omega) \cap H_0^2(\Omega)$ and $\Psi_m = (\psi_{mij})_{i,j=1,2}$ is an eigenpair of biharmonic eigenvalue problem in **H-H-M** mixed method formulation [2,13,14,22]:

$$\int_\Omega \psi_{mij}\phi_{ij}\mathrm{d}\Omega + \int_\Omega \phi_{ij,j}u_{m,i}\mathrm{d}x = 0\ \forall\Phi \in \mathbf{V},\quad -\int_\Omega \psi_{mij,j}v_{,i}\mathrm{d}\Omega = \lambda_m\langle u_m, v\rangle_{0,\Omega}\ \forall v \in W. \tag{2.27}$$

Remark 2.1. The associated biharmonic *source* problem corresponds to Stokes problem [34] of fluid mechanics in stream function-vorticity formulation and also to the bending problem of isotropic elastic plates with flexural rigidity $D = 1,\ \nu = 0$ (see (2.32)).

II. Eigenvalue problems associated with the vibration of elastic plates with variable/ constant thickness. (i) In **Anisotropic** case [4,27],

$$\begin{aligned} &a_{iiii} = D_{ii},\ a_{1212} = a_{1221} = a_{2121} = a_{2112} = D_{66},\ a_{1112} = a_{1211} = a_{2111} = a_{1121} = D_{16},\\ &a_{1222} = a_{2122} = a_{2212} = a_{2221} = D_{26},\ a_{2211} = a_{1122} = D_{12}, \end{aligned} \tag{2.28}$$

$D_{ij} = D_{ij}(x_1, x_2)\ \ \forall(x_1, x_2) \in \bar{\widetilde{\Omega}}$ being rigidities [8,27] for which (**A2**–**A4**) hold, and the anisotropic plate bending operator Λ is given by:

$$\begin{aligned} \Lambda u \equiv\ &(D_{11}u_{,11} + 2D_{16}u_{,12} + D_{12}u_{,22})_{,11} + 2(D_{16}u_{,11} + 2D_{66}u_{,12} + D_{26}u_{,22})_{,12}\\ &+ (D_{12}u_{,11} + 2D_{26}u_{,12} + D_{22}u_{,22})_{,22}. \end{aligned} \tag{2.29}$$

Then, coefficients A_{ijkl} are defined in terms of D_{ij}'s [4,9,10] and the corresponding bilinear form $A(\cdot,\cdot)$ of ($\mathbf{Q^E}$) is given by: $\Psi, \Phi \in \mathbf{V}$,

$$\begin{aligned} A(\Psi,\Phi) = \int_\Omega \frac{4}{|A(x)|}\Big[&\{(D_{22}D_{66} - D_{26}^2)\psi_{11} + (D_{16}D_{26} - D_{12}D_{66})\psi_{22} + (D_{12}D_{26} - D_{16}D_{22})\psi_{12}\}\phi_{11}\\ &+ \{(D_{16}D_{26} - D_{12}D_{66})\psi_{11} + (D_{11}D_{66} - D_{16}^2)\psi_{22} + (D_{16}D_{12} - D_{11}D_{26})\psi_{12}\}\phi_{22}\\ &+ \{(D_{12}D_{26} - D_{16}D_{22})\psi_{11} + (D_{16}D_{12} - D_{11}D_{26})\psi_{22} + (D_{11}D_{22} - D_{12}^2)\psi_{12}\}\phi_{12}\Big]\mathrm{d}x; \end{aligned} \tag{2.30}$$

where $|A(x)| = 4(D_{11}D_{22}D_{66} - D_{11}D_{26}^2 - D_{66}D_{12}^2 - D_{22}D_{16}^2 + D_{12}D_{16}D_{26})(x)$.

(ii) The **Orthotropic** case [3,27,38] can be retrieved from the anisotropic case (i) by putting in (2.28)–(2.30),

$$\begin{aligned} &a_{iiii} = D_i;\ a_{1122} = a_{2211} = D_{12} = \nu_1 D_2 = \nu_2 D_1;\\ &a_{1212} = a_{2121} = a_{2112} = a_{1221} = D_\tau,\ a_{ijkl} = 0 \text{ otherwise,} \end{aligned} \tag{2.31}$$

with $D_i = D_i(x_1, x_2)$ and $D_\tau = D_\tau(x_1, x_2)$ $\forall (x_1, x_2) \in \bar{\bar{\Omega}}$, $H = D_1\nu_2 + 2D_\tau$, ν_i, $i = 1, 2$ being Poisson's coefficients respectively, D_i's and D_τ being rigidities, assumptions (**A3**–**A4**) hold [3] and
(iii) the **Isotropic** case is obtained from the Orthotropic case by putting $\nu_1 = \nu_2 = \nu$ and $D_1 = D_2 = D$ in all formulae (2.31). Then, the orthotropic (resp. isotropic) plate (bending) operator Λ and the corresponding bilinear form $A(\cdot, \cdot)$ of $(\mathbf{Q^E})$ are given by:
Orthotropic Case: $\Lambda u \equiv (D_1 u_{,11} + \nu_2 D_1 u_{,22})_{,11} + 4(D_\tau u_{,12})_{,12} + (\nu_1 D_2 u_{,11} + D_2 u_{,22})_{,22}$

$$A(\Psi, \Phi) = \int_\Omega \left[\frac{1}{D_1(1 - \nu_1\nu_2)} (\psi_{11} - \nu_1\psi_{22})\phi_{11} + \frac{1}{D_2(1 - \nu_1\nu_2)} (-\nu_2\psi_{11} + \psi_{22})\phi_{22} + \frac{1}{D_\tau}\psi_{12}\phi_{12} \right] \mathrm{d}x \ \ \forall \Psi, \Phi \in \mathbf{V}.$$

Isotropic Case: $\Lambda u \equiv (D(u_{,11} + \nu u_{,22}))_{,11} + 2(D(1 - \nu)u_{,12})_{,12} + (D(\nu u_{,11} + u_{,22}))_{,22}$.

$$\text{Then, for } D = \textit{constant}, \ \Lambda u \equiv D\Delta\Delta u, \ (\mathbf{A6}) \text{ will hold [18, 21, 24].} \tag{2.32}$$

$$A(\Psi, \Phi) = \int_\Omega \left[\frac{1}{D(1 - \nu^2)} (\psi_{11} - \nu\psi_{22})\phi_{11} + \frac{1}{D(1 - \nu^2)} (-\nu\psi_{11} + \psi_{22})\phi_{22} + \frac{2}{D(1 - \nu)}\psi_{12}\phi_{12} \right] \mathrm{d}x \ \forall \Psi, \Phi \in \mathbf{V}.$$

In aniso-/ortho-/isotropic cases (i–iii), $\forall$ eigenpair $(\lambda_m; (\Psi_m, u_m))$ of $(\mathbf{Q^E})$, u_m is the deflection mode of the vibrating plate, $\Psi_m = (\psi_{mij})_{1\le i,j\le 2}$ is the corresponding bending moment tensor, ψ_{mii} being the bending moment in the x_i direction and $\psi_{m12} = \psi_{m21}$, being the twisting moment, *i.e.*
Anisotropic Case: $\psi_{mij} = a_{ijkl}u_{,kl}$ with a_{ijkl}'s defined by (2.28), $i, j = 1, 2$;
Orthotropic Case: $\psi_{m11} = D_1(u_{m,11} + \nu_2 u_{m,22})$; $\psi_{m22} = D_2(\nu_1 u_{m,11} + u_{m,22})$, $\psi_{m12} = 2D_\tau u_{m,12}$;
Isotropic Case: $\psi_{m11} = D(u_{m,11} + \nu u_{m,22})$, $\psi_{m22} = D(\nu u_{m,11} + u_{m,22})$, $\psi_{m12} = \psi_{m21} = D(1 - \nu)u_{m,12}$.

Remark 2.2. In the orthotropic plates with constant thickness, D_1 = constant, D_2 = constant, $H = D_1\nu_2 + 2D_\tau$ = constant and $\Lambda u \equiv D_1 u_{,1111} + 2H u_{,1122} + D_2 u_{,2222}$.

$$\text{Then, for } \ H = \sqrt{D_1 D_2}, \ \ \Lambda u \equiv D_1 u_{,1111} + 2\sqrt{D_1 D_2}u_{,1122} + D_2 u_{,2222} \tag{2.33}$$

can be reduced to the form (2.32) by introducing a new variable $\xi_2 = x_2(D_1/D_2)^{1/4}$, $\xi_1 = x_1$ ([38], pp. 366–367), *i.e.* $\Lambda u^* \equiv D_1\Delta\Delta u^*$ with $u^* = u^*(\xi_1, \xi_2)$, $\Delta = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2}$. Hence, (**A6**) will also hold for (2.33) [18,21,24].

3. Isoparametric mixed finite element eigenvalue problem $(\mathbf{Q_h^E})$

Isoparametric Triangulation τ_h^{ISO}: Let $\{P_i\}_{i=1}^{N_c}$ be N_c corner points of Γ at which C^m-smoothness $(m \ge 3)$ does **not** hold and $\{P_j\}_{N_c}^{N_h}$ be the set of possible additional points suitably chosen on Γ such that $\nu_h = \{P_i\}_{i=1}^{N_c} \cup \{P_j\}_{j=N_c+1}^{N_h} \subset \Gamma$ denote the set of boundary vertices of the isoparametric triangulation of $\bar{\Omega}$ under consideration.
I. τ_h^{pol}: Let $\tau_h^{\mathrm{pol}} = \tilde{\tau}_h^b \cup \tau_h^0$ be the admissible, regular, quasi-uniform triangulation of the closed **polygonal** domain $\bar{\Omega}_h^{\mathrm{pol}} = \Omega_h^{\mathrm{pol}} \cup \Gamma_h^{\mathrm{pol}}$ with vertices $\{P_i\}_{i=1}^{N_h}$ into closed triangles $\tilde{T}$ with vertices $(a_{i,\tilde{T}})_{i=1}^3$ and the **straight** sides $(\partial\tilde{T}_i)_{i=1}^3$, $\partial\tilde{T}_i = [a_{i,\tilde{T}}, a_{i+1,\tilde{T}}]$ (modulo 3) such that $\Gamma_h^{\mathrm{pol}} = \cup_{\tilde{T}\in\tilde{\tau}_h^b} \partial\tilde{T}_1$, $\partial\tilde{T}_1 = [a_{1,\tilde{T}}, a_{2,\tilde{T}}]$ being the boundary side $\forall \tilde{T} \in \tilde{\tau}_h^b$,

$$\tilde{\tau}_h^b \ \ = \ \ \{\tilde{T} : \tilde{T} \in \tau_h^{\mathrm{pol}} \text{ is a boundary triangle with single boundary side } \partial\tilde{T}_1\}; \tag{3.1}$$

$$\tau_h^0 = \{\tilde{T} : \tilde{T} \in \tau_h^{\mathrm{pol}} \text{ is an interior triangle with } \textbf{at most} \text{ one of its vertices lying on } \Gamma\}; \tag{3.2}$$

II. τ_h^{exact}: Keeping all **vertices** of τ_h^{pol} and interior triangles $\tilde{T} \in \tau_h^0$ in (3.2) **undisturbed** and replacing **each boundary triangle** $\tilde{T} \in \tilde{\tau}_h^b$ by a **curved** boundary triangle $\bar{T}$ which is obtained by replacing the straight boundary side $\partial\tilde{T}_1$ of $\tilde{T} \in \tilde{\tau}_h^b$ by a part $\partial\bar{T}_1$ of the boundary Γ between the boundary vertices of $\tilde{T} \in \tilde{\tau}_h^b$. Let $\bar{\tau}_h^b$ denote all such curved boundary triangles $\bar{T}$. Then, $\tau_h^{\text{exact}} = \bar{\tau}_h^b \cup \tau_h^0$ with $\bar{\Omega} = \cup_{\bar{T}\in\tau_h^{\text{exact}}}\bar{T}$ and $\Gamma = \cup_{\bar{T}\in\bar{\tau}_h^b}\partial\bar{T}_1$.

III. τ_h^{ISO}: Again, we keep all vertices of τ_h^{pol} and interior triangles $\tilde{T} \in \tau_h^0$ **undisturbed**. $\forall$ boundary triangle $\tilde{T} \in \tilde{\tau}_h^b$ in (3.1), define mid-side points $a_{4,\tilde{T}} = (a_{1,\tilde{T}} + a_{2,\tilde{T}})/2,\ a_{5,\tilde{T}} = (a_{2,\tilde{T}} + a_{3,\tilde{T}})/2,\ a_{6,\tilde{T}} = (a_{3,\tilde{T}} + a_{1,\tilde{T}})/2$. To $a_{4,\tilde{T}} \in \Gamma_h^{\text{pol}}$, we associate the point $a_{4,T}^* \in \Gamma$ as the point of intersection of the perpendicular bisector of $\partial\tilde{T}_1$ at $a_{4,\tilde{T}}$ with Γ. Let $\hat{T}$ be the reference triangle with vertices $\hat{a}_1 = (1,0),\ \hat{a}_2 = (0,1),\ \hat{a}_3 = (0,0)$ and mid-side nodes $\hat{a}_4 = (1/2,1/2),\ \hat{a}_5 = (0,1/2),\ \hat{a}_6 = (1/2,0)$, sides $\partial\hat{T}_i = [\hat{a}_i,\ \hat{a}_{i+1}]$ (modulo 3) $1 \le i \le 3$. Then, with the help of canonical basis functions $(\hat{\phi}_i)_{i=1}^6$ of $P_2(\hat{T})$ (*i.e.* $\forall\hat{\phi}_i \in P_2(\hat{T}),\ \hat{\phi}_i(\hat{a}_j) = \delta_{ij},\ 1 \le i,j \le 2$), define the *invertible* isoparametric mapping by: $\forall\hat{x} \in \hat{T}$,

$$F_T(\hat{x}) = \sum_{i=1}^{3} a_{i,\tilde{T}}\hat{\phi}_i(\hat{x}) + \sum_{i=5}^{6} a_{i,\tilde{T}}\hat{\phi}_i(\hat{x}) + a_{4,T}^*\hat{\phi}_4(\hat{x}) = x \in T = F_T(\hat{T}), \tag{3.3}$$

such that $F_T(\hat{a}_i) = a_{i,\tilde{T}} \in \tilde{T} \in \tilde{\tau}_h^b,\ 1 \le i \ne 4 \le 6,\ F_T(\hat{a}_4) = a_{4,T}^* \in \Gamma,\quad F_T(\partial\hat{T}_i) = \partial T_i,\quad 1 \le i \le 3$. Then, $\forall\tilde{T} \in \tilde{\tau}_h^b$ in (3.1), we get a **curved** boundary triangle $T = F_T(\hat{T})$ with the single curved boundary side $\partial T_1 = F_T(\partial\hat{T}_1)$. Let τ_h^b be all such curved boundary triangles. Then,

$$\tau_h^{\text{ISO}} = \tau_h^b \cup \tau_h^0 \ \textit{with}\ \tau_h^0\ \textit{defined by}\ (3.2),\ \ \bar{\Omega}_h = \cup_{T\in\tau_h^{\text{ISO}}}T,\ \Gamma_h = \cup_{T\in\tau_h^b}\partial T_1 \tag{3.4}$$

is the **Isoparametric Triangulation** of $\bar{\Omega}$, Γ_h being the approximation of the boundary Γ. For other methods of approximation of boundary Γ, we refer to [7, 41]. Such a τ_h^{ISO} is regular in the sense of [16]. Ω_h is **not** convex, $\Omega_h \not\subset \Omega,\ \Omega \not\subset \Omega_h$ in general. But by construction, the distance of Γ from Γ_h tends to 0 as $h \longrightarrow 0$ and from (**A1**), $\exists\widetilde{\Omega}$ with boundary $\widetilde{\Gamma}$, which is piecewise of C^k-class, $k \ge 3$, such that $\bar{\Omega} \subset \widetilde{\Omega}$. Hence,

(**A7**): $\exists h_0 > 0$ such that $\forall h \in]0,h_0[,\ \bar{\Omega}_h \subset \widetilde{\Omega}$.

Then, from (**A1**) and (**A7**) $\forall h \in]0,h_0[,\ \bar{\Omega}_h \subset \widetilde{\Omega},\ \bar{\Omega} \subset \widetilde{\Omega}$ with $(\bar{\Omega}_h \cup \bar{\Omega}) \subset \widetilde{\Omega}$ and define

$$\epsilon_h = \cup_{T\in\tau_h^b,\bar{T}\in\bar{\tau}_h^b}(T^{\text{int}} - (\bar{T}^{\text{int}} \cap T^{\text{int}})) = \Omega_h - (\Omega \cap \Omega_h) \text{ with meas}(\epsilon_h) = O(h^3) \qquad [10,17]; \tag{3.5}$$

$$\omega_h = \cup_{\bar{T}\in\bar{\tau}_h^b,\ T\in\tau_h^b}(\bar{T}^{\text{int}} - (\bar{T}^{\text{int}} \cap T^{\text{int}})) = \Omega - (\Omega \cap \Omega_h) \text{ with meas}(\omega_h) = O(h^3) \qquad [10,17], \tag{3.6}$$

where $T^{\text{int}} = int(T),\ \bar{T}^{\text{int}} = int(\bar{T}),\ \tau_h^b \subset \tau_h^{\text{ISO}},\ \bar{\tau}_h^b \subset \tau_h^{\text{exact}}\ \ \forall h \in]0,h_0[$ with $h_0 > 0$.
$\forall h$ boundary Γ_h of Ω_h is piecewise of C^∞-class, ν_h being the set of boundary vertices of τ_h^{ISO} at which C^∞ smoothness does *not* hold. For the properties of the invertible F_T (resp. F_T^{-1}) and its Jacobian $J(F_T) \in P_1(\hat{T})$ with important estimates, we refer to [10, 16, 17].
We will need extensions to $\mathbb{R}^2$ of functions defined in Ω_h (resp. Ω).

Theorem 3.1 ([32, 39])**.** *Let $\mathcal{D}$ be a bounded,* ***two-dimensional*** *domain with Lipschitz continuous boundary $\partial\mathcal{D}$, which is piecewise of C^k-class, $k \ge 1$. Then,*
(a) $\exists$ a ***continuous, linear*** *extension operator $E : H^k(\mathcal{D}) \longrightarrow H^k(\mathbb{R}^2)$, i.e. $\exists C > 0$ such that*

$$\|Eu\|_{k,\mathbb{R}^2} = \|\tilde{u}\|_{k,\mathbb{R}^2} \le C\|u\|_{k,\mathcal{D}}\ \ \forall \text{ fixed } k \ge 1 \text{ with } Eu\downarrow_{\mathcal{D}} = \tilde{u}\downarrow_{\mathcal{D}} = u \in H^k(\mathcal{D}). \tag{3.7}$$

(b) The operator E is also a linear and bounded extension operator from $H^{(k-i)}(\mathcal{D})$ into $H^{(k-i)}(\mathbb{R}^2)$, $1 \le i \le k$, i.e. $\exists C > 0$ such that $\|Eu\|_{k-i,\mathbb{R}^2} = \|\tilde{u}\|_{k-i,\mathbb{R}^2} \le C\|u\|_{k-i,\mathcal{D}}$, $1 \le i \le k$, with $Eu\downarrow_{\mathcal{D}} = \tilde{u}\downarrow_{\mathcal{D}} = u \in H^{k-i}(\mathcal{D})$, $1 \le i \le k$, and in particular,

$$\|\tilde{u}\|_{0,\mathbb{R}^2} \le C\|u\|_{0,\mathcal{D}}\ \ \forall u \in L^2(\mathcal{D}) \text{ with } \tilde{u} \in L^2(\mathbb{R}^2). \tag{3.8}$$

Boundaries $\Gamma_h, \Gamma, \tilde{\Gamma}$ of Ω_h, Ω and $\widetilde{\Omega}$ respectively satisfy the smoothness conditions in Theorem 3.1. We can choose $\mathcal{D} = \Omega_h$ (resp. Ω) and find the corresponding extension $\tilde{u}_h = Eu_h$ (resp. $\tilde{u} = Eu$).

Corollary 3.1. *For $\mathcal{D} = \Omega_h$ (resp.Ω), let $E : H^k(\mathcal{D}) \longrightarrow H^k(\mathbb{R}^2)$, be the continuous linear extension operator defined in (3.7) and (3.8) and $\rho : H^k(\mathbb{R}^2) \longrightarrow H^k(\widetilde{\Omega})$ be the continuous, linear restriction operator i.e. $\forall \tilde{u} \in H^k(\mathbb{R}^2)$, $\|\rho\tilde{u}\|_{k,\widetilde{\Omega}} = \|\tilde{u}\|_{k,\widetilde{\Omega}} \leq C\|\tilde{u}\|_{k,\mathbb{R}^2}$ with $\tilde{u} \downarrow_{\widetilde{\Omega}} = \tilde{u}$.* ***(For restriction to $\widetilde{\Omega}$ of $\widetilde{u}$, the same notation $\widetilde{u}$ has been used and will be used in the sequel).***
Then, $\rho \cdot E : H^k(\mathcal{D}) \longrightarrow H^k(\widetilde{\Omega})$ is a continuous, linear extension operator from $H^k(\mathcal{D})$ into $H^k(\widetilde{\Omega})$, i.e. $\forall u \in H^k(\mathcal{D})$, $\|\rho \cdot Eu\|_{k,\widetilde{\Omega}} = \|\tilde{u}\|_{k,\widetilde{\Omega}} \leq C\|u\|_{k,\mathcal{D}}$ for some $C > 0$

$$\text{and } \|\tilde{u}\|_{k-i,\widetilde{\Omega}} \leq C\|u\|_{k-i,\mathcal{D}} \quad \forall i = 1, 2, \cdots, k. \tag{3.9}$$

(In (3.7) and (3.9), the same letter $C > 0$, having different strictly positive values has been used and this convention of using the same letter $C > 0$ with different strictly positive values at different steps will be followed in the sequel unless stated otherwise).
With the help of Theorem 3.1 and Corollary 3.1, we define $\widetilde{\mathbf{V}}$ and $\widetilde{W}$ as follows:

$$\begin{aligned}\widetilde{\mathbf{V}} = \{\widetilde{\Phi} :\ \widetilde{\Phi} = (\widetilde{\phi}_{ij})_{i,\, j=1,\, 2}; \widetilde{\phi}_{ij} = \widetilde{\phi}_{ji} \in H^1(\widetilde{\Omega}) \text{ such that } \widetilde{\phi}_{ij} \downarrow_{\Omega} = \phi_{ij} \in H^1(\Omega),\\ \Phi = (\phi_{ij})_{i,j=1,2} \in \mathbf{V}\} \text{ with } \|\widetilde{\phi}_{ij}\|_{1,\widetilde{\Omega}} \leq C\|\phi_{ij}\|_{1,\,\Omega}\ \ \forall i, j = 1, 2, \text{ for some } C > 0;\end{aligned} \tag{3.10}$$

$$\widetilde{W} = \{\widetilde{\chi} : \widetilde{\chi} \in H_0^1(\widetilde{\Omega}),\ \widetilde{\chi}\downarrow_{\widetilde{\Omega}-\Omega} = 0\ \} \text{ with } \|\widetilde{\chi}\|_{\widetilde{W}} = \|\widetilde{\chi}\|_{H_0^1(\widetilde{\Omega})} = \|\chi\|_{1,\Omega}. \tag{3.11}$$

To every $\bar{\Omega}_h$ we associate Hilbert spaces $\mathbf{V}(\Omega_h)$ and $H_0^1(\Omega_h)$ defined by:
$\mathbf{V}(\Omega_h) = \{\Phi : \Phi = (\phi_{ij})_{i,j=1,2}, \phi_{ij} = \phi_{ji} \in H^1(\Omega_h) \ \ \forall i, j = 1, 2\}$ with $\|\Phi\|^2_{\mathbf{V}(\Omega_h)} = \|\Phi\|^2_{1,\Omega_h} = \sum_{i=1}^{2}\sum_{j=1}^{2} \|\phi_{ij}\|^2_{1,\Omega_h}$,
$H_0^1(\Omega_h) = \{v : v \in H^1(\Omega_h),\ v|_{\Gamma_h} = 0\}$ with $\|v\|_{H_0^1(\Omega_h)} = \|v\|_{1,\Omega_h}$,
and define the auxiliary continuous, bilinear forms $\widetilde{A}_h(\cdot,\ \cdot)$ and $\widetilde{b}_h(\cdot,\ \cdot)$ by:

$$\widetilde{A}_h(\Psi,\ \Phi) = \int_{\Omega_h} A_{ijkl}\psi_{ij}\phi_{kl}\ \mathrm{d}x = \widetilde{A}_h(\Phi, \Psi)\ \ \forall \Psi, \Phi \in \mathbf{V}(\Omega_h), \tag{3.12}$$

$$\widetilde{b}_h(\Phi,\ v) = \int_{\Omega_h} \phi_{ij,\ j} v_{,\ i}\ \mathrm{d}x\ \ \forall \Phi \in \mathbf{V}(\Omega_h)\ \forall v \in H_0^1(\Omega_h) \tag{3.13}$$

with $|\tilde{A}_h(\Phi, \Psi)| \leq \tilde{M}\|\Phi\|_{0,\Omega_h}\|\Psi\|_{0,\Omega_h}$ and $|\tilde{b}_h(\Phi, v)| \leq \tilde{m}\|\Phi\|_{1,\Omega_h}\|v\|_{1,\Omega_h}$ for some $\tilde{M}, \tilde{m} > 0$.

Quadrature schemes: For $\hat{\phi} \in C^0(\hat{T})$, the following **two** quadrature schemes over the reference triangle $\hat{T}$ corresponding to $i = 0$ and 1 will be considered:

$$\int_{\hat{T}} \hat{\phi}(\hat{x})\ \mathrm{d}\hat{x} \approx \sum_{n=1}^{N_i} \hat{w}_n^i \hat{\phi}(\hat{b}_n^i) \text{ with weights } \hat{w}_n^i > 0,\ \hat{b}_n^i \in \hat{T},\ 1 \leq n \leq N_i,\ i = 0, 1 \text{ such that} \tag{3.14}$$

$\hat{E}_0(\hat{p}) = \int\limits_{\hat{T}} \hat{p}(\hat{x})\ \mathrm{d}\hat{x} - \sum_{n=1}^{N_0} \hat{w}_n^0 \hat{p}(\hat{b}_n^0) = 0\ \ \forall \hat{p} \in P_3(\hat{T});\ \hat{E}_1(\hat{p}) = \int\limits_{\hat{T}} \hat{p}(\hat{x})\ \mathrm{d}\hat{x} - \sum_{n=1}^{N_1} \hat{w}_n^1 \hat{p}(\hat{b}_n^1) = 0\ \ \forall \hat{p} \in P_6(\hat{T}).$

Then, $\forall T \in \tau_h^{\mathrm{ISO}}, J(F_T) > 0,\ J(F_T) \in P_1(\hat{T}),\ \forall \phi \in C^0(T),$

$$\int\limits_{T} \phi(x)\ \mathrm{d}x = \int\limits_{\hat{T}} \hat{\phi}(\hat{x}) J(F_T)(\hat{x})\ \mathrm{d}\hat{x} \approx \sum_{n=1}^{N_i} w_{n,T}^i \phi(b_{n,T}^i) \text{ with } \phi(x) = \phi \cdot F_T(\hat{x}) = \hat{\phi}(\hat{x}), \tag{3.15}$$

where $w_{n,T}^i = \hat{w}_n^i J(F_T)(\hat{b}_n^i) > 0,\ \ b_{n,T}^i = F_T(\hat{b}_n^i) \in T$ satisfying the assumption:

(**A8**): Evaluation points $\hat{b}_n^i \in \hat{T}$ are vertices $\{\hat{a}_i\}_{i=1}^3$, midside nodes $\{\hat{a}_i\}_{i=4}^6$ and possibly also interior points $\hat{b}_n^i \in int(\hat{T})$ such that $F_T(\hat{b}_n^i) = b_{n,T}^i \in T \cap \bar{\Omega}\ \ \forall T \in \tau_h^{\mathrm{ISO}},\ 1 \le n \le N_i,\ (i = 0, 1).$

Then, to each τ_h^{ISO} of $\bar{\Omega}$, we associate the following finite dimensional subspaces:

$$X_h = \{\phi_h : \phi_h \in C^0(\bar{\Omega}_h),\ \phi_h \downarrow_T = \hat{\phi} \cdot F_T^{-1} \text{ with } \hat{\phi} \in P_2(\hat{T}) \quad \forall T \in \tau_h^{\mathrm{ISO}}\} \subset H^1(\Omega_h); \tag{3.16}$$

$$\mathbf{V}_h = \{\Phi_h : \Phi_h = (\phi_{hij})_{i,\ j=1,\ 2} \text{ with } \phi_{hij} = \phi_{hji} \in X_h\ \ \forall i,\ j = 1,\ 2\} \subset \mathbf{V}(\Omega_h); \tag{3.17}$$

$$W_h = \{\chi_h : \chi_h \in X_h,\ \chi_h|_{\Gamma_h} = 0\} \subset H_0^1(\Omega_h) \text{ with } \|\chi_h\|_{W_h} = \|\chi_h\|_{1,\Omega_h}, \tag{3.18}$$

(in which the essential boundary condition $\chi|_\Gamma = 0$ has been replaced by $\chi_h|_{\Gamma_h} = 0$).
Corresponding to $\widetilde{A}_h(\cdot,\cdot)$ and $\widetilde{b}_h(\cdot,\cdot)$ in (3.12) and (3.13) respectively, we define new continuous bilinear forms $A_h^{\mathrm{NI}}(\cdot,\ \cdot),\ b_h^{\mathrm{NI}}(\cdot,\ \cdot)$ with the help of numerical integration schemes (3.14)–(3.15) satisfying (**A8**):

$$\forall \Psi_h, \Phi_h \in \mathbf{V}_h,\ \ A_h^{\mathrm{NI}}(\Phi_h,\ \Psi_h) = \sum_{T \in \tau_h^{\mathrm{ISO}}} \sum_{n=1}^{N_1} w_{n,\ T}^1 (A_{ijkl}\phi_{hij}\psi_{hkl})(b_{n,\ T}^1) = A_h^{\mathrm{NI}}(\Psi_h,\ \Phi_h) \tag{3.19}$$

with $|A_h^{\mathrm{NI}}(\Phi_h, \Psi_h)| \le M_0 \|\Psi_h\|_{0,\Omega_h} \|\Phi_h\|_{0,\Omega_h} \le M_0 \|\Psi_h\|_{1,\Omega_h} \|\Phi_h\|_{0,\Omega_h}$ for some $M_0 > 0$;

$$b_h^{\mathrm{NI}}(\Phi_h,\ v_h) = \sum_{T \in \tau_h^{\mathrm{ISO}}} \sum_{n=1}^{N_0} w_{n,\ T}^0 (\phi_{hij,\ j} v_{h,\ i})(b_{n,\ T}^0) \qquad \forall \Phi_h \in \mathbf{V}_h,\ \forall v_h \in W_h \tag{3.20}$$

with $|b_h^{\mathrm{NI}}(\Phi_h, v_h)| \le m_0 \|\Phi_h\|_{1,\Omega_h} \|v_h\|_{1,\Omega_h}$ for some $m_0 > 0$. Now, we make the assumption:

(**A9**): $A_h^{\mathrm{NI}}(\cdot,\cdot)$ in (3.19) (resp. $b_h^{\mathrm{NI}}(\cdot,\cdot)$ in (3.20)) is evaluated using quadrature scheme (3.14)–(3.15) for $i = 1$ (resp. $i = 0$), which is exact for $P_6(\hat{T})$ resp. $P_3(\hat{T})$).

Now, to the eigenvalue problem $(\mathbf{Q^E})$, we associate the **Isoparametric Mixed Finite Element Eigenvalue Problem** $(\mathbf{Q_h^E})$:
$\forall h \in]0, h_0[$ with $h_0 > 0$, find $\lambda_h \in \mathbb{R}$ for which $\exists$ non-null $(\Psi_h, u_h) \in \mathbf{V}_h \times W_h$ such that

$$(\mathbf{Q_h^E}) : A_h^{\mathrm{NI}}(\Psi_h,\ \Phi_h) + b_h^{\mathrm{NI}}(\Phi_h,\ u_h) = 0 \quad \forall \Phi_h \in \mathbf{V}_h,\ \ -b_h^{\mathrm{NI}}(\Psi_h,\ v_h) = \lambda_h \langle u_h,\ v_h \rangle_{0,\ \Omega_h} \quad \forall v_h \in W_h, \tag{3.21}$$

and the associated **Isoparametric Mixed Finite Element Source Problem** $(\mathbf{Q_h})$, which corresponds to $(\mathbf{Q})$ in (2.23) is defined by: For given $\tilde{f} \in L^2(\Omega_h)$, find $(\Psi_h,\ u_h) \in \mathbf{V}_h \times W_h$ such that

$$(\mathbf{Q_h}) : A_h^{\mathrm{NI}}(\Psi_h,\ \Phi_h) + b_h^{\mathrm{NI}}(\Phi_h,\ u_h) = 0 \quad \forall \Phi_h \in \mathbf{V}_h,\ \ -b_h^{\mathrm{NI}}(\Psi_h,\ v_h) = \langle \tilde{f},\ v_h\rangle_{0,\ \Omega_h} \quad \forall v_h \in W_h, \tag{3.22}$$

$$\text{where } \langle \tilde{f},\ v_h\rangle_{0,\ \Omega_h} = \int_{\Omega_h} \tilde{f} v_h \mathrm{d}\Omega_h \quad \forall v_h \in W_h \text{ with } \tilde{f} \in L^2(\widetilde{\Omega}),\ \tilde{f} = \tilde{f}\downarrow_{\Omega_h} \tag{3.23}$$

(denoted by the same notation $\tilde{f}$), $\tilde{f}$ being the extension to $\widetilde{\Omega}$ of f with $\tilde{f} = \Lambda\tilde{u}$ (see [10]).
Now, based on Theorem 3.1 and Corollary 3.1, we define $\widetilde{X}_h,\ \widetilde{\mathbf{V}}_h$ and $\widetilde{W}_h$:

$$\widetilde{X}_h = \{\widetilde{\phi}_h : \widetilde{\phi}_h \in H^1(\widetilde{\Omega}),\ \widetilde{\phi}_h\downarrow_{\Omega_h} = \phi_h \in X_h\} \text{ with } \|\widetilde{\phi}_h\|_{1,\widetilde{\Omega}} \le C\|\phi_h\|_{1,\Omega_h} \text{ for some } C > 0; \tag{3.24}$$

$$\widetilde{\mathbf{V}}_h = \{\widetilde{\Phi}_h : \widetilde{\Phi}_h = (\widetilde{\phi}_{hij})_{i,\ j=1,\ 2} \text{ with } \widetilde{\phi}_{hij} = \widetilde{\phi}_{hji} \in \widetilde{X}_h \ \ \forall i,j = 1,2,\ \widetilde{\Phi}_h\downarrow_{\Omega_h} = \Phi_h \in \mathbf{V}_h\} \tag{3.25}$$

$$\text{with } \|\widetilde{\phi}_{hij}\|_{1,\widetilde{\Omega}} \le C\|\phi_{hij}\|_{1,\Omega_h} \ \ \forall i,j = 1,2, \ \ \|\widetilde{\Phi}_h\|_{1,\widetilde{\Omega}} \le C\|\Phi_h\|_{1,\Omega_h}; \tag{3.26}$$

$$\widetilde{W}_h = \{\widetilde{\chi}_h : \widetilde{\chi}_h \in H_0^1(\widetilde{\Omega}) \ \text{ with } \widetilde{\chi}\downarrow_{\widetilde{\Omega}/\Omega_h} = 0\} \text{ with } \|\widetilde{\chi}_h\|_{1,\widetilde{\Omega}} = \|\chi_h\|_{1,\Omega_h}. \tag{3.27}$$

Theorem 3.2 ([10])**.** *Let assumptions* (**A1**–**A9**) *hold. Then,*

(i) $\exists\bar{\alpha}_0 >$, *independent of* h, *such that* $A_h^{\mathrm{NI}}(\Phi_h,\ \Phi_h) \ge \bar{\alpha}_0\|\Phi_h\|^2_{0,\ \Omega_h} \ \ \forall \Phi_h \in \mathbf{V}_h$; (3.28)

(ii) $\exists\bar{\beta}_0 > 0$, *independent of* h, *such that* $\sup\limits_{\Phi_h \in \mathbf{V}_h - \{0\}} \dfrac{|b_h^{\mathrm{NI}}(\Phi_h,\ \chi_h)|}{\|\Phi_h\|_{\mathbf{V}_h}} \ge \bar{\beta}_0\|\chi_h\|_{1,\Omega_h} \ \ \forall \chi_h \in W_h$ *[10]*; (3.29)

(iii) The isoparametric mixed finite element source problem $(\mathbf{Q_h})$ *defined in* (3.22) *has a unique solution.*

Remark 3.1. $\mathbf{V}_h$- ellipticity of $A_h^{\mathrm{NI}}(\cdot,\cdot)$ in (3.28) will hold even if the quadrature scheme (3.14) with i=1 be exact for $P_4(\hat{T})$ (instead of $P_6(\hat{T})$ in (**A9**)) (see [10]). But this assumption (**A9**) will be necessary in Proposition 4.1 (see Rem. 5.4 for more details).

Hence, we can define $\vec{\mathbf{T}}_\mathbf{h} : \tilde{f} \in L^2(\Omega_h) \mapsto \vec{\mathbf{T}}_\mathbf{h}\tilde{f} = (S_h\tilde{f}, T_h\tilde{f}) = (\Psi_h, u_h) \in \mathbf{V}_h \times W_h$ such that

$$A_h^{\mathrm{NI}}(S_h\tilde{f}, \Phi_h) + b_h^{\mathrm{NI}}(\Phi_h, T_h\tilde{f}) = 0 \ \forall \Phi_h \in \mathbf{V}_h;\ -b_h^{\mathrm{NI}}(S_h\tilde{f}, v_h) = \langle \tilde{f}, v_h\rangle_{0,\Omega_h} \ \ \forall v_h \in W_h, \tag{3.30}$$

where $S_h \in \mathcal{L}(L^2(\Omega_h); \mathbf{V}_h)$ and $T_h \in \mathcal{L}(L^2(\Omega_h); W_h)$ with $S_h\tilde{f} = \Psi_h \in \mathbf{V}_h,\ \ T_h\tilde{f} = u_h \in W_h$ and $\|S_h\tilde{f}\|_{0,\Omega_h} + \|T_h\tilde{f}\|_{1,\Omega_h} \le C\|\tilde{f}\|_{0,\Omega_h}$ for some $C > 0$, independent of h [8].

Theorem 3.3 ([10])**.** *Let assumptions* (**A1**–**A9**) *hold and* $(\Psi, u) \in \mathbf{V} \times W$ *with* $u \in H^3(\Omega) \cap H_0^2(\Omega)$ *(resp.* $(\Psi_h, u_h) \in \mathbf{V}_h \times W_h$*) be the unique solution of* $(\mathbf{Q})$ *(resp.* $(\mathbf{Q_h})$*).*
Let $\tilde{u} \in H^3(\widetilde{\Omega})$ *be an extension to* $\widetilde{\Omega}$ *of* $u \in H^3(\Omega) \cap H_0^2(\Omega)$ *such that* $\tilde{\psi}_{ij} = a_{ijkl}\tilde{u}_{,kl} \in H^1(\widetilde{\Omega}) \ \ \forall i,j = 1,2$ *with* $\tilde{\Psi} = (\tilde{\psi}_{ij})_{1\le i,j\le 2}$ *and* $(\tilde{\Psi}, \tilde{u}) \in \tilde{\mathbf{V}} \times \tilde{W}$ *(resp.* $(\tilde{\Psi}_h, \tilde{u}_h) \in \tilde{\mathbf{V}}_h \times \tilde{W}_h$*) be an extension to* $\widetilde{\Omega}$ *of* $(\Psi, u) \in \mathbf{V} \times W$ *(resp.* $(\Psi_h, u_h) \in \mathbf{V}_h \times W_h$*). Let* $\tilde{f} \in L^2(\widetilde{\Omega})$ *be an extension to* $\widetilde{\Omega}$ *of* $f \in L^2(\Omega)$ *such that* $\tilde{f} = \Lambda\tilde{u}$*. Then,* $\exists C > 0$, *independent of h, such that*

$$\|\tilde{\Psi} - \Psi_h\|_{0,\Omega_h} \le Ch\big[\|\tilde{u}\|_{3,\tilde{\Omega}} + \|\tilde{\Psi}\|_{1,\tilde{\Omega}}\big], \qquad \|\tilde{u} - u_h\|_{1,\Omega_h} \le Ch\big[\|\tilde{u}\|_{3,\tilde{\Omega}} + \|\tilde{\Psi}\|_{1,\tilde{\Omega}}\big]. \tag{3.31}$$

Since $W_h \subset H_0^1(\Omega_h) \subset L^2(\Omega_h)$, we consider $T_h \downarrow_{W_h} = T_h \in \mathcal{L}(W_h)$ as the finite dimensional linear operator on $(W_h; \langle \cdot, \cdot \rangle_{0,\Omega_h})$ defined by: $T_h : \chi_h \in W_h \mapsto T_h\chi_h = w_h \in W_h$ such that $\forall \chi_h \in W_h$,

$$A_h^{\rm NI}(S_h\, \chi_h,\ \Phi_h) + b_h^{\rm NI}(\Phi_h,\ T_h\chi_h) = 0 \quad \forall \Phi_h \in \mathbf{V}_h,\ \ -b_h^{\rm NI}(S_h\, \chi_h,\ v_h) = \langle \chi_h,\ v_h \rangle_{0,\,\Omega_h}\ \forall v_h \in W_h. \tag{3.32}$$

Then, $(\mu_h; u_h) \in \mathbb{R}^+ \times W_h$ is an eigenpair of the symmetric, positive-definite operator T_h on $W_h \iff (\lambda_h; (\Psi_h, u_h)) \in \mathbb{R}^+ \times (\mathbf{V}_h \times W_h)$ is an eigenpair of $(\mathbf{Q_h^E})$ with $\lambda_h = 1/\mu_h$ and $\Psi_h = S_h(\lambda_h u_h)$, and we have:

Theorem 3.4 ([8])**.** $(\mathbf{Q_h^E})$ *has strictly positive, possibly repeated, real eigenvalues: For* $N_h = \dim\ W_h$, $0 < \lambda_{1,h} \le \lambda_{2,h} \le \cdots \le \lambda_{N_h,h}$ *with* $\lambda_{k,h} = 1/\mu_{k,h},\ 1 \le k \le N_h$, $(\mu_{1,h} \ge \mu_{2,h} \ge \cdots \ge \mu_{N_h,h} > 0$ *being eigenvalues of* T_h*), and* $\exists$ *corresponding eigensolutions* $(\Psi_{m,h}, u_{m,h}) \in \mathbf{V}_h \times W_h,\ 1 \le m \le N_h$, *of* $(\mathbf{Q_h^E})$ *i.e.* $A_h^{\rm NI}(\Psi_{m,h}, \Phi_h) + b_h^{\rm NI}(\Phi_h, u_{m,h}) = 0\ \forall \Phi_h \in \mathbf{V}_h,\ -b_h^{\rm NI}(\Psi_{m,h}, v_h) = \lambda_{m,h}\langle u_{m,h}, v_h\rangle_{0,\Omega_h}\ \forall v_h \in W_h$ *such that* $(u_{m,h})_{m=1}^{N_h}$ *is an orthonormal basis in* $(W_h, \langle \cdot, \cdot \rangle_{0,\Omega_h})$ *and* $\left(\frac{\Psi_{m,h}}{\sqrt{\lambda_{m,h}}}\right)_{m=1}^{N_h}$ *is an orthonormal system in* $(\mathbf{V}_h, [\cdot,\cdot]_{A_h^{\rm NI}(\cdot,\cdot)})$*; i.e. in* $\mathbf{V}_h$ *equipped with inner product* $[\cdot,\cdot]_{A_h^{\rm NI}(\cdot,\cdot)}$.

By virtue of (3.28), applying Lax-Milgram lemma, we can define $\mathcal{I}_h : v_h \in W_h \mapsto \mathcal{I}_h v_h = \underline{\sigma}_h \in \mathbf{V}_h$ such that $A_h^{\rm NI}(\mathcal{I}_h v_h, \Phi_h) + b_h^{\rm NI}(\Phi_h, v_h) = 0\quad \forall \Phi_h \in \mathbf{V}_h$, and set

$$\begin{aligned} \mathcal{E}_h = \mathcal{I}_h(W_h) \quad &= \quad \{\underline{\sigma}_h : \underline{\sigma}_h \in \mathbf{V}_h \text{ for which } \exists v_h \in W_h \text{ such that} \\ A_h^{\rm NI}(\underline{\sigma}_h, \Phi_h) + b_h^{\rm NI}(\Phi_h, v_h) \quad &= \quad 0\ \forall \Phi_h \in \mathbf{V}_h\} = \mathrm{Span}\left\{\left(\frac{\Psi_{m,h}}{\sqrt{\lambda_{m,h}}}\right)_{m=1}^{N_h}\right\}. \end{aligned} \tag{3.33}$$

Then, $\mathcal{I}_h : (W_h, \langle \cdot, \cdot \rangle_{0,\Omega_h}) \longrightarrow (\mathcal{E}_h, [\cdot,\cdot]_{A_h^{\rm NI}(\cdot,\cdot)})$ is **linear** and **bijective**. Then, $(\lambda_{m,h}; (\Psi_{m,h}, u_{m,h})) \in \mathbb{R}^+ \times (\mathbf{V}_h \times W_h)$ is an eigenpair of $(\mathbf{Q_h^E}) \Longrightarrow \Psi_{m,h} = \mathcal{I}_h u_{m,h},\ 1 \le m \le N_h\ =\ \dim W_h$.
Define N_h- dimensional space $\mathcal{M}_h$ of **linked pairs** $(\underline{\sigma}_h, v_h) = (\mathcal{I}_h v_h, v_h)$ by: $\mathcal{M}_h = \mathcal{E}_h \times W_h$. Then, $(\Psi_{m,h}, u_{m,h}) \in \mathcal{M}_h$ for $1 \le m \le N_h$.

Rayleigh quotient characterization of approximate eigenvalues.
As in the continuous case, $\forall$ linked pair $(\underline{\sigma}_h, v_h) \in \mathcal{M}_h$, we define the new Rayleigh quotient

$$\Re_h(\cdot,\cdot) \text{ by}:\ \ \Re_h(\underline{\sigma}_h, v_h) = \frac{A_h^{\rm NI}(\underline{\sigma}_h, \underline{\sigma}_h)}{\langle v_h, v_h\rangle_{0,\Omega_h}}\ \ \forall (\underline{\sigma}_h, v_h) \in \mathcal{M}_h. \tag{3.34}$$

$$\text{Define}\quad U_{p,h} = \mathrm{Span}\{(u_{m,h})_{m=1}^p\} \subset W_h,\ \mathcal{M}_{p,h} = \mathrm{Span}\{(\Psi_{m,h}, u_{m,h})_{m=1}^p\} \subset \mathcal{M}_h, \tag{3.35}$$

where $(\lambda_{m,h}, (\Psi_{m,h}, u_{m,h}))_{m=1}^p$ are the first 'p' eigenpairs of $(\mathbf{Q_h^E})$ with $0 < \lambda_{1,h} \le \lambda_{2,h} \le \cdots \le \lambda_{p,h}$ with $p \le N_h$, $\langle u_{m,h}, u_{n,h}\rangle_{0,\Omega_h} = \delta_{mn}$, $\left[\frac{\Psi_{m,h}}{\sqrt{\lambda_{m,h}}}, \frac{\Psi_{n,h}}{\sqrt{\lambda_{n,h}}}\right] = \delta_{mn}$.

Theorem 3.5 (Min-Max Principle, [2, 37])**.**

(*i*) *Eigensolutions of* $(\mathbf{Q_h^E})$ *are the stationary points of* $\Re_h(\cdot,\cdot)$ *on* $\mathcal{M}_h$*, the corresponding eigenvalues of* $(\mathbf{Q_h^E})$ *being the values of* $\Re_h(\cdot,\cdot)$ *at these stationary points*;

(*ii*) $\lambda_{p,h} = \min\limits_{\substack{S_{p,h} \subset \mathcal{M}_h \\ \dim S_{p,h} = p}} \max\limits_{(\underline{\sigma}_h, v_h) \in S_{p,h}} \Re_h(\underline{\sigma}_h, v_h) = \max\limits_{(\underline{\sigma}_h, v_h) \in \mathcal{M}_{p,h}} \Re_h(\underline{\sigma}_h, v_h) = \Re_h(\Psi_{p,h}, u_{p,h})\ \forall p = 1, 2, \cdots, N_h$

$((\lambda_{p,h}, (\Psi_{p,h}, u_{p,h}))$ *being an eigenpair of* $(\mathbf{Q_h^E}))$.

Since $T_h \in \mathcal{L}(W_h)$ is a symmetric, positive-definite, linear operator, we can define another

$$\text{Rayleigh quotient } Q_h(\cdot) \text{ by } Q_h(v_h) = \frac{\langle T_h v_h, v_h\rangle_{0,\Omega_h}}{\langle v_h, v_h\rangle_{0,\Omega_h}} \ \ \forall v_h \in W_h. \tag{3.36}$$

Theorem 3.6 (Max-Min Principle, [2,37])**.** *For $1 \le p \le N_h = \dim W_h$,*

$$\mu_{p,h} = \max_{\substack{S^*_{p,h}\subset W_h \\ \dim S^*_{p,h}=p}} \ \min_{v_h\in S^*_{p,h}} Q_h(v_h) = \min_{v_h \in U_{p,h}\subset W_h} Q_h(v_h) = Q_h(u_{p,h}), \tag{3.37}$$

$(\mu_{p,h};\, u_{p,h})$ being the p-th eigenpair of T_h with $\mu_{p,h} = 1/\lambda_{p,h}$.

For $\chi^*_p \in H^3(\Omega)\cap H^2_0(\Omega)$ defined in (2.26), let $\widetilde{\chi}^*_p \in H^3(\widetilde{\Omega})$ be its extension to $\widetilde{\Omega}$ and $\widetilde{\chi}^*_p \downarrow_{\Omega_h}$ be the restriction to Ω_h of $\widetilde{\chi}^*_p$, which will be denoted by the same notation $\widetilde{\chi}^*_p$, such that

$$\|\widetilde{\chi}^*_p\|_{3,\Omega_h} \le C\|\widetilde{\chi}^*_p\|_{3,\widetilde{\Omega}} \le C\|\chi^*_p\|_{3,\Omega} \ \text{ for some } C>0 \ \text{ (see (3.9))}. \tag{3.38}$$

Then, $\exists$ a unique $\vec{\mathbf{T}}_{\mathbf{h}}\,\widetilde{\chi}^*_p = (S_h\,\widetilde{\chi}^*_p, T_h\widetilde{\chi}^*_p) \in \mathbf{V}_h \times W_h$ defined by (3.30), *i.e.*

$$A^{\rm NI}_h(S_h\widetilde{\chi}^*_p, \Phi_h) + b^{\rm NI}_h(\Phi_h, T_h\widetilde{\chi}^*_p) = 0 \ \forall \Phi_h \in \mathbf{V}_h;\ -b^{\rm NI}_h(S_h\widetilde{\chi}^*_p, v_h) = \langle \widetilde{\chi}^*_p, v_h\rangle_{0,\Omega_h} \ \forall v_h \in W_h. \tag{3.39}$$

Since from (2.26) $\vec{\mathbf{T}}\chi^*_p = (\underline{\sigma}_p, \chi_p) \in \mathcal{M}_p$, it suggests to define a new linear operator $\vec{\Pi}_{\mathbf{h}} : \mathcal{M}_p \longrightarrow \mathcal{M}_h \subset \mathbf{V}_h \times W_h$ by: $(\underline{\sigma}_p, \chi_p) \in \mathcal{M}_p \mapsto \underline{\vec{\Pi}}_{\mathbf{h}}(\underline{\sigma}_p, \chi_p) = (\underline{\Pi}_{1h}\underline{\sigma}_p, \Pi_{2h}\chi_p) = \vec{\mathbf{T}}_h\widetilde{\chi}^*_p = (S_h\widetilde{\chi}^*_p, T_h\widetilde{\chi}^*_p)$ such that $\underline{\Pi}_{1h}\underline{\sigma}_p = S_h\widetilde{\chi}^*_p$, $\Pi_{2h}\chi_p = T_h\widetilde{\chi}^*_p$ *i.e.*

$$A^{\rm NI}_h(\underline{\Pi}_{1h}\underline{\sigma}_p, \Phi_h) + b^{\rm NI}_h(\Phi_h, \Pi_{2h}\chi_p) = 0 \ \ \forall \Phi_h \in \mathbf{V}_h;\ -b^{\rm NI}_h(\underline{\Pi}_{1h}\underline{\sigma}_p, v_h) = \langle \widetilde{\chi}^*_p, v_h\rangle_{0,\Omega_h} \ \ \forall v_h \in W_h. \tag{3.40}$$

Then we have: $(\underline{\vec{\Pi}}_{\mathbf{h}}\cdot\vec{\mathbf{T}})\chi^*_p = (\vec{\mathbf{T}}_h\cdot\rho_h\cdot E)\chi^*_p$ with $\rho_h(E\chi^*_p) = \rho_h\widetilde{\chi}^*_p = \widetilde{\chi}^*_p\downarrow_{\Omega_h}, \underline{\vec{\Pi}}_{\mathbf{h}}$ being a linear operator, E (resp. ρ_h) being the extension (resp. restriction) operator satisfying (3.38) (see also Cor. 3.1). Applying Theorem 3.3, we get the following result:

Corollary 3.2. *Let assumptions* (**A1**–**A9**) *hold. Let $\widetilde{\chi}^*_p \in H^3(\widetilde{\Omega})$ be an extension to $\widetilde{\Omega}$ of $\chi^*_p \in H^3(\Omega)\cap H^2_0(\Omega)$ defined in (2.26) such that (3.38) holds and $(\underline{\sigma}_p, \chi_p) \in \mathcal{M}_p$ with $\chi_p \in H^3(\Omega)\cap H^2_0(\Omega)$ and $\|\chi_p\|_{0,\Omega} = 1$ be defined by:*

$$A(\underline{\sigma}_p, \Phi) + b(\Phi, \chi_p) = 0 \ \ \forall \Phi \in \mathbf{V}; \qquad -b(\underline{\sigma}_p, v) = \langle \chi^*_p, v\rangle_{0,\Omega} \ \ \forall v \in W. \tag{3.41}$$

Let $\underline{\vec{\Pi}}_{\mathbf{h}}(\underline{\sigma}_p, \chi_p) = (\underline{\Pi}_{1h}\underline{\sigma}_p, \Pi_{2h}\chi_p) \in \mathcal{M}_h \subset \mathbf{V}_h \times W_h$ be defined by (3.40).

$$\begin{aligned} \text{Then, } \exists\, C>0, \text{ independent of `}h\text{', } \textit{such that } &\|\widetilde{\underline{\sigma}}_p - \underline{\Pi}_{1h}\underline{\sigma}_p\|_{0,\Omega_h} \le Ch(\|\widetilde{\chi}_p\|_{3,\widetilde{\Omega}} + \|\widetilde{\underline{\sigma}}_p\|_{1,\widetilde{\Omega}}),\\ &\|\widetilde{\chi}_p - \Pi_{2h}\chi_p\|_{1,\Omega_h} \le Ch(\|\widetilde{\chi}_p\|_{3,\widetilde{\Omega}} + \|\widetilde{\underline{\sigma}}_p\|_{1,\widetilde{\Omega}}), \end{aligned} \tag{3.42}$$

where $\widetilde{\underline{\sigma}}_p \in \widetilde{\mathbf{V}}$ (resp. $\widetilde{\chi}_p \in H^3(\widetilde{\Omega})$) is the extension to $\widetilde{\Omega}$ of $\underline{\sigma}_p \in \mathbf{V}$ (resp. $\chi_p \in H^3(\Omega)\cap H^2_0(\Omega)$).

4. Error estimates

Here, we shall develop error estimates for the case of **simple** eigenvalues.

Theorem 4.1. *Let assumptions* (**A1–A9**) *hold. Let* $(\lambda_p; (\Psi_p, u_p)) \in \mathbb{R}^+ \times (\mathbf{V} \times W)$ *with* $u_p \in H^3(\Omega) \cap H_0^2(\Omega)$ *(resp.* $(\lambda_{p,h}; (\Psi_{p,h}, u_{p,h})) \in \mathbb{R}^+ \times (\mathbf{V}_h \times W_h)$*) be an eigenpair of* $(\mathbf{Q^E})$ *(resp.* $(\mathbf{Q^E_h})$*),* λ_p *(resp.* $\lambda_{p,h}$*) being a* ***simple*** *eigenvalue of* $(\mathbf{Q^E})$ *(resp.* $(\mathbf{Q^E_h})$*) and* $(\widetilde{\Psi}_p, \widetilde{u}_p) \in \widetilde{\mathbf{V}} \times H^3(\widetilde{\Omega})$ *(resp.*$(\widetilde{\Psi}_{p,h}, \widetilde{u}_{p,h}) \in \widetilde{\mathbf{V}}_\mathbf{h} \times \widetilde{W}_h$*) be the extension to* $\widetilde{\Omega}$ *of the eigensolution* (Ψ_p, u_p) *of* $(\mathbf{Q^E})$ *(resp.* $(\Psi_{p,h}, u_{p,h})$ *of* $(\mathbf{Q^E_h})$*),* $1 \le p \le N_h = \dim W_h$*, satisfying (3.7)–(3.9). Then,* $\exists C > 0$*, independent of 'h' and 'p', such that*

$$\begin{aligned}\|\widetilde{u}_p - u_{p,h}\|_{1,\Omega_h} \le C \Bigg[&\|\widetilde{u}_p - \chi_h\|_{1,\Omega_h} + \|\widetilde{\Psi}_p - \Psi_{p,h}\|_{0,\Omega_h} + \sup_{\Phi_h \in \mathbf{V}_h - \{0\}} \frac{\left|b_h^{\mathrm{NI}}(\Phi_h, \chi_h) - \widetilde{b_h}(\Phi_h, \chi_h)\right|}{\|\Phi_h\|_{1,\Omega_h}} \\ &+ \sup_{\Phi_h \in \mathbf{V}_h - \{0\}} \frac{\left|A_h^{\mathrm{NI}}(\Psi_{p,h}, \Phi_h) - \widetilde{A_h}(\Psi_{p,h}, \Phi_h)\right|}{\|\Phi_h\|_{1,\Omega_h}} + \sup_{\Phi_h \in \mathbf{V}_h - \{0\}} \frac{\left|\tilde{A}_h(\widetilde{\Psi}_p, \Phi_h) - A(\Psi_p, \widetilde{\Phi}_h)\right|}{\|\Phi_h\|_{1,\Omega_h}} \\ &+ \sup_{\Phi_h \in \mathbf{V}_h - \{0\}} \frac{\left|\tilde{b}_h(\Phi_h, \widetilde{u}_p) - b(\widetilde{\Phi}_h, u_p)\right|}{\|\Phi_h\|_{1,\Omega_h}} \Bigg] \quad \forall \chi_h \in W_h, \ (\widetilde{\Phi}_h \in \widetilde{\mathbf{V}}_h \text{ with } \widetilde{\Phi}_h \downarrow_{\Omega_h} = \Phi_h \in \mathbf{V}_h). \end{aligned} \tag{4.1}$$

Proof.

$$\|\widetilde{u}_p - u_{p,h}\|_{1,\Omega_h} \le \|\widetilde{u}_p - \chi_h\|_{1,\Omega_h} + \|\chi_h - u_{p,h}\|_{1,\Omega_h} \quad \forall \chi_h \in W_h. \tag{4.2}$$

From (3.29), $\exists\, \bar{\beta}_0 > 0$, independent of '$h$' and '$p$', such that

$$\|\chi_h - u_{p,h}\|_{1,\Omega_h} \le \frac{1}{\bar{\beta}_0} \sup_{\Phi_h \in \mathbf{V}_h - \{0\}} \frac{|b_h^{\mathrm{NI}}(\Phi_h, \chi_h - u_{p,h})|}{\|\Phi_h\|_{1,\Omega_h}}. \tag{4.3}$$

But $b_h^{\mathrm{NI}}(\Phi_h, \chi_h - u_{p,h}) = \widetilde{b_h}(\Phi_h, \chi_h - \widetilde{u}_p) + \left[b_h^{\mathrm{NI}}(\Phi_h, \chi_h) - \widetilde{b_h}(\Phi_h, \chi_h)\right] + \left[\widetilde{b_h}(\Phi_h, \widetilde{u}_p) - b_h^{\mathrm{NI}}(\Phi_h, u_{p,h})\right]$.

Now, using (2.20) (resp. (3.21)), we have

$$\begin{aligned}|b_h^{\mathrm{NI}}(\Phi_h, \chi_h - u_{p,h})| \quad \le \quad & |\tilde{b}_h(\Phi_h, \chi_h - \widetilde{u}_p)| + |\widetilde{A}_h(\widetilde{\Psi}_p - \Psi_{p,h}, \Phi_h)| + |b_h^{\mathrm{NI}}(\Phi_h, \chi_h) - \tilde{b}_h(\Phi_h, \chi_h)| \\ & + |\tilde{A}_h(\widetilde{\Psi}_p, \Phi_h) - A(\Psi_p, \widetilde{\Phi}_h)| + |A_h^{\mathrm{NI}}(\Psi_{p,h}, \Phi_h) - \tilde{A}_h(\Psi_{p,h}, \Phi_h)| \\ & + |\tilde{b}_h(\Phi_h, \widetilde{u}_p) - b(\widetilde{\Phi}_h, u_p)| \ \ \forall \chi_h \in W_h \ (\widetilde{\Phi}_h \in \widetilde{\mathbf{V}}_h \text{ with } \widetilde{\Phi}_h \downarrow_{\Omega_h} \in \mathbf{V}_h). \end{aligned} \tag{4.4}$$

Applying the continuity of $\tilde{A}_h(\cdot,\cdot)$ and $\widetilde{b_h}(\cdot,\cdot)$ in (4.4) and using it in (4.3) and (4.2), (4.1) follows. □

Remark 4.1. In (4.1), the third and fourth terms on the right hand side are due to numerical integration and the fifth and sixth terms appear owing to the approximation of the boundary.

For finding estimates, we will need the following important results.

Lemma 4.1 ([41])**.** *Let* Γ *be Lipschitz-continuous curved boundary of the convex domain* Ω*, which is piecewise of* C^k*-class,* $k \ge 3$*.* $\forall h \in]0, h_0[$ *with* $h_0 > 0$*, let* τ_h^{ISO} *be the quasi-uniform regular isoparametric triangulation of* $\bar{\Omega}$ *defined in (3.4) and* $\widetilde{\Omega}$ *be the domain satisfying* (**A2**) *and* (**A7**)*. Let* ϵ_h *and* ω_h *be defined by (3.5) and (3.6) respectively. Then,* $\exists C > 0$*, independent of* h*, such that*

$$(a) \quad \|\widetilde{v}\|_{0,\epsilon_h} \le C h^{3/2} \|\widetilde{v}\|_{1,\widetilde{\Omega}}; \qquad \|v\|_{0,\omega_h} \le C h^{3/2} \|\widetilde{v}\|_{1,\widetilde{\Omega}} \ \ \forall \widetilde{v} \in H^1(\widetilde{\Omega}) \text{ with } \widetilde{v} \downarrow_{\Omega} = v. \tag{4.5}$$

(b) *Moreover, if $\tilde{u} \in H^3(\widetilde{\Omega})$ be the extension to $\widetilde{\Omega}$ of $u \in H^3(\Omega) \cap H_0^2(\Omega)$, we have*

$$\|\tilde{u}_{,i}\|_{0,\epsilon_h} \leq Ch^3 |\tilde{u}|_{2,\widetilde{\Omega}} \text{ and } \|u_{,i}\|_{0,\omega_h} \leq Ch^3 |\tilde{u}|_{2,\widetilde{\Omega}} \ (i=1,2). \tag{4.6}$$

Inverse inequalities [16]: $\forall \phi_h \in X_h$ (*resp.* $\Phi_h \in \mathbf{V}_h$), $\exists C > 0$,

$$|\phi_h|_{1,\Omega_h} \leq (C/h)\|\phi_h\|_{0,\Omega_h} \quad (\textit{resp. } |\Phi_h|_{1,\Omega_h} \leq (C/h)\|\Phi_h\|_{0,\Omega_h}). \tag{4.7}$$

Now, $\forall h \in]0, h_0[$ with $h_0 > 0$, we define X_h **-interpolation operator** $\mathcal{P}_h : H^s(\widetilde{\Omega}) \longrightarrow X_h$:

$$\text{For } \widetilde{\chi} \in H^s(\widetilde{\Omega}),\ s \geq 2,\ \mathcal{P}_h\widetilde{\chi} \in X_h,\ \mathcal{P}_h\widetilde{\chi}(a_{i,\,T}) = \widetilde{\chi}(a_{i,\,T}) = \chi(a_{i,\,T}),\ 1 \leq i \leq 6,\ \forall T \in \tau_h^{\mathrm{ISO}}. \tag{4.8}$$

Then, $\|\widetilde{\chi} - \mathcal{P}_h\widetilde{\chi}\|_{r,\,\Omega_h} \leq Ch^{s-r}|\widetilde{\chi}|_{s,\,\Omega_h}$ $(s \geq 2)$, and $\widetilde{\chi} \in H^3(\widetilde{\Omega})$, $\widetilde{\chi} \downarrow_\Gamma = 0$ (*resp.* $\widetilde{\chi} \downarrow_{\Gamma_h} = 0$)

$$\Longrightarrow \mathcal{P}_h\widetilde{\chi} \in W_h \text{ with } \|\widetilde{\chi} - \mathcal{P}_h\widetilde{\chi}\|_{r,\Omega_h} \leq Ch^{3-r}|\widetilde{\chi}|_{3,\Omega_h}\ (r=0,1) \qquad [16]. \tag{4.9}$$

We have the following results:

• Under (**A7**–**A9**). $\forall \Phi \in \mathbf{V}(\Omega_h)\ \exists$ a tensor-valued function $\Theta_h \in \mathbf{V}_h$ such that $\widetilde{b_h}(\Phi, \chi_h) = b_h^{\mathrm{NI}}(\Theta_h, \chi_h)\ \ \forall \chi_h \in W_h$, and $\exists\ C > 0$, independent of h, such that

$$\|\Phi - \Theta_h\|_{r,\Omega_h} \leq Ch^{1-r}\|\Phi\|_{1,\Omega_h} \ \ (r=0,1) \qquad [10]. \tag{4.10}$$

• Let $\widetilde{f}$ be an extension to $\widetilde{\Omega}$ of $f \in L^2(\Omega)$ with $\widetilde{f} = \Lambda\widetilde{u} = (a_{ijkl}\widetilde{u}_{,kl})_{,ij} \in L^2(\widetilde{\Omega})$. Let $\tilde{u} \in H^3(\widetilde{\Omega})$ be an extension to $\widetilde{\Omega}$ of the solution $u \in H^3(\Omega) \cap H_0^2(\Omega)$ of ($\mathbf{P_G}$) such that $\widetilde{\psi}_{ij} = a_{ijkl}\widetilde{u}_{,kl} \in H^1(\widetilde{\Omega})\ \ \forall i,j = 1,2$ and $\widetilde{\Psi} = (\widetilde{\psi}_{ij})_{1\leq i,j \leq 2} \in \widetilde{\mathbf{V}}$ with $\widetilde{u} \downarrow_\Omega = u \in H^3(\Omega) \cap H_0^2(\Omega)$, $\widetilde{\Psi} \downarrow_\Omega = \Psi \in \mathbf{V}$, $\widetilde{\Psi} \downarrow_{\Omega_h} \in \mathbf{V}(\Omega_h)$. Then, for $\widetilde{\Psi}$, $\exists \Theta_h \in \mathbf{V}_h$ such that $b_h^{\mathrm{NI}}(\Theta_h, \chi_h) = -\langle \widetilde{f}, \chi_h \rangle_{0,\Omega_h}$ and

$$\|\widetilde{\Psi} - \Theta_h\|_{r,\,\Omega_h} \leq Ch^{1-r}\|\widetilde{\Psi}\|_{1,\,\Omega_h} \ \ (r = 0,\ 1) \text{ for some C } > 0 \qquad [10]. \tag{4.11}$$

Proposition 4.1. *Suppose that* (**A5**) *holds i.e. coefficients $A_{ijkl} \in W^{2,\infty}(\widetilde{\Omega})\ \ \forall i,j,k,l = 1,2$. Let assumptions* (**A1**–**A9**) *hold. Then, $\exists C > 0$, independent of h, such that $\forall \Phi_h, \underline{\sigma}_h \in \mathbf{V}_h$,*

$$|\tilde{A}_h(\Phi_h, \underline{\sigma}_h) - A_h^{\mathrm{NI}}(\Phi_h, \underline{\sigma}_h)| \leq Ch^2\|A\|_{2,\infty,\widetilde{\Omega}}\|\Phi_h\|_{0,\Omega_h}\|\underline{\sigma}_h\|_{0,\Omega_h} \tag{4.12}$$

where $\|A\|_{2,\infty,\widetilde{\Omega}} \geq \|A\|_{2,\infty,\Omega_h} = \sum_{T \in \tau_h^{\mathrm{ISO}}} \sum_{i,j,k,l=1}^{2} \|A_{ijkl}\|_{2,\infty,T}.$

Proof. The proof is similar to that given in [10] for $A_{ijkl} \in W^{1,\infty}(\widetilde{\Omega})$. □

Remark 4.2. (4.12) gives an estimate of the error due to numerical integration associated with the definition of $A_h^{\mathrm{NI}}(\cdot,\cdot)$ in (3.19) (see also Rem. 4.1).

Proposition 4.2. *Suppose that assumptions* (**A1**–**A9**) *hold. Let $(\lambda_p; (\Psi_p, u_p)) \in \mathbb{R}^+ \times (\mathbf{V} \times W)$ be an eigenpair of* ($\mathbf{Q^E}$) *corresponding to the simple eigenvalue λ_p with $u_p \in H^3(\Omega) \cap H_0^2(\Omega)$, $\Psi_p = (\psi_{pij})_{1 \leq i,j \leq 2}$, $\psi_{pij} = a_{ijkl}u_{p,kl} \in H^1(\Omega)\ \ \forall i,j = 1,2$. Let $\widetilde{u}_p \in H^3(\widetilde{\Omega})$ be the extension to $\widetilde{\Omega}$ of $u_p \in H^3(\Omega) \cap H_0^2(\Omega)$ such that $\widetilde{\psi}_{pij} = a_{ijkl}\widetilde{u}_{p,kl} \in H^1(\widetilde{\Omega})\ \forall i,j = 1,2$, $\widetilde{\Psi}_p = (\widetilde{\psi}_{pij})_{1\leq i,j\leq 2} \in \widetilde{\mathbf{V}}$, and let $\widetilde{\Phi}_h \in \widetilde{\mathbf{V}}_h$ be an extension to $\widetilde{\Omega}$ of $\Phi_h \in \mathbf{V}_h$ defined in (3.25). Then, the following estimates hold:*

$$\mathbf{I}.|A(\Psi_p, \Psi_p) - \tilde{A}_h(\widetilde{\Psi}_p, \widetilde{\Psi}_p)| \leq Ch^3\|\widetilde{u}_p\|_{3,\widetilde{\Omega}}\|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}};\ \ \mathbf{II}.|A(\Psi_p, \tilde{\Phi}_h) - \tilde{A}_h(\widetilde{\Psi}_p, \Phi_h)| \leq Ch^3\|\widetilde{u}_p\|_{3,\widetilde{\Omega}}\|\Phi_h\|_{1,\Omega_h};$$

$$\mathbf{III}.|b(\tilde{\Phi}_h, u_p) - \widetilde{b_h}(\Phi_h, \widetilde{u}_p)| \leq Ch^3\|\widetilde{u}_p\|_{2,\widetilde{\Omega}}\|\Phi_h\|_{1,\Omega_h};\ \ \mathbf{IV}.|\widetilde{b_h}(\Phi_h, \chi_h) - b_h^{\mathrm{NI}}(\Phi_h, \chi_h)| \leq Ch^2\|\Phi_h\|_{1,\Omega_h}\|\chi_h\|_{1,\Omega_h}.$$

Following [37], we prepare some new results to be used in the sequel.

Proposition 4.3. *Let* $(\underline{\sigma}_p, \chi_p) \in \mathcal{M}_p$ *be a linked pair with* $\chi_p \in H^3(\Omega) \cap H_0^2(\Omega)$, $\|\chi_p\|_{0,\Omega} = 1$ *and* $\underline{\sigma}_p \in \mathbf{V}$ *be defined by (3.41) and* $\widetilde{\chi}_p \in H^3(\widetilde{\Omega})$ *(resp.* $\widetilde{\underline{\sigma}}_p \in \widetilde{\mathbf{V}}$*) be extension to* $\widetilde{\Omega}$ *of* χ_p *(resp.* $\underline{\sigma}_p$*). Let* $\vec{\underline{\Pi}}_{\mathbf{h}}(\underline{\sigma}_p, \chi_p) = (\underline{\Pi}_{1h}\underline{\sigma}_p, \Pi_{2h}\chi_p) \in \mathcal{M}_h \subset \mathbf{V}_h \times W_h$ *be defined by (3.40) such that the estimates (3.42) hold. Then,* $\exists h_0 \in]0,1[$ *such that* $\forall h \in]0, h_0[$

$$\langle \Pi_{2h}\chi_p, \Pi_{2h}\chi_p\rangle^{-1}_{0,\Omega_h} < (1 + 2\,|\widetilde{\alpha}_{p,h}|), \qquad \textit{where } \widetilde{\alpha}_{p,h} = \|\chi_p\|^2_{0,\omega_h} - \|\widetilde{\chi}_p\|^2_{0,\epsilon_h} + \alpha_{p,h} \textit{ with} \tag{4.13}$$

$$\alpha_{p,h} = \max_{(\underline{\sigma}_p,\chi_p)\in\mathcal{M}_p, \|\chi_p\|_{0,\Omega}=1} \left\{ \left| 2\langle \widetilde{\chi}_p, \widetilde{\chi}_p - \Pi_{2h}\chi_p\rangle_{0,\Omega_h} - \|\widetilde{\chi}_p - \Pi_{2h}\chi_p\|^2_{0,\Omega_h} \right| \right\}. \tag{4.14}$$

Proof.

$$\langle \Pi_{2h}\chi_p, \Pi_{2h}\chi_p\rangle_{0,\Omega_h} = \|\chi_p\|^2_{0,\Omega} - \|\chi_p\|^2_{0,\omega_h} + \|\widetilde{\chi}_p\|^2_{0,\epsilon_h} - \left[2\langle \widetilde{\chi}_p, \widetilde{\chi}_p - \Pi_{2h}\chi_p\rangle_{0,\Omega_h} - \|\widetilde{\chi}_p - \Pi_{2h}\chi_p\|^2_{0,\Omega_h}\right]. \tag{4.15}$$

Also, from (3.42) and (4.14),

$$\alpha_{p,h} \quad \le \quad Ch \max_{(\underline{\sigma}_p,\chi_p)\in\mathcal{M}_p, \|\chi_p\|_{0,\Omega}=1} \left\{ (\|\widetilde{\chi}_p\|_{3,\widetilde{\Omega}} + \|\widetilde{\underline{\sigma}}_p\|_{1,\widetilde{\Omega}})\left[2\,\|\widetilde{\chi}_p\|_{0,\widetilde{\Omega}} + h(\|\widetilde{\chi}_p\|_{3,\widetilde{\Omega}} + \|\widetilde{\underline{\sigma}}_p\|_{1,\widetilde{\Omega}})\right] \right\} \longrightarrow 0$$

$$\text{as } h \longrightarrow 0 \text{ and } \exists h_0 \in]0,1[\text{ such that } \forall h \in]0,h_0[,\ \ \alpha_{p,h} < 1/4. \tag{4.16}$$

$$\text{Again, from (4.5), we have : } \|\chi_p\|_{0,\omega_h} \le Ch^{3/2}\|\widetilde{\chi}_p\|_{1,\widetilde{\Omega}},\ \|\widetilde{\chi}_p\|_{0,\epsilon_h} \le Ch^{3/2}\|\widetilde{\chi}_p\|_{1,\widetilde{\Omega}}\ \ \forall \widetilde{\chi}_p \in H^3(\widetilde{\Omega}) \tag{4.17}$$

with $\chi_p \in H^3(\Omega) \cap H_0^2(\Omega)$, $\|\chi_p\|_{0,\Omega} = 1$ and $(\underline{\sigma}_p, \chi_p) \in \mathcal{M}_p$. Then, the right hand sides of these two inequalities in (4.17) tend to 0 as $h \longrightarrow 0$. Hence, $\exists h_0 \in]0,1[$ such that

$$\forall h \in]0,h_0[,\ \ \|\chi_p\|_{0,\omega_h} < 1/2,\ \ \|\widetilde{\chi}_p\|_{0,\epsilon_h} < 1/\sqrt{2}. \tag{4.18}$$

Thus, $\exists h_0 \in]0,1[$ such that $\forall h \in]0,h_0[$,

$$0 \le \|\chi_p\|^2_{0,\omega_h} < 1/4,\ 0 \le \|\widetilde{\chi}_p\|^2_{0,\epsilon_h} < 1/2,\ 0 \le \alpha_{p,h} < 1/4 \Longrightarrow |\tilde{\alpha}_{p,h}| < 1/2, \text{ and } (1 + 2|\tilde{\alpha}_{p,h}|) < 2. \tag{4.19}$$

Thus, from (4.15) and (4.19), $\forall h \in]0,h_0[$, $h_0 \in]0,1[$,
$\langle \Pi_{2h}\chi_p, \Pi_{2h}\chi_p\rangle_{0,\Omega_h} \ge 1 - (\|\chi_p\|^2_{0,\omega_h} - \|\widetilde{\chi}_p\|^2_{0,\epsilon_h} + \alpha_{p,h}) \ge (1 - |\widetilde{\alpha}_{p,h}|) \Longrightarrow \langle \Pi_{2h}\chi_p, \Pi_{2h}\chi_p\rangle^{-1}_{0,\Omega_h} < (1 + 2|\widetilde{\alpha}_{p,h}|)$.
(In (4.16), (4.18) and (4.19), the *same* $h_0 \in]0,1[$ **has been used to denote *different* small positive numbers on** $]0,1[$ **and this convention of using the same** h_0 **to denote different small numbers on** $]0,1[$ **at different steps will be followed also in the sequel).** □

Lemma 4.2. *:* $\forall h \in]0,h_0[$ *with* $h_0 \in]0,1[$ *for which (4.19) holds, (i) linear operator* $\vec{\underline{\Pi}}_{\mathbf{h}} : \mathcal{M}_p \longrightarrow \vec{\underline{\Pi}}_{\mathbf{h}}\mathcal{M}_p \subset \mathcal{M}_h$ *defined by (3.40) is injective, (ii)* $\dim(\vec{\underline{\Pi}}_{\mathbf{h}}\mathcal{M}_p) = \dim \mathcal{M}_p = p$.

Proof. For (i), we are to show that $\vec{\underline{\Pi}}_{\mathbf{h}}(\underline{\sigma}_p, \chi_p) = (\underline{\Pi}_{1h}\underline{\sigma}_p, \Pi_{2h}\chi_p) = (\mathbf{0}, 0) \Longrightarrow (\underline{\sigma}_p, \chi_p) = (\mathbf{0}, 0)$. Assume the contrary, *i.e.* $\exists(\widehat{\underline{\sigma}}_p, \widehat{\chi}_p) \neq (\mathbf{0}, 0)$ in $\mathcal{M}_p$ with $\widehat{\chi}_p \in H^3(\Omega) \cap H_0^2(\Omega)$ and $\|\widehat{\chi}_p\|_{0,\Omega} = 1$, for which (4.19) holds

$\forall h \in]0,h_0[$ with $h_0 > 0$, and $\vec{\underline{\Pi}}_{\mathbf{h}}(\widehat{\underline{\sigma}}_p, \widehat{\chi}_p) = (\underline{\Pi}_{1h}\widehat{\underline{\sigma}}_p, \Pi_{2h}\widehat{\chi}_p) = (\mathbf{0}, 0)$ *i.e.* $\underline{\Pi}_{1h}\widehat{\underline{\sigma}}_p = \mathbf{0}$, $\Pi_{2h}\widehat{\chi}_p = 0$ such that $\widetilde{\widehat{\chi}}_p \in H^3(\widetilde{\Omega})$ is its extension to $\widetilde{\Omega}$.

$$\begin{aligned} \alpha_{p,h} &\geq \left\{ \left| 2\langle \widetilde{\widehat{\chi}}_p, \widetilde{\widehat{\chi}}_p - \Pi_{2h}\widehat{\chi}_p >_{0,\Omega_h} - \langle \widetilde{\widehat{\chi}}_p - \Pi_{2h}\widehat{\chi}_p, \widetilde{\widehat{\chi}}_p - \Pi_{2h}\widehat{\chi}_p \rangle_{0,\Omega_h} \right| \right\} \text{ (see (4.14))} \\ &= 1 - \|\widehat{\chi}_p\|^2_{0,\omega_h} + \|\widetilde{\widehat{\chi}}_p\|^2_{0,\epsilon_h} \end{aligned} \tag{4.20}$$

with $\omega_h = \Omega - (\Omega \cap \Omega_h)$, $\epsilon_h = \Omega_h - (\Omega \cap \Omega_h) \Longrightarrow \alpha_{p,h} + \|\widehat{\chi}_p\|^2_{0,\omega_h} - \|\widetilde{\widehat{\chi}}_p\|^2_{0,\epsilon_h} \geq 1$, which contradicts the hypothesis that (4.19) holds: $\forall h \in]0,h_0[$ with $h_0 \in]0,1[$, $\alpha_{p,h} + \|\widehat{\chi}_p\|^2_{0,\omega_h} - \|\widetilde{\widehat{\chi}}_p\|^2_{0,\epsilon_h} < 1/2$. Hence, our assumption that $\exists(\widehat{\underline{\sigma}}_p, \widehat{\chi}_p) \neq (\mathbf{0}, 0)$ is wrong *i.e.* $(\widehat{\underline{\sigma}}_p, \widehat{\chi}_p) = (\mathbf{0}, 0) \Longrightarrow$ Linear operator $\vec{\underline{\Pi}}_{\mathbf{h}}$ is injective.
(ii) $\vec{\underline{\Pi}}_{\mathbf{h}}$ is linear and injective from p-dimensional space $\mathcal{M}_p$ onto $\vec{\underline{\Pi}}_{\mathbf{h}}\mathcal{M}_p \subset \mathcal{M}_h \Longrightarrow \dim(\vec{\underline{\Pi}}_{\mathbf{h}}\mathcal{M}_p) = \dim \mathcal{M}_p = p$. □

Now, first of all, we will prove that $\lim_{h\to 0} \lambda_{p,h} = \lambda_p$, λ_p (resp. $\lambda_{p,h}$) being a simple eigenvalue of $(\mathbf{Q^E})$ (resp. $(\mathbf{Q^E_h})$), and using this, we will find the estimate for $\|\widetilde{u}_p - u_{p,h}\|_{0,\Omega_h}$ in order to find the "optimal" estimate for $|\lambda_p - \lambda_{p,h}|$ and finally, for $\|\widetilde{\Psi}_p - \Psi_{p,h}\|_{0,\Omega_h}$ and $\|\widetilde{u}_p - u_{p,h}\|_{1,\Omega_h}$ in this order (see also [14]). The proofs are highly technical in nature. For the sake of brevity, we state the outline of the proof and the final results (for details of proofs, see [8]).

Theorem 4.2. *Let assumptions* (**A1**–**A9**) *and assumptions of Proposition 4.2 hold. Let* $\vec{\underline{\Pi}}_{\mathbf{h}}$ *be the bijective operator defined by (3.40) such that Lemma 4.2 and estimates (3.42) hold* $\forall h \in]0,h_0[$. *Then,* $\lim_{h\to 0} \lambda_{p,h} = \lambda_p$.

Proof. From Theorem 3.5,

$$\lambda_{p,h} = \min_{\substack{S_{p,h} \subset \mathcal{M}_h \\ \dim S_{p,h} = p}} \max_{(\underline{\sigma}_h, v_h) \in S_{p,h}} \left[\frac{A_h^{\mathrm{NI}}(\underline{\sigma}_h, \underline{\sigma}_h)}{\langle v_h, v_h \rangle_{0,\Omega_h}} \right] \leq \max_{(\underline{\sigma}_h, v_h) \in \vec{\underline{\Pi}}_{\mathbf{h}}\mathcal{M}_p} \left[\frac{A_h^{\mathrm{NI}}(\underline{\sigma}_h, \underline{\sigma}_h)}{\langle v_h, v_h \rangle_{0,\Omega_h}} \right], \tag{4.21}$$

$$\Longrightarrow \lambda_{p,h} \leq \max_{\substack{(\underline{\sigma}_p, \chi_p) \in \mathcal{M}_p \\ \|\chi_p\|_{0,\Omega} = 1}} \left[\frac{A_h^{\mathrm{NI}}(\underline{\Pi}_{1h}\underline{\sigma}_p, \underline{\Pi}_{1h}\underline{\sigma}_p)}{\langle \Pi_{2h}\chi_p, \Pi_{2h}\chi_p \rangle_{0,\Omega_h}} \right], \text{ since } \vec{\underline{\Pi}}_{\mathbf{h}}(\underline{\sigma}_p, \chi_p) = (\underline{\Pi}_{1h}\underline{\sigma}_p, \Pi_{2h}\chi_p) \in \vec{\underline{\Pi}}_{\mathbf{h}}\mathcal{M}_p. \tag{4.22}$$

Then $\forall h \in]0,h_0[$ with some $h_0 \in]0,1[$, for $(\underline{\sigma}_p, \chi_p) \in \mathcal{M}_p$ with $\|\chi_p\|_{0,\Omega} = 1$,

$$\langle \Pi_{2h}\chi_p, \Pi_{2h}\chi_p >^{-1}_{0,\Omega_h} < 1 + 2\,|\widetilde{\alpha}_{p,h}| < 2 \text{ (see (4.13) and (4.19))} \tag{4.23}$$

$$\begin{aligned} A_h^{\mathrm{NI}}(\underline{\Pi}_{1h}\underline{\sigma}_p, \underline{\Pi}_{1h}\underline{\sigma}_p) \leq{}& A(\underline{\sigma}_p, \underline{\sigma}_p) + |\tilde{A}_h(\tilde{\underline{\sigma}}_p, \tilde{\underline{\sigma}}_p) - A(\underline{\sigma}_p, \underline{\sigma}_p)| + |\tilde{A}_h(\underline{\Pi}_{1h}\underline{\sigma}_p, \underline{\Pi}_{1h}\underline{\sigma}_p) - \tilde{A}_h(\widetilde{\underline{\sigma}}_p, \widetilde{\underline{\sigma}}_p)| \\ &+ |A_h^{\mathrm{NI}}(\underline{\Pi}_{1h}\underline{\sigma}_p, \underline{\Pi}_{1h}\underline{\sigma}_p) - \tilde{A}_h(\underline{\Pi}_{1h}\underline{\sigma}_p, \underline{\Pi}_{1h}\underline{\sigma}_p)| \text{ with } \widetilde{\underline{\sigma}}_p \in \widetilde{\mathbf{V}},\ (\underline{\sigma}_p, \chi_p) \in \mathcal{M}_p, \end{aligned} \tag{4.24}$$

$$\text{where } \bullet\ A(\underline{\sigma}_p, \underline{\sigma}_p) \leq \max_{\substack{(\bar{\underline{\sigma}}_p, \bar{\chi}_p) \in \mathcal{M}_p \\ \|\bar{\chi}_p\|_{0,\Omega} = 1}} \left[\frac{A(\bar{\underline{\sigma}}_p, \bar{\underline{\sigma}}_p)}{\langle \bar{\chi}_p, \bar{\chi}_p \rangle_{0,\Omega}} \right] = \lambda_p \text{ for linked pair } (\underline{\sigma}_p, \chi_p) \in \mathcal{M}_p\,; \tag{4.25}$$

$$\bullet|\tilde{A}_h(\widetilde{\underline{\sigma}}_p,\widetilde{\underline{\sigma}}_p)-A(\underline{\sigma}_p,\underline{\sigma}_p)|\le Ch^3\|\widetilde{\underline{\sigma}}_p\|_{1,\widetilde{\Omega}}\|\widetilde{\chi}_p\|_{3,\widetilde{\Omega}};\ \ \text{(see (I), Prop. 4.2)} \tag{4.26}$$

$$\begin{aligned}\bullet|\tilde{A}_h(\underline{\Pi}_{1h}\underline{\sigma}_p,\underline{\Pi}_{1h}\underline{\sigma}_p)-\tilde{A}_h(\widetilde{\underline{\sigma}}_p,\widetilde{\underline{\sigma}}_p)| &= |\tilde{A}_h(\widetilde{\underline{\sigma}}_p-\underline{\Pi}_{1h}\underline{\sigma}_p,\widetilde{\underline{\sigma}}_p-\underline{\Pi}_{1h}\underline{\sigma}_p)-2\tilde{A}_h(\widetilde{\underline{\sigma}}_p,\widetilde{\underline{\sigma}}_p-\underline{\Pi}_{1h}\underline{\sigma}_p)|\\ &\le Ch(\|\widetilde{\underline{\sigma}}_p\|_{1,\widetilde{\Omega}}+\|\widetilde{\chi}_p\|_{3,\widetilde{\Omega}})[h(\|\widetilde{\underline{\sigma}}_p\|_{1,\widetilde{\Omega}}+\|\widetilde{\chi}_p\|_{3,\widetilde{\Omega}})+\|\widetilde{\underline{\sigma}}_p\|_{0,\widetilde{\Omega}}] \qquad (4.27)\\ &\text{(using continuity of } \tilde{A}_h(\cdot,\cdot) \text{ and (3.42));}\end{aligned}$$

$$\bullet\quad |A_h^{\rm NI}(\underline{\Pi}_{1h}\underline{\sigma}_p,\underline{\Pi}_{1h}\underline{\sigma}_p)-\tilde{A}_h(\underline{\Pi}_{1h}\underline{\sigma}_p,\underline{\Pi}_{1h}\underline{\sigma}_p)|\le Ch^2\|\widetilde{A}\|_{2,\infty,\widetilde{\Omega}}[(1+h)\|\widetilde{\underline{\sigma}}_p\|_{1,\widetilde{\Omega}}+h\|\widetilde{\chi}_p\|_{3,\widetilde{\Omega}}]^2, \tag{4.28}$$

from Proposition 4.1 and $\|\underline{\Pi}_{1h}\underline{\sigma}_p\|_{0,\Omega_h}\le\|\widetilde{\underline{\sigma}}_p\|_{0,\Omega_h}+\|\widetilde{\underline{\sigma}}_p-\underline{\Pi}_{1h}\underline{\sigma}_p\|_{0,\Omega_h}\le C[(1+h)\|\widetilde{\underline{\sigma}}_p\|_{1,\widetilde{\Omega}}+h\|\widetilde{\chi}_p\|_{3,\widetilde{\Omega}}]$. Hence, from (4.22)–(4.28),

$$\begin{aligned}\lambda_{p,h}\le&\max_{\substack{(\underline{\sigma}_p,\chi_p)\in\mathcal{M}_p\\ \|\chi_p\|_{0,\Omega}=1}}\left[\frac{A_h^{\rm NI}(\underline{\Pi}_{1h}\underline{\sigma}_p,\underline{\Pi}_{1h}\underline{\sigma}_p)}{\langle\Pi_{2h}\chi_p,\Pi_{2h}\chi_p\rangle_{0,\Omega_h}}\right]\le\ \lambda_p+2\max_{\substack{(\underline{\sigma}_p,\chi_p)\in\mathcal{M}_p\\ \|\chi_p\|_{0,\Omega}=1}}\{|\widetilde{\alpha}_{p,h}|\}\lambda_p\\ &+Ch\max_{\substack{(\underline{\sigma}_p,\chi_p)\in\mathcal{M}_p\\ \|\chi_p\|_{0,\Omega}=1}}\Big\{(1+2\,|\widetilde{\alpha}_{p,h}|)[h^2\|\widetilde{\underline{\sigma}}_p\|_{1,\widetilde{\Omega}}\|\widetilde{\chi}_p\|_{3,\widetilde{\Omega}}+(\|\widetilde{\underline{\sigma}}_p\|_{1,\widetilde{\Omega}}+\|\widetilde{\chi}_p\|_{3,\widetilde{\Omega}})\\ &\times(h(\|\widetilde{\underline{\sigma}}_p\|_{1,\widetilde{\Omega}}+\|\widetilde{\chi}_p\|_{3,\widetilde{\Omega}})+\|\widetilde{\underline{\sigma}}_p\|_{0,\widetilde{\Omega}})+\|\widetilde{A}\|_{2,\infty,\widetilde{\Omega}}((1+h)\|\widetilde{\underline{\sigma}}_p\|_{1,\widetilde{\Omega}}+h\|\widetilde{\chi}_p\|_{3,\widetilde{\Omega}})^2]\Big\}\end{aligned} \tag{4.29}$$

where $|\widetilde{\alpha}_{p,h}|\le(\|\chi_p\|^2_{0,\omega_h}+\|\widetilde{\chi}_p\|^2_{0,\epsilon_h}+\alpha_{p,h})\longrightarrow 0$ as $h\longrightarrow 0$ by virtue of (4.16) and (4.17)

$$\Longrightarrow\lim_{h\to 0}\max_{\substack{(\underline{\sigma}_p,\chi_p)\in\mathcal{M}_p\\ \|\chi_p\|_{0,\Omega}=1}}|\widetilde{\alpha}_{p,h}|=0\Longrightarrow\lim_{h\to 0}\lambda_{p,h}\le\lambda_p. \tag{4.30}$$

Now, we will show that $\lim_{h\to 0}\lambda_{p,h}\ge\lambda_p$. Let $(\mu_{m,h};u_{m,h})\in\mathbb{R}^+\times W_h$ be the eigenpairs of $T_h\in\mathcal{L}(W_h)$ with $\widetilde{u}_{m,h}\in\widetilde{W}_h\subset H_0^1(\widetilde{\Omega})$, $\widetilde{u}_{m,h}\downarrow_{\Omega_h}=u_{m,h}$. For $U_{p,h}=\text{Span}\{(u_{m,h})_{m=1}^p\}\subset W_h$, let $\widetilde{U}_{p,h}=\text{Span}\{(\widetilde{u}_{m,h})_{m=1}^p\}\subset\widetilde{W}_h\subset H_0^1(\widetilde{\Omega})$ be a p-dimensional subspace. Then, $v_h\in U_{p,h}\iff\widetilde{v}_h\in\widetilde{U}_{p,h}$, and from Theorem 3.6, $\mu_{p,h}=\min\limits_{v_h\in U_{p,h}}\dfrac{\langle T_hv_h,v_h\rangle_{0,\Omega_h}}{\langle v_h,v_h\rangle_{0,\Omega_h}}$. Under (**A5**), $T:\widetilde{v}_h\in L^2(\Omega)\mapsto T\widetilde{v}_h\in H^3(\Omega)\cap H_0^2(\Omega)$ with $\|T\widetilde{v}_h\|_{3,\Omega}\le C\|\widetilde{v}_h\|_{0,\Omega}$ and $\widetilde{T\widetilde{v}_h}\in H^3(\widetilde{\Omega})$ such that $\|\widetilde{T\widetilde{v}_h}\|_{3,\widetilde{\Omega}}\le C\|T\widetilde{v}_h\|_{3,\Omega}$ and $\widetilde{T\widetilde{v}_h}\downarrow_{\Omega_h}\in H^3(\Omega_h)$ will be denoted by $\widetilde{T\widetilde{v}_h}$ such that $\langle T_hv_h,v_h\rangle_{0,\Omega_h}=\langle T_hv_h-\widetilde{T\widetilde{v}_h}+\widetilde{T\widetilde{v}_h},v_h\rangle_{0,\Omega_h}=\langle\widetilde{T\widetilde{v}_h},v_h\rangle_{0,\Omega_h}+\langle T_hv_h-\widetilde{T\widetilde{v}_h},v_h\rangle_{0,\Omega_h}$

$$\begin{aligned}\Longrightarrow\mu_{p,h}&=\min_{v_h\in U_{p,h},\widetilde{v}_h\in\widetilde{U}_{p,h}}\left[\frac{\langle\widetilde{T\widetilde{v}_h},v_h\rangle_{0,\Omega_h}}{\langle v_h,v_h\rangle_{0,\Omega_h}}+\frac{\langle T_hv_h-\widetilde{T\widetilde{v}_h},v_h\rangle_{0,\Omega_h}}{\langle v_h,v_h\rangle_{0,\Omega_h}}\right]\\ &\le\min_{v_h\in U_{p,h},\widetilde{v}_h\in\widetilde{U}_{p,h}}\left[\frac{\langle\widetilde{T\widetilde{v}_h},v_h\rangle_{0,\Omega_h}}{\langle v_h,v_h\rangle_{0,\Omega_h}}\right]+\max_{v_h\in U_{p,h},\widetilde{v}_h\in\widetilde{U}_{p,h}}\left[\frac{\langle T_hv_h-\widetilde{T\widetilde{v}_h},v_h\rangle_{0,\Omega_h}}{\langle v_h,v_h\rangle_{0,\Omega_h}}\right]\end{aligned} \tag{4.31}$$

Since $\widetilde{v_h}\downarrow_{\omega_h=\Omega-(\Omega\cap\Omega_h)}=0$, $\langle\widetilde{T\widetilde{v}_h},v_h\rangle_{0,\Omega_h}=\int\limits_{\Omega_h}(\widetilde{T\widetilde{v}_h})v_h\,\mathrm{d}x=\langle T\widetilde{v}_h,\widetilde{v_h}\rangle_{0,\Omega}+\langle\widetilde{T\widetilde{v}_h},v_h\rangle_{0,\epsilon_h}$

$$\Longrightarrow\frac{\langle\widetilde{T\widetilde{v}_h},v_h\rangle_{0,\Omega_h}}{\langle v_h,v_h\rangle_{0,\Omega_h}}=\frac{\langle T\widetilde{v}_h,\widetilde{v}_h\rangle_{0,\Omega}}{\langle v_h,v_h\rangle_{0,\Omega_h}}+\frac{\langle\widetilde{T\widetilde{v}_h},v_h\rangle_{0,\epsilon_h}}{\langle v_h,v_h\rangle_{0,\Omega_h}}\ \ \forall v_h\in U_{p,h}\text{ with }\widetilde{v}_h\in\widetilde{U}_{p,h}, \tag{4.32}$$

From (3.8), (4.5) and $\|\tilde{v}_h\|_{0,\Omega}\le\|v_h\|_{0,\Omega_h}$,

$$\|\widetilde{T\widetilde{v}_h}\|_{0,\epsilon_h}\le Ch^{3/2}\|\widetilde{T\widetilde{v}_h}\|_{1,\widetilde{\Omega}}\le Ch^{3/2}\|\widetilde{T\widetilde{v}_h}\|_{3,\widetilde{\Omega}}\le Ch^{3/2}\|T\widetilde{v}_h\|_{3,\Omega}\le Ch^{3/2}\|\widetilde{v}_h\|_{0,\Omega}\le Ch^{3/2}\|v_h\|_{0,\Omega_h}$$

$$\text{and } |\langle \widetilde{T\widetilde{v}_h}, v_h\rangle_{0,\epsilon_h}| \le \|\widetilde{T\widetilde{v}_h}\|_{0,\epsilon_h}\|v_h\|_{0,\epsilon_h} \le Ch^{3/2}\|v_h\|^2_{0,\Omega_h}, \tag{4.33}$$

$$\min_{v_h\in U_{p,h},\widetilde{v}_h\in\widetilde{U}_{p,h}} \frac{\langle \widetilde{T\widetilde{v}_h}, v_h\rangle_{0,\Omega_h}}{\langle v_h, v_h\rangle_{0,\Omega_h}} \le \min_{\widetilde{v}_h\in\widetilde{U}_{p,h}\subset L^2(\Omega)} \frac{\langle T\widetilde{v}_h, \widetilde{v}_h\rangle_{0,\Omega}}{\langle \widetilde{v}_h, \widetilde{v}_h\rangle_{0,\Omega}} + Ch^{3/2} \le \mu_p + Ch^{3/2}$$

(using $\|\tilde{v}_h\|_{0,\Omega} \le \|v_h\|_{0,\Omega_h}$ and (4.33) in (4.32)).

$$\text{Hence, from (4.31)}, \ \mu_{p,h} \le \mu_p + Ch^{3/2} + \max_{v_h\in U_{p,h} \textit{ with } \widetilde{v}_h\in\widetilde{U}_{p,h}} \frac{|\langle T_h v_h - \widetilde{T\widetilde{v}_h}, v_h\rangle_{0,\Omega_h}|}{\langle v_h, v_h\rangle_{0,\Omega_h}}. \tag{4.34}$$

$$\text{But } |\langle T_h v_h - \widetilde{T\widetilde{v}_h}, v_h\rangle_{0,\Omega_h}| \le \|T_h v_h - \widetilde{T\widetilde{v}_h}\|_{0,\Omega_h}\|v_h\|_{0,\Omega_h}, \tag{4.35}$$

and from Theorem 3.3, $\|T_h v_h - \widetilde{T\widetilde{v}_h}\|_{0,\Omega_h} \le Ch(\|\widetilde{T\widetilde{v}_h}\|_{3,\widetilde{\Omega}} + \|\widetilde{S\widetilde{v}_h}\|_{1,\widetilde{\Omega}})$. Then, using (3.7)–(3.9),
$\|\widetilde{T\widetilde{v}_h}\|_{3,\widetilde{\Omega}} \le C\|T\widetilde{v}_h\|_{3,\Omega} \le C\|v_h\|_{0,\Omega_h}$ and $\|\widetilde{S\widetilde{v}_h}\|_{1,\widetilde{\Omega}} \le C\|S\widetilde{v}_h\|_{1,\Omega} \le C\|v_h\|_{0,\Omega_h}$
$\Longrightarrow |\langle T_h v_h - \widetilde{T\widetilde{v}_h}, v_h\rangle_{0,\Omega_h}| \le Ch\|v_h\|^2_{0,\Omega_h}$ and from (4.34), $\mu_{p,h} \le \mu_p + Ch^{3/2} + Ch$
$\Longrightarrow \dfrac{1}{\lambda_{p,h}} - \dfrac{1}{\lambda_p} \le Ch(1+\sqrt{h}) \Longrightarrow \lim_{h\to 0}\lambda_{p,h} \ge \lambda_p$, which together with (4.30), gives the result. □

Theorem 4.3. *Under the assumption that Theorem 4.2 holds and λ_p (resp. $\lambda_{p,h}$) is a simple eigenvalue of $(\mathbf{Q^E})$ (resp. $(\mathbf{Q^E_h})$), $\exists C > 0$, independent of 'h' and 'p', such that $\forall h \in]0, h_0[$ with $h_0 \in]0,1[$,*

$$\|\widetilde{u}_p - u_{p,h}\|_{0,\Omega_h} \le Ch\Big[h^2\|\widetilde{u}_p\|^2_{1,\widetilde{\Omega}} + (\|\widetilde{u}_p\|_{3,\widetilde{\Omega}} + \|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}}) \times \Big\{1 + 2\frac{\lambda_p}{d_p} + (h+2)\|\widetilde{u}_p\|_{3,\widetilde{\Omega}} + h\|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}}\Big\}\Big]. \tag{4.36}$$

$$\begin{aligned}\|\widetilde{\Psi}_p - \Psi_{p,h}\|_{0,\Omega_h} \le Ch\Big[&(\|\widetilde{u}_p\|_{3,\widetilde{\Omega}} + \|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}})\big(1 + \frac{2\lambda_p\sqrt{2}}{\sqrt{\lambda_1\bar{\alpha}_0}}\big\{1 + 2\frac{\lambda_p}{d_p} + (h+2)\|\widetilde{u}_p\|_{3,\widetilde{\Omega}} + h\|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}}\big\}\big)\\ &+\frac{2\sqrt{2}\lambda_p}{\sqrt{\lambda_1\bar{\alpha}_0}}h^2\|\widetilde{u}_p\|^2_{1,\widetilde{\Omega}}\Big] + \frac{\sqrt{2}}{\sqrt{\lambda_1\bar{\alpha}_0}}|\lambda_p - \lambda_{p,h}| \textit{ with parameter } d_p > 0 \textit{ defined in } (4.41).\end{aligned} \tag{4.37}$$

Proof. Let $\underline{\vec{\Pi}}_{\mathbf{h}} : \mathcal{M}_p \longrightarrow \underline{\vec{\Pi}}_{\mathbf{h}}\mathcal{M}_p \subset \mathcal{M}_h$ be defined by (3.40) with $\underline{\sigma}_p = \Psi_p,\ \chi_p = u_p$ and $\widetilde{\chi}^*_p = \lambda_p\widetilde{u}_p$.

$$\text{Then, choose } u_{p,h} \text{ such that } \langle \Pi_{2h}u_p, u_{p,h}\rangle_{0,\Omega_h} > 0. \tag{4.38}$$

$$\begin{aligned}\|\widetilde{u}_p - u_{p,h}\|_{0,\Omega_h} \quad \le \quad & \|\widetilde{u}_p - \Pi_{2h}u_p\|_{0,\Omega_h} + \big\|\Pi_{2h}u_p - \langle \Pi_{2h}u_p, u_{p,h}\rangle_{0,\Omega_h}u_{p,h}\big\|_{0,\Omega_h}\\ &+\|\langle \Pi_{2h}u_p, u_{p,h}\rangle_{0,\Omega_h}u_{p,h} - u_{p,h}\|_{0,\Omega_h}.\end{aligned} \tag{4.39}$$

We are to find estimates only for the second and third terms on the right hand side of (4.39), since (3.42) gives the estimate for the first term.
$\Pi_{2h}u_p \in W_h \Longrightarrow \Pi_{2h}u_p = \sum_{j=1}^{N_h}\langle \Pi_{2h}u_p, u_{j,h}\rangle_{0,\Omega_h}u_{j,h}$ with $\langle u_{j,h}, u_{k,h}\rangle_{0,\Omega_h} = \delta_{jk},\ 1 \le j,k \le N_h$.
From (3.21) and definition of $\underline{\vec{\Pi}}_{\mathbf{h}}(\Psi_p, u_p) = (\underline{\Pi}_{1h}\Psi_p, \Pi_{2h}u_p)$ in (3.40), we have:

$$\begin{aligned}\lambda_{j,h}\langle \Pi_{2h}u_p, u_{j,h}\rangle_{0,\Omega_h} &= -\,b_h^{\mathrm{NI}}(\Psi_{j,h}, \Pi_{2h}u_p) = A_h^{\mathrm{NI}}(\underline{\Pi}_{1h}\Psi_p, \Psi_{j,h})\\ &= -\,b_h^{\mathrm{NI}}(\underline{\Pi}_{1h}\Psi_p, u_{j,h}) = \lambda_p\langle \widetilde{u}_p, u_{j,h}\rangle_{0,\Omega_h} \ \text{ with } \ \widetilde{\chi}^*_p = \lambda_p\widetilde{u}_p.\\ \Longrightarrow (\lambda_{j,h} - \lambda_p)\langle \Pi_{2h}u_p, u_{j,h}\rangle_{0,\Omega_h} &= \lambda_p\big[\langle \widetilde{u}_p, u_{j,h}\rangle_{0,\Omega_h} - \langle \Pi_{2h}u_p, u_{j,h}\rangle_{0,\Omega_h}\big]\\ \Longrightarrow \langle \Pi_{2h}u_p, u_{j,h}\rangle_{0,\Omega_h} &= \frac{\lambda_p}{(\lambda_{j,h} - \lambda_p)}\big[\langle \widetilde{u}_p - \Pi_{2h}u_p, u_{j,h}\rangle_{0,\Omega_h}\big] \ \ (j \ne p).\end{aligned} \tag{4.40}$$

Since we are considering the case of *simple* eigenvalues, set $2d_p = \min\{\lambda_p - \lambda_{p-1}, \lambda_{p+1} - \lambda_p\} > 0.$ (4.41)

From Theorem 4.2, $\lim_{h\to 0} \lambda_{j,h} = \lambda_j$ $(j \neq p)$. Hence, $\forall h \in]0, h_0[$ with some $h_0 \in]0,1[$,

$$|\lambda_{j,h} - \lambda_p| \geq d_p \Longrightarrow |\langle \Pi_{2h} u_p, u_{j,h}\rangle_{0,\Omega_h}| \leq \frac{\lambda_p}{d_p} |\langle \widetilde{u}_p - \Pi_{2h} u_p, u_{j,h}\rangle_{0,\Omega_h}| \ \ \forall j \neq p.$$

But $(u_{j,h})_{j=1}^{N_h}$ is orthonormal in W_h. Then,

$$\big\|\Pi_{2h} u_p - \langle \Pi_{2h} u_p, u_{p,h}\rangle_{0,\Omega_h} u_{p,h}\big\|_{0,\Omega_h} = \Big(\sum_{\substack{j=1\\ j\neq p}}^{N_h} |\langle \Pi_{2h} u_p, u_{j,h}\rangle_{0,\Omega_h}|^2\Big)^{1/2}$$

$$\leq \Big(\frac{\lambda_p}{d_p}\Big)\Big(\sum_{\substack{j=1\\ j\neq p}}^{N_h} |\langle \widetilde{u}_p - \Pi_{2h} u_p, u_{j,h}\rangle_{0,\Omega_h}|^2\Big)^{1/2} \leq \Big(\frac{\lambda_p}{d_p}\Big)\|\widetilde{u}_p - \Pi_{2h} u_p\|_{0,\Omega_h} \ \forall h \in]0, h_0[\text{ with } h_0 > 0. \quad (4.42)$$

Finally,

$$\begin{aligned}
\big\|\big\langle \Pi_{2h} u_p, u_{p,h}\big\rangle_{0,\Omega_h} u_{p,h} - u_{p,h}\big\|_{0,\Omega_h} &= \big|\ \big\|\big\langle \Pi_{2h} u_p, u_{p,h}\big\rangle_{0,\Omega_h} u_{p,h}\big\|_{0,\Omega_h} - 1\big| \\
&\leq \big|\ \big\|\big\langle \Pi_{2h} u_p, u_{p,h}\big\rangle_{0,\Omega_h} u_{p,h}\big\|_{0,\Omega_h} - \big\|\Pi_{2h} u_p\big\|_{0,\Omega_h}\big| + \big|\ \big\|\Pi_{2h} u_p\big\|_{0,\Omega_h} - 1\big| \\
&\leq \big\|\big\langle \Pi_{2h} u_p, u_{p,h}\big\rangle_{0,\Omega_h} u_{p,h} - \Pi_{2h} u_p\big\|_{0,\Omega_h} + \big|\ \big\|\Pi_{2h} u_p\big\|^2_{0,\Omega_h} - 1\big|.
\end{aligned}$$

$$i.e.\ \big\|\big\langle \Pi_{2h} u_p, u_{p,h}\big\rangle_{0,\Omega_h} u_{p,h} - u_{p,h}\big\|_{0,\Omega_h} \leq \frac{\lambda_p}{d_p}\|\widetilde{u}_p - \Pi_{2h} u_p\|_{0,\Omega_h} + \big|\|\Pi_{2h} u_p\|^2_{0,\Omega_h} - 1\big| \ \text{(using (4.42))}. \quad (4.43)$$

$$\begin{aligned}
\text{But } \big|\|\Pi_{2h} u_p\|^2_{0,\Omega_h} - 1\big| &\leq \big|\|\Pi_{2h} u_p\|^2_{0,\Omega_h} - \|\widetilde{u}_p\|^2_{0,\Omega_h}\big| + \big|\|\widetilde{u}_p\|^2_{0,\Omega_h} - \|u_p\|^2_{0,\Omega}\big| \\
&\leq \big(\|\Pi_{2h} u_p\|_{0,\Omega_h} + \|\widetilde{u}_p\|_{0,\Omega_h}\big)\big|\|\Pi_{2h} u_p\|_{0,\Omega_h} - \|\widetilde{u}_p\|_{0,\Omega_h}\big| + \|u_p\|^2_{0,\omega_h} + \|\widetilde{u}_p\|^2_{0,\epsilon_h} \\
&\leq \big(\|\Pi_{2h} u_p - \widetilde{u}_p\|_{0,\Omega_h} + 2\|\widetilde{u}_p\|_{0,\Omega_h}\big)\|\Pi_{2h} u_p - \widetilde{u}_p\|_{0,\Omega_h} + Ch^3\|\widetilde{u}_p\|^2_{1,\widetilde{\Omega}}. \quad (4.44)
\end{aligned}$$

From (4.43)–(4.44),

$$\begin{aligned}
&\|\langle \Pi_{2h} u_p, u_{p,h}\rangle_{0,\Omega_h} u_{p,h} - u_{p,h}\|_{0,\Omega_h} \\
&\qquad \leq \|\widetilde{u}_p - \Pi_{2h} u_p\|_{0,\Omega_h}\Big[\frac{\lambda_p}{d_p} + \|\widetilde{u}_p - \Pi_{2h} u_p\|_{0,\Omega_h} + 2\|\widetilde{u}_p\|_{0,\Omega_h}\Big] + Ch^3\|\widetilde{u}_p\|^2_{1,\widetilde{\Omega}}. \quad (4.45)
\end{aligned}$$

Finally, from (4.39), (4.42) and (4.45), we get

$$\|\widetilde{u}_p - u_{p,h}\|_{0,\Omega_h} \leq \|\widetilde{u}_p - \Pi_{2h} u_p\|_{0,\Omega_h}\Big[1 + 2\frac{\lambda_p}{d_p} + \|\widetilde{u}_p - \Pi_{2h} u_p\|_{0,\Omega_h} + 2\|\widetilde{u}_p\|_{1,\widetilde{\Omega}}\Big] + Ch^3\|\widetilde{u}_p\|^2_{1,\widetilde{\Omega}}. \quad (4.46)$$

Then, using (3.42) with $\chi_p = u_p$, $\underline{\sigma}_p = \Psi_p$ in (4.46), we get the result (4.36).

Now, we proceed to prove (4.37). $\|\widetilde{\Psi}_p - \Psi_{p,h}\|_{0,\Omega_h} \ \leq \ \|\widetilde{\Psi}_p - \underline{\Pi}_{1h}\Psi_p\|_{0,\Omega_h} + \|\underline{\Pi}_{1h}\Psi_p - \Psi_{p,h}\|_{0,\Omega_h}.$ (4.47)

$$\text{From (3.28), } \|\underline{\Pi}_{1h}\Psi_p - \Psi_{p,h}\|_{0,\Omega_h} \leq \frac{1}{\sqrt{\bar{\alpha}_0}}\left(A_h^{\rm NI}(\underline{\Pi}_{1h}\Psi_p - \Psi_{p,h}, \underline{\Pi}_{1h}\Psi_p - \Psi_{p,h})\right)^{1/2} = \frac{1}{\sqrt{\bar{\alpha}_0}}\,|||\underline{\Pi}_{1h}\Psi_p - \Psi_{p,h}|||_{A_h^{\rm NI}(\cdot,\cdot)}\,. \tag{4.48}$$

Setting $\widehat{\Psi}_{j,h} = \dfrac{\Psi_{j,h}}{\sqrt{\lambda_{j,h}}}$, we get

$$|||\underline{\Pi}_{1h}\Psi_p - \Psi_{p,h}|||_{A_h^{\rm NI}(\cdot,\cdot)} \leq \left|\left|\left|\underline{\Pi}_{1h}\Psi_p - \left[\underline{\Pi}_{1h}\Psi_p, \widehat{\Psi}_{p,h}\right]_{A_h^{\rm NI}(\cdot,\cdot)}\widehat{\Psi}_{p,h}\right|\right|\right|_{A_h^{\rm NI}(\cdot,\cdot)} + \left|\left[\underline{\Pi}_{1h}\Psi_p, \widehat{\Psi}_{p,h}\right]_{A_h^{\rm NI}(\cdot,\cdot)} - \sqrt{\lambda_{p,h}}\right|, \text{ since } \left[\widehat{\Psi}_{j,h}, \widehat{\Psi}_{p,h}\right]_{A_h^{\rm NI}(\cdot,\cdot)} = \delta_{jp}. \tag{4.49}$$

But $\underline{\Pi}_{1h}\Psi_p \in \mathcal{E}_h \Longrightarrow \underline{\Pi}_{1h}\Psi_p = \sum_{j=1}^{N_h}\left[\underline{\Pi}_{1h}\Psi_p, \widehat{\Psi}_{j,h}\right]_{A_h^{\rm NI}(\cdot,\cdot)}\widehat{\Psi}_{j,h}$ with

$$\left[\underline{\Pi}_{1h}\Psi_p, \widehat{\Psi}_{j,h}\right]_{A_h^{\rm NI}(\cdot,\cdot)} = \frac{1}{\sqrt{\lambda_{j,h}}}A_h^{\rm NI}(\underline{\Pi}_{1h}\Psi_p, \Psi_{j,h}) = \frac{-1}{\sqrt{\lambda_{j,h}}}b_h^{\rm NI}(\underline{\Pi}_{1h}\Psi_p, u_{j,h}) = \frac{\lambda_p}{\sqrt{\lambda_{j,h}}}\langle\widetilde{u}_p, u_{j,h}\rangle_{0,\Omega_h} \tag{4.50}$$

with $\lambda_p\widetilde{u}_p = \widetilde{\chi}_p^*$ (using (3.21) and (3.40))

$$\Longrightarrow \left[\underline{\Pi}_{1h}\Psi_p, \widehat{\Psi}_{j,h}\right]_{A_h^{\rm NI}(\cdot,\cdot)} = \frac{\lambda_p}{\sqrt{\lambda_{j,h}}}\left[\langle\widetilde{u}_p - u_{p,h}, u_{j,h}\rangle_{0,\Omega_h}\right] \quad 1 \leq j \neq p \leq N_h. \tag{4.51}$$

From Theorem 4.2, $\lim_{h\to 0}\lambda_{j,h} = \lambda_j,\ 1 \leq j \leq N_h \Longrightarrow \exists h_0 \in]0,1[$ such that

$$\lambda_{j,h} \geq \frac{\lambda_1}{2} \quad \forall j = 1,2,\cdots,N_h \Longrightarrow \sqrt{\lambda_{j,h}} \geq \sqrt{\frac{\lambda_1}{2}}. \tag{4.52}$$

Hence, from (4.51) and (4.52), $\forall j \neq p,\ \ 1 \leq j \leq N_h,\ \ \forall h \in]0,h_0[$ with $h_0 \in]0,1[$,

$$\left|\left[\underline{\Pi}_{1h}\Psi_p, \widehat{\Psi}_{j,h}\right]_{A_h^{\rm NI}(\cdot,\cdot)}\right| \leq \lambda_p\sqrt{\frac{2}{\lambda_1}}\left|\langle\widetilde{u}_p - u_{p,h}, u_{j,h}\rangle_{0,\Omega_h}\right|$$

and

$$\left|\left|\left|\underline{\Pi}_{1h}\Psi_p - \left[\underline{\Pi}_{1h}\Psi_p, \widehat{\Psi}_{p,h}\right]_{A_h^{\rm NI}(\cdot,\cdot)}\widehat{\Psi}_{p,h}\right|\right|\right|_{A_h^{\rm NI}(\cdot,\cdot)} = \left(\sum_{\substack{j=1\\ j\neq p}}^{N_h}\left|\left|\left|\left[\underline{\Pi}_{1h}\Psi_p, \widehat{\Psi}_{j,h}\right]_{A_h^{\rm NI}(\cdot,\cdot)}\widehat{\Psi}_{j,h}\right|\right|\right|^2_{A_h^{\rm NI}(\cdot,\cdot)}\right)^{1/2}$$

$$\leq \frac{\sqrt{2}}{\sqrt{\lambda_1}}(\lambda_p)\Big(\sum_{\substack{j=1\\ j\neq p}}^{N_h}|\langle\widetilde{u}_p - u_{p,h}, u_{j,h}\rangle_{0,\Omega_h}|^2\Big)^{1/2} \leq \frac{\sqrt{2}}{\sqrt{\lambda_1}}(\lambda_p)\|\widetilde{u}_p - u_{p,h}\|_{0,\Omega_h} \text{ (by Bessel's inequality).} \tag{4.53}$$

Putting $j = p$ in (4.50), $\left[\underline{\Pi}_{1h}\Psi_p, \widehat{\Psi}_{p,h}\right]_{A_h^{\rm NI}(\cdot,\cdot)} = \dfrac{\lambda_p}{\sqrt{\lambda_{p,h}}}\langle\widetilde{u}_p, u_{p,h} >_{0,\Omega_h}$.

Therefore, $\left|\left[\underline{\Pi}_{1h}\Psi_p, \widehat{\Psi}_{p,h}\right]_{A_h^{\rm NI}(\cdot,\cdot)} - \sqrt{\lambda_{p,h}}\right| = \dfrac{1}{\sqrt{\lambda_{p,h}}}\left|\lambda_p\langle\widetilde{u}_p, u_{p,h}\rangle_{0,\Omega_h} - \lambda_{p,h}\right|$

$$\begin{aligned}
&= \frac{1}{\sqrt{\lambda_{p,h}}}\left|\lambda_p\langle\widetilde{u}_p - u_{p,h}, u_{p,h}\rangle_{0,\Omega_h} + (\lambda_p - \lambda_{p,h})\right| \\
&\Longrightarrow \left|\left[\underline{\Pi}_{1h}\Psi_p, \widehat{\Psi}_{p,h}\right]_{A_h^{\rm NI}(\cdot,\cdot)} - \sqrt{\lambda_{p,h}}\right| \le \frac{1}{\sqrt{\lambda_{p,h}}}\left[\lambda_p\|\widetilde{u}_p - u_{p,h}\|_{0,\Omega_h} + |\lambda_p - \lambda_{p,h}|\,\right] \\
&\le \sqrt{\frac{2}{\lambda_1}}\left[\lambda_p\|\widetilde{u}_p - u_{p,h}\|_{0,\Omega_h} + |\lambda_p - \lambda_{p,h}|\,\right] \text{ (by (4.52)).} \qquad (4.54)
\end{aligned}$$

Finally, from (4.47),(4.48), (4.49), (4.53) and (4.54), we get: $\forall h \in]0, h_0[$ with $h_0 \in]0, 1[$,

$$\|\widetilde{\Psi}_p - \Psi_{p,h}\|_{0,\Omega_h} \le \|\widetilde{\Psi}_p - \underline{\Pi}_{1h}\Psi_p\|_{0,\Omega_h} + \frac{1}{\sqrt{\widetilde{\alpha}_0}}\sqrt{\frac{2}{\lambda_1}}\left[2\lambda_p\|\widetilde{u}_p - u_{p,h}\|_{0,\Omega_h} + |\lambda_p - \lambda_{p,h}|\right]. \qquad (4.55)$$

Using the estimates (3.42) and (4.36) for $\|\widetilde{\Psi}_p - \underline{\Pi}_{1h}\Psi_p\|_{0,\Omega_h}$ and $\|\widetilde{u}_p - u_{p,h}\|_{0,\Omega_h}$ respectively in (4.55), we get the result (4.37).

□

Now, we proceed to find the estimate for the term $|\lambda_p - \lambda_{p,h}|$ occurring in (4.37).

Theorem 4.4. *Under the assumptions that Theorems 4.2 and 4.3 hold and λ_p (resp. $\lambda_{p,h}$) is a simple eigenvalue of $(\mathbf{Q^E})$ (resp. $(\mathbf{Q_h^E})$), $1 \le p \le N_h = \dim W_h$, $\forall h \in]0, h_0[$ with $h_0 \in]0, 1[$, $\exists C > 0$, independent of 'h' and 'p' such that*

$$\begin{aligned}
|\lambda_p - \lambda_{p,h}| \le Ch^2\Big\{&(\|\widetilde{u}_p\|_{3,\widetilde{\Omega}} + \|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}})\big[\lambda_p\big(h^2\|\widetilde{u}_p\|^2_{1,\widetilde{\Omega}} + (\|\widetilde{u}_p\|_{3,\widetilde{\Omega}} + \|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}})(1 + \frac{2\lambda_p}{d_p} + (h+2)\|\widetilde{u}_p\|_{3,\widetilde{\Omega}} \\
&+ h\|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}})\big) + (h+2)\|\widetilde{u}_p\|_{3,\widetilde{\Omega}} + \|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}}\big] + \big(\lambda_p h\|\widetilde{u}_p\|_{3,\widetilde{\Omega}} + (h+2)\|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}}\big)\|\widetilde{u}_p\|_{3,\widetilde{\Omega}}\Big\} = O(h^2). \qquad (4.56)
\end{aligned}$$

Proof. Let $\vec{\underline{\Pi}}_{\mathbf{h}} : \mathcal{M}_p \longrightarrow \vec{\underline{\Pi}}_{\mathbf{h}}\mathcal{M}_p \subset \mathcal{M}_h$ with $\vec{\underline{\Pi}}_{\mathbf{h}}(\Psi_p, u_p) = (\underline{\Pi}_{1h}\Psi_p, \Pi_{2h}u_p)$ be defined by (3.40) such that the estimates (3.42) hold, $(\lambda_p; (\Psi_p, u_p)) \in \mathbb{R}^+ \times (\mathbf{V} \times W)$(*resp.* $(\lambda_{p,h}; (\Psi_{p,h}, u_{p,h})) \in \mathbb{R}^+ \times (\mathbf{V}_h \times W_h)$) with $u_p \in H^3(\Omega) \cap H_0^2(\Omega)$, $\|u_p\|_{0,\Omega} = 1$ being an eigenpair of $(\mathbf{Q^E})$(*resp.* $(\mathbf{Q_h^E})$). We have from (3.40) and (3.21), $b_h^{\rm NI}(\Psi_{p,h}, \Pi_{2h}u_p) = -\lambda_p\langle\widetilde{u}_p, u_{p,h}\rangle_{0,\Omega_h}$, and from (3.21),

$$\begin{aligned}
\lambda_p - \lambda_{p,h} &= \lambda_p - \frac{\lambda_p\langle\widetilde{u}_p, u_{p,h}\rangle_{0,\Omega_h}}{\langle u_{p,h}, \Pi_{2h}u_p\rangle_{0,\Omega_h}} = \frac{\lambda_p}{\langle u_{p,h}, \Pi_{2h}u_p\rangle_{0,\Omega_h}}\Big[-\langle\widetilde{u}_p - \Pi_{2h}u_p, u_{p,h}\rangle_{0,\Omega_h}\Big] \qquad (4.57)\\
&\quad \text{with } \tilde{u}_p \in H^3(\widetilde{\Omega}), \tilde{u}_p\downarrow_{\Omega} = u_p \in H^3(\Omega) \cap H_0^2(\Omega).
\end{aligned}$$

$$\text{But } -\langle\widetilde{u}_p - \Pi_{2h}u_p, u_{p,h} >_{0,\Omega_h} = \langle\widetilde{u}_p - \Pi_{2h}u_p, \widetilde{u}_p - u_{p,h}\rangle_{0,\Omega_h} - \langle\widetilde{u}_p - \Pi_{2h}u_p, \widetilde{u}_p\rangle_{0,\Omega_h}. \qquad (4.58)$$

Using (3.40) with $\widetilde{\chi}_p^* = \widetilde{u}_p$ and (2.20), we have:
$-\langle\widetilde{u}_p - \Pi_{2h}u_p, \widetilde{u}_p\rangle_{0,\Omega_h} = -\langle u_p, u_p\rangle_{0,\Omega} - \langle\widetilde{u}_p, \widetilde{u}_p\rangle_{0,\epsilon_h} + \langle u_p, u_p\rangle_{0,\omega_h} - \lambda_p^{-1}b_h^{\rm NI}(\underline{\Pi}_{1h}\Psi_p, \Pi_{2h}u_p)$

$$= \lambda_p^{-1}\big[\widetilde{b_h}(\widetilde{\Psi}_p, \widetilde{u}_p) - b_h^{\rm NI}(\underline{\Pi}_{1h}\Psi_p, \Pi_{2h}u_p)\big] + \big[\|u_p\|^2_{0,\omega_h} - \|\widetilde{u}_p\|^2_{0,\epsilon_h}\big] + \lambda_p^{-1}\big[b(\Psi_p, u_p) - \widetilde{b_h}(\widetilde{\Psi}_p, \widetilde{u}_p)\big]. \qquad (4.59)$$

From (4.57)–(4.59),

$$\lambda_p - \lambda_{p,h} = \frac{1}{\langle u_{p,h}, \Pi_{2h} u_p\rangle_{0,\Omega_h}} \Big\{ \lambda_p \langle \widetilde{u}_p - \Pi_{2h} u_p, \widetilde{u}_p - u_{p,h}\rangle_{0,\Omega_h} + \lambda_p (\|u_p\|^2_{0,\omega_h} - \|\widetilde{u}_p\|^2_{0,\epsilon_h}) + \big[b(\Psi_p, u_p) - \widetilde{b_h}(\widetilde{\Psi}_p, \widetilde{u}_p)\big] + \big[\widetilde{b_h}(\widetilde{\Psi}_p, \widetilde{u}_p) - b_h^{\mathrm{NI}}(\underline{\Pi}_{1h}\Psi_p, \Pi_{2h} u_p)\big] \Big\}. \tag{4.60}$$

$$\text{But } \widetilde{b_h}(\widetilde{\Psi}_p, \widetilde{u}_p) - b_h^{\mathrm{NI}}(\underline{\Pi}_{1h}\Psi_p, \Pi_{2h} u_p) = \widetilde{b_h}(\widetilde{\Psi}_p - \underline{\Pi}_{1h}\Psi_p, \widetilde{u}_p - \chi_h) + \big[\widetilde{b_h}(\widetilde{\Psi}_p, \chi_h) - \widetilde{b_h}(\underline{\Pi}_{1h}\Psi_p, \chi_h)\big] + \big[\widetilde{b_h}(\underline{\Pi}_{1h}\Psi_p, \widetilde{u}_p) - b_h^{\mathrm{NI}}(\underline{\Pi}_{1h}\Psi_p, \Pi_{2h} u_p)\big] \ \ \forall \chi_h \in W_h. \tag{4.61}$$

Rewriting one by one the expressions in square brackets in (4.61) using (2.20) and (3.21):

$$\begin{aligned}\big[\widetilde{b_h}(\widetilde{\Psi}_p, \chi_h) - \widetilde{b_h}(\underline{\Pi}_{1h}\Psi_p, \chi_h)\big] &= \Big\{ \widetilde{b_h}(\widetilde{\Psi}_p, \chi_h - \widetilde{u}_p) + \widetilde{b_h}(\widetilde{\Psi}_p, \widetilde{u}_p) - b_h^{\mathrm{NI}}(\underline{\Pi}_{1h}\Psi_p, \chi_h) \\ &\quad + \big[b_h^{\mathrm{NI}}(\underline{\Pi}_{1h}\Psi_p, \chi_h) - \widetilde{b_h}(\underline{\Pi}_{1h}\Psi_p, \chi_h)\big] \Big\} \\ &= \Big\{ \widetilde{b_h}(\widetilde{\Psi}_p, \chi_h - \widetilde{u}_p) + \lambda_p \langle \widetilde{u}_p, \chi_h - \widetilde{u}_p\rangle_{0,\Omega_h} + \big[\widetilde{b_h}(\widetilde{\Psi}_p, \widetilde{u}_p) - b(\Psi_p, u_p)\big] \\ &\quad + \lambda_p \big[\|\widetilde{u}_p\|^2_{0,\epsilon_h} - \|u_p\|^2_{0,\omega_h}\big] + \big[b_h^{\mathrm{NI}}(\underline{\Pi}_{1h}\Psi_p, \chi_h) - \widetilde{b_h}(\underline{\Pi}_{1h}\Psi_p, \chi_h)\big] \Big\}. \end{aligned} \tag{4.62}$$

Let $\widetilde{\underline{\Pi}_{1h}\Psi_p} \in \widetilde{\mathbf{V}}_h \subset \widetilde{\mathbf{V}}$ be an extension to $\widetilde{\Omega}$ of $\underline{\Pi}_{1h}\Psi_p \in \mathbf{V}_h$ defined in (3.7)–(3.9) with the help of Corollary 3.1. Then, using (3.21) and (2.20), we have

$$\begin{aligned}\widetilde{b_h}(\underline{\Pi}_{1h}\Psi_p, \widetilde{u}_p) - b_h^{\mathrm{NI}}(\underline{\Pi}_{1h}\Psi_p, \Pi_{2h} u_p) &= \tilde{A}_h(\widetilde{\Psi}_p - \underline{\Pi}_{1h}\Psi_p, \widetilde{\Psi}_p - \underline{\Pi}_{1h}\Psi_p) + \big[-\widetilde{A_h}(\widetilde{\Psi}_p, \widetilde{\Psi}_p) + \widetilde{A_h}(\underline{\Pi}_{1h}\Psi_p, \widetilde{\Psi}_p)\big] \\ &\quad + \big[\tilde{A}_h(\widetilde{\Psi}_p, \underline{\Pi}_{1h}\Psi_p) - A(\Psi_p, \widetilde{\underline{\Pi}_{1h}\Psi_p})\big] + \big[A_h^{\mathrm{NI}}(\underline{\Pi}_{1h}\Psi_p, \underline{\Pi}_{1h}\Psi_p) - \widetilde{A_h}(\underline{\Pi}_{1h}\Psi_p, \underline{\Pi}_{1h}\Psi_p)\big] \\ &\quad + \big[\widetilde{b_h}(\underline{\Pi}_{1h}\Psi_p, \widetilde{u}_p) - b(\widetilde{\underline{\Pi}_{1h}\Psi_p}, u_p)\big] \text{ in which} \end{aligned} \tag{4.63}$$

$$\begin{aligned}\big[-\tilde{A}_h(\widetilde{\Psi}_p, \widetilde{\Psi}_p) + \tilde{A}_h(\underline{\Pi}_{1h}\Psi_p, \widetilde{\Psi}_p)\big] &= \widetilde{b_h}(\widetilde{\Psi}_p - \underline{\Pi}_{1h}\Psi_p, \widetilde{u}_p - \chi_h) + \big[\widetilde{b_h}(\widetilde{\Psi}_p, \chi_h) - \widetilde{b_h}(\underline{\Pi}_{1h}\Psi_p, \chi_h)\big] \\ &\quad + \big[b(\Psi_p, u_p) - \widetilde{b_h}(\widetilde{\Psi}_p, \widetilde{u}_p)\big] + \big[\widetilde{b_h}(\underline{\Pi}_{1h}\Psi_p, \widetilde{u}_p) - b(\widetilde{\underline{\Pi}_{1h}\Psi_p}, u_p)\big] \\ &\quad + \big[A(\Psi_p, \Psi_p) - \tilde{A}_h(\widetilde{\Psi}_p, \widetilde{\Psi}_p)\big] + \big[\tilde{A}_h(\underline{\Pi}_{1h}\Psi_p, \widetilde{\Psi}_p) - A(\widetilde{\underline{\Pi}_{1h}\Psi_p}, \Psi_p)\big] \quad \forall \chi_h \in W_h \end{aligned} \tag{4.64}$$

with the term $\big[\widetilde{b_h}(\widetilde{\Psi}_p,\chi_h)-\widetilde{b_h}(\underline{\Pi}_{1h}\Psi_p,\chi_h)\big]$ in (4.64) being the same one considered earlier in (4.62). Hence, substituting (4.62)–(4.64) in (4.61), using it in (4.60) and applying triangular inequality, we have

$$
\begin{aligned}
|\lambda_p-\lambda_{p,h}| \le \frac{1}{|\langle u_{p,h},\Pi_{2h}u_p\rangle_{0,\Omega_h}|}\Bigg\{&\lambda_p|\langle\widetilde{u}_p-\Pi_{2h}u_p,\widetilde{u}_p-u_{p,h}\rangle_{0,\Omega_h}|+\lambda_p(\|u_p\|^2_{0,\omega_h}+\|\widetilde{u}_p\|^2_{0,\epsilon_h})\\
&+2\Big(\big|\tilde{A}_h(\widetilde{\Psi}_p,\underline{\Pi}_{1h}\Psi_p)-A(\Psi_p,\widetilde{\underline{\Pi}_{1h}\Psi_p})\big|+\big|\widetilde{b_h}(\widetilde{\Psi}_p-\underline{\Pi}_{1h}\Psi_p,\widetilde{u}_p-\chi_h)\big|+\big|\widetilde{b_h}(\underline{\Pi}_{1h}\Psi_p,\widetilde{u}_p)-b(\widetilde{\underline{\Pi}_{1h}\Psi_p},u_p)\big|\\
&+\big|{b_h}^{\mathrm{NI}}(\underline{\Pi}_{1h}\Psi_p,\chi_h)-\widetilde{b_h}(\underline{\Pi}_{1h}\Psi_p,\chi_h)\big|+\big|\widetilde{b_h}(\widetilde{\Psi}_p,\chi_h-\widetilde{u}_p)\big|+\lambda_p\big|\langle\widetilde{u}_p,\chi_h-\widetilde{u}_p\rangle_{0,\Omega_h}\big|\Big)\\
&+\big|A_h^{\mathrm{NI}}(\underline{\Pi}_{1h}\Psi_p,\underline{\Pi}_{1h}\Psi_p)-\tilde{A}_h(\underline{\Pi}_{1h}\Psi_p,\underline{\Pi}_{1h}\Psi_p)\big|+\big|A(\Psi_p,\Psi_p)-\tilde{A}_h(\widetilde{\Psi}_p,\widetilde{\Psi}_p)\big|\\
&+\big|\widetilde{A}(\widetilde{\Psi}_p-\underline{\Pi}_{1h}\Psi_p,\widetilde{\Psi}_p-\underline{\Pi}_{1h}\Psi_p)\big|\Bigg\}\quad\forall\chi_h\in W_h\ \ \text{(see [8] for details).}
\end{aligned}\tag{4.65}
$$

First of all, we will prove that $\langle u_{p,h},\Pi_{2h}u_p\rangle_{0,\Omega_h}\to 1$ as $h\to 0$. In fact,

$$
|\langle u_p,u_p\rangle_{0,\Omega}-\langle u_{p,h},\Pi_{2h}u_p\rangle_{0,\Omega_h}|\le|\int\limits_{\Omega\cap\Omega_h} u_p^2-u_{p,h}\Pi_{2h}u_p\ \ \mathrm{d}x|+\int\limits_{\omega_h}u_p^2\mathrm{d}x+|\int\limits_{\epsilon_h}(u_{p,h})(\Pi_{2h}u_p)\mathrm{d}x|. \tag{4.66}
$$

But using (4.5), $\int\limits_{\omega_h}u_p^2\mathrm{d}x=\|u_p\|^2_{0,\omega_h}\le Ch^3\|\widetilde{u}_p\|^2_{1,\widetilde{\Omega}}\to 0$ as $h\to 0\Longrightarrow \lim_{h\to 0}\int\limits_{\omega_h}u_p^2\mathrm{d}x=0$.
$|\int\limits_{\epsilon_h}(u_{p,h})(\Pi_{2h}u_p)\mathrm{d}x|\le\|u_{p,h}\|_{0,\epsilon_h}\|\Pi_{2h}u_p\|_{0,\epsilon_h}\to 0$ as $h\to 0$. (using (4.5) and (4.36))

$$
\begin{aligned}
\text{and }|\int\limits_{\Omega\cap\Omega_h}(u_p^2-u_{p,h}\Pi_{2h}u_p)\mathrm{d}x| &\le |\int\limits_{\Omega\cap\Omega_h}(u_p^2-u_p.u_{p,h})\ \mathrm{d}x|+|\int\limits_{\Omega\cap\Omega_h}u_pu_{p,h}-u_{p,h}\Pi_{2h}u_p\ \mathrm{d}x|\\
&\le\|u_p\|_{0,\Omega\cap\Omega_h}\|u_p-u_{p,h}\|_{0,\Omega\cap\Omega_h}+\|u_{p,h}\|_{0,\Omega\cap\Omega_h}\|u_p-\Pi_{2h}u_p\|_{0,\Omega\cap\Omega_h}\\
&\le\big[\|u_p\|_{0,\Omega}\|\widetilde{u}_p-u_{p,h}\|_{0,\Omega_h}+\|u_{p,h}\|_{0,\Omega_h}\|\widetilde{u}_p-\Pi_{2h}u_p\|_{0,\Omega_h}\big]\to 0\text{ as }h\to 0,
\end{aligned}
$$

(from (4.36) and (3.42)). Hence, from (4.66),

$$
\begin{aligned}
\lim_{h\to 0}|\langle u_p,u_p\rangle_{0,\Omega}-\langle u_{p,h},\Pi_{2h}u_p\rangle_{0,\Omega_h}|=0 &\Longrightarrow \lim_{h\to 0}\langle u_{p,h},\Pi_{2h}u_p\rangle_{0,\Omega_h}=1\\
&\Longrightarrow \exists h_0\in]0,1[\text{such that }\forall h\in]0,h_0[|\langle u_{p,h},\Pi_{2h}u_p\rangle_{0,\Omega_h}|>1/2\\
&\Longrightarrow \frac{1}{|\langle u_{p,h},\Pi_{2h}u_p\rangle_{0,\Omega_h}|}<2.
\end{aligned}\tag{4.67}
$$

Using (3.42) and (4.36), we have

$$\bullet\lambda_p|\langle\widetilde{u}_p-\Pi_{2h}u_p,\widetilde{u}_p-u_{p,h}\rangle_{0,\Omega_h}|\leq\lambda_p\|\widetilde{u}_p-\Pi_{2h}u_p\|_{0,\Omega_h}\|\widetilde{u}_p-u_{p,h}\|_{0,\Omega_h}$$
$$\leq C\lambda_p h^2\Bigg\{(\|\widetilde{u}_p\|_{3,\widetilde{\Omega}}+\|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}})\Big[h^2\|\widetilde{u}_p\|^2_{1,\widetilde{\Omega}}+(\|\widetilde{u}_p\|_{3,\widetilde{\Omega}}$$
$$+\|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}})(1+2\frac{\lambda_p}{d_p}+(2+h)\|\widetilde{u}_p\|_{3,\widetilde{\Omega}}+h\|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}})\Big]\Bigg\}; \quad (4.68)$$

$$\bullet\lambda_p(\|\widetilde{u}_p\|^2_{0,\epsilon_h}+\|u_p\|^2_{0,\omega_h})\leq C\lambda_p h^3\|\widetilde{u}_p\|^2_{1,\widetilde{\Omega}}\ \ (\text{using}(4.5)); \quad (4.69)$$

$$\bullet|A(\Psi_p,\widetilde{\underline{\Pi}_{1h}\Psi_p})-\tilde{A}_h(\widetilde{\Psi}_p,\underline{\Pi}_{1h}\Psi_p)|\leq Ch^3\|\widetilde{u}_p\|_{3,\widetilde{\Omega}}\|\underline{\Pi}_{1h}\Psi_p\|_{1,\Omega_h}\quad(\text{see (II) in Prop. 4.2}). \quad (4.70)$$

Estimate for $\|\underline{\Pi}_{1h}\Psi_p\|_{1,\Omega_h}$ is given now: $\|\underline{\Pi}_{1h}\Psi_p\|_{1,\Omega_h}\leq\|\underline{\Pi}_{1h}\Psi_p-\widetilde{\Psi}_p\|_{1,\Omega_h}+\|\widetilde{\Psi}_p\|_{1,\Omega_h}$
But from (4.11), for $\widetilde{\Psi}_p\in\widetilde{\mathbf{V}}$, $\exists(\Theta_h)_p\in\mathbf{V}_h$ such that $\|\widetilde{\Psi}_p-(\Theta_h)_p\|_{r,\Omega_h}\leq Ch^{1-r}\|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}}$ $(r=0,1)$ with $\|\widetilde{\Psi}_p-\underline{\Pi}_{1h}\Psi_p\|_{1,\Omega_h}\leq\|\underline{\Pi}_{1h}\Psi_p-(\Theta_h)_p\|_{1,\Omega_h}+\|\widetilde{\Psi}_p-(\Theta_h)_p\|_{1,\Omega_h}\leq C/h\|\underline{\Pi}_{1h}\Psi_p-(\Theta_h)_p\|_{0,\Omega_h}+C\|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}}$

and $\|\underline{\Pi}_{1h}\Psi_p-(\Theta_h)_p\|_{0,\Omega_h}\leq\|\underline{\Pi}_{1h}\Psi_p-\widetilde{\Psi}_p\|_{0,\Omega_h}+\|\widetilde{\Psi}_p-(\Theta_h)_p\|_{0,\Omega_h}$. Then, using (3.40),

$$\|\underline{\Pi}_{1h}\Psi_p-(\Theta_h)_p\|_{0,\Omega_h}\leq Ch(\|\widetilde{u}_p\|_{3,\widetilde{\Omega}}+\|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}}),$$
$$\Longrightarrow\|\underline{\Pi}_{1h}\Psi_p-\widetilde{\Psi}_p\|_{1,\Omega_h}\leq C(\|\widetilde{u}_p\|_{3,\widetilde{\Omega}}+\|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}}),\ \ \|\underline{\Pi}_{1h}\Psi_p\|_{1,\Omega_h}\leq C(\|\widetilde{u}_p\|_{3,\widetilde{\Omega}}+\|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}}). \quad (4.71)$$

Finally from (4.70) and (4.71), we have

$$\bullet\ \ |A(\Psi_p,\widetilde{\underline{\Pi}_{1h}\Psi_p})-\tilde{A}_h(\widetilde{\Psi}_p,\underline{\Pi}_{1h}\Psi_p)|\leq Ch^3\|\widetilde{u}_p\|_{3,\widetilde{\Omega}}\big(\|\widetilde{u}_p\|_{3,\widetilde{\Omega}}+\|\Psi_p\|_{1,\widetilde{\Omega}}\big). \quad (4.72)$$

Using the continuity of $\tilde{A}_h(\cdot,\cdot)$, $\widetilde{b}_h(\cdot,\cdot)$, Proposition 4.2, estimates (3.42), (4.5), (4.9), (4.12), (4.37) and (4.71), we have: For $\chi_h=\mathcal{P}_h\widetilde{u}_p\in W_h$

- $|\widetilde{b}_h(\widetilde{\Psi}_p-\underline{\Pi}_{1h}\Psi_p,\widetilde{u}_p-\mathcal{P}_h\widetilde{u}_p)|\leq Ch^2\|\widetilde{u}_p\|_{3,\widetilde{\Omega}}(\|\widetilde{u}_p\|_{3,\widetilde{\Omega}}+\|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}});$ (4.73)
- $|\widetilde{b}_h(\underline{\Pi}_{1h}\Psi_p,\widetilde{u}_p)-b(\widetilde{\underline{\Pi}_{1h}\Psi_p},u_p)|\ \ \leq Ch^3(\|\widetilde{u}_p\|_{3,\widetilde{\Omega}}+\|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}})\|\widetilde{u}_p\|_{2,\widetilde{\Omega}};$ (4.74)
- $|b_h^{\mathrm{NI}}(\underline{\Pi}_{1h}\Psi_p,\mathcal{P}_h\widetilde{u}_p)-\widetilde{b}_h(\underline{\Pi}_{1h}\Psi_p,\mathcal{P}_h\widetilde{u}_p)|\leq Ch^2(\|\widetilde{u}_p\|_{3,\widetilde{\Omega}}+\|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}})\|\widetilde{u}_p\|_{3,\widetilde{\Omega}};$ (4.75)
- $|\widetilde{b}_h(\widetilde{\Psi}_p,\mathcal{P}_h\widetilde{u}_p-\widetilde{u}_p)|\leq Ch^2\|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}}\|\widetilde{u}_p\|_{3,\widetilde{\Omega}};$ (4.76)
- $\lambda_p|\langle\widetilde{u}_p,\mathcal{P}_h\widetilde{u}_p-\widetilde{u}_p\rangle_{0,\Omega_h}|\leq C\lambda_p h^3\|\widetilde{u}_p\|^2_{3,\widetilde{\Omega}};$ (4.77)
- $|A(\Psi_p,\Psi_p)-\widetilde{A}_h(\widetilde{\Psi}_p,\widetilde{\Psi}_p)|\leq Ch^3\|\widetilde{u}_p\|_{3,\widetilde{\Omega}}\|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}};$ (4.78)
- $|A_h^{\mathrm{NI}}(\underline{\Pi}_{1h}\Psi_p,\underline{\Pi}_{1h}\Psi_p)-\widetilde{A}_h(\underline{\Pi}_{1h}\Psi_p,\underline{\Pi}_{1h}\Psi_p)|\leq Ch^2(\|\widetilde{u}_p\|_{3,\widetilde{\Omega}}+\|\Psi_p\|_{1,\widetilde{\Omega}})^2;$ (4.79)
- $|\widetilde{A}_h(\widetilde{\Psi}_p-\underline{\Pi}_{1h}\Psi_p,\widetilde{\Psi}_p-\underline{\Pi}_{1h}\Psi_p)|\leq Ch^2(\|\widetilde{u}_p\|_{3,\widetilde{\Omega}}+\|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}})^2.$ (4.80)

Then, from (4.65) and the estimates in (4.67)–(4.80), we get the required result: $|\lambda_p-\lambda_{p,h}|=O(h^2)$. □

Theorem 4.5. *Under the assumptions that Theorems 4.2, 4.3 and 4.4 hold, and* λ_p *(resp.* $\lambda_{p,h}$*) is a simple eigenvalue of* $(\mathbf{Q^E})$ *(resp.* $(\mathbf{Q^E_h})$*),* $1\leq p\leq N_h=\dim W_h$,

$$\|\widetilde{\Psi}_p-\Psi_{p,h}\|_{0,\Omega_h}=O(h),\ \ \|\widetilde{u}_p-u_{p,h}\|_{1,\Omega_h}=O(h). \quad (4.81)$$

Proof. From (4.37) and (4.56), we have

$$\|\widetilde{\Psi}_p - \Psi_{p,h}\|_{0,\Omega_h} \leq Ch\Bigg(\Bigg[(\|\widetilde{u}_p\|_{3,\widetilde{\Omega}} + \|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}})\Bigg(1 + \frac{2\lambda_p\sqrt{2}}{\sqrt{\lambda_1}\,\bar{\alpha}_0}\Bigg\{1 + 2\frac{\lambda_p}{d_p} + (h+2)\|\widetilde{u}_p\|_{3,\widetilde{\Omega}} + h\|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}}\Bigg\}\Bigg) + \frac{2\sqrt{2}\lambda_p}{\sqrt{\lambda_1}\,\bar{\alpha}_0}h^2\|\widetilde{u}_p\|^2_{1,\widetilde{\Omega}}\Bigg] + \frac{\sqrt{2}}{\sqrt{\lambda_1}\bar{\alpha}_0}h\Bigg\{\cdots\Bigg\}\Bigg) = O(h) \tag{4.82}$$

where $\Big\{\cdots\Big\}$ denotes the expression within the curly brackets on the right-hand side of (4.56). Putting $\chi_h = \mathcal{P}_h\widetilde{u}_p$ in (4.1), we now find the estimates for the terms on the right-hand side of (4.1).

$$\text{Indeed, from (4.9), } \|\widetilde{u}_p - \mathcal{P}_h\widetilde{u}_p\|_{1,\Omega_h} \leq Ch^2\|\widetilde{u}_p\|_{3,\widetilde{\Omega}}. \tag{4.83}$$

Estimate for $\|\widetilde{\Psi}_p - \Psi_{p,h}\|_{0,\Omega_h}$ is obtained from (4.82). From Proposition 4.2, we have

- $$\sup_{\Phi_h\in\mathbf{V}_h-\{0\}}\frac{\big|b_h^{\mathrm{NI}}(\Phi_h,\mathcal{P}_h\widetilde{u}_p) - \widetilde{b_h}(\Phi_h,\mathcal{P}_h\widetilde{u}_p)\big|}{\|\Phi_h\|_{1,\Omega_h}} \leq \sup_{\Phi_h\in\mathbf{V}_h-\{0\}}\frac{Ch^2\|\Phi_h\|_{1,\Omega_h}\|\mathcal{P}_h\widetilde{u}_p\|_{1,\Omega_h}}{\|\Phi_h\|_{1,\Omega_h}} \leq Ch^2\|\widetilde{u}_p\|_{3,\widetilde{\Omega}}; \tag{4.84}$$
- $$\sup_{\Phi_h\in\mathbf{V}_h-\{0\}}\frac{\big|\tilde{A}_h(\widetilde{\Psi}_p,\Phi_h) - A(\Psi_p,\widetilde{\Phi}_h)\big|}{\|\Phi_h\|_{1,\Omega_h}} \leq Ch^3\|\widetilde{u}_p\|_{3,\widetilde{\Omega}} \tag{4.85}$$
- $$\sup_{\Phi_h\in\mathbf{V}_h-\{0\}}\frac{\big|A_h^{\mathrm{NI}}(\Psi_{p,h},\Phi_h) - \widetilde{A_h}(\Psi_{p,h},\Phi_h)\big|}{\|\Phi_h\|_{1,\Omega_h}} \leq \sup_{\Phi_h\in\mathbf{V}_h-\{0\}}\frac{Ch^2\|A\|_{2,\infty,\widetilde{\Omega}}\|\Psi_{p,h}\|_{0,\Omega_h}\|\Phi_h\|_{0,\Omega_h}}{\|\Phi_h\|_{1,\Omega_h}} \quad \text{(From (4.12))}$$
$$\leq Ch^2(\|\widetilde{\Psi}_p - \Psi_{p,h}\|_{0,\Omega_h} + \|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}}); \tag{4.86}$$
- $$\sup_{\Phi_h\in\mathbf{V}_h-\{0\}}\frac{\big|\widetilde{b_h}(\Phi_h,\widetilde{u}_p) - b(\widetilde{\Phi}_h,u_p)\big|}{\|\Phi_h\|_{1,\Omega_h}} \leq \sup_{\Phi_h\in\mathbf{V}_h-\{0\}}\frac{Ch^3\|\Phi_h\|_{1,\Omega_h}\|\widetilde{u}_p\|_{2,\widetilde{\Omega}}}{\|\Phi_h\|_{1,\Omega_h}} \leq Ch^3\|\widetilde{u}_p\|_{2,\widetilde{\Omega}}. \tag{4.87}$$

Substituting (4.82)–(4.87) in (4.1) with $\chi_h = \mathcal{P}_h\widetilde{u}_p \in W_h$, we have

$$\|\widetilde{u}_p - u_{p,h}\|_{1,\Omega_h} \leq Ch\Bigg[h\|\widetilde{u}_p\|_{3,\widetilde{\Omega}} + h^2\|\widetilde{u}_p\|_{3,\widetilde{\Omega}} + (1+h)\Bigg(\Bigg[(\|\widetilde{u}_p\|_{3,\widetilde{\Omega}} + \|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}})\Bigg(1 + \frac{2\lambda_p\sqrt{2}}{\sqrt{\lambda_1}\,\bar{\alpha}_0}\Bigg\{1 + 2\frac{\lambda_p}{d_p}(h+2)\|\widetilde{u}_p\|_{3,\widetilde{\Omega}} + h\|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}}\Bigg\}\Bigg) + \frac{2\sqrt{2}\lambda_p}{\sqrt{\lambda_1}\,\bar{\alpha}_0}h^2\|\widetilde{u}_p\|^2_{1,\widetilde{\Omega}}\Bigg] + \frac{\sqrt{2}}{\sqrt{\lambda_1}\bar{\alpha}_0}h\Bigg\{\cdots\Bigg\}\Bigg) + h\|\widetilde{\Psi}_p\|_{1,\widetilde{\Omega}}\Bigg],$$

where $\Big\{\cdots\Big\}$ denotes the expression within the curly brackets on the right-hand side of (4.56), from which the result follows.

Remark 5.3. As in the case of usual elliptic eigenvalue problems, the exponent of 'h' in (4.56) is *optimal* in the sense that it is *twice* the order of convergence for the corresponding source/steady state problem, *i.e.* for $\|\widetilde{u} - u_h\|_{1,\Omega_h} = O(h)$, $\|\widetilde{\Psi} - \Psi_h\|_{0,\Omega_h} = O(h)$ of the corresponding source /steady state problem [10], $|\lambda_p - \lambda_{p,h}| = O(h^2)$.

Remark 5.4. In the case of eigenvalue problems, the estimates for simple eigenvalues and corresponding eigenelements: $|\lambda_p - \lambda_{p,h}| = O(h^2)$, $\|\widetilde{u}_p - u_{p,h}\|_{1,\Omega_h} = O(h)$ and $\|\widetilde{\Psi}_p - \Psi_{p,h}\|_{0,\Omega_h} = O(h)$ have been obtained in (4.56) and (4.81) respectively under the assumptions that

(i) coefficients A_{ijkl} have *additional regularity* (*i.e.* $A_{ijkl} \in W^{2,\infty}(\widetilde{\Omega})\ \forall i,j,k,l = 1,2$), and

(ii) the quadrature scheme (3.14) with i=1 having *higher* algebraic degree of accuracy (*i.e.* exact for $P_6(\hat{T})$), has been used in the definition (3.19) of $A_h^{\rm NI}(\cdot,\cdot)$, since the error estimates of the same order *i.e.* $\|\widetilde{u} - u_h\|_{1,\Omega_h} = O(h)$, $\|\widetilde{\Psi} - \Psi_h\|_{0,\Omega_h} = O(h)$ have been obtained in [10] for the corresponding *source problem* for (i') $A_{ijkl} \in W^{1,\infty}(\widetilde{\Omega})$ $\forall i,j,k,l = 1,2$ and (i'i') the quadrature scheme (3.14) with i=1 exact for $P_4(\hat{T})$, which has been used in the definition (3.19) of $A_h^{\rm NI}(\cdot,\cdot)$ (see Th. 5.2 of [10]).

But these estimates $\|\widetilde{u}_p - u_{p,h}\|_{1,\Omega_h} = O(h)$, $\|\widetilde{\Psi}_p - \Psi_{p,h}\|_{0,\Omega_h} = O(h)$ in (4.81) for the eigenvalue problem (resp. $\|\widetilde{u} - u_h\|_{1,\Omega_h} = O(h)$, $\|\widetilde{\Psi} - \Psi_h\|_{0,\Omega_h} = O(h)$ for the corresponding source problem in [10]) *cannot* be improved upon by assuming still more regularity *i.e.* $A_{ijkl} \in W^{m,\infty}(\widetilde{\Omega})$ with $m > 2$ (resp. $W^{m,\infty}(\widetilde{\Omega})$ with $m > 1$) and using quadrature scheme (3.14) with i=1 exact for $P_m(\hat{T})$ with $m > 6$ (resp. $P_m(\hat{T})$ with $m > 4$) in the definition (3.19) of $A_h^{\rm NI}(\cdot,\cdot)$.

5. Numerical examples

In this section, we would consider numerical examples on eigenvalue problems defined in (2.1), the coefficients a_{ijkl} for which satisfy (**A1**–**A2**). The convex domains Ω with curved boundary considered are approximated by a polygonal boundary $\Gamma_h^{\rm pol}$ and a curved boundary Γ_h constructed with the help of an isoparametric mapping. The fundamental and a few higher frequencies and mode shapes of a class of orthotropic plates with clamped boundary conditions are computed and the results obtained are compared with the existing results.
For the plate bending operator Λ, the eigenvalue problem (2.1) is obtained from the equation of motion for the small transverse displacement $UU(x_1, x_2; t)$ of the vibrating elastic plate under consideration:

$$\Lambda U + \rho \frac{\partial^2 U}{\partial t^2} = 0 \qquad \forall ((x_1, x_2); t) \in \Omega \times]0, T] \tag{5.1}$$

with $U|_\Gamma = 0, \dfrac{\partial U}{\partial n}|_\Gamma = 0 \quad \forall t \in]0, T]$, ρ being the mass density of the elastic plate per unit area measure of $\bar{\Omega}$, when free natural vibrations are assumed and the motion is defined by:

$$U(x_1, x_2; t) = u(x_1, x_2) \cos \omega t, \tag{5.2}$$

ω being the circular frequency expressed in radians/unit time, *i.e.* a substitution of (5.2) into (5.1) will yield (2.1) with $\lambda = \rho\omega^2$.
In the practical applications (examples considered below), dimensionless coordinates are introduced and instead of $\lambda = \rho\omega^2$, some new parameter of convenience which will depend on ρ, ω, characteristic plate size parameter, flexural rigidity of the plate etc will be introduced and **will still be denoted by the same notation** λ by giving its new definition **without deduction**,for which we refer to [26].

• For constant coefficients A_{ijkl} (or equivalently a_{ijkl}), which will be considered in the examples, introducing suitable canonical bases $\{\Phi_h^i\}_{i=1}^{3\,N_1}$ in $\mathbf{V}_h$ and $\{\chi_h^i\}_{i=1}^{N_0}$ in W_h, the isoparametric mixed finite element eigenvalue problem $(\mathbf{Q_h^E})$ can be reduced to the following problem in matrix form (see [31] for details):
Find $(\lambda_h; (\underline{\alpha}; \underline{\beta})) = (\lambda_h; (\underline{\alpha}^1, \underline{\alpha}^2, \underline{\alpha}^3, \underline{\beta})) \in \mathbb{R}^+ \times \mathbb{R}^{3\,N_1 + N_0}$ such that

$$\begin{aligned} c_{11}[A]\underline{\alpha}^1 + c_{12}[A]\underline{\alpha}^2 + c_{13}[A]\underline{\alpha}^3 + [B_1]\underline{\beta} &= \mathbf{0} \\ c_{12}[A]\underline{\alpha}^1 + c_{22}[A]\underline{\alpha}^2 + c_{23}[A]\underline{\alpha}^3 + [B_2]\underline{\beta} &= \mathbf{0} \\ c_{13}[A]\underline{\alpha}^1 + c_{23}[A]\underline{\alpha}^2 + c_{33}[A]\underline{\alpha}^3 + [B_3]\underline{\beta} &= \mathbf{0} \\ [B_1]^t\underline{\alpha}^1 + [B_2]^t\underline{\alpha}^2 + [B_3]^t\underline{\alpha}^3 &= \lambda_h\,[M]\,\underline{\beta} \end{aligned} \tag{5.3}$$

where $[A]_{N_1\times N_1}$ is a symmetric, positive-definite matrix of order N_1 [31]; $c_{ij}\in\mathbb{R}$ with $c_{12}=0,\ c_{23}=0$ **for the class of Orthotropic Plates considered in the examples**, $\underline{\alpha}^i\in\mathbb{R}^{N_1}$, $\underline{\beta}\in\mathbb{R}^{N_0}$ such that $\Psi_h=\sum_{j=1}^{3N_1}\alpha_j\Phi_h^j,\ u_h=\sum_{j=1}^{N_0}\beta_j\chi_h^j$ with $\underline{\alpha}^1=(\alpha_j)_{j=1}^{N_1}$, $\underline{\alpha}^2=(\alpha_j)_{j=N_1+1}^{2\,N_1}$, $\underline{\alpha}^3=(\alpha_j)_{j=2\,N_1+1}^{3\,N_1}$, $\underline{\beta}=(\beta_i)_{i=1}^{N_0}$, $[B_i]_{N_1\times N_0}$ is a rectangular matrix of size $N_1\times N_0$ with its transpose denoted by $[B_i]^t$; $[M]_{N_0\times N_0}$ is the symmetric, positive-definite global mass matrix of order $N_0\times N_0$ got after assembling the element mass matrices $[M_T]$.

Then, $\underline{\alpha}^1,\ \underline{\alpha}^2,\ \underline{\alpha}^3$ can be eliminated from the first, second and third equations in (5.3) and substituting the expressions for $\underline{\alpha}^i$ in the fourth equation in (5.3), we get

$$[K]_{N_0\times N_0}\ \ \underline{\beta}=\lambda_h\,[M]_{N_0\times N_0}\,\underline{\beta}, \tag{5.4}$$

where $[K]$ is the symmetric, positive-definite, global stiffness matrix of order $N_0\times N_0$.
Solving (5.4) for $(\lambda_h;\ \underline{\beta})$, we can find $\underline{\alpha}^i$ using the expression used in the elimination of $\underline{\alpha}^i$, $1\le i\le 3$.

• (5.4) has been solved by Subspace Iteration Method, although Lanczos method can also be efficiently used.

Example: Clamped Orthotropic Elliptic Plate Problem. The coefficients a_{ijkl} for the orthotropic case are:

$$\begin{aligned}&a_{iiii}=D_{ii};\ a_{1122}=a_{2211}=D_{12}=\nu_1D_{22}=\nu_2D_{11};\\&a_{1212}=a_{2121}=a_{2112}=a_{1221}=D_t,\ a_{ijkl}=0 \text{ otherwise,}\end{aligned} \tag{5.5}$$

$$\begin{aligned}\text{where }\ D_{ii}&=\frac{E_it^3}{12(1-\nu_1\nu_2)}\ (i=1,2)\\D_t&=\frac{Gh^3}{12}>0,\ \ H=\nu_2D_1+2D_t\\G&=\frac{E_1E_2}{E_1+(1+2\nu_1)E_2}>0,\ E_1\nu_2=E_2\nu_1,\end{aligned}$$

E_i and ν_i are Young's moduli and Poisson's coefficients respectively, $t(x_1,x_2)$ being the thickness function.
We consider the following cases where the Poisson's coefficient $\nu_1=1/3$ and the flexural rigidities are given by:

Case I: $D_{12}/D_{22}=1/3,\ D_{11}/D_{22}=1$
Case II: $D_{12}/D_{22}=1/3,\ D_{11}/D_{22}=1/3$
Case III: $D_{12}/D_{22}=1,\quad D_{11}/D_{22}=1$
Case IV: $D_{12}/D_{22}=1,\quad D_{11}/D_{22}=1/3$.

The eigenvalue problem (2.1) with $\lambda^{\mathbf{2}}=\omega\mathbf{a^2}\sqrt{\rho\,\mathbf{t}/\mathbf{D_{22}}}$ corresponding to the natural vibrations of the **clamped orthotropic elliptic plate** with $b/a=0.5$, 'a' being the semi-major axis and 'b' being the semi-minor axis is considered. The first few eigenvalues and the corresponding eigenvectors for both polygonal and curved approximations have been computed and only the eigenvalues, which are compared with those given in [19], are shown in Tables I–IV below. □

TABLE 1. Case I: $D_{12}/D_{22} = 1/3, D_{11}/D_{22} = 1, \nu = 1/3, b/a = 0.5$.

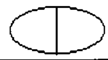
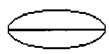
Eigenvalues	*Polygonal Approximation*	*Isoparametric Approximation*	*[19]*	*Nature of the Mode*	*Nodal Pattern*
$\lambda_{1,h}$	27.90	27.41	27.38	Doubly Symmetric	
$\lambda_{2,h}$	40.45	39.68	39.49	Symmetric Antisymmetric	
$\lambda_{3,h}$	57.73	56.55	55.97	Second Doubly Symmetric	
$\lambda_{4,h}$	71.32	70.07	69.87	Antisymmetric Symmetric	
$\lambda_{5,h}$	80.00	78.25	76.99	Second Symmetric Antisymmetric	

TABLE 2. Case II: $D_{12}/D_{22} = 1/3, D_{11}/D_{22} = 1/3, \nu = 1/3, b/a = 0.5$.

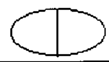
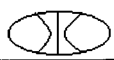
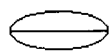
Eigenvalues	*Polygonal Approximation*	*Isoparametric Approximation*	*[19]*	*Nature of the Mode*	*Nodal Pattern*
$\lambda_{1,h}$	27.24	26.76	26.73	Doubly Symmetric	
$\lambda_{2,h}$	37.33	36.64	36.43	Symmetric Antisymmetric	
$\lambda_{3,h}$	49.78	48.82	48.23	Second Doubly Symmetric	
$\lambda_{4,h}$	64.91	63.55	62.30	Second Symmetric Antisymmetric	
$\lambda_{5,h}$	70.81	69.59	69.38	Antisymmetric Symmetric	

TABLE 3. Case III: $D_{12}/D_{22} = 1, D_{11}/D_{22} = 1, \nu = 1/3, b/a = 0.5$.

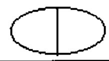
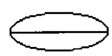
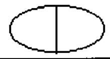
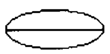
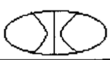
Eigenvalues	*Polygonal Approximation*	*Isoparametric Approximation*	*[19]*	*Nature of the Mode*	*Nodal Pattern*
$\lambda_{1,h}$	30.46	29.93	29.88	Doubly Symmetric	
$\lambda_{2,h}$	47.45	46.56	46.34	Symmetric Antisymmetric	
$\lambda_{3,h}$	68.04	66.73	66.09	Second Doubly Symmetric	
$\lambda_{4,h}$	76.10	74.77	74.56	Antisymmetric Symmetric	
$\lambda_{5,h}$	93.11	91.19	89.73	Second Symmetric Antisymmetric	

TABLE 4. Case IV: $D_{12}/D_{22} = 1, D_{11}/D_{22} = 1/3, \nu = 1/3, b/a = 0.5$.

Eigenvalues	***Polygonal Approximation***	***Isoparametric Approximation***	***[19]***	***Nature of the Mode***	***Nodal Pattern***
$\lambda_{1,h}$	29.93	29.41	29.38	Doubly Symmetric	
$\lambda_{2,h}$	44.91	44.09	43.87	Symmetric Antisymmetric	
$\lambda_{3,h}$	61.43	60.29	59.68	Second Doubly Symmetric	
$\lambda_{4,h}$	75.74	74.44	74.24	Antisymmetric Symmetric	
$\lambda_{5,h}$	80.37	78.74	77.29	Second Symmetric Antisymmetric	

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